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RESEARCH PAPER

NONLOCAL FRACTIONAL BOUNDARY VALUE PROBLEMS WITH SLIT-STRIPS BOUNDARY CONDITIONS

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Abstract

In this paper, we study nonlocal boundary value problems of fractional differential equations and inclusions with slit-strips integral boundary conditions. We show the existence and uniqueness of solutions for the single valued case (equations) by means of classical contraction mapping principle while the existence result is obtained via a fixed point theorem due to D. O'Regan. The existence of solutions for the multivalued case (inclusions) is established via nonlinear alternative for contractive maps. The results are well illustrated with the aid of examples.

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Key Words and Phrases: fractional differential equations, fractional differential inclusions, nonlocal boundary value problems, integral boundary conditions, fixed point theorem

1. Introduction

Fractional differential equations have been extensively investigated by several researchers in recent years. The sphere of study for these equations covers theoretical treatment, analytic and numerical methods of solutions, and applications in a variety of disciplines in physical and technical sciences. Examples include biophysics, blood flow phenomena, control theory, wave

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propagation, signal and image processing, viscoelasticity, percolation, identification, fitting of experimental data, economics, etc. [28, 20, 29, 31, 19]. As a matter of fact, fractional calculus has evolved into an interesting field of research and its tools have significantly improved the modelling techniques. It has been mainly due to the fact that fractional-order operators are nonlocal in nature and take into account the history of many important materials and processes. The literature on linear and nonlinear boundary value problems of fractional order, involving boundary conditions of diverse nature, is also quite enriched now. For some recent work on the topic, see [7, 22, 30, 14, 6, 1, 10, 4, 25, 15, 32] and the references therein.

Nonlocal Cauchy problems are regarded as more practical than the classical Cauchy problems with the initial conditions, see [9, 8]. Many kinds of nonlocal problems have been studied in the last few decades. The topic of nonlocal integral boundary conditions has also attracted a considerable attention. More recently, the authors [2] discussed the existence of solutions for fractional differential equations with slit-strips type integral boundary conditions. In the present work, we consider a modified version of the problem investigated in [2] by replacing the initial condition with a nonlocal boundary condition. Precisely, we study the following boundary value problem:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ t \in [0,1], \ 1 < q \leq 2, \\ x(0) = \mathfrak{h}(x), \\ x(\mu) = a \displaystyle\int_0^\alpha x(s)ds + b \int_\beta^1 x(s)ds, \ 0 < \alpha < \mu < \beta < 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\mathfrak{h} : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and a, b are real constants.

We emphasize that the integral boundary condition in (1.1) can be interpreted as the sum of the influences due to finite strips of arbitrary lengths is proportional to the value of the unknown function at an arbitrary position (nonlocal point) in the slit (a part of the boundary off the two strips), and the nonlocal term $\mathfrak{h}(x)$ in (1.1) may be understood as $\mathfrak{h}(x) = \sum_{j=1}^{p} \kappa_j x(t_j)$ where $\kappa_j, j = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \leq 1$. For some real world problems and engineering applications involving the strip conditions similar to the ones considered in the present study, we refer the reader to the works [18, 26, 5, 23].

As a second problem, we study the multivalued analogue of problem (1.1) given by

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \ t \in [0,1], \ 1 < q \leq 2, \\ x(0) = \mathfrak{h}(x), \\ x(\mu) = a\displaystyle\int_0^\alpha x(s)ds + b\int_\beta^1 x(s)ds, \ 0 < \alpha < \mu < \beta < 1, \end{cases} \tag{1.2}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $\mathfrak{h}(x)$ and a, b are the same as defined in the problem (1.1). For some recent work on fractional multivalued problems, see ([16, 12, 11, 3]) and the references therein.

The paper is organized as follows. In Section 2, we recall some basic definitions from fractional calculus and establish a lemma which plays a pivotal role in the sequel. Section 3 deals with the existence results for the problem (1.1) which are shown by applying Banach's contraction principle and a fixed point theorem due to D. O'Regan. In Section 4, we discuss the existence of solutions for the problem (1.2) by means of the nonlinear alternative for contractive maps.

2. **Preliminaries**

In this section, some basic definitions on fractional calculus and an auxiliary lemma are presented [28, 20].

Definition 2.1. The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \ q > 0,$$

provided the integral exists.

Definition 2.2. For at least n-times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \ n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

LEMMA 2.1. *For any $y \in C([0,1],\mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \ 1 < q \le 2, \ t \in [0,1] \\ x(0) = \mathfrak{h}(x), \\ x(\mu) = a\int_0^\alpha x(s)ds + b\int_\beta^1 x(s)ds, \ 0 < \alpha < \mu < \beta < 1, \end{cases} \tag{2.1}$$

is

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + \frac{t}{A}\Bigg\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} y(\tau)d\tau ds \\ &+ b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} y(\tau)d\tau ds - \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} y(s)d\ ds \Bigg\} \\ &+ \Big[1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big]\mathfrak{h}(x), \end{aligned} \tag{2.2}$$

where

$$A = \mu - \frac{a\alpha^2}{2} - \frac{b(1-\beta^2)}{2} \neq 0. \tag{2.3}$$

P r o o f. It is well known that the general solution of the fractional differential equation in (2.1) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds, \tag{2.4}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Applying the given boundary conditions, we find that $c_0 = \mathfrak{h}(x)$, and

$$\begin{aligned} c_1 &= \frac{1}{A}\Bigg\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} y(\tau)d\tau ds + b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} y(\tau)d\tau ds \\ &- \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} y(s)d\ ds - \{1 - a\alpha - b(1-\beta)\}\mathfrak{h}(x) \Bigg\}. \end{aligned}$$

Substituting the values of c_0, c_1 in (2.4), we get (2.2). This completes the proof. □

3. Existence results for single-valued problem

We denote by $\mathcal{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Also by $L^1([0,1],\mathbb{R})$ we denote the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

In view of Lemma 2.1, we define an operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &+ \frac{t}{A}\Bigg\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau,x(\tau))d\tau ds \\ &+ b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau,x(\tau))d\tau ds \\ &- \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \Bigg\} \\ &+ \Big[1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big]\mathfrak{h}(x) \ t \in [0,1].\end{aligned} \tag{3.1}$$

Let us define $\mathcal{P}_{1,2} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{P}_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &+ \frac{t}{A}\Bigg\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau,x(\tau))d\tau ds \\ &+ b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau,x(\tau))d\tau ds \\ &- \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \Bigg\},\end{aligned} \tag{3.2}$$

and

$$(\mathcal{P}_2x)(t) = \Big[1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big]\mathfrak{h}(x). \tag{3.3}$$

Clearly

$$(\mathcal{P}x)(t) = (\mathcal{P}_1x)(t) + (\mathcal{P}_2x)(t), \ \ t \in [0,1]. \tag{3.4}$$

For convenience we introduce the notations:

$$\eta := \frac{1}{\Gamma(q+1)} + \frac{1}{|A|}\Bigg\{ |a|\frac{\alpha^{q+1}}{\Gamma(q+2)} + |b|\frac{1-\beta^{q+1}}{\Gamma(q+2)} + \frac{\mu^q}{\Gamma(q+1)} \Bigg\}, \tag{3.5}$$

and

$$\delta := 1 + \frac{1}{|A|}\Big|1 - a\alpha - b(1-\beta)\Big|. \tag{3.6}$$

Theorem 3.1. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A_1) $|f(t,x)-f(t,y)| \le L\|x-y\|, \forall t \in [0,1],\ L>0,\ x,y \in \mathbb{R}$;
(A_2) $\mathfrak{h} : C([0,1],\mathbb{R}) \to \mathbb{R}$ *is a continuous function satisfying the condition:*

$$|\mathfrak{h}(u)-\mathfrak{h}(v)| \le \ell\|u-v\|,\ \ell < \delta^{-1},\ \forall\, u,v \in C([0,1],\mathbb{R}),\ \ell > 0;$$

(A_3) $\gamma := L\eta + \delta\ell < 1$.

Then the boundary value problem (1.1) has a unique solution.

P r o o f. For $x,y \in \mathcal{C}$ and for each $t \in [0,1]$, from the definition of $\mathcal{P}$ and assumptions (A_1) and (A_2), we obtain

$$\begin{aligned}
&|(\mathcal{P}x)(t)-(\mathcal{P}y)(t)| \\
\le\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds \\
&+\frac{t}{|A|}\Bigg\{|a|\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}|f(\tau,x(\tau))-f(\tau,y(\tau))|d\tau ds \\
&+|b|\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}|f(\tau,x(\tau))-f(\tau,y(\tau))|d\tau ds \\
&+\int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\Bigg\} \\
&+\left|1-\frac{t}{A}(1-a\alpha-b(1-\beta))\right||\mathfrak{h}(x)-\mathfrak{h}(y)| \\
\le\ & L\|x-y\|\Bigg[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{1}{|A|}\Bigg\{|a|\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}d\tau ds \\
&+|b|\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}d\tau ds + \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)}ds\Bigg\}\Bigg] \\
&+\left|1-\frac{t}{A}(1-a\alpha-b(1-\beta))\right|\ell\|x-y\| \\
\le\ & L\|x-y\|\left[\frac{1}{\Gamma(q+1)}+\frac{1}{|A|}\left\{|a|\frac{\alpha^{q+1}}{\Gamma(q+2)}+|b|\frac{1-\beta^{q+1}}{\Gamma(q+2)}+\frac{\mu^q}{\Gamma(q+1)}\right\}\right] \\
&+\left\{1+\frac{1}{|A|}\Big|1-a\alpha-b(1-\beta)\Big|\right\}\ell\|x-y\| \\
=\ & (L\eta+\delta\ell)\|x-y\|.
\end{aligned}$$

Hence

$$\|(\mathcal{P}x)-(\mathcal{P}y)\| \le \gamma\|x-y\|.$$

As $\gamma < 1$ by (A_3), the operator $\mathcal{P}$ is a contraction map from the Banach space $\mathcal{C}$ into itself. Hence the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Our next existence result relies on a fixed point theorem due to O'Regan in [24].

LEMMA 3.1. *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \to E$ is continuous and completely continuous and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) *F has a fixed point $u \in \bar{U}$; or*

(C2) *there exist a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.*

Let
$$\Omega_r = \{x \in C([0, 1], \mathbb{R}) : \|x\| < r\},$$
and denote the maximum number by
$$M_r = \max\{|f(t, x)| : (t, x) \in [0, 1] \times [-r, r]\}.$$

THEOREM 3.2. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $(A_1), (A_2)$ hold. In addition we assume that*

(A_4) $\mathfrak{h}(0) = 0$;

(A_5) *there exists a nonnegative function $p \in C([0, 1], \mathbb{R})$ and a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that*
$$|f(t, u)| \leq p(t)\psi(\|u\|) \text{ for any } (t, u) \in [0, 1] \times \mathbb{R};$$

(A_6) $\sup_{r \in (0, \infty)} \dfrac{r}{\eta\psi(r)\|p\|} > \dfrac{1}{1 - \delta\ell}$, *where η and δ are defined in (3.5) and (3.6) respectively.*

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

P r o o f. By the assumption (A_6), there exists a number $r_0 > 0$ such that
$$\frac{r_0}{\eta\psi(r_0)\|p\|} > \frac{1}{1 - \delta\ell}. \tag{3.7}$$
We shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by (3.2) and (3.3) respectively, satisfy all the conditions of Lemma 3.1.

Step 1. *The operator $\mathcal{P}_1$ is continuous and completely continuous.* We first show that $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$\begin{aligned}
\|\mathcal{P}_1 x\| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&+\frac{1}{|A|}\Bigg\{|a|\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}|f(\tau,x(\tau))|d\tau ds \\
&+|b|\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}|f(\tau,x(\tau))|d\tau ds \\
&+\int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Bigg\} \\
&\leq M_r\|p\|\Bigg[\frac{1}{\Gamma(q+1)}+\frac{1}{|A|}\Bigg\{|a|\frac{\alpha^{q+1}}{\Gamma(q+2)}+|b|\frac{1-\beta^{q+1}}{\Gamma(q+2)} \\
&+\frac{\mu^q}{\Gamma(q+1)}\Bigg\}\Bigg] \\
&= M_r\eta\|p\|.
\end{aligned}$$

Thus the operator $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is uniformly bounded. For any $t_1, t_2 \in [0,1], t_1 < t_2$, we have

$$\begin{aligned}
&|(\mathcal{P}_1 x)(t_2)-(\mathcal{P}_1 x)(t_1)| \\
\leq\; & \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]|f(s,x(s))|ds \\
&+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}|f(s,x(s))|ds \\
&+\frac{|t_2-t_1|}{|A|}\Bigg\{|a|\int_0^\alpha\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}|f(\tau,x(\tau))|d\tau ds \\
&+|b|\int_\beta^1\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}|f(\tau,x(\tau))|d\tau ds+\int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Bigg\} \\
\leq\; & \frac{M_r}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]ds+\frac{M_r}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}ds \\
&+\frac{M_r|t_2-t_1|}{|A|}\Bigg[\frac{1}{\Gamma(q+1)}+\frac{1}{|A|}\Bigg\{|a|\frac{\alpha^{q+1}}{\Gamma(q+2)}+|b|\frac{1-\beta^{q+1}}{\Gamma(q+2)}+\frac{\mu^q}{\Gamma(q+1)}\Bigg\}\Bigg],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{P}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ is uniformly valid on $[0,1]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,1] \times [-r_0, r_0]$, it follows that $\|f(t,x_n(t)) - f(t,x(t))\| \to 0$ is uniformly valid on $[0,1]$. Hence $\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{P}_1$. This completes the proof of Step 1.

Step 2. *The operator* $\mathcal{P}_2 : \bar{\Omega}_{r_0} \to C([0,1],\mathbb{R})$ *is contractive.* This is a consequence of (A_2). For $x, y \in C([0,1],\mathbb{R})$, we have

$$\begin{aligned}|\mathcal{P}_2 x(t) - \mathcal{P}_2 y(t)| &= \Big|1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big| |\mathfrak{h}(x) - \mathfrak{h}(y)| \\ &\le \Big\{1 + \frac{1}{|A|}\big|a(1-\alpha) - 1\big|\Big\} |\mathfrak{h}(x) - \mathfrak{h}(y)|, \\ &\le \delta\ell \|x - y\|,\end{aligned}$$

which, on taking supremum over $t \in [0,1]$, yields

$$\|\mathcal{P}_2 x - \mathcal{P}_2 y\| \le L_0 \|x - y\|, \quad L_0 = \delta\ell < 1.$$

This shows that $\mathcal{P}_2$ is a contraction as $L_0 < 1$.

Step 3. *The set* $\mathcal{P}(\bar{\Omega}_{r_0})$ *is bounded.* The assumptions (A_2) and (A_4) imply that

$$\|\mathcal{P}_2(x)\| \le \delta\ell r_0,$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{P}_1(\bar{\Omega}_{r_0})$ implies that the set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 3.1 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\lambda \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathcal{P} x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned}x(t) &= \lambda \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &\quad + \lambda \frac{t}{A} \Big\{ a \int_0^{\alpha} \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau, x(\tau)) d\tau ds \\ &\quad + b \int_{\beta}^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau, x(\tau)) d\tau ds \\ &\quad - \int_0^{\mu} \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \Big\} \\ &\quad + \lambda \Big[1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big] \mathfrak{h}(x), \ t \in [0,1].\end{aligned}$$

Using the assumptions $(A_4)-(A_6)$, we get

$$r_0 \leq \eta\psi(r_0)\|p\| + \delta\ell r_0.$$

Thus, we get a contradiction:

$$\frac{r_0}{\eta\psi(r_0)\|p\|} \leq \frac{1}{1-\delta\ell}.$$

Thus the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all the conditions of Lemma 3.1. Hence, the operator $\mathcal{P}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the problem (1.1). This completes the proof. □

Example 3.1. Consider the following fractional boundary value problem

$$\begin{cases} {}^cD^{8/5}x(t) = f(t, x(t), \ t \in [0,1], \\ x(0) = \dfrac{1}{4} + \dfrac{|x(1/5)|}{12(1+|x(1/5)|)}, \\ x\,(2/3) = \dfrac{1}{2}\displaystyle\int_0^{1/4} x(s)ds + \int_{3/4}^{1} x(s)ds. \end{cases} \tag{3.8}$$

Here, $q = 8/5$, $a = 1/2, b = 1$, $\alpha = 1/4$, $\mu = 2/3$ $\beta = 3/4$, and $\mathfrak{h}(x) = \dfrac{1}{4} + \dfrac{|x(1/5)|}{12(1+|x(1/5)|)}$. Obviously $|\mathfrak{h}(x) - \mathfrak{h}(y)| \leq \ell\|x-y\|$ with $\ell = 1/12$. Further, $A = 83/192$, $\eta \approx 1.881496, \delta \approx 2.445783$.

(a) Let us take

$$f(t,x) = \frac{1}{\sqrt{t^2+16}}\left(\cos x + \frac{1}{6}\tan^{-1}(x/2)\right) + \frac{t}{t^3+8} \tag{3.9}$$

in the problem (3.8). Then $|f(t,x) - f(t,y)| \leq L\|x-y\|$ with $L = 1/3$ and $\gamma = L\eta + \delta\ell \approx 0.830981 < 1$. Thus, the all the conditions of Theorem 3.1 with $f(t,x)$ given by (3.9) are satisfied and hence it follows by the conclusion of Theorem 3.1 that there exists a unique solution for the problem (3.8).

The next example is concerned with the illustration of Theorem 3.2.

(b) Consider problem (3.8) with

$$f(t,x) = \frac{1}{9}\left(\frac{1}{\sqrt{t+1}} + 1\right)\left(t + |x| + \frac{|\sin x|}{1+|\sin x|}\right), \tag{3.10}$$

and $\mathfrak{h}(x) = \dfrac{|x(1/5)|}{12(1+|x(1/5)|)}$. Clearly $|f(t,x)| \leq p(t)\psi(\|x\|)$, where

$$p(t) = \frac{1}{9}\left(\frac{1}{\sqrt{t+1}} + 1\right), \ \psi(\|x\|) = 2 + \|x\|.$$

Moreover, $\sup_{r\in(0,\infty)} \dfrac{r}{\eta\psi(r)\|p\|} \approx 2.391714$, $\dfrac{1}{1-\delta\ell} \approx 1.255990$, that is, the condition (A_6) holds. Hence the hypothesis of Theorem 3.2 holds. In consequence, by the conclusion of Theorem 3.2, the problem (3.8) with chosen values of $f(t,x)$ and $\mathfrak{h}(x)$ has a solution on $[0,1]$.

4. **Existence results for multivalued problem (1.2)**

Let us recall some basic definitions on multi-valued maps [13, 17]. For a normed space $(X,\|\cdot\|)$, let $P_{cl}(X)=\{Y\in\mathcal{P}(X): Y \text{ is closed}\}$, $P_b(X)=\{Y\in\mathcal{P}(X): Y \text{ is bounded}\}$, $P_{cp}(X)=\{Y\in\mathcal{P}(X): Y \text{ is compact}\}$, and $P_{cp,c}(X)=\{Y\in\mathcal{P}(X): Y \text{ is compact and convex}\}$. A multi-valued map $G: X\to\mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. The map G is bounded on bounded sets if $G(\mathbb{B})=\cup_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B}\in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y|: y\in G(x)\}\}<\infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0)\subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n\to x_*$, $y_n\to y_*$, $y_n\in G(x_n)$ imply $y_*\in G(x_*)$. G has a fixed point if there is $x\in X$ such that $x\in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G:[0;1]\to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y\in\mathbb{R}$, the function

$$t\longmapsto d(y,G(t))=\inf\{|y-z|: z\in G(t)\}$$

is measurable.

Definition 4.1. A function $x\in AC^2([0,1],\mathbb{R})$ is a solution of the problem (1.2) if $x(0)=\mathfrak{h}(x)$, $x(\mu)=a\int_0^\alpha x(s)ds+b\int_\beta^1 x(s)ds$, and there exists a function $f\in L^1([0,1],\mathbb{R})$ such that $f(t)\in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\Bigg\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \\ &+ b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds - \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s)ds \Bigg\} \\ &+ \Big[1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big]\mathfrak{h}(x). \end{aligned} \tag{4.1}$$

Here $AC^1([0,1],\mathbb{R})$ will denote the space of functions $x:[0,1]\to\mathbb{R}$ that are absolutely continuous and whose second derivative is absolutely continuous.

Definition 4.2. A multivalued map $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(*i*) $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$;
(*ii*) $x\longmapsto F(t,x)$ is upper semicontinuous for almost all $t\in[0,1]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(*iii*) for each $a>0$, there exists $\varphi_a\in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v\in F(t,x)\} \le \varphi_a(t)$$

for all $\|x\|\le a$ and for a. e. $t\in[0,1]$.

For each $y\in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v\in L^1([0,1],\mathbb{R}) : v(t)\in F(t,y(t)) \text{ for a.e. } t\in[0,1]\}.$$

The following lemma will be used in the sequel.

Lemma 4.1. ([21]) *Let X be a Banach space. Let $F:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta\circ S_F : C([0,1],X)\to\mathcal{P}_{cp,c}(C([0,1],X)),\ x\mapsto(\Theta\circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X)\times C([0,1],X)$.

To prove our main result in this section, we use the following form of the Nonlinear Alternative for contractive maps [27, Corollary 3.8].

Theorem 4.1. *Let X be a Banach space, and D a bounded neighborhood of $0\in X$. Let $Z_1 : X\to\mathcal{P}_{cp,c}(X)$ and $Z_2:\bar{D}\to\mathcal{P}_{cp,c}(X)$ two multi-valued operators satisfying*

(a) *Z_1 is contraction, and*
(b) *Z_2 is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$ or*
(ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

THEOREM 4.2. *Assume that (A_2) holds. In addition we suppose that:*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*
(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,1]\times\mathbb{R};$$

(H_3) *there exists a number $M > 0$ such that*

$$\frac{(1-\delta\ell)M}{\psi(M)\eta\|p\|} > 1, \tag{4.2}$$

where η, δ are defined in (3.5) and (3.6) respectively.

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

P r o o f. To transform the problem (1.2) to a fixed point, we introduce an operator $\mathcal{N} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ +\frac{t}{A}\Big\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \\ \displaystyle +b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \\ -\int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s)ds \Big\} \\ \displaystyle +\Big[1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big]\mathfrak{h}(x), \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Now, we define two operators $\mathcal{A}_1 : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\mathcal{A}_1 x(t) = \Big[1 - \frac{t}{A}(1 - a\alpha - b(1-\beta))\Big]\mathfrak{h}(x), \tag{4.3}$$

and a multi-valued operator $\mathcal{A}_2 : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\mathcal{A}_2(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle +\frac{t}{A}\left\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \right. \\ \displaystyle +b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \\ \displaystyle \left. -\int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s)ds \right\}. \end{array} \right. \end{array} \right\} \qquad (4.4)$$

Observe that $\mathcal{N} = \mathcal{A}_1 + \mathcal{A}_2$. We shall show that the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all the conditions of Theorem 4.1 on $[0,1]$. The proof consists of several steps and claims.

Step 1: We show that $\mathcal{A}_1$ *is a contraction on* $C([0,1],\mathbb{R})$. The proof is similar to the one for operator $\mathcal{P}_2$ in Step 2 of Theorem 3.2.

Step 2: $\mathcal{A}_2$ *is compact and convex valued and it is completely continuous.* This will be established in several claims.

Claim I: $\mathcal{A}_2$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. Let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \mathcal{A}_2(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\left\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \right. \\ &\quad \left. +b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds - \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s)ds \right\}. \end{aligned}$$

Then, for $t \in [0,1]$, we have

$$\begin{aligned} &|h(t)| \\ \le\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)|ds + \frac{1}{|A|}\left\{ |a|\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} |f(\tau)|d\tau ds \right. \\ & \left. +|b|\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} |f(\tau)|d\tau ds + \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} |f(s)|ds \right\} \\ \le\ & \psi(\|x\|)\|p\| \left[\frac{1}{\Gamma(q+1)} + \frac{1}{|A|}\left\{ |a|\frac{\alpha^{q+1}}{\Gamma(q+2)} + |b|\frac{1-\beta^{q+1}}{\Gamma(q+2)} + \frac{\mu^q}{\Gamma(q+1)} \right\} \right]. \end{aligned}$$

Thus,

$$\|h\| \le \psi(r)\eta\|p\|.$$

Claim II: *$\mathcal{A}_2$ maps bounded sets into equi-continuous sets.* Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\rho$. Then, for each $h \in \mathcal{A}_2(x)$, we obtain

$$\begin{aligned} |h(t_2) - h(t_1)| \le\ & \frac{\psi(\|x\|)\|p\|}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds \\ & + \frac{\psi(\|x\|)\|p\|}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\ & + \frac{\psi(\|x\|)\|p\||t_2 - t_1|}{|A|} \left[\frac{1}{\Gamma(q+1)} + \frac{1}{|A|} \left\{ |a| \frac{\alpha^{q+1}}{\Gamma(q+2)} \right.\right. \\ & \left.\left. + |b| \frac{1 - \beta^{q+1}}{\Gamma(q+2)} + \frac{\mu^q}{\Gamma(q+1)} \right\} \right]. \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t_2 - t_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{A}_2 : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ is completely continuous.
Claim III: *$\mathcal{A}_2$ has a closed graph.* Let $x_n \to x_*, h_n \in \mathcal{A}_2(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{A}_2(x_*)$. Associated with $h_n \in \mathcal{A}_2(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) =\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s) ds + \frac{t}{A} \left\{ a \int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f_n(\tau) d\tau ds \right. \\ & \left. + b \int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau) d\tau ds - \int_0^\mu \frac{(\mu - s)^{q-1}}{\Gamma(q)} f_n(s) ds \right\}. \end{aligned}$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) =\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s) ds + \frac{t}{A} \left\{ a \int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f_*(\tau) d\tau ds \right. \\ & \left. + b \int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau) d\tau ds - \int_0^\mu \frac{(\mu - s)^{q-1}}{\Gamma(q)} f_*(s) ds \right\}. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\Bigg\{ a\int_0^{\alpha}\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \\ &\quad + b\int_{\beta}^1\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds - \int_0^{\mu}\frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s)ds \Bigg\}. \end{aligned}$$

Observe that

$$\begin{aligned} & \|h_n(t) - h_*(t)\| \\ = & \Bigg\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s) - f_*(s))ds \\ & + \frac{t}{A}\Bigg\{ a\int_0^{\alpha}\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)}(f_n(\tau) - f_*(\tau))d\tau ds \\ & + b\int_{\beta}^1\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds - \int_0^{\mu}\frac{(\mu-s)^{q-1}}{\Gamma(q)}(f_n(s) - f_*(s))ds \Bigg\} \Bigg\| \end{aligned}$$

$\to 0$, as $n \to \infty$.

Thus, it follows by Lemma 4.1 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds + \frac{t}{A}\Bigg\{ a\int_0^{\alpha}\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f_*(\tau)d\tau ds \\ &\quad + b\int_{\beta}^1\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds - \int_0^{\mu}\frac{(\mu-s)^{q-1}}{\Gamma(q)} f_*(s)ds \Bigg\}, \end{aligned}$$

for some $f_* \in S_{F,x_*}$. Hence $\mathcal{A}_2$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{A}_2$ is compact valued.

Thus the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all the conditions of Theorem 4.1 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathcal{A}_1(x) + \lambda\mathcal{A}_2(x)$

for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$\begin{aligned} x(t) &= \lambda \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \lambda \frac{t}{A}\bigg\{ a\int_0^\alpha \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds \\ &+b\int_\beta^1 \int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau ds - \int_0^\mu \frac{(\mu-s)^{q-1}}{\Gamma(q)} f(s)ds\bigg\} \\ &+\lambda\Big[1-\frac{t}{A}(1-a\alpha-b(1-\beta))\Big]\mathfrak{h}(x),\ t\in[0,1]. \end{aligned}$$

Following the method for proof of Claim I, we can obtain

$$\begin{aligned} |x(t)| &\le \psi(\|x\|)\|p\|\bigg[\frac{1}{\Gamma(q+1)} + \frac{1}{|A|}\bigg\{|a|\frac{\alpha^{q+1}}{\Gamma(q+2)} + |b|\frac{1-\beta^{q+1}}{\Gamma(q+2)} \\ &+\frac{\mu^q}{\Gamma(q+1)}\bigg\}\bigg] + \Big\{1+\frac{1}{|A|}\big|a(1-\alpha)-1\big|\Big\}\ell\|x\|. \end{aligned}$$

Thus

$$\|x\| \le \psi(\|x\|)\eta\|p\| + \delta\ell\|x\|. \tag{4.5}$$

If condition (ii) of Theorem 4.1 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_r$ with $x = \lambda\mathcal{N}(x)$. Then, x is a solution of (1.2) with $\|x\| = M$. Now, by the inequality (4.5), we get

$$\frac{(1-\delta\ell)M}{\psi(M)\eta\|p\|} \le 1$$

which contradicts (4.2). Hence, $\mathcal{N}$ has a fixed point in $[0,1]$ by Theorem 4.1, and consequently the problem (1.2) has a solution. This completes the proof. □

Example 4.1. Consider the following boundary value problem

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t,x),\ 0<t<1, \\ x(0) = \dfrac{1}{4} + \dfrac{|x(1/5)|}{12(1+|x(1/5)|)}, \\ x\,(2/3) = \dfrac{1}{2}\displaystyle\int_0^{1/4} x(s)ds + \int_{3/4}^1 x(s)ds. \end{cases} \tag{4.6}$$

Here, $q = 8/5,\ a = 1/2, b = 1,\ \alpha = 1/4,\ \mu = 2/3\ \beta = 3/4$, and $\mathfrak{h}(x) = \dfrac{1}{4} + \dfrac{|x(1/5)|}{12(1+|x(1/5)|)}$. Obviously $|\mathfrak{h}(x)-\mathfrak{h}(y)| \le \ell\|x-y\|$ with $\ell = 1/12$. Further, $A = 83/192,\ \eta \approx 1.881496, \delta \approx 2.445783$. Let $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t,x) = \left[\frac{1}{12}\frac{\sin^3 x}{(\sin^3 x + 3)} + \frac{1}{24}(t+1), \frac{1}{2}\cos x\right].$$

For $f \in F$, we have

$$|f| \le \max\left[\frac{1}{12}\frac{\sin^3 x}{(\sin^3 x + 3)} + \frac{1}{24}(t+1), \frac{1}{2}\cos x\right] \le \frac{1}{2}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|),\ \ x \in \mathbb{R},$$

with $p(t) = 1,\ \psi(\|x\|) = 1/2$. By the condition (H_3), we find that $M > M_1,\ M_1 \approx 1.81570$. Clearly, all the conditions of Theorem 4.2 are satisfied and hence the problem (4.6) has at least one solution on $[0,1]$.

References

[1] R.P. Agarwal, D. O'Regan, S. Stanek, Positive solutions for mixed problems of singular fractional differential equations. *Math. Nachr.* **285** (2012), 27–41.

[2] B. Ahmad, R.P. Agarwal, Some new versions of fractional boundary value problems with slit-strips conditions. *Bound. Value Probl.* **2014** (2014), 175.

[3] B. Ahmad, S.K. Ntouyas, Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2013** (2013), Article No 20, 19 pp.

[4] B. Ahmad, S.K. Ntouyas, A. Alsaedi, A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions. *Math. Probl. Eng.* **2013** (2013), Article ID 320415, 9 pp.

[5] S. Asghar, T. Hayat, B. Ahmad, Acoustic diffraction from a slit in an infinite absorbing sheet. *Japan J. Indust. Appl. Math.* **13** (1996), 519–532.

[6] Z.B. Bai, W. Sun, Existence and multiplicity of positive solutions for singular fractional boundary value problems. *Comput. Math. Appl.* **63** (2012), 1369–1381.

[7] D. Baleanu, O.G. Mustafa, R.P. Agarwal, On L^p-solutions for a class of sequential fractional differential equations. *Appl. Math. Comput.* **218** (2011), 2074–2081.

[8] L. Byszewski, Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **162** (1991), 494–505.

[9] L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Appl. Anal.* **40** (1991), 11–19.

[10] A. Cabada, G. Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. *J. Math. Anal. Appl.* **389** (2012), 403–411.

[11] A. Cernea, On the existence of solutions for nonconvex fractional hyperbolic differential inclusions. *Commun. Math. Anal.* **9**, No 1 (2010), 109–120.

[12] Y.-K. Chang, J.J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions. *Math. Comput. Modelling* **49** (2009), 605–609.

[13] K. Deimling, *Multivalued Differential Equations*. Walter De Gruyter, Berlin-New York, 1992.

[14] J.R. Graef, L. Kong, Q. Kong, Application of the mixed monotone operator method to fractional boundary value problems. *Fract. Differ. Calc.* **2** (2011), 554–567.

[15] J.R. Graef, L. Kong, M. Wang, Existence and uniqueness of solutions for a fractional boundary value problem on a graph. *Fract. Calc. Appl. Anal.* **17**, No 2 (2014), 499–510; DOI: 10.2478/s13540-014-0182-4; http://link.springer.com/article/10.2478/s13540-014-0182-4.

[16] J. Henderson, A. Ouahab, Fractional functional differential inclusions with finite delay. *Nonlinear Anal.* **70** (2009), 2091–2105.

[17] Sh. Hu, N. Papageorgiou, *Handbook of Multivalued Analysis, Theory*, I. Kluwer, Dordrecht, 1997.

[18] R.A. Hurd, Y. Hayashi, Low-frequency scattering by a slit in a conducting plane. *Radio Sci.* **15** (1980), 1171–1178.

[19] V. Keyantuo, C. Lizama, A characterization of periodic solutions for time-fractional differential equations in UMD spaces and applications. *Math. Nach.* **284** (2011), 494–506.

[20] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[21] A. Lasota, Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781–786.

[22] S. Liang, J. Zhang, Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval. *Math. Comput. Modelling* **54** (2011), 1334–1346.

[23] S.P. Lipshitz, T.C. Scott, B. Salvy, On the acoustic impedance of baffled strip radiators. *J. Audio Eng. Soc.* **43** (1995), 573–580.

[24] D. O'Regan, Fixed-point theory for the sum of two operators. *Appl. Math. Lett.* **9** (1996), 1–8.
[25] D. O'Regan, S. Stanek, Fractional boundary value problems with singularities in space variables. *Nonlinear Dynam.* **71** (2013), 641–652.
[26] T. Otsuki, Diffraction by multiple slits. *JOSA A* **7** (1990), 646–652.
[27] W.V. Petryshyn, P. M. Fitzpatric, A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps. *Trans. Amer. Math. Soc.* **194** (1974), 1–25.
[28] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego, 1999.
[29] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering.* Springer, Dordrecht, 2007.
[30] X. Su, Solutions to boundary value problem of fractional order on unbounded domains in a Banach space. *Nonlinear Anal.* **74** (2011), 2844–2852.
[31] Z. Tomovski, R. Hilfer, H.M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. *Integral Transforms Spec. Funct.* **21** (2010), 797–814.
[32] G. Wang, S. Liu, L. Zhang, Eigenvalue problem for nonlinear fractional differential equations with integral boundary conditions. *Abstr. Appl. Anal.* **2014** (2014), Article ID 916260, 6 pp.

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