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New Hardy-type integral inequalities

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Abstract. The proofs of generalized Hardy, Copson, Bennett, Leindler-type, and Levinson integral inequalities are revisited. It is contemplated to establish new proof of these classical inequalities using probability density function. New integral inequalities of Hardy-type involving the r^{th} order *Generalized Riemann–Liouville*, *Generalized Weyl*, *Erdélyi–Kober*, (k,ν)-*Riemann–Liouville*, and (k,ν)-*Weyl fractional integrals* are established through a probabilistic approach. The Kullback–Leibler inequality has been applied to compute the best possible constant factor associated with each of these inequalities.

1. Introduction

In 1920, G. H. Hardy [11] published a note that says that if $a > 0$, $f(x) \geq 0$, $p > 1$ and $\int_a^\infty f^p(x)dx$ is convergent, then

$$\int_a^\infty \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx \leq \Big(\frac{p}{p-1}\Big)^p \int_a^\infty f^p(x)dx \text{ holds.} \tag{1}$$

In 1925, the same author published another note ([12], see also [10, Theorem 327, p. 240]) in which the *continuous Hardy inequality* was established as stated below:

Theorem 1. *Suppose $p > 1$, $f(x) \geq 0$, and that $f(x)$ is integrable over any finite interval $(0,x)$ and $f^p(x)$ is integrable over $(0,\infty)$. Then*

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx \leq \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p(x)dx \textit{ holds.} \tag{2}$$

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The *discrete Hardy inequality* asserts that if $(a_k)_{k=1}^{\infty}$ is a non-negative sequence of real numbers and $p > 1$, then

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=1}^{n} a_k\Big)^p \le \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{\infty} a_n^p. \tag{3}$$

The constant term $\big(\frac{p}{p-1}\big)^p$ in both inequalities (2) and (3) is sharp. It is easy to prove inequality (3) by restricting inequality (2) to the class of step functions. Landau ([12, p. 154]) first discovered this important fact. These inequalities are known as the standard form of *Hardy's inequality* and which occupy prominent positions in different branches of mathematics. Based on this inequality, several applications in the form of generalizations, extensions, refinements, etc. have been studied by many mathematicians during the past nine decades. For further information regarding Hardy inequalities and their extensions and applications, the readers are referred to the book by Hardy et al. [10] and to the articles [3, 5–7, 15, 21] and references cited therein.

In case when $0 < p < 1$, $f(x) \ge 0$, $f(x)$ is integrable over any interval (x, ∞) and $f^p(x)$ is integrable over $(0, \infty)$ then

$$\int_0^{\infty} \Big(\frac{1}{x} \int_x^{\infty} f(t)dt\Big)^p dx > \Big(\frac{p}{1-p}\Big)^p \int_0^{\infty} f^p(x)dx \text{ holds,} \tag{4}$$

unless $f \equiv 0$. The term $\big(\frac{p}{1-p}\big)^p$ is a best possible constant factor in (4) [10, Theorem 337, p. 251].

An important extension of inequality (2) was proved by Hardy [13] which is given below:

Theorem 2. *Suppose that* $p > 1$, $m \ne 1$, $f(x) \ge 0$ *and* $f^p(x)$ *is integrable on* $(0, \infty)$. *Define* $F(x)$ *as*

$$F(x) = \int_0^{x} f(t)dt \textit{ for } m > 1 \textit{ and } F(x) = \int_x^{\infty} f(t)dt \textit{ for } m < 1.$$

Then

$$\int_0^{\infty} x^{-m} F^p(x)dx \le \Big(\frac{p}{|m-1|}\Big)^p \int_0^{\infty} x^{-m}(xf)^p dx \textit{ holds,} \tag{5}$$

where the constant term $\big(\frac{p}{|m-1|}\big)^p$ *is best possible. It may be noted that when* $p = 1$, *inequality* (5) *becomes an equality* [10].

The reversed version of inequality (5) is also due to Hardy [13], and is stated below:

Theorem 3. *Suppose that m, and F satisfy the conditions as provided in Theorem 2 but $0 < p < 1$. Then the inequality*

$$\int_0^\infty x^{-m} F^p(x)dx > \Big(\frac{p}{|m-1|}\Big)^p \int_0^\infty x^{-m}(xf)^p dx \text{ holds.} \tag{6}$$

The constant term $\big(\frac{p}{|m-1|}\big)^p$ is best possible.

Further extensions of inequalities (5) and (6) are due to Copson [8] as given below:

Theorem 4. *Suppose that $p \geq 1$, $m \neq 1$, $f(x) \geq 0$ and $f^p(x)$ is integrable over $[a, \infty)$, $a > 0$. For a positive real-valued function $\lambda(x)$ (> 0), denote $\Lambda(x) = \int_a^x \lambda(l)dl$ such that $\Lambda(\infty) = \infty$. Define $F(x)$ as*

$$F(x) = \int_a^x \lambda(t)f(t)dt \text{ for } m > 1 \text{ and } F(x) = \int_x^\infty \lambda(t)f(t)dt \text{ for } m < 1.$$

Then the following inequality holds:

$$\int_a^\infty \lambda(x)\Lambda^{-m}(x)F^p(x)dx \leq \Big(\frac{p}{|m-1|}\Big)^p \int_a^\infty \lambda(x)\Lambda^{p-m}(x)f^p(x)dx. \tag{7}$$

In case when $0 < p < 1$, the following inequality

$$\int_a^\infty \lambda(x)\Lambda^{-m}(x)F^p(x)dx \geq \Big(\frac{p}{|m-1|}\Big)^p \int_a^\infty \lambda(x)\Lambda^{p-m}(x)f^p(x)dx \text{ holds.} \tag{8}$$

In both the cases, the constant term $\big(\frac{p}{|m-1|}\big)^p$ is sharp.

A variant of the *Hardy–Copson inequality* was studied by Leindler [16] who obtained the *discrete version* of the following *continuous inequality*:

Theorem 5. *For a positive real-valued function $\lambda(x)$ (> 0), denote $\Lambda_*(x) = \int_x^\infty \lambda(l)dl$, where $x \in [a, \infty)$ with $a > 0$ such that $\Lambda_*(\infty) = 0$. Suppose that $p > 1$ and $0 \leq m < 1$. Then the following inequality holds:*

$$\int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\Big(\int_a^x \lambda(t)f(t)dt\Big)^p dx \leq \Big(\frac{p}{1-m}\Big)^p \int_a^\infty \frac{\lambda(x)}{\Lambda_*^{m-p}(x)} f^p(x)dx. \tag{9}$$

Bennett [4] established another variant of the *Hardy–Copson inequality* and obtained the *discrete version* of the following *continuous inequality*:

Theorem 6. *Denote $\Lambda_*(x) = \int_x^\infty \lambda(l)dl$, where $\lambda(x) > 0$, $x \in [a, \infty)$ with $a > 0$ such that $\Lambda_*(\infty) = 0$. Suppose that $1 < m \leq p$. Then the following result is true:*

$$\int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\Big(\int_x^\infty \lambda(t)f(t)dt\Big)^p dx \leq \Big(\frac{p}{m-1}\Big)^p \int_a^\infty \frac{\lambda(x)}{\Lambda_*^{m-p}(x)} f^p(x)dx. \tag{10}$$

In case when $0 < p < 1$, the signs of both inequalities (9) and (10) are reversed, and associated constant terms are sharp.

Apart from the discrete version of inequality (9), Leindler [16] further obtained the discrete version of the following integral inequalities, called as *Leindler-type inequalities*:

Theorem 7. *Suppose that $p > 1$, $\lambda(x) > 0$, $g(x)$ is integrable over the intervals (a, x) and (x, ∞). Then the following inequalities are true:*

$$\int_a^\infty \lambda(x)\Big(\int_a^x g(t)dt\Big)^p dx \le p^p \int_a^\infty \lambda^{1-p}(x)\Big(\int_x^\infty \lambda(t)dt\Big)^p g^p(x)dx, \tag{11}$$

$$\int_a^\infty \lambda(x)\Big(\int_x^\infty g(t)dt\Big)^p dx \le p^p \int_a^\infty \lambda^{1-p}(x)\Big(\int_a^x \lambda(t)dt\Big)^p g^p(x)dx. \tag{12}$$

The term p^p in both cases is best possible. In case when $0 < p < 1$, the signs of the above inequalities are reversed as shown by Leindler [18] in the discrete case.

Corollary 1. ([10, Theorem 328]) *If $p > 1$, $\lambda(x) \equiv 1$ and $a \to 0^+$, then inequality (12) gives*

$$\int_0^\infty \Big(\int_x^\infty g(t)dt\Big)^p dx \le p^p \int_0^\infty x^p g^p(x)dx.$$

The inequalities (2), (4), (5) and (6) are known as the basic models of the theory of classical integral inequalities. These inequalities and their generalizations and variants are dealt with in this paper. Many mathematicians have published proofs of these inequalities in the literature. Recently, Walker [24] presented proof of inequality (2) by a fully probabilistic approach.

In a continuation to the work presented by Walker [24], we report here a probabilistic proof of the other important classical inequalities (7)–(12). It is noteworthy that inequality (4) is a special case of inequality (6) and inequalities (5) & (6) are special case of inequalities (7) & (8), respectively. So, it is needless to present the proofs of inequalities (4), (5) & (6) separately. Some of the other important inequalities established earlier by Leindler [17] and Levinson [20] have been reinvestigated here to obtain proofs via probabilistic approach and employing a certain class of functions in the process. Finally, using *Generalized Riemann–Liouville integral*, *Generalized Weyl integral*, *Erdélyi–Kober*, (k,ν)-*Riemann–Liouville*, and (k,ν)-*Weyl* fractional integrals of order r, new integral inequalities are derived through a probabilistic approach.

The Kullback–Leibler inequality [9], stated below, has been applied to prove that the constant term is *sharp*: Let η and ζ be two probability density functions

($p.\,d.\,f.$) on a set A of non-zero Lebesgue measure in $(0,1)$. Then the Kullback–Leibler inequality (or '*Information inequality*') states that

$$\int_0^1 \zeta(u)\log\Big\{\frac{\zeta(u)}{\eta(u)}\Big\}du \geq 0 \tag{13}$$

and that equality holds if and only if $\zeta(u)=\eta(u)$ for all $u\in(0,1)$.

Now, multiplying both sides of inequality (13) by $(p-1)$ and hence using $\log x < x-1$, one immediately obtains

$$\int_0^1 \frac{\zeta^p(u)}{\eta^{p-1}(u)}du \geq 1, \tag{14}$$

which is used frequently for proving all results.

2. Proofs of the main results and more

Let us begin with the proof of Theorem 4 that is the proof of inequality (7).

Proof of Theorem 4. Two different cases are considered to prove it.

Case I: $m>1$. Let $\eta(t;x)$ be a $p.\,d.\,f.$ in the domain (a,x) for $x>a>0$. Then by definition, one gets $\int_a^x \eta(t;x)dt = 1$. Applying the Jensen's inequality, the following is obtained:

$$\begin{aligned}\Big(\int_a^x \lambda(t)f(t)dt\Big)^p &= \Big(\int_a^x \frac{\lambda(t)f(t)}{\eta(t;x)}\eta(t;x)dt\Big)^p \\ &\leq \int_a^x \frac{\lambda^p(t)f^p(t)}{\eta^p(t;x)}\eta(t;x)dt = \int_a^x \frac{\lambda^p(t)f^p(t)}{\eta^{p-1}(t;x)}dt.\end{aligned}$$

Using Fubini's theorem, one obtains

$$\begin{aligned}\int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\Big(\int_a^x\lambda(t)f(t)dt\Big)^p dx &\leq \int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\Big(\int_a^x\frac{\lambda^p(t)f^p(t)}{\eta^{p-1}(t;x)}dt\Big)dx \\ &= \int_a^\infty \lambda^p(t)f^p(t)\Big(\int_t^\infty \frac{\lambda(x)}{\Lambda^m(x)\eta^{p-1}(t;x)}dx\Big)dt. \end{aligned} \tag{15}$$

Now assuming that $\eta(t;x)$ is a scale distribution in the domain (a,x) as defined by

$$\eta(t;x) = \frac{\lambda(t)}{\Lambda(x)}\eta\Big(\frac{\Lambda(t)}{\Lambda(x)}\Big)$$

for some $p.\,d.\,f.$ $\eta(u)$ in the domain $(0,1)$. Inserting the above value of $\eta(t;x)$ in the inner integral of right-hand side of inequality (15) and changing the variable using transformation $u = \frac{\Lambda(t)}{\Lambda(x)}$, one gets

$$\begin{aligned}\int_t^\infty \frac{\lambda(x)}{\Lambda^m(x)\eta^{p-1}(t;x)}dx &= \int_t^\infty \frac{\lambda(x)\Lambda^{p-1}(x)}{\Lambda^m(x)\lambda^{p-1}(t)\eta^{p-1}\big(\frac{\Lambda(t)}{\Lambda(x)}\big)}dx \\ &= \frac{1}{\lambda^{p-1}(t)}\int_0^1 \frac{\lambda(x)}{\Lambda^{m-p+1}(x)\eta^{p-1}\big(\frac{\Lambda(t)}{\Lambda(x)}\big)}dx \\ &= \frac{1}{\lambda^{p-1}(t)\Lambda^{m-p}(t)}\int_0^1 \frac{u^{m-p-1}}{\eta^{p-1}(u)}du.\end{aligned}$$

Therefore inequality (15) becomes

$$\int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\Big(\int_a^x \lambda(t)f(t)dt\Big)^p dx \le \int_0^1 \frac{u^{m-p-1}}{\eta^{p-1}(u)}du \int_a^\infty \frac{\lambda(x)}{\Lambda^{m-p}(x)}f^p(x)dx, \tag{16}$$

which proves inequality (7) with the constant term

$$C_{p,m}(\eta) = \int_0^1 \frac{u^{m-p-1}}{\eta^{p-1}(u)}du.$$

For sharpness of the associated constant term $C_{p,m}(\eta)$, it is required to show that

$$\Big(\frac{p}{m-1}\Big)^p = C_{p,m} \le \int_0^1 \frac{u^{m-p-1}}{\eta^{p-1}(u)}du = C_{p,m}(\eta).$$

By choosing a *p. d. f.* $\zeta(u) = \big(\frac{m-1}{p}\big)u^{\frac{m-1}{p}-1}$ in the domain $(0,1)$, inequality (14) after simplifications assumes the form

$$\int_0^1 \frac{u^{m-p-1}}{\eta^{p-1}(u)}du \ge \Big(\frac{p}{m-1}\Big)^p. \tag{17}$$

In addition, if one chooses $\eta(u) = \big(\frac{m-1}{p}\big)u^{\frac{m-1}{p}-1}$ *p. d. f.* on $(0,1)$, then the equality sign occurs in inequality (17) and hence the proof of Case I is complete.

Case II: $0 \le m < 1$. Now assuming that $\eta(t;x)$ is a *p. d. f.* in the domain (x,∞), one gets by definition $\int_x^\infty \eta(t;x)dt = 1$. Applying Jensen's inequality, the following is obtained:

$$\begin{aligned}\Big(\int_x^\infty \lambda(t)f(t)dt\Big)^p &= \Big(\int_x^\infty \frac{\lambda(t)f(t)}{\eta(t;x)}\eta(t;x)dt\Big)^p \\ &\le \int_x^\infty \frac{\lambda^p(t)f^p(t)}{\eta^p(t;x)}\eta(t;x)dt = \int_x^\infty \frac{\lambda^p(t)f^p(t)}{\eta^{p-1}(t;x)}dt.\end{aligned}$$

Then Fubini's theorem implies that

$$\int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\Big(\int_x^\infty \lambda(t)f(t)dt\Big)^p dx \le \int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\Big(\int_x^\infty \frac{\lambda^p(t)f^p(t)}{\eta^{p-1}(t;x)}dt\Big)dx$$
$$= \int_a^\infty \lambda^p(t)f^p(t)\Big(\int_a^t \frac{\lambda(x)}{\Lambda^m(x)\eta^{p-1}(t;x)}dx\Big)dt. \quad (18)$$

Now a scale distribution $\eta(t;x)$ in the domain (x,∞) is chosen as follows:

$$\eta(t;x) = \frac{\lambda(t)\Lambda(x)}{\Lambda^2(t)}\eta\Big(\frac{\Lambda(x)}{\Lambda(t)}\Big)$$

for some *p. d. f.* $\eta(u)$ in the domain $(0,1)$. Then the inner integral of the right-hand side of inequality (18) simplifies to the following form:

$$\int_a^t \frac{\lambda(x)}{\Lambda^m(x)\eta^{p-1}(t;x)}dx = \int_a^t \frac{\lambda(x)\Lambda^{2(p-1)}(t)}{\lambda^{p-1}(t)\Lambda^{m+p-1}(x)\eta^{p-1}\Big(\frac{\Lambda(x)}{\Lambda(t)}\Big)}dx.$$

Changing the variable by using transformation $u = \frac{\Lambda(x)}{\Lambda(t)}$, one gets

$$\int_a^t \frac{\lambda(x)}{\Lambda^m(x)\eta^{p-1}(t;x)}dx = \frac{1}{\lambda^{p-1}(t)\Lambda^{m-p}(t)}\int_0^1 \frac{du}{u^{m+p-1}\eta^{p-1}(u)}.$$

Finally, inequality (18) assumes the form

$$\int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\Big(\int_x^\infty \lambda(t)f(t)dt\Big)^p dx \le \int_0^1 \frac{u^{-m-p+1}}{\eta^{p-1}(u)}du \int_a^\infty \frac{\lambda(x)}{\Lambda^{m-p}(x)}f^p(x)dx, \quad (19)$$

which proves inequality (7) with the constant term

$$C_{p,m}(\eta) = \int_0^1 \frac{u^{-m-p+1}}{\eta^{p-1}(u)}du.$$

To obtain the sharp constant term $C_{p,m}(\eta)$, it is enough to show that

$$\Big(\frac{p}{1-m}\Big)^p = C_{p,m} \le \int_0^1 \frac{u^{-m-p+1}}{\eta^{p-1}(u)}du = C_{p,m}(\eta).$$

Let us choose a *p. d. f.* $\zeta(u) = \big(\frac{1-m}{p}\big)u^{\frac{1-m}{p}-1}$ in the domain $(0,1)$. Then inequality (14) takes the form

$$\int_0^1 \frac{u^{-m-p+1}}{\eta^{p-1}(u)}du \ge \Big(\frac{p}{1-m}\Big)^p, \quad (20)$$

and that equality holds when $\eta(u)=\left(\frac{1-m}{p}\right)u^{\frac{1-m}{p}-1}$ is chosen in the domain $(0,1)$. This completes the proof of inequality (7).

It may be noted that when $0<p<1$, the function $F(x)$ defined earlier is concave for $m>1$ as well as for $m<1$. Consequently the signs of inequalities (15) and (18) are reversed and hence inequality (8) can be proved easily and subsequent derivations are left for the readers. ■

Now it is planned to present the proof of Theorem 5 that is the proof of inequality (9).

Proof of Theorem 5. Proceeding in the parallel lines of the earlier theorem, $\eta(t;x)$ denotes $p.\,d.\,f.$ in the domain (a,x) for $x>a>0$. Using Jensen's inequality and Fubini's theorem, one gets

$$\int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\Big(\int_a^x \lambda(t)f(t)dt\Big)^p dx \le \int_a^\infty \lambda^p(t)f^p(t)\Big(\int_t^\infty \frac{\lambda(x)}{\Lambda_*^m(x)\eta^{p-1}(t;x)}dx\Big)dt. \tag{21}$$

Now $\eta(t;x)$ is chosen as a scale distribution in the domain (a,x) as defined below:

$$\eta(t;x)=\frac{\lambda(t)\Lambda_*(x)}{\Lambda_*^2(t)}\eta\Big(\frac{\Lambda_*(x)}{\Lambda_*(t)}\Big)$$

for some density function $\eta(u)$ in the domain $(0,1)$. Then the inner integral of right-hand side of inequality (21) simplifies to

$$\int_t^\infty \frac{\lambda(x)}{\Lambda_*^m(x)\eta^{p-1}(t;x)}dx=\int_t^\infty \frac{\lambda(x)\Lambda_*^{2(p-1)}(t)}{\lambda^{p-1}(t)\Lambda_*^{m+p-1}(x)\eta^{p-1}\left(\frac{\Lambda_*(x)}{\Lambda_*(t)}\right)}dx,$$

which reduces to the following by changing the variable $u=\frac{\Lambda_*(x)}{\Lambda_*(t)}$:

$$\int_t^\infty \frac{\lambda(x)}{\Lambda_*^m(x)\eta^{p-1}(t;x)}dx=\frac{1}{\lambda^{p-1}(t)\Lambda_*^{m-p}(t)}\int_0^1 \frac{du}{u^{m+p-1}\eta^{p-1}(u)}.$$

Finally, inequality (21) can be written as

$$\int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\Big(\int_a^x \lambda(t)f(t)dt\Big)^p dx \le \int_0^1 \frac{u^{-m-p+1}}{\eta^{p-1}(u)}du\int_a^\infty \frac{\lambda(x)}{\Lambda_*^{m-p}(x)}f^p(x)dx, \tag{22}$$

which is inequality (9) with the constant term

$$C_{p,m}(\eta)=\int_0^1 \frac{u^{-m-p+1}}{\eta^{p-1}(u)}du.$$

It has already been proved in Theorem 4 (Case II) that the term $C_{p,m}(\eta)\ge C_{p,m}=\left(\frac{p}{1-m}\right)^p$ is best possible constant. Hence inequality (9) is established. ■

Proof of Theorem 6. Considering $\eta(t;x)$ as a $p.\,d.\,f.$ in the domain (x,∞) for $x\in[a,\infty)$, $a>0$ and then applying Jensen's inequality and Fubini's theorem, one obtains

$$\int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\Big(\int_x^\infty \lambda(t)f(t)dt\Big)^p dx \le \int_a^\infty \lambda^p(t)f^p(t)\Big(\int_a^t \frac{\lambda(x)}{\Lambda_*^m(x)\eta^{p-1}(t;x)}dx\Big)dt. \quad (23)$$

Choosing $\eta(t;x)$ is a scale distribution in the domain (x,∞) as below:

$$\eta(t;x)=\frac{\lambda(t)}{\Lambda_*(x)}\eta\Big(\frac{\Lambda_*(t)}{\Lambda_*(x)}\Big)$$

for some $p.\,d.\,f.$ $\eta(u)$ in the domain $(0,1)$. Then inner integral of the right-hand side of inequality (23) becomes

$$\begin{aligned}\int_a^t \frac{\lambda(x)}{\Lambda_*^m(x)\eta^{p-1}(t;x)}dx &= \int_t^\infty \frac{\lambda(x)\Lambda_*^{2(p-1)}(t)}{\lambda^{p-1}(t)\Lambda_*^{m+p-1}(x)\eta^{p-1}\big(\frac{\Lambda_*(x)}{\Lambda_*(t)}\big)}dx,\\ &= \frac{1}{\lambda^{p-1}(t)\Lambda_*^{m-p}(t)}\int_{\frac{\Lambda_*(t)}{\Lambda_*(a)}}^1 \frac{du}{u^{-m+p+1}\eta^{p-1}(u)}\\ &\le \frac{1}{\lambda^{p-1}(t)\Lambda_*^{m-p}(t)}\int_0^1 \frac{du}{u^{-m+p+1}\eta^{p-1}(u)},\end{aligned}$$

where $u=\frac{\Lambda_*(t)}{\Lambda_*(x)}$. Finally, inequality (23) reduces to

$$\int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\Big(\int_x^\infty \lambda(t)f(t)dt\Big)^p dx \le \int_0^1 \frac{u^{m-p-1}}{\eta^{p-1}(u)}du\int_a^\infty \frac{\lambda(x)}{\Lambda_*^{m-p}(x)}f^p(x)dx, \quad (24)$$

which proves inequality (10) with the constant term

$$C_{p,m}(\eta)=\int_0^1 \frac{u^{m-p-1}}{\eta^{p-1}(u)}du.$$

The constant $\left(\frac{p}{m-1}\right)^p = C_{p,m} \le C_{p,m}(\eta)$ is sharp as Case I shows in the proof of Theorem 4, hence its derivation is omitted. Therefore inequality (10) is proved. ∎

It is now contemplated to give a probabilistic proof of the Leindler-type inequalities (11) and (12) of Theorem 7.

Proof of Theorem 7. Choosing the substitutions $g(t)=\frac{f(t)}{\Lambda_*(t)}$ for inequality (11) and $g(t)=\frac{f(t)}{\Lambda(t)}$ for inequality (12), respectively, then these inequalities reduce to the inequalities

$$\int_a^\infty \lambda(x)\Big(\int_a^x \frac{f(t)}{\Lambda_*(t)}dt\Big)^p dx \le p^p\int_a^\infty \lambda^{1-p}(x)f^p(x)dx \quad (25)$$

and

$$\int_a^\infty \lambda(x)\Big(\int_x^\infty \frac{f(t)}{\Lambda(t)}dt\Big)^p dx \le p^p \int_a^\infty \lambda^{1-p}(x) f^p(x)dx. \tag{26}$$

Therefore it is adequate to prove inequalities (25) and (26).

Proof of (25). Let $\eta(t;x)$ be a *p. d. f.* in the domain (a,x) for $x > a > 0$. Then by applying Jensen's inequality and Fubini's theorem, one can directly obtain

$$\int_a^\infty \lambda(x)\Big(\int_a^x \frac{f(t)}{\Lambda_*(t)}dt\Big)^p dx \le \int_a^\infty \frac{f^p(t)}{\Lambda_*^p(t)} \int_t^\infty \frac{\lambda(x)}{\eta^{p-1}(t;x)} dxdt. \tag{27}$$

Select a scale distribution $\eta(t;x)$ in the domain (a,x) as follows:

$$\eta(t;x) = \frac{\lambda(t)\Lambda_*(x)}{\Lambda_*^2(t)}\eta\Big(\frac{\Lambda_*(x)}{\Lambda_*(t)}\Big)$$

for some density function $\eta(u)$ in the domain $(0,1)$. Then inequality (27) after substituting $u = \frac{\Lambda_*(x)}{\Lambda_*(t)}$ with $\Lambda_*(\infty) = 0$ assumes the form

$$\int_a^\infty \lambda(x)\Big(\int_a^x \frac{f(t)}{\Lambda_*(t)}dt\Big)^p dx \le \int_0^1 \frac{du}{u^{p-1}\eta^{p-1}(u)} \int_a^\infty \lambda^{1-p}(x) f^p(x)dx,$$

which proves inequality (25) with the constant term

$$C_p(\eta) = \int_0^1 \frac{du}{u^{p-1}\eta^{p-1}(u)} \ge p^p = C_p.$$

The term p^p is best possible which can be easily deduced from Theorem 5 by putting $m = 0$. This completes the proof of inequality (25).

Proof of (26). Suppose that $\eta(t;x)$ is a *p. d. f.* in the domain (x,∞) for $x \in [a,\infty)$, $a > 0$. Applying Jensen's inequality and Fubini's theorem, the following is obtained:

$$\int_a^\infty \lambda(x)\Big(\int_x^\infty \frac{f(t)}{\Lambda(t)}dt\Big)^p dx \le \int_a^\infty \frac{f^p(t)}{\Lambda^p(t)} \int_a^t \frac{\lambda(x)}{\eta^{p-1}(t;x)} dxdt. \tag{28}$$

Defining a scale distribution $\eta(t;x)$ in the domain (x,∞) as follows:

$$\eta(t;x) = \frac{\lambda(t)\Lambda(x)}{\Lambda^2(t)}\eta\Big(\frac{\Lambda(x)}{\Lambda(t)}\Big)$$

for some density function $\eta(u)$ in the domain $(0,1)$. Then by using the transformation $u = \frac{\Lambda(x)}{\Lambda(t)}$, inequality (28) assumes the form

$$\int_a^\infty \lambda(x)\Big(\int_x^\infty \frac{f(t)}{\Lambda(t)}dt\Big)^p dx \le \int_0^1 \frac{du}{u^{p-1}\eta^{p-1}(u)} \int_a^\infty \lambda^{1-p}(x) f^p(x)dx,$$

which is inequality (26) with the constant term

$$C_p(\eta) = \int_0^1 \frac{du}{u^{p-1}\eta^{p-1}(u)} \geq p^p = C_p,$$

which is again best possible as shown earlier. This completes the proof of inequality (26).

■

In addition to the above inequalities, some generalizations were made by Leindler [17] in the discrete case. Before putting a list of generalized inequalities, the following is recalled: for $p > 0$, $\varphi \in D(p)$ if φ is a non-negative increasing function on $[a, \infty)$, $a \geq 0$, $\varphi(0) = 0$, $\frac{\varphi(x)}{x^p}$ is non-increasing. Then the following results are true:

Theorem 8. *Suppose $p \geq 1$, $m \neq 1$ and $\varphi \in D(p)$. Notations $\Lambda(x)$ and $F(x)$ carry the same meaning as in Theorem 4. Then the following holds:*

$$\int_a^\infty \lambda(x)\Lambda^{-m}(x)\varphi(F(x))dx \leq \Big(\frac{p}{|m-1|}\Big)^p \int_a^\infty \lambda^{1-p}(x)\Lambda^{p-m}(x)\varphi(\lambda(x)f(x))dx. \tag{29}$$

The constant term $\big(\frac{p}{|m-1|}\big)^p$ is best possible.

Corollary 2. *By choosing $\varphi(x) = x^p$, $p \geq 1$, one gets inequality (7).*

Similarly Theorems 5 & 6 can be generalized in the following way:

Theorem 9. *Suppose $p > 1$ and $m \neq 1$. Define $F^*(x)$ as*

$$F^*(x) = \int_a^x \lambda(t)f(t)dt \text{ for } m < 1 \text{ and } F^*(x) = \int_x^\infty \lambda(t)f(t)dt \text{ for } m > 1.$$

If $\varphi \in D(p)$, then the following inequality is true:

$$\int_a^\infty \lambda(x)\Lambda_*^{-m}(x)\varphi\big(F^*(x)\big)dx \leq \Big(\frac{p}{|m-1|}\Big)^p \int_a^\infty \lambda^{1-p}(x)\Lambda_*^{p-m}(x)\varphi(\lambda(x)f(x))dx. \tag{30}$$

Corollary 3. *By choosing $\varphi(x) = x^p$, $p \geq 1$, one gets inequalities (9) and (10).*

To prove the above theorems, the following lemma proved by Leindler [19] is recalled:

Lemma 1. ([19]) *If $p > 0$, $\varphi \in D(p)$ and $x \geq 0$, then*

$$t^p\varphi(x) \leq \varphi(tx) \text{ for } 0 \leq t \leq 1 \text{ and } \varphi(tx) \leq t^p\varphi(x) \text{ for } t \geq 1.$$

Proof. The proof immediately follows from the definition of the class $D(p)$ and is hence omitted. ∎

Next we present the proofs of Theorem 8 and Theorem 9.

Proof of Theorem 8. The proof of this theorem is quite similar to that of Theorem 4. Suppose that $p \geq 1$, $m > 1$ and $\varphi \in D(p)$. Applying Jensen's inequality and using Fubini's theorem, one can directly establish the reduced form of inequality (15) as shown below:

$$\begin{aligned}\int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\varphi\Big(F(x)\Big)dx &\leq \int_a^\infty \frac{\lambda(x)}{\Lambda^m(x)}\int_a^x \varphi\Big(\frac{\lambda(t)f(t)}{\eta(t;x)}\Big)\eta(t;x)dtdx \\ &= \int_a^\infty \varphi(\lambda(t)f(t))\Big(\int_t^\infty \frac{\lambda(x)}{\Lambda^m(x)\eta^{p-1}(t;x)}dx\Big)dt, \qquad (31)\end{aligned}$$

where the notations have their usual meanings and the last line is the application of Lemma 1 for $0 \leq \eta(t;x) \leq 1$. Remaining of the proof runs similar lines as developed in Theorem 4. The case $m < 1$ will be treated similarly as the Case II of Theorem 4. Therefore inequality (29) immediately follows by joining these two cases. The procedure for obtaining the best possible constant will also be same as that of Theorem 4, and hence omitted. ∎

The proof of Theorem 9 is also quite similar to those of Theorems 5 & 6.

Proof of Theorem 9. Suppose that $p \geq 1, m < 1$ and $\varphi \in D(p)$. Applying Jensen's inequality and using Fubini's theorem, one can directly establish the reduced form of inequality (21) as follows:

$$\begin{aligned}\int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\varphi\big(F^*(x)\big)dx &\leq \int_a^\infty \frac{\lambda(x)}{\Lambda_*^m(x)}\int_a^x \varphi\Big(\frac{\lambda(t)f(t)}{\eta(t;x)}\Big)\eta(t;x)dtdx \\ &= \int_a^\infty \varphi(\lambda(t)f(t))\Big(\int_t^\infty \frac{\lambda(x)}{\Lambda_*^m(x)\eta^{p-1}(t;x)}dx\Big)dt, \qquad (32)\end{aligned}$$

where the notations have their usual meanings and the last line is the application of Lemma 1 for $0 \leq \eta(t;x) \leq 1$. The rest of the proof runs similar lines as developed in Theorem 5. Similarly the case $m > 1$ will be treated as presented in Theorem 6. Hence inequality (30) follows immediately by combining these two cases. The procedure for obtaining the best possible constant is again the same as those of Theorems 5 & 6, and is hence skipped. ∎

Levinson proved the following:

Levinson inequality. ([20, Theorem 2]) *Let φ be defined on an open interval (finite or infinite) such that $\varphi(u) \geq 0$, $\varphi'' \geq 0$ and satisfies the following inequality for $p > 1$:*

$$\varphi\varphi'' \geq \Big(1 - \frac{1}{p}\Big)(\varphi')^2. \tag{33}$$

Suppose that $f(x)$ is defined in the domain $(0, \infty)$, range of which lies in the closed interval of definition of φ. For a positive real-valued function $\lambda(x)$ (> 0), let $\Lambda(x) = \int_a^x \lambda(l)dl$ such that $\Lambda(\infty) = \infty$. Then we have

$$\int_0^\infty \varphi\Big(\frac{1}{\Lambda(x)} \int_0^x \lambda(t)f(t)dt\Big)dx \leq \Big(\frac{p}{p-1}\Big)^p \int_0^\infty \varphi(f(x))dx. \tag{34}$$

The reduced inequality ($\lambda \equiv 1$) is also due to Levinson ([20, Theorem 1]).

Now inequality (34) will be proved using a probabilistic approach as discussed in the earlier proofs. But this time only the condition $\varphi \in D(p)$ will be used instead of condition (33). The statement of the theorem reads now as follows:

Theorem 10. *Let $p > 1$, $\varphi \in D(p)$ and $f(x)$ be defined on (a, ∞), $a > 0$. Then the following*

$$\int_a^\infty \varphi\Big(\frac{1}{\Lambda(x)} \int_a^x \lambda(t)f(t)dt\Big)dx \leq \Big(\frac{p}{p-1}\Big)^p \int_a^\infty \varphi(f(x))dx \quad \textit{holds}. \tag{35}$$

Also the constant factor $\big(\frac{p}{p-1}\big)^p$ is sharp.

Proof. Let $\eta(t; x)$ be a $p.\,d.\,f.$ in the domain (a, x) for $x > a > 0$. Proceeding similarly as in Theorem 4, by applying Jensen's inequality and Fubini's theorem, one gets

$$\int_a^\infty \varphi\Big(\frac{1}{\Lambda(x)} \int_a^x \lambda(t)f(t)dt\Big)dx \leq \int_a^\infty \int_t^\infty \varphi\Big(\frac{\lambda(t)f(t)}{\Lambda(x)\eta(t;x)}\Big)\eta(t;x)dxdt. \tag{36}$$

Choosing the scale distribution $\eta(t; x)$ in the domain (a, x) as in Theorem 4 (Case I) given by $\eta(t; x) = \frac{\lambda(t)}{\Lambda(x)}\eta\big(\frac{\Lambda(t)}{\Lambda(x)}\big)$, inequality (36) takes the form

$$\int_a^\infty \varphi\Big(\frac{1}{\Lambda(x)} \int_a^x \lambda(t)f(t)dt\Big)dx \leq \int_a^\infty \int_t^\infty \varphi\Big(\frac{f(t)}{\eta\big(\frac{\Lambda(t)}{\Lambda(x)}\big)}\Big)\frac{\lambda(t)}{\Lambda(x)}\eta\Big(\frac{\Lambda(t)}{\Lambda(x)}\Big)dxdt. \tag{37}$$

Using the transformation $u = \frac{\Lambda(t)}{\Lambda(x)}$, Lemma 1 with $0 \leq \eta(t; x) \leq 1$ and then applying the non-decreasing property of λ, inequality (37) is reduced to the form

$$\int_a^\infty \varphi\Big(\frac{1}{\Lambda(x)} \int_a^x \lambda(t)f(t)dt\Big)dx \leq \int_0^1 \frac{1}{u\eta^{p-1}(u)}du \int_a^\infty \varphi(f(x))dx,$$

which proves the inequality (35) with the constant term $\int_0^1 \frac{1}{u\eta^{p-1}(u)}du$. To prove that the associated constant term $C_p = \left(\frac{p}{p-1}\right)^p$ sharp, it is enough to show that $C_p \leq \int_0^1 \frac{1}{u\eta^{p-1}(u)}du = C_p(\eta)$ and which again follows from Theorem 4. This completes the proof of inequality (35). ∎

3. New integral inequalities

In fractional calculus, the *Riemann–Liouville integral* plays an essential role, whereas the *Weyl integral* finds its application in the theory of Fourier series. A natural extension of Hardy's inequality (2) using the *Riemann–Liouville integral* is due to Knopp [14], and another extension of Hardy's inequality involving the *Weyl integral* can be found in the book of Hardy et al. ([10], Theorem 329).

In this section, a generalized version of the *Riemann–Liouville integral*, and the *Weyl integral* are introduced and using both these generalized integrals, new fractional integral inequalities are established. Further, applying the same techniques, new integral inequalities involving the r^{th} order *Erdélyi–Kober*, (k,ν)-*Riemann–Liouville*, and (k,ν)-*Weyl* fractional integrals are presented. Finally, it will be shown that the best possible constants attached with each of these integral inequalities may be obtained through a probabilistic approach. In the remaining portion of the paper, it is assumed that the right-hand side integral in each of the inequalities (39), (43), (46), (49), (53) and (58) is finite.

3.1. Generalized Riemann-Liouville integral

Let $r > 0$ be a real number and $\lambda(x)$, $\Lambda(x)$ carry the same meaning as mentioned in Section 1. Then *Riemann–Liouville integral* of $\lambda(x)f(x)$ of order r with origin at 0 is defined as

$$f_r(x) = \frac{1}{\Gamma(r)} \int_0^x (\Lambda(x) - \Lambda(t))^{r-1} \lambda(t) f(t) dt. \tag{38}$$

Then we have the following theorem.

Theorem 11. *Suppose that $p > 1$. Then the following inequality*

$$\int_0^\infty \lambda(x) \Big(\frac{f_r(x)}{\Lambda^r(x)}\Big)^p dx < \left\{ \frac{\Gamma\big(1-\frac{1}{p}\big)}{\Gamma\big(r+1-\frac{1}{p}\big)} \right\}^p \int_0^\infty \lambda(x) f^p(x) dx \text{ holds,} \tag{39}$$

unless $f \equiv 0$. The constant term $\left\{ \frac{\Gamma(1-\frac{1}{p})}{\Gamma(r+1-\frac{1}{p})} \right\}^p$ is sharp.

Corollary 4. *In particular when $\lambda(x) = 1$ for all $x \in (0, \infty)$, then expression (38) for $f_r(x)$ is known as the* Riemann–Liouville integral *of $f(x)$ of order r with origin at 0 and inequality (39) reduces to*

$$\int_0^\infty \Big(\frac{f_r}{x^r}\Big)^p dx < \left\{\frac{\Gamma\big(1-\frac{1}{p}\big)}{\Gamma\big(r+1-\frac{1}{p}\big)}\right\}^p \int_0^\infty f^p(x)dx,$$

which further reduces to Hardy's inequality (2) for $r = 1$ (see [10, Theorem 329]).

Naturally, inequality (39) is more general than its earlier form. It is now decided to provide a probabilistic proof of Theorem 11 that is inequality (39).

Proof. Let $\eta(t; x)$ be a probability density function in the domain $(0, x)$ for $x > 0$, so that one gets $\int_0^x \eta(t; x)dt = 1$. Applying Jensen's inequality and Fubini's theorem, one obtains

$$\begin{aligned}
&\int_0^\infty \lambda(x)\Big(\frac{f_r(x)}{\Lambda^r(x)}\Big)^p dx \\
&\qquad \le \int_0^\infty \frac{\lambda(x)}{\Lambda^{rp}(x)\{\Gamma(r)\}^p}\Big(\int_0^x \frac{(\Lambda(x)-\Lambda(t))^{p(r-1)}\lambda^p(t)f^p(t)}{\eta^{p-1}(t;x)}dt\Big)dx \\
&\qquad = \int_0^\infty \frac{\lambda^p(t)f^p(t)}{\{\Gamma(r)\}^p}\Big(\int_t^\infty \frac{\lambda(x)(\Lambda(x)-\Lambda(t))^{p(r-1)}}{\Lambda^{rp}(x)\eta^{p-1}(t;x)}dx\Big)dt. \qquad (40)
\end{aligned}$$

Let $\eta(t; x)$ be a scale distribution in the domain $(0, x)$ and is defined as $\eta(t; x) = \frac{\lambda(t)}{\Lambda(x)}\eta\big(\frac{\Lambda(t)}{\Lambda(x)}\big)$ for some density function $\eta(u)$ in the domain $(0, 1)$. Replacing the value of $\eta(t; x)$ in inequality (40), one gets

$$\begin{aligned}
&\int_0^\infty \lambda(x)\Big(\frac{f_r(x)}{\Lambda^r(x)}\Big)^p dx \\
&\quad \le \int_0^\infty \frac{\lambda^p(t)f^p(t)}{\{\Gamma(r)\}^p}\Big(\int_t^\infty \frac{\lambda(x)(\Lambda(x)-\Lambda(t))^{p(r-1)}\Lambda^{p-1}(x)}{\lambda^{p-1}(t)\Lambda^{rp}(x)\eta^{p-1}(\frac{\Lambda(t)}{\Lambda(x)})}dx\Big)dt \\
&\quad = \int_0^\infty \lambda(t)f^p(t)\Big(\frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-u)^{p(r-1)}}{u\eta^{p-1}(u)}du\Big)dt \quad \text{where } u = \frac{\Lambda(t)}{\Lambda(x)} \\
&\quad = \frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-u)^{p(r-1)}}{u\eta^{p-1}(u)}du \int_0^\infty \lambda(x)f^p(x)dx. \qquad (41)
\end{aligned}$$

This proves inequality (39) with the constant term

$$C_{p,r}(\eta) = \frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-u)^{p(r-1)}}{u\eta^{p-1}(u)}du.$$

To show that the associated constant term $C_{p,r} = \left\{\frac{\Gamma(1-\frac{1}{p})}{\Gamma(r+1-\frac{1}{p})}\right\}^p$ is sharp, one needs to prove that

$$C_{p,r}(\eta) \geq \left\{\frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(r+1-\frac{1}{p}\right)}\right\}^p = C_{p,r}.$$

By choosing a *p. d. f.* $\zeta(u)$ in the domain $(0,1)$ as $\zeta(u) = \frac{u^{-\frac{1}{p}}(1-u)^{r-1}}{B(1-\frac{1}{p},r)}$, where $B(\cdot,\cdot)$ denotes the well-known *Beta function*, the inequality (14) becomes

$$\begin{aligned} C_{p,r}(\eta) = \frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-u)^{p(r-1)}}{u\eta^{p-1}(u)}du &\geq \frac{1}{\{\Gamma(r)\}^p}\left\{B(1-\frac{1}{p},r)\right\}^p \\ &= \left\{\frac{\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(r+1-\frac{1}{p}\right)}\right\}^p, \end{aligned}$$

hence the constant $C_{p,r}(\eta) \equiv C_{p,r}$ in inequality (41) is sharp. Equality holds when $\eta(u) = \frac{u^{-\frac{1}{p}}(1-u)^{r-1}}{B(1-\frac{1}{p},r)}$. If f is non-nul, then there is an equality in (39) for $f(x) = \{\lambda(x)\}^{-1/p}$. But this leads to the divergence of $\int_0^\infty \lambda(x)f^p(x)$, and this completes the proof of inequality (39). ∎

3.2. Generalized Weyl integral

For $r > 0$, the *Weyl integral* of $\lambda(x)f(x)$ of order r is now defined as follows:

$$f_r(x) = \frac{1}{\Gamma(r)}\int_x^\infty (\Lambda(t)-\Lambda(x))^{r-1}\lambda(t)f(t)dt. \tag{42}$$

Then the following is true:

Theorem 12. *Suppose that $p > 1$. Then the following result*

$$\int_0^\infty \lambda(x)\big(f_r(x)\big)^p dx < \left\{\frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(r+\frac{1}{p}\right)}\right\}^p \int_0^\infty \lambda(x)(\Lambda^r(x)f(x))^p dx \text{ is holds,} \tag{43}$$

unless $f \equiv 0$. The term $\left\{\frac{\Gamma(\frac{1}{p})}{\Gamma(r+\frac{1}{p})}\right\}^p$ is best possible constant.

Corollary 5. *In particular, when $\lambda(x) = 1$ for all $x \in (0,\infty)$, then (42) turns out to be the* Weyl integral *of $f(x)$ of order r so that inequality (43) reduces to*

$$\int_0^\infty \big(f_r(x)\big)^p dx < \left\{\frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(r+\frac{1}{p}\right)}\right\}^p \int_0^\infty (x^r f(x))^p dx,$$

which is another form of Hardy's inequality (see [10, Theorem 329]) *and further reduces to another well-known inequality (see* [10, Theorem 328]) *when $r = 1$.*

It is to be noted that the inequality (43) is an extension of Hardy's inequality and here a probabilistic proof of inequality (43), i.e. Theorem 12 is presented.

Proof. Let $\eta(t;x)$ be a probability density function in the domain (x,∞) for $x>0$. Hence by definition $\int_x^\infty \eta(t;x)dt = 1$. Then applying Jensen's inequality and Fubini's theorem, one obtains

$$\begin{aligned}\int_0^\infty \lambda(x)\big(f_r(x)\big)^p dx &\le \int_0^\infty \frac{\lambda(x)}{\{\Gamma(r)\}^p}\Big(\int_x^\infty \frac{(\Lambda(t)-\Lambda(x))^{p(r-1)}\lambda^p(t)f^p(t)}{\eta^{p-1}(t;x)}dt\Big)dx\\ &= \int_0^\infty \frac{\lambda^p(t)f^p(t)}{\{\Gamma(r)\}^p}\Big(\int_0^t \frac{\lambda(x)(\Lambda(t)-\Lambda(x))^{p(r-1)}}{\eta^{p-1}(t;x)}dx\Big)dt. \quad (44)\end{aligned}$$

Choosing $\eta(t;x)$ is a scale distribution in the domain (x,∞) as $\eta(t;x) = \frac{\lambda(t)\Lambda(x)}{\Lambda^2(t)}\eta\big(\frac{\Lambda(x)}{\Lambda(t)}\big)$ for some density function $\eta(v)$ in the domain $(0,1)$. Replacing the value of $\eta(t;x)$ in inequality (44), one gets

$$\begin{aligned}&\int_0^\infty \lambda(x)\big(f_r(x)\big)^p dx\\ &\le \int_0^\infty \frac{\lambda^p(t)f^p(t)}{\{\Gamma(r)\}^p}\Big(\int_0^t \frac{\lambda(x)(\Lambda(t)-\Lambda(x))^{p(r-1)}\Lambda^{2(p-1)}(t)}{\lambda^{p-1}(t)\Lambda^{p-1}(x)\eta^{p-1}(\frac{\Lambda(x)}{\Lambda(t)})}dx\Big)dt\\ &= \int_0^\infty \lambda(t)(\Lambda^r(t)f(t))^p\Big(\frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-v)^{p(r-1)}}{v^{p-1}\eta^{p-1}(v)}dv\Big)dt \ \text{ by putting } v=\frac{\Lambda(x)}{\Lambda(t)}\\ &= \frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-v)^{p(r-1)}}{v^{p-1}\eta^{p-1}(v)}dv\int_0^\infty \lambda(t)(\Lambda^r(t)f(t))^p dt, \quad (45)\end{aligned}$$

which proves inequality (43) with the constant term

$$C_{p,r}(\eta) = \frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-v)^{p(r-1)}}{v^{p-1}\eta^{p-1}(v)}dv.$$

In order to demonstrate that term $C_{p,r}$ is best possible only when it equals $\big\{\frac{\Gamma(\frac{1}{p})}{\Gamma(r+\frac{1}{p})}\big\}^p$, one needs to prove that

$$C_{p,r}(\eta) \ge \Big\{\frac{\Gamma\big(\frac{1}{p}\big)}{\Gamma\big(r+\frac{1}{p}\big)}\Big\}^p = C_{p,r}.$$

By choosing a *p. d. f.* $\zeta(v)$ in the domain $(0,1)$ as $\zeta(v) = \frac{v^{\frac{1}{p}-1}(1-v)^{r-1}}{B(\frac{1}{p},r)}$, then inequality (14) assumes the form

$$C_{p,r}(\eta) = \frac{1}{\{\Gamma(r)\}^p}\int_0^1 \frac{(1-v)^{p(r-1)}}{v^{p-1}\eta^{p-1}(v)}dv \ge \frac{1}{\{\Gamma(r)\}^p}\Big\{B(\frac{1}{p},r)\Big\}^p = \left\{\frac{\Gamma\big(\frac{1}{p}\big)}{\Gamma\big(r+\frac{1}{p}\big)}\right\}^p,$$

which shows that the constant $C_{p,r}(\eta) \equiv C_{p,r}$ in inequality (45) is best possible.

Equality holds when $\eta(v) = \frac{v^{\frac{1}{p}-1}(1-v)^{r-1}}{B(\frac{1}{p},r)}$ and this completes the proof of inequality (43).

■

3.3. Erdélyi–Kober fractional integral

The Erdélyi–Kober fractional integral operators, $I^{\nu}_{r,\xi}$ and $J^{\nu}_{r,\eta}$, are respectively the generalized form of the *Riemann–Liouville* and *Weyl* fractional integrals with power weights and are defined as follows:

$$(I^{\nu}_{r,\xi}f)(x) = \frac{\nu x^{-\nu(\xi+r)+\nu}}{\Gamma(r)} \int_0^x (x^{\nu} - t^{\nu})^{r-1} t^{\nu\xi-1} f(t)dt$$

and

$$(J^{\nu}_{r,\eta}f)(x) = \frac{\nu x^{\nu\eta}}{\Gamma(r)} \int_x^{\infty} (t^{\nu} - x^{\nu})^{r-1} t^{-\nu(r+\eta)+\nu-1} f(t)dt.$$

These operators have many applications in different parts of fractional integral and differential calculus. The operators and their generalizations also find numerous applications in mathematical physics [23]. Several authors have also obtained results involving these operators (see [1]). Here new Hardy-type integral inequalities involving the Erdélyi–Kober fractional integral operators are established, and it is proved that the best possible constants can be obtained by a probabilistic approach.

Let us begin with the following theorem.

Theorem 13. *Suppose that $p > 1$, $r > 0$ and $\xi > 0$. Then the following inequality*

$$\int_0^{\infty} ((I^{\nu}_{r,\xi}f)(x))^p dx < \left\{ \frac{\Gamma\big(\xi - \frac{1}{\nu p}\big)}{\Gamma\big(r + \xi - \frac{1}{\nu p}\big)} \right\}^p \int_0^{\infty} f^p(x)dx \quad \textit{holds,} \tag{46}$$

unless $f \equiv 0$. The constant term $\left\{ \frac{\Gamma(\xi - \frac{1}{\nu p})}{\Gamma(r+\xi-\frac{1}{\nu p})} \right\}^p$ is best possible.

Proof. Let $\eta(t;x)$ be a probability density function on $(0,x)$ for $x > 0$, so by definition $\int_0^x \eta(t;x)dt = 1$. Applying Jensen's inequality and Fubini's theorem, one obtains

$$\begin{aligned}
&\int_0^{\infty} ((I^{\nu}_{r,\xi}f)(x))^p dx \\
&\quad \le \int_0^{\infty} \frac{\nu^p x^{p(-\nu(\xi+\alpha)+\nu)}}{\{\Gamma(r)\}^p} \Big(\int_0^x \frac{(x^{\nu} - t^{\nu})^{p(r-1)} t^{p(\nu\xi-1)} f^p(t)}{\eta^{p-1}(t;x)} dt \Big) dx \\
&\quad = \int_0^{\infty} \frac{\nu^p f^p(t)}{\{\Gamma(r)\}^p} \Big(\int_t^{\infty} \frac{\big(1 - \frac{t^{\nu}}{x^{\nu}}\big)^{p(r-1)} x^{p\nu(r-1)+p\nu(1-\xi-r)} t^{p(\nu\xi-1)}}{\eta^{p-1}(t;x)} dx \Big) dt.
\end{aligned} \tag{47}$$

Choosing $\eta(t;x)$ is a scale distribution on $(0,x)$ as $\eta(t;x)=\frac{1}{x}\eta\big(\frac{t}{x}\big)$ for some density function $\eta(u)$ on $(0,1)$. Replacing the value of $\eta(t;x)$ in (47), one gets

$$\begin{aligned}
&\int_0^\infty \Big((I^\nu_{r,\xi}f)(x)\Big)^p dx \\
&\quad\le \int_0^\infty \frac{\nu^p f^p(t)}{\{\Gamma(r)\}^p}\bigg(\int_t^\infty \frac{x^{p-1+p\nu(r-1)+p\nu(1-\xi-r)}}{\eta^{p-1}\big(\frac{t}{x}\big)} t^{p(\nu\xi-1)}\Big(1-\frac{t^\nu}{x^\nu}\Big)^{p(r-1)} dx\bigg)dt \\
&\quad= \int_0^\infty \frac{\nu^p f^p(t)}{\{\Gamma(r)\}^p}\bigg(\int_0^1 \frac{u^{p(\nu\xi-1)-1}(1-u^\nu)^{p(r-1)}}{\eta^{p-1}(u)} du\bigg)dt \ \text{ by putting } u=\frac{t}{x} \\
&\quad= \frac{\nu^p}{\{\Gamma(r)\}^p}\int_0^1 \frac{u^{p(\nu\xi-1)-1}(1-u^\nu)^{p(r-1)}}{\eta^{p-1}(u)} du \int_0^\infty f^p(t)dt. \qquad (48)
\end{aligned}$$

Let us consider a *p. d. f.* $\zeta(u)$ on $(0,1)$ as $\zeta(u)=\frac{\nu u^{\nu\xi-\frac{1}{p}-1}(1-u^\nu)^{r-1}}{B(\xi-\frac{1}{p\nu},r)}$. Then inequality (14) becomes

$$\begin{aligned}
&\frac{\nu^p}{\{\Gamma(r)\}^p}\int_0^1 \frac{u^{p(\nu\xi-1)-1}(1-u^\nu)^{p(r-1)}}{\eta^{p-1}(u)} du \\
&\qquad\ge \frac{1}{\{\Gamma(r)\}^p}\Big\{B\Big(\xi-\frac{1}{p\nu},r\Big)\Big\}^p = \left\{\frac{\Gamma\big(\xi-\frac{1}{\nu p}\big)}{\Gamma\big(r+\xi-\frac{1}{\nu p}\big)}\right\}^p,
\end{aligned}$$

which shows that the associated constant term in inequality (48) is best possible. Equality holds when $\eta(u)=\frac{\nu u^{\nu\xi-\frac{1}{p}-1}(1-u^\nu)^{r-1}}{B(\xi-\frac{1}{p\nu},r)}$ and this completes the proof of inequality (46), that is Theorem 13. ■

Now the following theorem involving the fractional integral operator $J^\nu_{r,\eta}$ will be proved.

Theorem 14. *Suppose that $p>1$, $r>0$ and $\eta>0$. Then*

$$\int_0^\infty ((J^\nu_{r,\eta}f)(x))^p dx < \left\{\frac{\Gamma\big(\eta+\frac{1}{\nu p}\big)}{\Gamma\big(r+\eta+\frac{1}{\nu p}\big)}\right\}^p \int_0^\infty f^p(x)dx \ \textit{holds}, \qquad (49)$$

unless $f\equiv 0$. The constant term $\left\{\frac{\Gamma(\eta+\frac{1}{\nu p})}{\Gamma(r+\eta+\frac{1}{\nu p})}\right\}^p$ is best possible.

Proof. Let $\eta(t;x)$ be a probability density function on (x,∞) for $x>0$, so by definition $\int_x^\infty \eta(t;x)dt=1$.

Applying Jensen's inequality and Fubini's theorem, one obtains

$$\begin{aligned}\int_0^\infty &((J^{\nu}_{r,\eta}f)(x))^p dx\\ &\le \int_0^\infty \frac{\nu^p x^{p\nu\eta}}{\{\Gamma(r)\}^p}\Big(\int_x^\infty \frac{(t^\nu - x^\nu)^{p(r-1)} t^{p(-\nu(r+\eta)+\nu-1)} f^p(t)}{\eta^{p-1}(t;x)} dt\Big) dx\\ &= \int_0^\infty \frac{\nu^p f^p(t)}{\{\Gamma(r)\}^p}\Big(\int_0^t \frac{\big(1-\frac{x^\nu}{t^\nu}\big)^{p(r-1)} x^{p\nu\eta} t^{p\nu(r-1)-p\nu(r+\eta)+p\nu-p}}{\eta^{p-1}(t;x)} dx\Big) dt. \qquad (50)\end{aligned}$$

Choosing $\eta(t;x)$ is a scale distribution on (x,∞) as $\eta(t;x) = \frac{x}{t^2}\eta\big(\frac{x}{t}\big)$ for some density function $\eta(u)$ on $(0,1)$. Inserting the value of $\eta(t;x)$ in (50), one obtains

$$\begin{aligned}\int_0^\infty &((J^{\nu}_{r,\eta}f)(x))^p\\ &< \int_0^\infty \frac{\nu^p f^p(t)}{\{\Gamma(r)\}^p}\Big(\int_0^t \frac{x^{p\nu\eta-p+1}}{\eta^{p-1}(\frac{t}{x})} t^{-p\nu\eta+p-2}\Big(1-\frac{x^\nu}{t^\nu}\Big)^{p(r-1)} dx\Big) dt\\ &= \int_0^\infty \frac{\nu^p f^p(t)}{t\{\Gamma(r)\}^p}\Big(\int_0^1 \frac{u^{p\nu\eta-p+1}(1-u^\nu)^{p(r-1)}}{\eta^{p-1}(u)} du\Big) dt \ \text{ by putting } u = \frac{x}{t}\\ &= \frac{\nu^p}{\{\Gamma(r)\}^p}\int_0^1 \frac{u^{p\nu\eta-p+1}(1-u^\nu)^{p(r-1)}}{\eta^{p-1}(u)} du \int_0^\infty f^p(t) dt. \qquad (51)\end{aligned}$$

Consider a $p.\,d.\,f.$ $\zeta(u)$ on $(0,1)$ as $\zeta(u) = \frac{\nu u^{\nu\eta+\frac{1}{p}-1}(1-u^\nu)^{r-1}}{B(\eta+\frac{1}{p\nu},r)}$. Then inequality (14) assumes the form

$$\begin{aligned}\frac{\nu^p}{\{\Gamma(r)\}^p}\int_0^1 \frac{u^{p\nu\eta-p+1}(1-u^\nu)^{p(r-1)}}{\eta^{p-1}(u)} du &\ge \frac{1}{\{\Gamma(r)\}^p}\Big\{B\Big(\eta+\frac{1}{p\nu},r\Big)\Big\}^p\\ &= \left\{\frac{\Gamma\big(\eta+\frac{1}{\nu p}\big)}{\Gamma\big(r+\eta+\frac{1}{\nu p}\big)}\right\}^p. \qquad (52)\end{aligned}$$

Inequality (52) shows that the constant term $\left\{\frac{\Gamma(\eta+\frac{1}{\nu p})}{\Gamma(r+\eta+\frac{1}{\nu p})}\right\}^p$ in inequality (51) and hence in inequality (49) is best possible. Equality holds when $\eta(u) = \zeta(u)$ and this completes proof of the theorem. ■

3.4. (k,ν)-Riemann-Liouville fractional integral of order r

Let $k > 0$, $r > 0$, ν be a non-zero real number and $f(x)$ be a continuous function defined on $[0,x]$. The (k,ν)-Riemann–Liouville fractional integral of $f(x)$ of order

r is then denoted by ${}^{\nu}_{k}I^r f(x)$ and defined as follows [22]:

$$ {}^{\nu}_{k}I^r f(x) = \frac{\nu^{1-\frac{r}{k}} x^{-\frac{\nu r}{k}}}{k\Gamma_k(r)} \int_0^x (x^\nu - t^\nu)^{\frac{r}{k}-1} t^{\nu-1} f(t)dt, $$

where $\Gamma_k(x)$ is a k-Gamma function as defined below:

$$ \Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt. $$

The relation between the k-Beta function $B_k(x,y)$ and the k-Gamma function is $B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$, where $B_k(x,y) = \frac{1}{k}B\big(\frac{x}{k}, \frac{y}{k}\big)$, the original Beta function. Now a new Hardy-type inequalty involving the r-th order (k,ν)-Riemann–Liouville fractional integral operator is presented. Indeed, the following theorem is proved:

Theorem 15. *Suppose that $f(x)$ $(f \not\equiv 0)$ is a continuous function defined on $[0,x]$ and $p > 1$. Then the following holds:*

$$ \int_0^\infty ({}^{\nu}_{k}I^r f(x))^p dx < \left\{ \frac{\nu^{-\frac{r}{k}} \Gamma_k\big(k(1-\frac{1}{p\nu})\big)}{\Gamma_k\big(r + k(1-\frac{1}{p\nu})\big)} \right\}^p \int_0^\infty f^p(x)dx \quad \textit{unless } f \equiv 0. \tag{53} $$

The constant term $\left\{ \frac{\nu^{-\frac{r}{k}} \Gamma_k(k(1-\frac{1}{p\nu}))}{\Gamma_k(r+k(1-\frac{1}{p\nu}))} \right\}^p$ *is sharp.*

Proof. Let $\eta(t;x)$ be a probability density function on $(0,x)$ for $x > 0$. Applying Jensen's inequality and Fubini's theorem, one obtains

$$ \begin{aligned} &\int_0^\infty ({}^{\nu}_{k}I^r f(x))^p dx \\ &\quad \le \int_0^\infty \frac{\nu^{p(1-\frac{r}{k})} x^{-\frac{p\nu r}{k}}}{k^p \Gamma_k^p(r)} \int_0^x (x^\nu - t^\nu)^{p(\frac{r}{k}-1)} t^{p(\nu-1)} \frac{f^p(t)}{\eta^{p-1}(t;x)} dtdx \\ &\quad = \int_0^\infty \frac{\nu^{p(1-\frac{r}{k})} f^p(t)}{k^p \Gamma_k^p(r)} \bigg(\int_t^\infty \frac{\big(1 - \frac{t^\nu}{x^\nu}\big)^{p(\frac{r}{k}-1)} x^{-\frac{p\nu r}{k} + \frac{p\nu r}{k} - p\nu} t^{p(\nu-1)}}{\eta^{p-1}(t;x)} dx \bigg) dt. \end{aligned} \tag{54} $$

Now choosing $\eta(t;x)$ is a scale distribution on $(0,x)$ as $\eta(t;x) = \frac{1}{x}\eta(t/x)$ for some density function $\eta(u)$ on $(0,1)$. Substituting the value of $\eta(t;x)$ in (54), one gets

$$ \begin{aligned} &\int_0^\infty ({}^{\nu}_{k}I^r f(x))^p dx \\ &\quad \le \int_0^\infty \frac{\nu^{p(1-\frac{r}{k})} f^p(t)}{k^p \Gamma_k^p(r)} \bigg(\int_t^\infty \frac{\big(1 - \frac{t^\nu}{x^\nu}\big)^{p(\frac{r}{k}-1)} x^{-p\nu+p-1} t^{p(\nu-1)}}{\eta^{p-1}(t/x)} dx \bigg) dt. \end{aligned} \tag{55} $$

Changing the variable in inequality (55) using the transformation $u = t/x$, one gets

$$\int_0^\infty ({}_k^\nu I^r f(x))^p dx \le \int_0^\infty \frac{\nu^{p(1-\frac{r}{k})} f^p(t)}{k^p \Gamma_k^p(r)} \Big(\int_0^1 \frac{u^{p\nu-p-1}(1-u^\nu)^{p(\frac{r}{k}-1)}}{\eta^{p-1}(u)} du \Big) dt$$
$$= \frac{\nu^{p(1-\frac{r}{k})}}{k^p \Gamma_k^p(r)} \int_0^1 \frac{u^{p\nu-p-1}(1-u^\nu)^{p(\frac{r}{k}-1)}}{\eta^{p-1}(u)} du \int_0^\infty f^p(t) dt. \qquad (56)$$

This proves that inequality (53) holds with the constant term

$$\frac{\nu^{p(1-\frac{r}{k})}}{k^p \Gamma_k^p(r)} \int_0^1 u^{p\nu-p-1}(1-u^\nu)^{p(\frac{r}{k}-1)} du.$$

To prove that the associated constant term is sharp, we choose a *p. d. f.* $\zeta(u)$ on $(0,1)$ as $\zeta(u) = \frac{\nu u^{\nu-\frac{1}{p}-1}(1-u^\nu)^{\frac{r}{k}-1}}{B(1-\frac{1}{p\nu},\frac{r}{k})}$. Then inequality (14) reduces to the following form:

$$\frac{\nu^{p(1-\frac{r}{k})}}{k^p \Gamma_k^p(r)} \int_0^1 \frac{u^{p\nu-p-1}(1-u^\nu)^{p(\frac{r}{k}-1)}}{\eta^{p-1}(u)} du \ge \frac{\nu^{p(1-\frac{r}{k})}}{k^p \Gamma_k^p(r)} \left\{ \frac{B\big(1-\frac{1}{p\nu},\frac{r}{k}\big)}{\nu} \right\}^p$$
$$= \left\{ \frac{\nu^{-\frac{r}{k}} \Gamma_k\big(k(1-\frac{1}{\nu p})\big)}{\Gamma_k\big(r + k(1-\frac{1}{\nu p})\big)} \right\}^p. \qquad (57)$$

Inequality (57) says that the constant term $\left\{ \frac{\nu^{-\frac{r}{k}} \Gamma_k(k(1-\frac{1}{\nu p}))}{\Gamma_k(r+k(1-\frac{1}{\nu p}))} \right\}^p$ in inequality (56) and hence in inequality (53) is sharp. This completes the proof. ∎

3.5. (k,ν)-Weyl fractional integral of order r

Let $k > 0$, $r > 0$, ν be a non-zero real number and $f(x)$ be a continuous function defined on $[x,\infty)$, $x > 0$. Then the (k,ν)-Weyl fractional integral ${}_k^\nu J^r f(x)$ of $f(x)$ of order r is introduced in [2] and defined as follows:

$${}_k^\nu J^r f(x) = \frac{\nu^{1-\frac{r}{k}}}{k\Gamma_k(r)} \int_x^\infty (t^\nu - x^\nu)^{\frac{r}{k}-1} t^{\nu-1} f(t) dt,$$

where $\Gamma_k(x)$ is the k-Gamma function as defined in the previous sub-section. Now a Hardy-type inequality involving the (k,ν)-Weyl fractional integral of order r is presented in the following theorem.

Theorem 16. *Suppose that $f(x)$ $(f \not\equiv 0)$ is a continuous function defined on $[x,\infty)$ for $x > 0$ and $p > 1$. Then*

$$\int_0^\infty ({}_k^\nu J^r f(x))^p dx < \left\{ \frac{\nu^{-\frac{r}{k}} \Gamma_k\big(\frac{k}{p\nu}\big)}{\Gamma_k\big(r + \frac{k}{p\nu}\big)} \right\}^p \int_0^\infty \big(x^{\frac{\nu r}{k}} f(x)\big)^p dx \quad \textit{holds unless } f \equiv 0, \qquad (58)$$

where the constant term $\left\{\frac{\nu^{-\frac{r}{k}}\Gamma_k\left(\frac{k}{p\nu}\right)}{\Gamma_k\left(r+\frac{k}{p\nu}\right)}\right\}^p$ *is best possible.*

Proof. Let $\eta(t;x)$ be a *p. d. f.* on (x,∞) for $x>0$. Applying Jensen's inequality and Fubini's theorem, one gets

$$\begin{aligned}\int_0^\infty &({}_k^\nu J^r f(x))^p dx\\ &\le \int_0^\infty \frac{\nu^{p(1-\frac{r}{k})}}{k^p\Gamma_k^p(r)}\int_x^\infty (t^\nu-x^\nu)^{p(\frac{r}{k}-1)}t^{p(\nu-1)}\frac{f^p(t)}{\eta^{p-1}(t;x)}dtdx\\ &=\int_0^\infty \frac{\nu^{p(1-\frac{r}{k})}f^p(t)}{k^p\Gamma_k^p(r)}\Big(\int_0^t \frac{\left(1-\frac{x^\nu}{t^\nu}\right)^{p(\frac{r}{k}-1)}t^{p\nu(\frac{r}{k}-1)}t^{p(\nu-1)}}{\eta^{p-1}(t;x)}dx\Big)dt. \qquad (59)\end{aligned}$$

Selecting a scale distribution $\eta(t;x)$ on (x,∞) as $\eta(t;x)=\frac{x}{t^2}\eta(x/t)$ for some density function $\eta(u)$ on $(0,1)$ and substituting the value of $\eta(t;x)$ in (59), one obtains

$$\begin{aligned}\int_0^\infty &({}_k^\nu J^r f(x))^p dx\\ &\le \int_0^\infty \frac{\nu^{p(1-\frac{r}{k})}f^p(t)}{k^p\Gamma_k^p(r)}\Big(\int_0^t \frac{\left(1-\frac{x^\nu}{t^\nu}\right)^{p(\frac{r}{k}-1)}t^{p\nu(\frac{r}{k}-1)+p(\nu-1)+2(p-1)}}{x^{p-1}\eta^{p-1}\left(\frac{x}{t}\right)}dx\Big)dt. \qquad (60)\end{aligned}$$

Using the variable transformation $u=\frac{x}{t}$ in inequality (60), one gets

$$\int_0^\infty ({}_k^\nu J^r f(x))^p dx \le \frac{\nu^{p(1-\frac{r}{k})}}{k^p\Gamma_k^p(r)}\int_0^1 \frac{u^{1-p}(1-u^\nu)^{p(\frac{r}{k}-1)}}{\eta^{p-1}(u)}du\int_0^\infty \left(t^{\frac{\nu r}{k}}f(t)\right)^p dt, \qquad (61)$$

which proves inequality (58) with the constant factor

$$\frac{\nu^{p(1-\frac{r}{k})}}{k^p\Gamma_k^p(r)}\int_0^1 \frac{u^{1-p}(1-u^\nu)^{p(\frac{r}{k}-1)}}{\eta^{p-1}(u)}du.$$

For proving that the associated constant factor is best possible, a *p. d. f.* $\zeta(u)=\frac{\nu u^{\frac{1}{p}-1}(1-u^\nu)^{\frac{r}{k}-1}}{B(\frac{1}{p\nu},\frac{r}{k})}$ on $(0,1)$ is chosen. Then inequality (14) simplifies to the following form:

$$\frac{\nu^{p(1-\frac{r}{k})}}{k^p\Gamma_k^p(r)}\int_0^1 \frac{u^{1-p}(1-u^\nu)^{p(\frac{r}{k}-1)}}{\eta^{p-1}(u)}du \ge \frac{\nu^{p(1-\frac{r}{k})}}{k^p\Gamma_k^p(r)}\left\{\frac{B\left(\frac{1}{p\nu},\frac{r}{k}\right)}{\nu}\right\}^p=\left\{\frac{\nu^{-\frac{r}{k}}\Gamma_k\left(\frac{k}{p\nu}\right)}{\Gamma_k\left(r+\frac{k}{p\nu}\right)}\right\}^p,$$

which implies that the constant term $\left\{\frac{\nu^{-\frac{r}{k}}\Gamma_k(\frac{k}{p\nu})}{\Gamma_k(r+\frac{k}{p\nu})}\right\}^p$ in inequality (61), and hence inequality (58) is best possible. ∎

4. Conclusions

An attempt has been made to give a probabilistic proof of the classical integral inequalities of Hardy [10,13], Copson [8], Bennett [4], Leindler [16,18] and Levinson [20]. Several new integral inequalities of Hardy-type are also established using a probabilistic approach. The associated constant factor in each of these inequalities is sharp, and derived by using the Kullback–Leibler inequality. The approach may be considered for proving and establishing other well-known and new integral inequalities.

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