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Lebesgue points and Cesàro summability of higher dimensional Fourier series over a cone

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To the memory of Professor László Leindler

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Abstract. We introduce a new concept of Lebesgue points, the so-called ω-Lebesgue points, where $\omega > 0$. As a generalization of the classical Lebesgue's theorem, we prove that the Cesàro means $\sigma_n^\alpha f$ of the Fourier series of a multi-dimensional function $f \in L_1(\mathbb{T}^d)$ converge to f at each ω-Lebesgue point $(0 < \omega < \alpha)$ as $n \to \infty$.

1. Introduction

It was proved by Lebesgue [10] that the Fejér means [2] of the trigonometric Fourier series of a one-dimensional integrable function converge almost everywhere to the function, i.e.,

$$\sigma_n f(x) := \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n}\Big) \widehat{f}(k) e^{\imath kx} \to f(x)$$

for almost every $x \in \mathbb{T}$, where $\mathbb{T}$ denotes the torus and $\widehat{f}(k)$ is the kth Fourier coefficient. More exactly, Lebesgue [10] introduced the concept of the so-called Lebesgue points and verified that almost every point is a Lebesgue point and the preceding convergence holds at every Lebesgue point. The set of Lebesgue points contains all continuity points of f. Some years later M. Riesz [15] generalized this theorem for the Cesàro means of one-dimensional integrable functions.

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In the two-dimensional case Marcinkiewicz and Zygmund [12] proved that the Fejér means

$$\sigma_{n_1,n_2} f(x,y) = \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \Big(1-\frac{|k_1|}{n_1}\Big)\Big(1-\frac{|k_2|}{n_2}\Big) \widehat{f}(k_1,k_2) e^{\imath k_1 x_1} e^{\imath k_2 x_2}$$

of a function $f \in L_1(\mathbb{T}^2)$ converge almost everywhere to f as $n \to \infty$, provided that n is in a cone, i.e., $\tau^{-1} \le n_1/n_2 \le \tau$ for some $\tau \ge 1$.

In this paper, we generalize these results to Cesàro means of multi-dimensional functions and partly characterize the set of convergence. The Cesàro summability was investigated in a great number of papers and books (see, e.g., Leindler [11], Gát [4–6], Goginava [7–9], Simon [16, 17], Nagy, Persson, Tephnadze and Wall [13, 14], Weisz [20, 21] and Zygmund [24]). We generalize the Lebesgue points and introduce the so-called ω-Lebesgue points, where $\omega > 0$. It is known that almost every point is an ω-Lebesgue point of $f \in L_1(\mathbb{T}^d)$ and if f is continuous at x, then x is also an ω-Lebesgue point of f. We introduce a new maximal function $\mathcal{M}^\omega f$ and show that the Cesàro means $\sigma_n^\alpha f$ of $f \in L_1(\mathbb{T}^d)$ can be estimated by $\mathcal{M}^\omega f$ pointwise. Next we prove that if $\mathcal{M}^\omega f(x)$ is finite and x is an ω-Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x), \tag{1.1}$$

whenever n is in a cone. This implies the convergence of the Cesàro means almost everywhere.

After I have submitted the paper, the reviewer called my attention to the paper of Gabisoniya [3]. There he introduced another concept of Lebesgue points for functions of two variables. However, not every continuity point of f is a Lebesgue point of f (see Remark 1). This means that our definition is different from the definition of Gabisoniya. In the two-dimensional case, he proved in [3] that almost every point is a Lebesgue point of $f \in L_1(\mathbb{T}^2)$ as well as the convergence (1.1).

2. Maximal functions and Lebesgue points

Let us fix $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d times. We briefly write $L_p^\omega(\mathbb{X}^d)$ $(\omega \ge 0)$ instead of the weighted Lebesgue space $L_p^\omega(\mathbb{X}^d, \lambda)$ equipped with the norm

$$\|f\|_{L_p^\omega(\mathbb{X}^d)} := \Big(\int_{\mathbb{X}^d} |f(x)(1+|x|)^\omega|^p \, dx \Big)^{1/p} \qquad (1 \le p < \infty),$$

with the usual modification for $p = \infty$ and with $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{T}$, where λ is the Lebesgue measure and $\mathbb{T} = [-\pi, \pi]$ is the torus. If $\omega = 0$, then we get back the usual $L_p(\mathbb{R}^d)$ spaces. Clearly, $L_p(\mathbb{R}^d) \supset L_p^\omega(\mathbb{R}^d)$.

For some $\omega > 0$ and $f \in L_1(\mathbb{T}^d)$, we define the Hardy–Littlewood maximal function

$$\mathcal{M}^\omega f(x) := \sup_{i\in\mathbb{N}^d,\ h>0} \frac{2^{-\omega\|i\|_1}}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-t)|\,dt.$$

If $\omega = 0$, we obtain the strong Hardy–Littlewood maximal function. Moreover, if $\omega = 0$ and $i_1 = \cdots = i_d$, then the usual Hardy–Littlewood maximal function.

In [22], we proved the next two inequalities:

$$\sup_{\rho>0} \rho\lambda(\mathcal{M}^\omega f > \rho) \le C\|f\|_{L_1(\mathbb{T}^d)} \qquad (f \in L_1(\mathbb{T}^d)) \tag{2.1}$$

and, for $1 < p \le \infty$,

$$\|\mathcal{M}^\omega f\|_p \le C_p\|f\|_{L_p(\mathbb{T}^d)} \qquad (f \in L_p(\mathbb{T}^d)). \tag{2.2}$$

In this paper the constants C and C_p may vary from line to line.

Based on the definition of $\mathcal{M}^\omega$, let

$$U_r^\omega f(x) := \sup_{\substack{i\in\mathbb{N}^d,\ h>0 \\ 2^{i_k}h<r,\ k=1,\dots,d}} \frac{2^{-\omega\|i\|_1}}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-t)-f(x)|\,dt.$$

For $\omega > 0$, a point $x \in \mathbb{T}^d$ is called an ω-Lebesgue point of $f \in L_1(\mathbb{T}^d)$ if

$$\lim_{r\to 0} U_r^\omega f(x) = 0.$$

Different versions of Lebesgue points were considered in Gabisoniya [3] and Skopina [18, 19] for two dimensions. If $\omega = 0$, then this definition is equivalent to the strong Lebesgue points, i.e.,

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)-f(x)|\,dt = 0.$$

If in addition $i_1 = \cdots = i_d$, then it is equivalent to the usual Lebesgue points, i.e.,

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t)-f(x)|\,dt = 0.$$

For the concept of the usual Lebesgue and strong Lebesgue points, see, e.g., Feichtinger and Weisz [1] and the references therein. Every ω_2-Lebesgue point is an ω_1-Lebesgue point ($0 < \omega_2 < \omega_1 < \infty$), because $U_r^{\omega_1} f \le U_r^{\omega_2} f$. Obviously, if f is continuous at x, then x is an ω-Lebesgue point of f. The next theorem was proved in [22].

Theorem 1. *For $\omega > 0$, almost every point $x \in \mathbb{T}^d$ is an ω-Lebesgue point of $f \in L_1(\mathbb{T}^d)$.*

3. Restricted Cesàro summability

For $\alpha \neq -1, -2, \dots$ and $n \in \mathbb{N}$, let

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

Then $A_0^\alpha = 1$, $A_n^0 = 1$ and $A_n^1 = n+1$ $(n \in \mathbb{N})$. The kth Fourier coefficient of a d-dimensional integrable function $f \in L_1(\mathbb{T}^d)$ is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-\imath k \cdot x}\, dx \qquad (k \in \mathbb{Z}^d),$$

where $u \cdot x := \sum_{k=1}^d u_k x_k$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$. To obtain better convergence properties, we consider Cesàro summability.

Let $f \in L_1(\mathbb{T}^d)$, $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}_+^d$. The nth rectangular Cesàro means $\sigma_n^\alpha f$ of the Fourier series of f and the Cesàro kernel K_n^α are introduced by

$$\sigma_n^\alpha f(x) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1| \le n_1} \cdots \sum_{|k_d| \le n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha \widehat{f}(k) e^{\imath k \cdot x}$$

and

$$K_n^\alpha(t) := \frac{1}{\prod_{i=1}^d A_{n_i-1}^\alpha} \sum_{|k_1| \le n_1} \cdots \sum_{|k_d| \le n_d} \prod_{i=1}^d A_{n_i-1-|k_i|}^\alpha e^{\imath k \cdot t},$$

respectively. It is easy to see that

$$\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x-t) K_n^\alpha(t)\, dt \quad \text{and} \quad K_n^\alpha = K_{n_1}^{\alpha_1} \otimes \cdots \otimes K_{n_d}^{\alpha_d},$$

where the functions $K_{n_i}^{\alpha_i}$ are the one-dimensional Cesàro or (C, α) kernels. If $\alpha_i = 1$ for all i, then we get back the rectangular Fejér means.

For the one-dimensional Cesàro kernels, it is known (see Zygmund [24]) that

$$K_n^\alpha(t) \le C \min\left(n, \frac{1}{n^\alpha |t|^{\alpha+1}}\right) \tag{3.1}$$

and $\sup_{n \in \mathbb{N}} \int_{\mathbb{T}} |K_n^\alpha|\, d\lambda \le C$, where $n \in \mathbb{N}$, $0 < \alpha \le 1$ and $t \in (-\pi, \pi)$. In this paper, we study the convergence of $\sigma_n^\alpha f$ over a cone and the corresponding restricted maximal operator

$$\sigma_\square^\alpha f := \sup_{n \in \mathbb{R}_\tau^d} |\sigma_n^\alpha f|,$$

where $\tau \ge 1$ is fixed and the cone is given by

$$\mathbb{R}_\tau^d := \{x \in \mathbb{R}_+^d : \tau^{-1} \le x_i / x_j \le \tau, i, j = 1, \dots, d\}.$$

4. Restricted convergence at Lebesgue points

For $\omega \geq 0$, the weighted Herz space $E_\infty^\omega(\mathbb{R}^d)$ contains all functions f for which

$$\|f\|_{E_\infty^\omega} := \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} 2^{(k_1+\cdots+k_d)(\omega+1)} \|f 1_{Q_k}\|_\infty < \infty,$$

where $Q_k := Q_{k_1} \times \cdots \times Q_{k_d}$ $(k \in \mathbb{N}^d)$ and

$$Q_i = \{x \in \mathbb{R} : 2^{i-1}\pi \leq |x| < 2^i \pi\} \qquad (i \in \mathbb{N}_+), \qquad Q_0 := (-\pi, \pi).$$

If $\omega = 0$, we get back the usual Herz spaces. Obviously, $L_1(\mathbb{R}^d) \supset L_1^\omega(\mathbb{R}^d) \supset E_\infty^\omega(\mathbb{R}^d)$. In the next proofs, we will use the functions

$$h^{\alpha_j}(t) := \min\{1, |t|^{-\alpha_j - 1}\} \ (t \in \mathbb{R}) \ \text{ and } \ h^\alpha := h^{\alpha_1} \otimes \cdots \otimes h^{\alpha_d}.$$

We get from (3.1) that

$$\frac{1}{n_j}\left|\left(1_{(-\pi,\pi)} K_{n_j}^{\alpha_j}\right)\left(\frac{t}{n_j}\right)\right| \leq \frac{C}{n_j} \min\left\{n_j, \frac{n_j}{|t|^{\alpha_j+1}}\right\} = C h^{\alpha_j}(t) \qquad (t \in \mathbb{R}). \tag{4.1}$$

It is easy to see that

$$\|h^\alpha\|_{E_\infty^\omega(\mathbb{R}^d)} = \prod_{j=1}^{d} \|h^{\alpha_j}\|_{E_\infty^\omega(\mathbb{R})} \leq C_\alpha, \tag{4.2}$$

whenever $\omega < \min(\alpha_j, j = 1, \ldots, d)$. First we will estimate pointwise the restricted maximal operator by the maximal function $\mathcal{M}^\omega f$.

Theorem 2. *Suppose that $0 < \omega < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. For all $f \in L_1(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$, $\sigma_\square^\alpha f(x) \leq C \mathcal{M}^\omega f(x)$.*

Proof. Observe that

$$\begin{aligned} |\sigma_n^\alpha f(x)| &= \frac{1}{(2\pi)^d}\left|\int_{\mathbb{R}^d} f(x-t)\left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\, dt\right| \\ &= \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t)| |(1_{(-\pi,\pi)^d} K_n^\alpha)(t)|\, dt, \end{aligned}$$

where $Q_i(n_j) := \{x \in \mathbb{R} : 2^{i-1}\pi/n_j \leq |x| < 2^i\pi/n_j\}$ $(i \in \mathbb{N}_+)$ and $Q_0(n_j) := (-\pi/n_j, \pi/n_j)$. Then

$$\begin{aligned} |\sigma_n^\alpha f(x)| \leq \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t)|\, dt \times \\ \times \sup_{t \in Q_{k_1}(n_1) \times \cdots \times Q_{k_d}(n_d)} \left|\left(1_{(-\pi,\pi)^d} K_n^\alpha\right)(t)\right| \end{aligned}$$

$$= \frac{1}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \int_{Q_{k_1}(n_1)} \cdots \int_{Q_{k_d}(n_d)} |f(x-t)|\, dt \times$$
$$\times \sup_{t \in Q_{k_1} \times \cdots \times Q_{k_d}} \left| \left(1_{(-\pi,\pi)^d} K_n^\alpha\right) \left(\frac{t_1}{n_1}, \ldots, \frac{t_d}{n_d}\right)\right|.$$

Choose $s \in \mathbb{N}$ such that $2^{s-1} < \tau \le 2^s$. Using the fact $n \in \mathbb{R}_\tau^d$ and (4.1), we conclude

$$|\sigma_n^\alpha f(x)| \le \frac{\prod_{j=1}^d n_j}{(2\pi)^d} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \int_{-2^{k_1+s}\pi/n_1}^{2^{k_1+s}\pi/n_1} \cdots \int_{-2^{k_d+s}\pi/n_1}^{2^{k_d+s}\pi/n_1} |f(x-t)|\, dt \times \qquad (4.3)$$
$$\times \|h^\alpha 1_{Q_k}\|_\infty.$$

Consequently,

$$|\sigma_n^\alpha f(x)| \le C \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} 2^{(k_1+\cdots+k_d)(1+\omega)} \mathcal{M}^\omega f(x) \sup_{t \in Q_k} |h^\alpha(t)|$$
$$= C \|h^\alpha\|_{E_\infty^\omega(\mathbb{R}^d)} \mathcal{M}^\omega f(x).$$

Inequality (4.2) finishes the proof. ∎

Inequalities (2.1) and (2.2) imply

Corollary 1. *Suppose that $0 < \omega < \alpha_j \le 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\mathbb{T}^d)$, then*

$$\sup_{\rho > 0} \rho \lambda(\sigma_\square^\alpha f > \rho) \le C \|f\|_{L_1(\mathbb{T}^d)}.$$

If $1 < p \le \infty$ and $f \in L_p(\mathbb{T}^d)$, then

$$\|\sigma_\square^\alpha f\|_p \le C_p \|f\|_{L_p(\mathbb{T}^d)}.$$

The usual density argument due to Marcinkiewicz and Zygmund [12] implies

Corollary 2. *Suppose that $0 < \omega < \alpha_j \le 1$ for all $j = 1, \ldots, d$. If $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n \to \infty,\, n \in \mathbb{R}_\tau^d} \sigma_n^\alpha f = f \qquad a.e.$$

Now we partly characterize the set of convergence.

Theorem 3. *Suppose that $0 < \omega < \alpha_j \le 1$ for all $j = 1, \ldots, d$. If $\mathcal{M}^\omega f(x)$ is finite and x is an ω-Lebesgue point of $f \in L_1(\mathbb{T}^d)$, then*

$$\lim_{n \to \infty,\, n \in \mathbb{R}_\tau^d} \sigma_n^\alpha f(x) = f(x).$$

Proof. Since

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} K_n^\alpha(t)\,dt = 1,$$

we have

$$\begin{aligned}|\sigma_n^\alpha f(x) - f(x)| &\le \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |f(x-t)-f(x)|\left|\left(1_{(-\pi,\pi)^d}K_n^\alpha\right)(t)\right|\,dt\\ &= A_1(x)+A_2(x),\end{aligned}$$

where

$$A_1(x) := \frac{1}{(2\pi)^d}\sum_{k_1=0}^{r_0}\cdots\sum_{k_d=0}^{r_0}\int_{Q_{k_1}(n_1)}\cdots\int_{Q_{k_d}(n_d)}|f(x-t)-f(x)||(1_{(-\pi,\pi)^d}K_n^\alpha)(t)|\,dt,$$

and

$$A_2(x) := \frac{1}{(2\pi)^d}\sum_{\pi_1,\ldots,\pi_d}\sum_{k_{\pi_1}=r_0+1}^{\infty}\cdots\sum_{k_{\pi_j}=r_0+1}^{\infty}\sum_{k_{\pi_{j+1}}=0}^{\infty}\cdots\sum_{k_{\pi_d}=0}^{\infty}$$
$$\int_{Q_{k_1}(n_1)}\cdots\int_{Q_{k_d}(n_d)}|f(x-t)-f(x)|\left|\left(1_{(-\pi,\pi)^d}K_n^\alpha\right)(t)\right|\,dt,$$

where $\{\pi_1,\ldots,\pi_d\}$ is a permutation of $\{1,\ldots,d\}$ and $1\le j\le d$.

Since x is an ω-Lebesgue point of f, we can fix a number $r<1$ such that $U^\omega_{r2^s\pi}f(x)<\epsilon$, where $2^{s-1}<\tau\le 2^s$. Let us denote by r_0 the largest number i, for which $r/2\le 2^i/n_1<r$. Observe that $n\in\mathbb{R}^d_\tau$ and $k_j\le r_0$ imply $2^{k_j+s}/n_1\le 2^{r_0+s}/n_1<r2^s$ $(j=1,\ldots,d)$. Denoting

$$G(u) := \int_{-u_1}^{u_1}\cdots\int_{-u_d}^{u_d}|f(x-t)-f(x)|\,dt \qquad (u\in\mathbb{R}^d_+),$$

we get that

$$2^{-\omega(k_1+\cdots+k_d)}n_1^d\frac{G(2^{k_1+s}\pi/n_1,\ldots,2^{k_d+s}\pi/n_1)}{(2\pi)^d2^{sd}2^{k_1+\cdots+k_d}}\le U^\omega_{r2^s\pi}f(x).$$

As in (4.3),

$$\begin{aligned}A_1(x) \le C\sum_{k_1=0}^{r_0}\cdots\sum_{k_d=0}^{r_0}&\int_{-2^{k_1+s}\pi/n_1}^{2^{k_1+s}\pi/n_1}\cdots\int_{-2^{k_d+s}\pi/n_1}^{2^{k_d+s}\pi/n_1}|f(x-t)-f(x)|\,dt\times\\ &\times\Big(\prod_{j=1}^d n_j\Big)\|h^\alpha 1_{Q_k}\|_\infty\end{aligned}$$

$$\leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} G(2^{k_1+s}\pi/n_1, \ldots, 2^{k_d+s}\pi/n_1) \Big(\prod_{j=1}^{d} n_j \Big) \|h^\alpha 1_{Q_k}\|_\infty$$

$$\leq C \sum_{k_1=0}^{r_0} \cdots \sum_{k_d=0}^{r_0} 2^{(k_1+\cdots+k_d)(\omega+1)} n_1^{-d} U^\omega_{r2^s\pi} f(x) \Big(\prod_{j=1}^{d} n_j \Big) \|h^\alpha 1_{Q_k}\|_\infty.$$

Since $n \in \mathbb{R}^d_\tau$, we conclude $A_1(x) \leq C\epsilon \|h^\alpha\|_{E^\omega_\infty(\mathbb{R}^d)} \leq C_\alpha \epsilon$. Similarly,

$$A_2(x) \leq \frac{1}{(2\pi)^d} \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty} \int_{-2^{k_1+s}\pi/n_1}^{2^{k_1+s}\pi/n_1} \cdots \int_{-2^{k_d+s}\pi/n_1}^{2^{k_d+s}\pi/n_1} |f(x-t) - f(x)| \, dt \Big(\prod_{j=1}^{d} n_j \Big) \|h^\alpha 1_{Q_k}\|_\infty$$

and

$$2^{-\omega(k_1+\cdots+k_d)} n_1^d \frac{G(2^{k_1+s}\pi/n_1, \ldots, 2^{k_d+s}\pi/n_1)}{(2\pi)^d 2^{sd} 2^{k_1+\cdots+k_d}} \leq \mathcal{M}^\omega f(x) + |f(x)|.$$

Hence

$$A_2(x) \leq C \sum_{\pi_1,\ldots,\pi_d} \sum_{k_{\pi_1}=r_0+1}^{\infty} \cdots \sum_{k_{\pi_j}=r_0+1}^{\infty} \sum_{k_{\pi_{j+1}}=0}^{\infty} \cdots \sum_{k_{\pi_d}=0}^{\infty} 2^{(k_1+\cdots+k_d)(\omega+1)} \|h^\alpha 1_{Q_k}\|_\infty \left(\mathcal{M}^\omega f(x) + |f(x)| \right).$$

Since $\mathcal{M}^\omega f(x)$ is finite and $r_0 \to \infty$ as $n_1 \to \infty$, we conclude that $A_2(x) \to 0$ as $n \to \infty$, which finishes the proof. ■

A different version of this result was shown in Gabisoniya [3] for two dimensions. Similar theorems are proved by the author [23] for the θ-means generated by a single function θ. However, those results and proofs do not contain the results for Cesàro means. If f is continuous at a point x, then x is also an ω-Lebesgue point. So we obtain

Corollary 3. *Suppose that $0 < \omega < \alpha_j \leq 1$ for all $j = 1, \ldots, d$. If $\mathcal{M}^\omega f(x)$ is finite and $f \in L_1(\mathbb{T}^d)$ is continuous at a point x, then*

$$\lim_{n\to\infty,\, n\in\mathbb{R}^d_\tau} \sigma^\alpha_n f(x) = f(x).$$

The condition that $\mathcal{M}^\omega f(x)$ is finite is important even if f is continuous at x. Indeed, for two dimensions let

$$f(x_1, x_2) := \begin{cases} 0 & \text{if } x \in [-\pi, \pi] \times [-\epsilon, \epsilon]; \\ |x_1|^{-\delta} & \text{if } x \in [-\pi, \pi]^2 \setminus ([-\pi, \pi] \times [-\epsilon, \epsilon]). \end{cases} \tag{4.4}$$

Let $\epsilon > 0$ be small enough, $\omega < \delta < 1$, $i_1 = 0$ and $h = 2^{-i_2}$. Then f is obviously continuous at 0, integrable and

$$\mathcal{M}^\omega f(0) \geq \sup_{i_2 \in \mathbb{N}} \frac{2^{-\omega(i_1+i_2)}}{(2h)^2 2^{i_1+i_2}} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-2^{i_2}h}^{2^{i_2}h} f(t)\,dt = \sup_{i_2 \in \mathbb{N}} \frac{2^{-\omega i_2}}{4 \cdot 2^{-i_2}} \int_{-2^{-i_2}}^{2^{-i_2}} \int_{-1}^{1} f(t)\,dt$$
$$\geq \frac{1}{2(1-\delta)} \sup_{i_2 \in \mathbb{N}} 2^{(\delta-\omega)i_2} = \infty.$$

Remark 1. In the two-dimensional case Gabisoniya [3] introduced basically the following concept of Lebesgue points. Let

$$M_h^\omega f(x) := \sup_{\substack{0 < i_1 \leq 3 ln 2/h \\ 0 < i_2 \leq 3 ln 2/h}} \frac{2^{-\omega(i_1+i_2)}}{(2h)^2 2^{i_1+i_2}} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-2^{i_2}h}^{2^{i_2}h} |f(x-t) - f(x)|\,dt.$$

x is called a Lebesgue point of f if

$$\lim_{h\to 0} M_h^\omega f(x) = 0. \tag{4.5}$$

Actually, the concept of Lebesgue points used by Gabisoniya [3] is equivalent to this definition. For two dimensions, he proved that almost every point is a Lebesgue point of f as well as Theorem 3, whenever $f \in L_1(\mathbb{T}^2)$. However, in contrast to our definition of ω-Lebesgue points, the definition (4.5) does not hold for all continuity points of f. Indeed, let $\epsilon > 0$ be small enough, $\omega < \delta < 1$, $i_1 = 0$ and $h = 2^{-i_2}$ and consider the function (4.4) as before. Then

$$M_h^\omega f(0) \geq \frac{2^{-\omega(i_1+i_2)}}{(2h)^2 2^{i_1+i_2}} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-2^{i_2}h}^{2^{i_2}h} f(t)\,dt \geq \frac{1}{2(1-\delta)} 2^{(\delta-\omega)i_2}$$

and so $\lim_{h\to 0} M_h^\omega f(0) = \infty$.

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