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A survey on Tingley's problem for operator algebras

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Abstract. We survey the most recent results on the extension of isometries between special subsets of the unit spheres of C^*-algebras, von Neumann algebras, trace class operators, preduals of von Neumann algebras, and p-Schatten–von Neumann spaces, with special interest on Tingley's problem.

1. Introduction

The problem of extending a surjective isometry between two subsets of the unit spheres of two operator algebras was treated in several talks during the conference on preservers problems held in Szeged in June 2017. The conference "Preservers Everywhere" gathered a substantial group of world experts on these topics. It became clear that the problems regarding the extension of this type of surjective isometries constitute an intensively studied line in recent times. Let us try to unify all these problems in the following statement.

Problem 1.1. Let X and Y be two Banach spaces whose unit spheres are denoted by $S(X)$ and $S(Y)$, respectively. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subsets of $S(X)$ and $S(Y)$, respectively. Suppose $\Delta\colon \mathcal{S}_1 \to \mathcal{S}_2$ is a surjective isometry. Does Δ extend to a real linear isometry from X onto Y?

Henceforth, we shall write $\mathbb{T}$ for the unit sphere of $\mathbb{C}$. The complex conjugation on $\mathbb{T}$ cannot be extended to a complex linear isometry on $\mathbb{C}$. So, in the case of complex Banach spaces, a complex linear extension is simply hopeless for all cases. Similar constraints will appear in subsequent results.

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These problems, whose origins are in geometry, are nowadays a central topic for those researchers working on preservers. If in Problem 1.1 we consider $\mathcal{S}_1 = S(X)$ and $\mathcal{S}_2 = S(Y)$, we meet the so-called *Tingley's problem*. This problem was named after the contribution of D. Tingley, who established that for any two finite dimensional Banach spaces X and Y, every surjective isometry $\Delta\colon S(X) \to S(Y)$ preserves antipodal points, that is, $\Delta(-x) = -\Delta(x)$, for every x in $S(X)$ (see [54, THEOREM in page 377]). Tingley's problem remains open even in the case of two-dimensional Banach spaces.

Let us observe that, given a surjective isometry $\Delta\colon S(X) \to S(Y)$, where X and Y are Banach spaces, we can always consider the natural (positively) homogeneous extension $F_\Delta\colon X \to Y$ given by $F_\Delta(0) = 0$, and $F_\Delta(x) = \|x\|\Delta\left(\frac{x}{\|x\|}\right)$ for $x \neq 0$. Clearly, F_Δ is a bijection, however it is a hard question to decide whether F_Δ is an isometry. Actually, the Mazur–Ulam theorem implies that F_Δ is real linear as soon as it is an isometry.

We have already found our first connection with the Mazur–Ulam theorem. Tingley's problem and Problem 1.1 can be considered as generalization of this pioneering result in Functional Analysis. P. Mankiewicz established in 1972 an intermediate result which provides a useful tool for our purposes.

Theorem 1.2. ([32, Theorem 5 and Remark 7]) *Every bijective isometry between convex sets in normed linear spaces with nonempty interiors admits a unique extension to a bijective affine isometry between the corresponding spaces.*

During the thirty years that elapsed after Tingley's paper, a lot of hard efforts from many authors, especially many Chinese mathematicians, and the elite Chinese group led by G.G. Ding, have been conducted in the seeking of a solution to Tingley's problem in concrete spaces. The huge contribution due to mathematicians like R.S. Wang, G.G. Ding, D. Tan, L. Cheng, Y. Dong, X.N. Fang, J.H. Wang, and R. Liu, among others, have been overviewed in full detail in the excellent surveys published by G.G. Ding [12] and X. Yang and X. Zhao [56].

A reborn interest in the problems concerning the extension of isometries between subsets of the unit spheres of two operator algebras has been materialized in a fruitful series of recent papers dealing with Tingley's problem and related questions for certain operator algebras, which have been published during the short interval determined by the last three years. The abundance of new results for operator algebras motivates and justifies the writing of this survey with the aim of completing and updating the surveys [12, 56], and providing a recent state of the art of these problems. The real "avalanche" of recent achievements provides enough material to write a new and detailed survey on this topic.

We strive for conciseness and restrict the results to the setting of operator algebras, despite the fact that some of the results have been already extended to the strictly wider setting of JB*-triples (compare [25, 27]). So, few or none of the proofs are explicitly included. The main tools and results are reviewed with a full bibliographic information. We shall also insert some new arguments to establish some additional statements.

In Section 2 we gather some of the key tools applied in many of the proofs given to solve Tingley's problem. Most of the studies make use of a result, which was originally established by L. Cheng, Y. Dong in [5], and proves that a surjective isometry $\Delta\colon S(X) \to S(Y)$ between the unit spheres of two Banach spaces maps maximal proper faces of the closed unit ball of X to maximal proper proper faces of the closed unit ball of Y (see Theorem 2.2). The section also contains a recent generalization of this result due to F.J. Fenández-Polo, J. Garcés, I. Villanueva and the author of this note, which assures the following: Let $\Delta\colon S(X) \to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces, and suppose that these spaces satisfy the following two properties:

(h.1) Every norm closed face of $\mathcal{B}_X$ (respectively, of $\mathcal{B}_Y$) is norm-semi-exposed.

(h.2) Every weak* closed proper face of $\mathcal{B}_{X^*}$ (respectively, of $\mathcal{B}_{Y^*}$) is weak*-semi-exposed.

Then the following statements hold:

(a) Let $\mathcal{F}$ be a convex set in $S(X)$. Then $\mathcal{F}$ is a norm closed face of $\mathcal{B}_X$ if and only if $\Delta(\mathcal{F})$ is a norm closed face of $\mathcal{B}_Y$.

(b) Let $e \in S(X)$. Then $e \in \partial_e(\mathcal{B}_X)$ if and only if $\Delta(e) \in \partial_e(\mathcal{B}_Y)$

(see Corollary 2.3).

It should be remarked that hypotheses (h.1) and (h.2) above hold whenever X and Y are C^*-algebras, hermitian parts of C^*-algebras, von Neumann algebra preduals, preduals of the hermitian part of a von Neumann algebra, JB*-triples, and JBW*-triple preduals (see [21] and the comments after Corollary 2.3).

In Section 2 we shall also survey the main results on the facial structure of the closed unit ball of a C^*-algebra due to C.A. Akemann and G.K. Pedersen [1] and C.M. Edwards and G.T. Rüttimann [17].

Section 3 is completely devoted to present the most recent achievements on Tingley's problem in the setting of C^*-algebras. In all sections we shall insert an introductory paragraph with the equivalent results in the commutative setting. We begin from the results by R. Tanaka, which assure that every surjective isometry from the unit sphere of a finite-dimensional C^*-algebra A onto the unit sphere of another C^*-algebra B extends to a unique surjective real linear isometry from A onto B, and the same conclusion holds when A and B are finite von Neumann algebras (see Theorem 3.4). In Theorem 3.8 we revisit the solution to Tingley's problem for surjective isometries between the unit spheres of two compact C^*-algebras found by R. Tanaka and the author of this survey in [44]. This solution also covers the case of a surjective isometry between the unit spheres of two $K(H)$ spaces. In this note $K(H)$ and $B(H)$ will denote the spaces of compact and bounded linear operators on a complex Hilbert space H, respectively.

Accordingly to the chronological order, the next step in the study of Tingley's problem on C^*-algebras is a result by F.J. Fernández-Polo and the author of this note, which shows that for any two complex Hilbert spaces H_1 and H_2, every surjective isometry $\Delta\colon S(B(H_1)) \to S(B(H_2))$ admits a unique extension to a surjective complex linear or conjugate linear surjective isometry T from $B(H_1)$ onto $B(H_2)$ (see Theorem 3.9). The most conclusive result on Tingley's problem has been also obtained by the same authors in a result showing that every surjective isometry $\Delta\colon S(M) \to S(N)$ between the unit spheres of two von Neumann algebras M and N, where M is not a factor, or is a finite factor, or a type I factor, or a type II_∞ factor on a separable Hilbert space, or a type III factor on a separable Hilbert space, admits a unique extension to a surjective real linear isometry $T\colon M \to N$. Furthermore, under these hypotheses, there exist a central projection p in N and a Jordan *-isomorphism $J\colon M \to N$ such that defining $T\colon M \to N$ by $T(x) = \Delta(1)\,(pJ(x) + (1-p)J(x)^*)$ $(x \in M)$, then T is a surjective real linear isometry and $T|_{S(M)} = \Delta$ (see Theorem 3.15).

Section 4 is devoted to survey the results on Tingley's problem for surjective isometries between the unit spheres of von Neumann algebra preduals. In [21], F.J. Fernández-Polo, J. Garcés, I. Villanueva and the author of this survey gave a complete solution to Tingley's problem for surjective isometries on the unit sphere of the space $C_1(H)$ of trace class operators on an arbitrary complex Hilbert space H (see Theorem 4.5).

It is well known that the space $C_1(H)$ can be identified with the dual of the space $K(H)$ and with the predual of $B(H)$. It seems a natural question whether the previous positive solution to Tingley's problem in the setting of trace class operators remains true for preduals of general von Neumann algebras.

When the writing of this survey was being completed (precisely, on December 27th, 2017), an alert message came to this author from arxiv. This alert was about a very recent preprint by M. Mori (see [34]), which has been an impressive discovery, and made this author change the original project to insert some nice achievements, one of them is a positive solution to Tingley's problem for preduals of general von Neumann algebras (see Theorem 4.6).

Henceforth, the hermitian part of a C^*-algebra A will be denoted by A_{sa}. As we commented before, when in Problem 1.1 the subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ are the unit spheres of two Banach spaces, we find the so-called Tingley's problem. Another interesting variant of Problem 1.1 is obtained when X and Y are von Neumann algebras or C^*-algebras and $\mathcal{S}_1$ and $\mathcal{S}_2$ are the unit spheres of their respective hermitian parts. In Section 5, we shall study the problem of extending a surjective isometry $\Delta\colon S(M_{sa}) \to S(N_{sa})$, where M and N are von Neumann algebras. In this section we shall show that the same tools given by F.J. Fernández-Polo and the author of this survey in [28] can be, almost literarily, applied to find a surjective complex linear isometry $T\colon M \to N$ satisfying $T(a^*) = T(a)^*$ for all a in M and $T(x) = \Delta(x)$ for all x in $S(M_{sa})$ (see Theorem 5.8).

It should be remarked here that, after completing the writing of this chapter, the preprint by M. Mori [34] became available in arxiv. Section 5 in [34] is devoted to the study of Tingley's problem for surjective isometries between the unit spheres of the hermitian parts of two von Neumann algebras, and our Theorem 5.8 is also established by M. Mori with an alternative proof.

The sixth and final section of this paper is devoted to review the main result on a topic which had its own protagonism in the meeting held in Szeged. We are talking about the problem of extending a surjective isometry between the sets of positive norm-one operators of two type I von Neumann factors $B(H_1)$ and $B(H_2)$. During the talk delivered by G. Nagy in this conference, he presented a recent achievement which shows that, for a finite-dimensional complex Hilbert space H, every isometry $\Delta\colon S(B(H)^+) \to S(B(H)^+)$ admits a (unique) extension to a surjective complex linear isometry $T\colon B(H) \to B(H)$ satisfying $T(x) = \Delta(x)$ for all $x \in S(B(H)^+)$ (see Theorem 6.5), where for a C^*-algebra A, the symbol A^+ will denote the cone of positive elements in A, and $S(A^+)$ will stand for the sphere of positive norm-one operators. It was conjectured by Nagy that the same conclusion holds for every complex Hilbert space H.

We culminate this section, and the results in this note, by surveying a recent work where we provide a proof to Nagy's conjecture. The main result is treated in Theorem 6.10, where it is shown that every surjective isometry $\Delta\colon S(B(H_1)^+) \to S(B(H_2)^+)$, where H_1 and H_2 are complex Hilbert spaces, admits an extension to a surjective complex linear isometry (actually, a *-isomorphism or a *-anti-automorphism) $T\colon B(H_1) \to B(H_2)$.

We shall revisit one of the main tools employed to establish the above result. This tool is a geometric characterization of projections in atomic von Neumann algebras. Let us recall some notation first. Suppose that E and P are non-empty subsets of a Banach space X. Following the notation employed in the recent paper

[43], the *unit sphere around E in P* is the set

$$\mathrm{Sph}(E;P) := \{x \in P : \|x - b\| = 1 \text{ for all } b \in E\}.$$

To simplify the notation, given a C^*-algebra A, and a subset $E \subset A$, we shall write $\mathrm{Sph}^+(E)$ or $\mathrm{Sph}^+_A(E)$ for the set $\mathrm{Sph}(E; S(A^+))$. The geometric characterization of projections reads as follows: Let M be an atomic von Neumann algebra, and let a be a positive norm-one element in M. Then the following statements are equivalent:

(a) a is a projection;

(b) $\mathrm{Sph}^+_M\left(\mathrm{Sph}^+_M(a)\right) = \{a\}$.

(see Theorem 6.6). This characterization also holds when M is replaced by $K(H_3)$, where H_3 is a separable complex Hilbert space (Theorem 6.8). Moreover, if a is a positive norm-one element in an arbitrary C^*-algebra A satisfying $\mathrm{Sph}^+_A\left(\mathrm{Sph}^+_A(a)\right) = \{a\}$, then a is a projection (see [43, Proposition 2.2]). ing This geometric characterization has been also applied to prove that if H_3 and H_4 are separable complex Hilbert spaces, then every surjective isometry

$$\Delta\colon S(K(H_3)^+) \to S(K(H_4)^+)$$

admits a unique extension to a surjective complex linear isometry T from $K(H_3)$ onto $K(H_4)$ (see Theorem 6.9).

2. Geometric background

In this section we survey the basic geometric tools which are frequently applied in most of the studies on extending isometries. The results gathered in this section are established in the general setting of Banach spaces.

A non-empty convex subset F of a convex set C is said to be a *face* of C if $\alpha x + (1-\alpha)y \in F$ with $x, y \in C$ and $0 < \alpha < 1$ implies $x, y \in F$. An element x in the unit sphere of a Banach space X is an *extreme point* of $\mathcal{B}_X$ precisely when the set $\{x\}$ is a face of $\mathcal{B}_X$. Accordingly to the standard notation, from now on, the extreme points of a convex set C will be denoted by $\partial_e(C)$. The Krein–Milman theorem is a fantastic tool to assure the existence and abundance of extreme points in any non-empty compact convex subset of a locally convex, Hausdorff, topological vector space.

Up to now, most of the studies on Tingley's problem have been based on a good and appropriate knowledge of the geometric properties of the involved spaces. This is because the most general geometric conclusion which can be derived from the existence of a surjective isometry between the unit spheres of two Banach spaces is the following result, which was originally established by L. Cheng and Y. Dong

[5], and later rediscovered by R. Tanaka [50, 51]. From now on, given a normed space X, the symbol $\mathcal{B}_X$ will stand for the closed unit ball of X.

Theorem 2.1. ([5, Lemma 5.1], [51, Lemma 3.3], [50, Lemma 3.5]) *Let $\Delta\colon S(X)\to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces, and let $\mathcal{M}$ be a convex subset of $S(X)$. Then $\mathcal{M}$ is a maximal proper face of $\mathcal{B}_X$ if and only if $\Delta(\mathcal{M})$ is a maximal proper (closed) face of $\mathcal{B}_Y$.*

As we commented in the introduction, Tingley's problem remains open even in the case of two-dimensional Banach spaces, the reason, probably, being the lack of a concrete description of the maximal convex subsets of the unit sphere of a general Banach space.

All strategies based on Theorem 2.1 above require a concrete description of the maximal proper norm-closed faces of $\mathcal{B}_X$ in terms of the algebraic or geometric properties of X. This is the point where the results by C. A. Akemann and G.K. Pedersen [1], C. M. Edwards and G. T. Rüttimann [17], C. M. Edwards, F. J. Fernández-Polo, C. Hoskin and A. M. Peralta [14], and F. J. Fernández-Polo and A. M. Peralta [24], describing the facial structure of the closed unit ball of C^*-algebras, von Neumann algebra preduals, JB*-triples and their dual spaces, and JBW*-triples and their preduals, become a useful tool.

We recall now the "facear" and "pre-facear" operations introduced in [17]. For each $F\subseteq\mathcal{B}_X$ and $G\subseteq\mathcal{B}_{X^*}$, we define

$$F' = \{a\in\mathcal{B}_{X^*} : a(x)=1\ \forall x\in F\},\quad G_{\prime} = \{x\in\mathcal{B}_X : a(x)=1\ \forall a\in G\}.$$

Then F' is a weak* closed face of $\mathcal{B}_{X^*}$ and $G_{\prime}$ is a norm closed face of $\mathcal{B}_X$. The subset F is said to be a *norm-semi-exposed face* of $\mathcal{B}_X$ if $F=(F')_{\prime}$, while the subset G is called a *weak*-semi-exposed face* of $\mathcal{B}_{X^*}$ if $G=(G_{\prime})'$. The mappings $F\mapsto F'$ and $G\mapsto G_{\prime}$ are anti-order isomorphisms between the complete lattices $\mathcal{S}_n(\mathcal{B}_X)$ of norm-semi-exposed faces of $\mathcal{B}_X$, and $\mathcal{S}_{w^*}(\mathcal{B}_{X^*})$ of weak*-semi-exposed faces of $\mathcal{B}_{X^*}$ and are inverses of each other.

If in Theorem 2.1 we assume a richer geometric structure on the spaces X and Y, then the conclusion of this result was improved in a recent paper by F.J. Fernández-Polo, J. Garcés, I. Villanueva and the author of this note in [21].

Theorem 2.2. ([21, Proposition 2.4]) *Let $\Delta\colon S(X)\to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces, and let C be a convex subset of $S(X)$. Suppose that for every extreme point ϕ_0 in $\partial_e(\mathcal{B}_{X^*})$, the set $\{\phi_0\}$ is a weak*-semi-exposed face of $\mathcal{B}_{X^*}$. Then C is a norm-semi-exposed face of $\mathcal{B}_X$ if and only if $\Delta(C)$ is a norm-semi-exposed face of $\mathcal{B}_Y$.*

The real interest of the previous theorem is the following corollary.

Corollary 2.3. ([21, Corollary 2.5]) *Let X and Y be Banach spaces satisfying the following two properties:*

(1) *every norm closed face of $\mathcal{B}_X$ (respectively, of $\mathcal{B}_Y$) is norm-semi-exposed;*
(2) *every weak* closed proper face of $\mathcal{B}_{X^*}$ (respectively, of $\mathcal{B}_{Y^*}$) is weak*-semi-exposed.*

Let $\Delta\colon S(X) \to S(Y)$ be a surjective isometry. The following statements hold:

(a) *Let $\mathcal{F}$ be a convex set in $S(X)$. Then $\mathcal{F}$ is a norm closed face of $\mathcal{B}_X$ if and only if $\Delta(\mathcal{F})$ is a norm closed face of $\mathcal{B}_Y$.*
(b) *Let $e \in S(X)$. Then $e \in \partial_e(\mathcal{B}_X)$ if and only if $\Delta(e) \in \partial_e(\mathcal{B}_Y)$.*

As it is observed in [21], the hypotheses of the above corollary hold whenever X and Y are C^*-algebras [1, Theorems 4.10 and 4.11], hermitian parts of C^*-algebras (see [16, Corollary 5.1] and [1, Theorem 3.11]), von Neumann algebra preduals [17, Theorems 5.3 and 5.4], preduals of the hermitian part of a von Neumann algebra (see [15, Theorem 4.4] and [17, Theorem 4.1]), or more generally, JB*-triples (cf. [14, Corollary 3.11] and [24, Corollary 1]), or JBW*-triple preduals [17, Corollaries 4.5 and 4.7].

By extending a result of D. Tingley [54, §4], M. Mori has recently added in [34, Proposition 2.3] more information to the conclusion of the above Corollary 2.3. Actually with similar arguments we can deduce the following result.

Proposition 2.4. *Let $\Delta\colon S(X) \to S(Y)$ be a surjective isometry between the unit spheres of two Banach spaces. Then the following statements hold:*

(a) *If $\mathcal{M}$ is a maximal proper face of $\mathcal{B}_X$, then $\Delta(-\mathcal{M}) = -\Delta(\mathcal{M})$.*
(b) *If X and Y satisfy the hypotheses of Corollary 2.3, then $\Delta(-F) = -\Delta(F)$ for every proper norm closed face of $\mathcal{B}_X$.*

The elements a, b in a C^*-algebra A are said to be orthogonal if $ab^* = b^*a = 0$. The set of partial isometries in A can be equipped with a partial order defined by $e \leq v$ if $v-e$ is a partial isometry orthogonal to e, equivalently, $v = e+(1-ee^*)v(1-v^*v)$.

This seems to be an optimal moment to recall the facial structure of the closed unit ball of a C^*-algebra. We recall first some basic notions required to understand the results. Let A be a C^*-algebra. It was shown by Akemann and Pedersen in [1] that norm closed faces of $\mathcal{B}_A$ are in one-to-one correspondence with the compact partial isometries in A^{**}. Let us recall that a projection p in A^{**} is said to be *open* if $A \cap (pA^{**}p)$ is weak* dense in $pA^{**}p$, equivalently, there exists an increasing net of positive elements in A, all of them bounded by p, converging to p in the strong* topology of A^{**} (see [42, §3.11], [47, §III.6 and Corollary III.6.20]). A projection

$p \in A^{**}$ is called *closed* if $1-p$ is open. A closed projection p in A^{**} is called *compact* if $p \leq x$ for some norm-one positive element $x \in A$.

Compact partial isometries in the bidual of a C*-algebra were studied by C.M. Edwards and G.T. Rüttimann in [18, §5] as an application of the more general notion of compact tripotent in the bidual of a JB*-triple. C.A. Akemann and G.K. Pedersen consider an alternative term for the same notion. A partial isometry $v \in A^{**}$ *belongs locally to* A if v^*v is a compact projection and there exists a norm-one element x in A satisfying $v = xv^*v$ (compare [1, Remark 4.7]). It was shown by C.A. Akemann and G.K. Pedersen that a partial isometry v in A^{**} belongs locally to A if and only if v^* belongs locally to A (see [1, Lemma 4.8]). We know from [18, Theorem 5.1] that a partial isometry v in A^{**} belongs locally to A if and only if it is compact in the sense introduced in [18].

Akemann and Pedersen gave in [1, Lemma 4.8 and Remark 4.11] an interesting procedure to understand well those partial isometries in A^{**} belonging locally to A. Borrowing a paragraph from the just quoted paper we recall that "the partial isometries v in A^{**} that belong locally to A are obtained by taking an element x in A with norm 1 and polar decomposition $x = u|x|$ (in A^{**}), and then letting $v = ue$ for some compact projection e contained in the spectral projection $\chi_{\{1\}}(|x|)$ of $|x|$ corresponding to the eigenvalue 1." Similarly to most of the basic references, for each element x in A we set $|x| = (x^*x)^{\frac{1}{2}}$.

We are now in a position to revisit the results by C.A. Akemann and G.K. Pedersen.

Theorem 2.5. ([1, Theorems 4.10 and 4.11]) *Let A be a C^*-algebra. The following statements hold:*

(a) *For each norm closed face F of $\mathcal{B}_A$ there exists a unique partial isometry v in A^{**} belonging locally to A such that*

$$F = F_v = \{v\}_{\prime\prime} = (v + (1-vv^*)\mathcal{B}_{A^{**}}(1-v^*v)) \cap \mathcal{B}_A = \{x \in \mathcal{B}_A : \ xv^* = vv^*\}.$$

Furthermore, the mapping $v \mapsto F_v$ is an anti-order isomorphism from the complete lattice of partial isometries in A^{**} belonging locally to A onto the complete lattice of norm closed faces of $\mathcal{B}_A$.

(b) *For each weak* closed face $\mathcal{G}$ of $\mathcal{B}_{A^*}$ there exists a unique partial isometry v in A^{**} belonging locally to A such that $\mathcal{G} = \{v\}_{\prime}$, and the mapping $v \mapsto \{v\}_{\prime}$ is an order isomorphism from the complete lattice of partial isometries in A^{**} belonging locally to A onto the complete lattice of weak* closed faces of $\mathcal{B}_{A^*}$.*

A non-zero projection p in a C^*-algebra A is called *minimal* if $pAp = \mathbb{C}p$. A non-zero partial isometry e in a C^*-algebra A is *minimal* if ee^* (equivalently, e^*e)

is a minimal projection in A. By Kadison's transitivity theorem, minimal partial isometries in A^{**} belong locally to A, and hence every maximal proper face of the unit ball of a C^*-algebra A is of the form

$$(v + (1 - vv^*)\mathcal{B}_{A^{**}}(1 - v^*v)) \cap \mathcal{B}_A \tag{1}$$

for a unique minimal partial isometry v in A^{**} (compare [1, Remark 5.4 and Corollary 5.5]).

Another technical result of geometric nature, which is frequently applied in the study of Tingley's problem and should be considered in any survey on this topic, was established by X.N. Fang, J.H. Wang and G.G. Ding in [20] and [11], respectively.

Theorem 2.6. ([20, Corollary 2.2], [11, Corollary 1]) *Let X and Y be normed spaces and let $\Delta\colon S(X) \to S(Y)$ be a surjective isometry. Then, for any x, y in $S(X)$, we have $\|x + y\| = 2$ if and only if $\|\Delta(x) + \Delta(y)\| = 2$.*

This result plays a role, for example, in some of the proofs in [21, 43].

2.1. A taste of Jordan structures

Many recent advances on Tingley's problem and its variants on C^*-algebras make an explicit use of the Jordan theory of JB*-triples (see, for example, the proofs in [26–28, 44] and [21]). Although we are not going to enter in the deep details of the proofs, it seems convenient to recall the basic notions and connections with this theory.

We recall that, according to the definition introduced in [31], a *JB*-triple* is a complex Banach space E admitting a continuous triple product $\{a, b, c\}$ which is conjugate linear in b and linear and symmetric in a and c, and satisfies the following axioms:

(JB*1) $L(a, b)L(c, d) - L(c, d)L(a, b) = L(L(a, b)(c), d) - L(c, L(b, a)(d))$, for every a, b, c, d in E, where $L(a, b)$ is the operator on E defined by $L(a, b)(x) = \{a, b, x\}$;

(JB*2) $L(a, a)$ is an hermitian operator on E with non-negative spectrum;

(JB*3) $\|\{a, a, a\}\| = \|a\|^3$, for every $a \in E$.

Examples of JB*-triples include the space $B(H, H')$ of bounded linear operators and the space $K(H, H')$ of compact operators between two complex Hilbert spaces, complex Hilbert spaces, and all C^*-algebras when equipped with the triple product defined by $\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x)$. JB*-triples constitute a category which produces a Jordan model valid to generalize C^*-algebras. Every JB*-algebra is a JB*-triple under the triple product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

For the basic notions and results on JB*-triples the reader is referred to the monograph [6].

A linear mapping between JB*-triples is called a triple homomorphism if it preserves triple products. Surjective real linear isometries between C^*-algebras and JB*-triples are deeply connected to triple isomorphisms (see [7, 8] and [23]). Many of the results in this survey can be complemented with a good description of the real triple isomorphisms between von Neumann algebras. Let us add that real linear triple isomorphisms play a fundamental role in the original proofs of the main results in [26–28, 44].

3. Tingley's problem on C^*-algebras

Tingley's problem for surjective isometries between the unit spheres of two commutative C^*-algebras are completely covered by the results for $C_0(L)$-spaces [55], $\ell^\infty(\Gamma)$-spaces [9], and $L^p(\Omega, \Sigma, \mu)$ spaces [48]. It should be remarked that in [9] and [48] the authors only consider real sequences and real-valued measurable functions, respectively, that is, their results are restricted to the hermitian parts of the corresponding C^*-algebras.

According to the chronological order, and for our own convenience, we highlight a pioneering result due to R.S. Wang. Let us recall the prototype example of commutative C^*-algebras. Given a locally compact Hausdorff space L, we shall write $C_0(L)$ for the commutative C^*-algebra of all complex-valued continuous functions on L which vanish at infinity.

Theorem 3.1. ([55]) *Let L_1 and L_2 be two locally compact Hausdorff spaces, and let $\Delta\colon S(C_0(L_1)) \to S(C_0(L_2))$ be a surjective isometry. Then there exists a real linear surjective isometry $T\colon C_0(L_1) \to C_0(L_2)$ satisfying $T|_{S(C_0(L_1))} = \Delta$. Furthermore, there exist two disjoint subsets A and B of L_1 such that $A \cup B = L_1$, $T|_{C_0(A)}$ is complex linear, and $T|_{C_0(B)}$ is conjugate linear, where $C_0(A) = \{f \in C_0(L_1) : f|_B \equiv 0\}$, and $C_0(B) = \{f \in C_0(L_1) : f|_A \equiv 0\}$.*

Wang's theorem, whose proof is based on Urysohn's lemma and fine geometric arguments, solves Tingley's problem for commutative C^*-algebras. Actually, if $\Delta\colon S(\ell_\infty(\Gamma_1)) \to S(\ell_\infty(\Gamma_2))$ (respectively, $\Delta\colon S(c(\Gamma_1)) \to S(c(\Gamma_2))$, or $\Delta\colon S(c_0(\Gamma_1)) \to S(c_0(\Gamma_2))$) is a surjective isometry, then we can always find an extension to a surjective real linear isometry between the corresponding spaces, where $c_0(\Gamma)$, $c(\Gamma_1)$, and $\ell_\infty(\Gamma)$ denote the spaces of all complex null, convergent, and bounded functions on Γ, respectively. A similar conclusion holds for a surjective isometry $\Delta\colon S(L^\infty(\Omega,\Sigma,\mu)) \to S(L^\infty(\Omega,\Sigma,\mu))$.

The previous result reveals the importance of considering real linear surjective isometries between $C_0(L)$ spaces. A generalization of the Banach–Stone theorem to real linear surjective isometries (see [19] and [33]) assures that for each surjective real linear isometry $T\colon C_0(L_1) \to C_0(L_2)$ there exist a homeomorphism $\varphi\colon L_2 \to L_1$, a clopen subset K_2 of L_2, and a unitary continuous function $u\colon L_2 \to \mathbb{C}$ such that

$$T(f)(s) = u(s)\ f(\varphi(s)), \quad \forall f \in C_0(L_1), s \in K_2,$$

and

$$T(f)(s) = u(s)\ \overline{f(\varphi(s))}, \quad \forall f \in C_0(L_1), s \in L_2\backslash K_2.$$

Having this theorem in mind, the conclusion in [9] can be explicitly obtained as a consequence of the above Theorem 3.1.

In 2014, 2016, and 2017, R. Tanaka publishes the first achievements on Tingley's problem for surjective isometries between the unit spheres of two non-commutative C^*-algebras; his results focus on finite-dimensional C^*-algebras, and more generally on finite von Neumann algebras (see [50–53]). From now on, we shall write $M_n(\mathbb{C})$ for the space of all $n\times n$ matrices with complex entries.

Theorem 3.2. ([51, Theorem 6.1]) *Let $\Delta\colon S(M_n(\mathbb{C})) \to S(M_n(\mathbb{C}))$ be a surjective isometry. Then Δ admits a (unique) extension to a complex linear or to a conjugate linear surjective isometry on $M_n(\mathbb{C})$. Furthermore, there exist a complex linear or conjugate linear *-automorphism $\Phi\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and a unitary matrix u in $M_n(\mathbb{C})$ such that one of the next statements holds:*
(a) $\Delta(x) = u\Phi(x)$, *for all* $x \in S(M_n(\mathbb{C}))$;
(b) $\Delta(x) = u\Phi(x)^*$, *for all* $x \in S(M_n(\mathbb{C}))$.

Again surjective real linear isometries seem to be behind the results. The proof of the above Theorem 3.2 is based on the following well-known fact: The extreme points of the closed unit ball of $M_n(\mathbb{C})$ are precisely the unitary matrices in $M_n(\mathbb{C})$. Let $\mathcal{U}_n$ denote the set of all unitary matrices in $M_n(\mathbb{C})$. It follows from the above fact and from Corollary 2.3 that a surjective isometry $\Delta\colon S(M_n(\mathbb{C})) \to S(M_n(\mathbb{C}))$ maps $\mathcal{U}_n$ onto itself, and thus the restriction $\Delta|_{\mathcal{U}_n} : \mathcal{U}_n \to \mathcal{U}_n$ gives a surjective isometry too. Similar conclusions also hold when $M_n(\mathbb{C})$ is replaced by a finite-dimensional C^*-algebra, or more generally, by a finite von Neumann algebra. We are naturally lead to an outstanding theorem due to O. Hatori and L. Molnár.

Theorem 3.3. ([29, Corollary 3]) *Every surjective isometry between the unitary groups of two von Neumann algebras extends to a surjective real linear isometry between the von Neumann algebras. More concretely, let M_1 and M_2 be von Neumann algebras whose unitary groups are denoted by $\mathcal{U}_1$ and $\mathcal{U}_2$. Let $\Upsilon\colon \mathcal{U}_1 \to \mathcal{U}_2$ be a bijection. Then Υ is a surjective isometry if and only if there exist a central projection $p \in M_2$ and a Jordan *-isomorphism $\Phi\colon M_1 \to M_2$ such that*

$$\Upsilon(u) = \Upsilon(1)(p\ \Phi(u) + (1-p)\ \Phi(u)^*), \quad \textit{for all } u \in \mathcal{U}_1.$$

R.V. Kadison and G.K. Pedersen showed in [30] that every element in a finite von Neumann algebra M can be expressed as the convex combination (actually as the midpoint) of two unitary elements in M. Tanaka's arguments rely on the facial structure of von Neumann algebras and the property of preservation of midpoints between unitary elements. By these arguments the above Theorem 3.2 was generalized by R. Tanaka in the following form:

Theorem 3.4. ([53, Theorem 4.2] and [52]) *Let $\Delta\colon S(M_1) \to S(M_2)$ be a surjective isometry, where M_1 and M_2 are finite von Neumann algebras. There exists a surjective real linear isometry $T\colon M_1 \to M_2$ satisfying $\Delta(a) = T(a)$ for all $a \in S(M_1)$. More concretely, we can find a central projection $p \in M_2$ and a Jordan *-isomorphism $\Phi\colon M_1 \to M_2$ such that*

$$\Delta(a) = \Delta(1)(p\ \Phi(a) + (1-p)\ \Phi(a)^*),$$

for all $a \in S(M_1)$. The same conclusion holds when $\Delta\colon S(A) \to S(B)$ is a surjective isometry from the unit sphere of a finite-dimensional C^*-algebra onto the unit sphere of another C^*-algebra.

The Hatori–Molnár theorem is applied by Tanaka to synthesize a surjective real linear isometry $T\colon M_1 \to M_2$.

The first results on Tingley's problem for (non-necessarily commutative) operator algebras opened the exploration of this problem for more general classes of operator algebras.

The next natural steps are perhaps, the C^*-algebras $K(H)$ and $B(H)$ of all compact and bounded linear operators on an infinite-dimensional complex Hilbert space H, respectively. There is a clear obstruction in the case of $K(H)$ because $\partial_e(\mathcal{B}_{K(H)}) = \emptyset$, even more, $K(H)$ contains no unitary elements, and hence Theorem 3.3 is meaningless to synthesize a surjective real linear isometry in this setting. Surprisingly, we shall get back to the Hatori–Molnár theorem (Theorem 3.3) when we survey the recent solution to Tingley's problem for a wide family of von Neumann algebras obtained in [28].

3.1. Tingley's problem for compact C^*-algebras

Along the paper, given a vector x_0 in a Banach space X, the translation with respect to x_0 will be denoted by $\mathcal{T}_{x_0}$.

Let us consider the C^*-algebra $K(H)$ of all compact operators on an arbitrary complex Hilbert space H. It is well known that $K(H)^{**} = B(H)$. There is a clear advantage in this case because minimal partial isometries in $K(H)^{**} = B(H)$ are precisely the rank-one partial isometries, which clearly belong to $K(H)$. Furthermore, compact partial isometries in $K(H)^{**}$ are all finite rank partial isometries in $K(H)$.

A C^*-algebra A is called compact if it can be written as a c_0-sum of the form $A = \oplus_j^{c_0} K(H_j)$, where each H_j is a complex Hilbert space (compare [2,57]). In this case $A^{**} = \oplus_j^{\infty} B(H_j)$, and every minimal partial isometry in A^{**} is a rank-one partial isometry in one of the factors, and hence belongs to A. Actually compact partial isometries in A^{**} are finite rank partial isometries, and hence they all belong to A. The following proposition was derived in [44] by combining these facts with Corollary 2.3, the Akemann–Pedersen theorem (see Theorem 2.5), the comments in (1), and Mankiewicz' theorem (see Theorem 1.2).

Proposition 3.5. ([44, Proposition 3.2]) *Let A and B be compact C^*-algebras, and suppose that $\Delta\colon S(A) \to S(B)$ is a surjective isometry. Then the following statements hold:*

(a) *Δ maps norm closed proper faces of $\mathcal{B}_A$ to norm closed proper faces of $\mathcal{B}_B$.*

(b) *For each (minimal) partial isometry e_1 in A there exists a unique (minimal) partial isometry u_1 in B such that $\Delta\left((e_1 + (1-e_1e_1^*)\mathcal{B}_{A^{**}}(1-e_1^*e_1)) \cap \mathcal{B}_A\right) = (u_1 + (1-u_1u_1^*)\mathcal{B}_{B^{**}}(1-u_1^*u_1)) \cap \mathcal{B}_B$. Moreover, there exists a surjective real linear isometry $T_{e_1}\colon (1-e_1e_1^*)A(1-e_1^*e_1) \to (1-u_1u_1^*)B(1-u_1^*u_1)$ such that*

$$\Delta(e_1 + x) = u_1 + T_{e_1}(x),$$

for every $x \in \mathcal{B}_{(1-e_1e_1^*)A(1-e_1^*e_1)}$.

(c) *The restriction of Δ to each norm closed proper face of $\mathcal{B}_A$ is an affine function.*

(d) *For each partial isometry e_1 in A there exists a unique partial isometry u_1 in B such that $\Delta(e_1) = u_1$. Moreover, the rank of e_1 coincides with the rank of u_1 and both are finite.*

The proof of the above result can be outlined and guessed by the reader from the previously commented results.

A result determining when a partial isometry is at distance two from another minimal partial isometry in a compact C^*-algebra was first considered in [44].

Lemma 3.6. ([44, Lemma 3.5]) *Let e and w be partial isometries in a compact C^*-algebra A. Suppose that e is minimal and $\|e-w\|=2$. Then*

$$w = -e + (1-ee^*)w(1-e^*e).$$

Let $\Delta\colon S(A)\to S(B)$ be a surjective isometry between the unit spheres of two compact C^*-algebras. Let us pick a minimal partial isometry e in A. Proposition 3.5 implies that $\Delta(e)$ and $\Delta(-e)$ are minimal partial isometries in B. Since $\|\Delta(e)-\Delta(-e)\| = \|e+e\| = 2$, Lemma 3.6 assures that

$$\Delta(-e) = -\Delta(e) + (1-\Delta(e)\Delta(e)^*)\Delta(-e)(1-\Delta(e)^*\Delta(e)),$$

and we derive from the minimality of $\Delta(-e)$ that $\Delta(-e)=-\Delta(e)$. A more elaborate argument was applied in [44], via similar arguments, to establish a version of the original theorem of Tingley [54] for finite rank partial isometries.

Theorem 3.7. ([44, Theorem 3.7]) *Let $\Delta\colon S(A)\to S(B)$ be a surjective isometry between the unit spheres of two compact C^*-algebras. The following statements hold:*

(a) *If e is a partial isometry in A, then $\Delta(-e)=-\Delta(e)$.*

(b) *If $e_1,\ldots,e_m$ are mutually orthogonal partial isometries in A, then $\Delta(e_1),\ldots,\Delta(e_m)$ are mutually orthogonal partial isometries in B and*

$$\Delta(e_1+\cdots+e_m) = \Delta(e_1)+\cdots+\Delta(e_m).$$

If we take a projection p in a C^*-algebra A, the subspace $(1-p)A(1-p)$ is a C^*-subalgebra of A. However, if we take a partial isometry e in A, the subspace $(1-ee^*)A(1-e^*e)$ need not be, in general, a C^*-subalgebra of A. However, the set $(1-ee^*)A(1-e^*e)$ is a norm closed subspace of A which is also closed under the triple product given by

$$\{a,b,c\} = \frac{1}{2}(ab^*c+cb^*a). \tag{2}$$

This is equivalent to saying that $(1-ee^*)A(1-e^*e)$ is a JB*-subtriple of A in the sense defined in [31] (see Subsection 2.1).

Suppose that e is a partial isometry in a compact C^*-algebra A, and let B be another compact C^*-algebra. Suppose $\Delta\colon S(A)\to S(B)$ is a surjective isometry. Let us consider the surjective real linear isometry

$$T_e\colon (1-ee^*)A(1-e^*e)\to(1-\Delta(e)\Delta(e)^*)A(1-\Delta(e)^*\Delta(e))$$

given by Proposition 3.5(b). Let e_1 be any partial isometry in $(1-p)A(1-p)$. By Propositions 3.5 and 3.7 we have

$$\Delta(e)+T_e(e_1) = \Delta(e+e_1) = \Delta(e)+\Delta(e_1),$$

we get $T_e(e_1) = \Delta(e_1)$. Furthermore, let $e_1, \ldots, e_m$ be mutually orthogonal partial isometries in $(1 - ee^*)A(1 - e^*e)$, and let $\alpha_1, \ldots, \alpha_m$ be positive real numbers with $\max\{\alpha_1, \ldots, \alpha_m\} \leq 1$. By the same arguments as above we deduce that

$$\Delta\Big(e+\sum_{j=1}\alpha_j e_j\Big) = \Delta(e)+T_e\Big(\sum_{j=1}\alpha_j e_j\Big) = \Delta(e)+\sum_{j=1}\alpha_j T_e(e_j) = \Delta(e)+\sum_{j=1}\alpha_j \Delta(e_j).$$

Furthermore, if w is a partial isometry in A such that $e, e_j \in (1 - ww^*)A(1 - w^*w)$ for all j, we also have

$$\begin{aligned}\Delta\Big(e + \sum_{j=1}\alpha_j e_j\Big) &= \Delta(e) + \sum_{j=1}\alpha_j \Delta(e_j)\\ &= T_w(e) + \sum_{j=1}\alpha_j T_w(e_j) = T_w\Big(e + \sum_{j=1}\alpha_j e_j\Big)\end{aligned} \tag{3}$$

A triple spectral resolution assures that every compact operator can be approximated in norm by finite linear combinations of mutually orthogonal minimal partial isometries, and the same statement holds for every element in a compact C^*-algebra. Therefore, under the above hypotheses, we deduce from the continuity of T_w and Δ that for each non-zero partial isometry $w \in A$ we have

$$\Delta(x) = T_w(x), \text{ for all } x \in S((1 - ww^*)A(1 - w^*w)). \tag{4}$$

A straight consequence of (4) gives the following: if w_1 and w_2 are non-zero partial isometries, we have

$$T_{w_2}(x) = \Delta(x) = T_{w_1}(x), \tag{5}$$

for all $x \in S((1 - w_1w_1^*)A(1 - w_1^*w_1)) \cap S((1 - w_2w_2^*)A(1 - w_2^*w_2))$.

The lack of possibility to apply the Hatori–Molnár theorem to synthesize a surjective real linear isometry between A and B forces us to apply a different strategy in [44]. This different approach is worth to be, at least, outlined here.

In the first step we assume that we can find a non-zero subfactor $K(H_1)$ of A such that A is the orthogonal sum of $K(H_1)$ and its orthogonal complement $J = K(H_1)^\perp$ and the latter is non-zero. Let us take two non-zero projections p_1 in $K(H_1)$ and $p_2 \in J$, and define the mapping $T\colon A = K(H_1) \oplus^\perp J \to B$ given by

$$T(x) = T_{p_1}(\pi_2(x)) + T_{p_2}(\pi_1(x))$$

where π_1 and π_2 denote the canonical projections of A onto $K(H_1)$ and J, respectively, and T_{p_1} and T_{p_2} are defined by Proposition 3.5. The mapping T is real linear because T_{p_1} and T_{p_2} are. Clearly T is bounded with $\|T\| \leq 2$. A minimal partial isometry in A either lies in $K(H_1)$ or in J. Let us pick an element x in $S(A)$ which can be written in the form $x = e + \sum_{j=1} \alpha_j e_j + \sum_{k=1} \beta_k e_k$, where e, e_j, e_k are mutually orthogonal minimal partial isometries in A, α_j, $\beta_k \in \mathbb{R}^+$, $e_j \in B(H_1)$ and $e_k \in J$ for all j, k, and e either lies in $B(H_1)$ or in J. If $e \in K(H_1)$ (respectively, $e \in J$), by (4), we have $\Delta(e) = T_{p_1}(e) = T(e)$ (respectively, $\Delta(e) = T_{p_2}(e) = T(e)$). Now, by (3) and (4), we have

$$\Delta(x) = \Delta(e) + \sum_{j=1} \alpha_j \Delta(e_j) + \sum_{k=1} \beta_k \Delta(e_k) = \Delta(e) + \sum_{j=1} \alpha_j T_{p_2}(e_j) + \sum_{k=1} \beta_k T_{p_1}(e_k)$$
$$= T(e) + \sum_{j=1} \alpha_j T(e_j) + \sum_{k=1} \beta_k T(e_k) = T(x).$$

The norm density of this kind of elements x in $S(A)$ together with the norm continuity of T and Δ prove that $T(x) = \Delta(x)$ for all $x \in S(A)$.

In the second case we assume that $A = K(H)$ for some complex Hilbert space H. If H is finite-dimensional, Theorem 3.4 proves that our mapping $\Delta\colon S(A) \to S(B)$ admits a unique extension to a surjective real linear isometry from A onto B. We can therefore assume that H is infinite-dimensional.

Let us take three mutually orthogonal minimal projections p_1, p_2 and p_3 in A, and the corresponding surjective real linear isometries T_{p_1}, T_{p_2}, and T_{p_3} given by Proposition 3.5. We can decompose A in the form

$$A = \mathbb{C}p_1 \oplus (p_1 A p_2 \oplus p_2 A p_1) \oplus ((1-p_2)Ap_1 \oplus p_1 A(1-p_2)) \oplus (1-p_1)A(1-p_1),$$

where $\mathbb{C}p_1 \oplus (p_1 A p_2 \oplus p_2 A p_1) \subset (1-p_3)A(1-p_3)$, and $((1-p_2)Ap_1 \oplus p_1 A(1-p_2)) \subset (1-p_2)A(1-p_2)$. Let π_1, π_2, and π_3 denote the corresponding projections of A onto $\mathbb{C}p_1 \oplus (p_1 A p_2 \oplus p_2 A p_1)$, $((1-p_2)Ap_1 \oplus p_1 A(1-p_2))$ and $(1-p_1)A(1-p_1)$, respectively. We synthesize the mapping $T\colon A \to B$ given by

$$T(x) = T_{p_3}(\pi_1(x)) + T_{p_2}(\pi_2(x)) + T_{p_1}(\pi_3(x)).$$

The mapping T is continuous and real linear because T_{p_1}, T_{p_2} and T_{p_3} are.

If we prove that

$$T(e) = \Delta(e), \text{ for every minimal partial isometry } e \text{ in } A,$$

then a similar argument to that given in the first step above, based on (3) and (4), the norm density in $S(A)$ of those elements which can be written as finite positive combinations of mutually orthogonal minimal partial isometries, and the continuity of T and Δ, shows that $T(x) = \Delta(x)$ for all $x \in S(A)$.

Let e be a minimal partial isometry in A. Since H is infinite-dimensional, we can find another minimal projection p_4 which is orthogonal to p_1, p_2, p_3, e.

Since $e \in (1-p_4)A(1-p_4)$, the statement in (4) implies that $\Delta(e) = T_{p_4}(e)$.

Let us write $e = p_1ep_1 + p_1ep_2 + p_2ep_1 + p_1e(1-p_2) + (1-p_2)ep_1 + (1-p_1)e(1-p_1)$. Clearly, $p_1ep_1, p_1ep_2, p_2ep_1 \in (1-p_4)A(1-p_4)$. Since $p_1, p_2, e \in (1-p_4)A(1-p_4)$, we also deduce that $p_1e(1-p_2), (1-p_2)ep_1, (1-p_1)e(1-p_1) \in (1-p_4)A(1-p_4)$. By applying (5) to T_{p_4} and T_{p_3} (respectively, to T_{p_4} and T_{p_2}, and T_{p_4} and T_{p_3}) we get

$$\begin{aligned} T(e) & \\ &= T_{p_3}(p_1ep_1+p_1ep_2+p_2ep_1)+T_{p_2}(p_1e(1-p_2)+(1-p_2)ep_1)+T_{p_1}((1-p_1)e(1-p_1)) \\ &= T_{p_4}(p_1ep_1+p_1ep_2+p_2ep_1)+T_{p_4}(p_1e(1-p_2)+(1-p_2)ep_1)+T_{p_4}((1-p_1)e(1-p_1)) \\ &= T_{e_4}(e) = \Delta(e). \end{aligned}$$

We have sketched the main arguments leading to one of the main achievements in [44].

Theorem 3.8. ([44, Theorem 3.14]) *Let $\Delta\colon S(A) \to S(B)$ be a surjective isometry between the unit spheres of two compact C^*-algebras. Then there exists a (unique) surjective real linear isometry $T\colon A \to B$ such that $T(x) = \Delta(x)$, for every x in $S(A)$. In particular, the same conclusion holds when $A = K(H_1)$ and $B = K(H_2)$, where H_1 and H_2 are arbitrary complex Hilbert spaces.*

Surjective real linear isometries between (real) C^*-algebras were studied in depth by Ch. H. Chu, T. Dang, B. Russo, B. Ventura in [7]. Theorem 6.4 in [7] proves that every surjective real linear isometry between (real) C^*-algebras is a triple isomorphism with respect to the triple product defined in (2). Studies on surjective real linear isometries on JB*-triples and real JB*-triples have been considered by T. Dang [8] and F. J. Fernández-Polo, J. Martínez and the author of this survey in [23].

3.2. Tingley's problem for $B(H)$

After having revisited the solution to Tingley's problem for compact C^*-algebras published in [44], the next natural challenge is to consider a surjective isometry $\Delta\colon S(B(H_1)) \to S(B(H_2))$, where H_1, H_2 are arbitrary complex Hilbert spaces. Let us observe that if H_1 or H_2 is finite-dimensional, then the extension of Δ to a surjective real linear isometry is guaranteed by Tanaka's theorem (see Theorem 3.4).

The problem in the setting of $B(H)$ spaces has been recently solved in a contribution by F.J. Fernández-Polo and the author of this survey in [26].

Theorem 3.9. ([26, Theorem 3.2]) *Let H_1 and H_2 be complex Hilbert spaces. Suppose that $\Delta\colon S(B(H_1)) \to S(B(H_2))$ is a surjective isometry. Then there exists a surjective complex linear or conjugate linear surjective isometry T from $B(H_1)$ onto $B(H_2)$ satisfying $\Delta(x) = T(x)$, for every $x \in S(B(H_1))$.*

Actually, a stronger conclusion has been achieved.

Theorem 3.10. ([26, Theorem 3.2]) *Let $(H_i)_{i\in I}$ and $(K_j)_{j\in J}$ be two families of complex Hilbert spaces. Suppose $\Delta\colon S\big(\bigoplus_j^{\ell_\infty} B(K_j)\big) \to S\big(\bigoplus_i^{\ell_\infty} B(H_i)\big)$ is a surjective isometry. Then there exists a surjective real linear isometry*

$$T\colon \bigoplus_j^{\ell_\infty} B(K_j) \to \bigoplus_i^{\ell_\infty} B(H_i) \quad \textit{satisfying } T|_{S\left(\bigoplus_j^{\ell_\infty} B(K_j)\right)} = \Delta.$$

The strategy to obtain the previous two theorems also begins with results based on the facial structure of the closed unit ball of $B(H)$, Theorem 2.1, Corollary 2.3 and the Akemann–Pedersen theorem (Theorem 2.5). The latter result forces us to face a serious additional obstacle which requires a completely new strategy. More concretely, we have already seen in the previous subsection that, for a compact C^*-algebra A, the norm closed faces of $\mathcal{B}_A$ are determined by finite rank partial isometries in A. However, for a general C^*-algebra A the maximal proper faces of $\mathcal{B}_A$ are determined by minimal partial isometries in A^{**}. This is a serious obstacle which makes invalid the arguments in previous subsections and in [25, 44] to the case of a surjective isometry $\Delta\colon S(B(H_1)) \to S(B(H_2))$, because, in principle, Δ cannot be applied to every minimal projection in $B(H_1)^{**}$. The novelties in [26] are based on certain technical results which provide an antidote to avoid these difficulties.

Two results from [26] deserve to be highlighted by their own right.

Theorem 3.11. ([26, Theorem 2.3]) *Let A and B be C^*-algebras, and suppose that $\Delta\colon S(A) \to S(B)$ is a surjective isometry. Let e be a minimal partial isometry in A. Then 1 is isolated in the spectrum of $|\Delta(e)|$.*

The consequences of the previous result are stronger after the next additional theorem.

Theorem 3.12. ([26, Theorem 2.5]) *Let A be a C^*-algebra, and let H be a complex Hilbert space. Suppose that $\Delta\colon S(A) \to S(B(H))$ is a surjective isometry. Let e be a minimal partial isometry in A. Then $\Delta(e)$ is a minimal partial isometry in $B(H)$. Moreover, there exists a surjective real linear isometry*

$$T_e\colon (1 - ee^*)A(1 - e^*e) \to (1 - \Delta(e)\Delta(e)^*)B(H)(1 - \Delta(e)^*\Delta(e))$$

such that $\Delta(e+x) = \Delta(e) + T_e(x)$*, for all* x *in* $\mathcal{B}_{(1-ee^*)A(1-e^*e)}$*. In particular, the restriction of* Δ *to the face* $F_e = e + (1-ee^*)\mathcal{B}_A(1-e^*e)$ *is a real affine function.*

Technical algebraic and geometric manipulations combined with the previous theorem determine a precise control of a surjective isometry $\Delta\colon S(B(K)) \to S(B(H))$ on algebraic elements in the sphere which can be expressed as finite positive linear combinations of mutually orthogonal minimal partial isometries. It should be remarked here that a traditional spectral resolution with finite linear combinations of mutually orthogonal projections is only valid to approximate hermitian elements in the sphere.

Theorem 3.13. ([26, Theorem 2.7]) *Let* $\Delta\colon S(B(H_1)) \to S(B(H_2))$ *be a surjective isometry where* H_1 *and* H_2 *are complex Hilbert spaces with dimension greater than or equal to three. Then the following statements hold:*

(a) *For each minimal partial isometry* v *in* $B(H_1)$*, the mapping*

$$T_v\colon (1-vv^*)B(H_1)(1-v^*v) \to (1-\Delta(v)\Delta(v)^*)B(H_2)(1-\Delta(v)^*\Delta(v))$$

given by Theorem 3.12 is complex linear or conjugate linear.

(b) *For each minimal partial isometry* v *in* $B(H_1)$ *we have* $\Delta(-v) = -\Delta(v)$ *and* $T_v = T_{-v}$*. Furthermore,* T_v *is weak***-continuous and* $\Delta(e) = T_v(e)$ *for every minimal partial isometry* $e \in (1-vv^*)B(H_1)(1-v^*v)$*.*

(c) *For each minimal partial isometry* v *in* $B(H_1)$ *the equality* $\Delta(w) = T_v(w)$ *holds for every partial isometry* $w \in (1-vv^*)B(H_1)(1-v^*v)\backslash\{0\}$*.*

(d) *Let* $w_1, \ldots, w_n$ *be mutually orthogonal non-zero partial isometries in* $B(H_1)$*, and let* $\lambda_1, \ldots, \lambda_n$ *be positive real numbers with* $\lambda_1 = 1$*, and* $\lambda_j \leq 1$ *for all* j*. Then* $\Delta\big(\sum_{j=1}^n \lambda_j w_j\big) = \sum_{j=1}^n \lambda_j \Delta(w_j)$*.*

(e) *For each minimal partial isometry* v *in* $B(H_1)$ *we have* $\Delta(x) = T_v(x)$ *for every* $x \in S(\mathcal{B}_{(1-vv^*)B(H_1)(1-v^*v)})$*.*

(f) *For each partial isometry* w *in* $B(H_1)$ *the element* $\Delta(w)$ *is a partial isometry.*

(g) *Suppose* v_1, v_2 *are mutually orthogonal minimal partial isometries in* $B(H_1)$ *then* $T_{v_1}(x) = T_{v_2}(x)$ *for every* x *in the intersection*

$$((1-v_1v_1^*)B(H_1)(1-v_1v_1^*)) \cap ((1-v_2v_2^*)B(H_1)(1-v_2v_2^*)).$$

(h) *Suppose* v_1, v_2 *are mutually orthogonal minimal partial isometries in* $B(H_1)$ *then exactly one of the following statements holds:*

  (1) *The mappings* T_{v_1} *and* T_{v_2} *are complex linear.*

  (2) *The mappings* T_{v_1} *and* T_{v_2} *are conjugate linear.*

The synthesis of a surjective real linear isometry in the proof of Theorem 3.9 (see [26, Theorem 3.2]) is given with similar arguments to those we sketched in page

97 with the obvious modifications and the new tools developed in Theorems 3.12 and 3.13. That is, assuming that H is infinite-dimensional, we pick three mutually orthogonal minimal projections p_1, p_2 and p_3 in A, and the corresponding surjective real linear isometries T_{p_1}, T_{p_2}, and T_{p_3} given by Theorem 3.12. By decomposing $B(H_1)$ in the form

$$B(H_1) = \mathbb{C}p_1 \oplus (p_1 B(H_1) p_2 \oplus p_2 B(H_1) p_1) \oplus \\ \oplus ((1-p_2)B(H_1)p_1 \oplus p_1 B(H_1)(1-p_2)) \oplus (1-p_1)B(H_1)(1-p_1),$$

with

$$\mathbb{C}p_1 \oplus (p_1 B(H_1) p_2 \oplus p_2 B(H_1) p_1) \subset (1-p_3)B(H_1)(1-p_3),$$

and

$$((1-p_2)B(H_1)p_1 \oplus p_1 B(H_1)(1-p_2)) \subset (1-p_2)B(H_1)(1-p_2),$$

and denoting by π_1, π_2, and π_3 the corresponding projections of $B(H_1)$ onto $\mathbb{C}p_1 \oplus (p_1 B(H_1) p_2 \oplus p_2 B(H_1) p_1)$, $((1-p_2)B(H_1)p_1 \oplus p_1 B(H_1)(1-p_2))$ and $(1-p_1)B(H_1)(1-p_1)$, respectively. We synthesize a mapping $T\colon B(H_1) \to B(H_2)$ given by $T(x) = T_{p_3}(\pi_1(x)) + T_{p_2}(\pi_2(x)) + T_{p_1}(\pi_3(x))$. The mapping T is weak* continuous and real linear because T_{p_1}, T_{p_2} and T_{p_3} are. By the new tools given by Theorem 3.13 it is shown in the proof of [26, Theorem 3.2] that $\Delta(e) = T(e)$ for every minimal partial isometry e in $B(H_1)$.

Contrary to the case of $K(H)$ spaces and compact C^*-algebras, where every element in the sphere can be approximated in norm by norm-one elements which are finite linear combinations of mutually orthogonal minimal partial isometries, elements in the sphere of $B(H)$ can be approximated only in the weak* topology by these kind of algebraic elements. To solve this additional obstacle, it is established in [26] an identity principle in the following terms.

Proposition 3.14. ([26, Proposition 3.1]) *Let H_1 and H_2 be complex Hilbert spaces. Suppose that $\Delta\colon S(B(H_1)) \to S(B(H_2))$ is a surjective isometry, and there exists a weak*-continuous real linear operator $T\colon B(H_1) \to B(H_2)$ such that $\Delta(v) = T(v)$, for every minimal partial isometry v in $B(H_1)$. Then T and Δ coincide on the whole $S(B(H_1))$.*

The above proposition, Theorem 3.13 and the above observation are, in essence, all the arguments required to prove Theorem 3.9. The proof of Theorem 3.10 required additional technical adaptations which can be found in [26].

3.3. Tingley's problem for von Neumann algebras

The most recent, and for the moment, the most general conclusion on Tingley's problem is an affirmative solution to this problem for surjective isometries between the unit spheres of two von Neumann algebras in a wide class, which has been recently obtained by F.J. Fernández-Polo and the author of this survey in [28]. The result reads as follows:

Theorem 3.15. ([28, Theorem 3.3]) *Let* $\Delta\colon S(M)\to S(N)$ *be a surjective isometry between the unit spheres of two von Neumann algebras. Suppose* M *is not a factor, or it is a finite factor, or a type I factor, or a type* II_∞ *factor on a separable Hilbert space, or a type III factor on a separable Hilbert space. Then there exists a surjective real linear isometry* $T\colon M\to N$ *whose restriction to* $S(M)$ *is* Δ*. More precisely, there exist a central projection* p *in* N *and a Jordan* **-isomorphism* $J\colon M\to N$ *such that, defining* $T\colon M\to N$ *by* $T(x)=\Delta(1)\,(pJ(x)+(1-p)J(x)^*)$ $(x\in M)$*, then* T *is a surjective real linear isometry and* $T|_{S(M)}=\Delta$*.*

The mathematical difficulties of the problem in this general setting are considerable. The techniques, procedures and strategies applied in the previous case to synthesize a surjective real linear isometry and to apply the facial structure are no longer valid under the new hypotheses.

Let $\Delta\colon S(A)\to S(B)$ be a surjective isometry between the unit spheres of two C^*-algebras. A combination of Theorem 2.5 and Corollary 2.3 (see also the subsequent comments) gives a one-to-one correspondence between compact partial isometries in the corresponding second duals.

Theorem 3.16. *Let* $\Delta\colon S(A)\to S(B)$ *be a surjective isometry between the unit spheres of two* C^**-algebras. Then the following statements hold:*

(a) *For each non-zero compact partial isometry* $e\in A^{**}$ *there exists a unique (non-zero) compact partial isometry* $\phi_\Delta(e)\in B^{**}$ *such that* $\Delta(F_e)=F_{\phi_\Delta(e)}$*, where* $F_e=(e+(1-ee^*)\mathcal{B}_{A^{**}}(1-e^*e))\cap\mathcal{B}_A$*.*

(b) *The mapping* $e\mapsto\phi_\Delta(e)$ *defines an order preserving bijection between the set of non-zero compact partial isometries in* A^{**} *and the set of non-zero compact partial isometries in* B^{**}*.*

(c) ϕ_Δ *maps minimal partial isometries in* A^{**} *to minimal partial isometries in* B^{**}*.*

The above result produces no alternative to our obstacles because compact partial isometries in the second dual cannot be transformed under Δ. Technical arguments based on ultraproducts techniques and a subtle uniform generalization of Lemma 3.6 are appropriately applied in [28] to obtain generalizations of the above Theorems 3.11 and 3.12.

Theorem 3.17. ([28, Theorem 2.7]) *Let* $\Delta\colon S(A)\to S(B)$ *be a surjective isometry between the unit spheres of two* C^**-algebras. Let* e *be a non-zero partial isometry in* A*. Then* 1 *is isolated in the spectrum of* $|\Delta(e)|$*.*

Mankiewicz's theorem (Theorem 1.2) plays a fundamental role in the second part of the statement of the next theorem.

Theorem 3.18. ([28, Theorem 2.8]) *Let $\Delta\colon S(A) \to S(B)$ be a surjective isometry between the unit spheres of two C^*-algebras. Then Δ maps non-zero partial isometries in A to non-zero partial isometries in B. Moreover, for each non-zero partial isometry e in A, we have $\phi_\Delta(e) = \Delta(e)$, and there exists a surjective real linear isometry $T_e\colon (1-ee^*)A(1-e^*e) \to (1-\Delta(e)\Delta(e)^*)B(1-\Delta(e)^*\Delta(e))$ such that*

$$\Delta(e+x) = \Delta(e) + T_e(x), \text{ for all } x \text{ in } \mathcal{B}_{(1-ee^*)A(1-e^*e)}.$$

In particular the restriction of Δ to the face $F_e = e + (1-ee^*)\mathcal{B}_A(1-e^*e)$ is a real affine function.

Another crucial step in the study of Tingley's problem on von Neumann algebras asserts that the mapping ϕ_Δ given by Theorem 3.16 preserves antipodal points.

Theorem 3.19. ([28, Theorem 2.11]) *Let $\Delta\colon S(A) \to S(B)$ be a surjective isometry between the unit spheres of two C^*-algebras. Then, for each non-zero compact partial isometry e in A^{**}, we have $\phi_\Delta(-e) = -\phi_\Delta(e)$, where ϕ_Δ is the mapping given by Theorem 3.16. Consequently, for each non-zero partial isometry $e \in A$, we have $\Delta(-e) = -\Delta(e)$.*

The orthogonal complement of a subset S in a C^*-algebra A is defined by

$$S^\perp := \{x \in A : x \perp a, \text{ for all } a \in S\}.$$

The previous theorems provide the key tools to extend Theorem 3.13 to the setting of von Neumann algebras.

Proposition 3.20. ([28, Proposition 2.12]) *Let $\Delta\colon S(A) \to S(B)$ be a surjective isometry between the unit spheres of two C^*-algebras. Then the following statements hold:*

(a) *For each non-zero partial isometry v in A, the surjective real linear isometry*

$$T_v\colon (1-vv^*)A(1-v^*v) \to (1-\Delta(v)\Delta(v)^*)B(1-\Delta(v)^*\Delta(v))$$

given by Theorem 3.18 satisfies $\Delta(e) = T_v(e)$, for every non-zero partial isometry $e \in (1-vv^*)A(1-v^*v)$.

(b) *Let $w_1,\ldots,w_n$ be mutually orthogonal non-zero partial isometries in A, and let $\lambda_1,\ldots,\lambda_n$ be real numbers with $1 = |\lambda_1| \geq \max\{|\lambda_j|\}$. Then*

$$\Delta\Big(\sum_{j=1}^n \lambda_j w_j\Big) = \sum_{j=1}^n \lambda_j \Delta(w_j).$$

(c) *Suppose v, w are mutually orthogonal non-zero partial isometries in A. Then $T_v(x) = T_w(x)$ for every $x \in \{v\}^\perp \cap \{w\}^\perp$.*

(d) *If A is a von Neumann algebra, then for each non-zero partial isometry v in A we have $\Delta(x) = T_v(x)$ for every $x \in S((1 - vv^*)A(1 - v^*v))$.*

Given a surjective isometry $\Delta\colon S(M) \to S(N)$ between the unit spheres of two von Neumann algebras, the synthesis of a surjective real linear extension to a surjective real linear isometry $T\colon M \to N$ follows completely different arguments than those in the cases of compact C^*-algebras and $B(H)$. The technique in this case relies again on the Hatori–Molnár theorem (Theorem 3.3). R. Tanaka proves in [53] that a surjective isometry between the unit spheres of two finite von Neumann algebras maps unitary elements to unitary elements. This result has been extended to general von Neumann algebras in [28, Theorem 3.2].

Theorem 3.21. ([28, Theorem 3.2]) *Let $\Delta\colon S(M) \to S(N)$ be a surjective isometry between the unit spheres of two von Neumann algebras. Then Δ maps unitaries in M to unitaries in N.*

From now on, let the symbol $\mathcal{U}(A)$ denote the unitary group of a C^*-algebra A. The above Theorem 3.21 opens the door to apply the Hatori–Molnár theorem (Theorem 3.3) to synthesize a surjective real linear isometry $T\colon M \to N$ satisfying $T(u) = \Delta(u)$ for all $u \in \mathcal{U}(M)$. The difficulties to finish the proof of Theorem 3.15 reside in proving that $\Delta(x) = T(x)$ for all $x \in S(M)$. This is solved in [28] with a convenient application of the theory of convex combinations of unitary operators in von Neumann algebras developed by C. L. Olsen and G. K. Pedersen in [39] and [40]. These are the main lines in the proof of Theorem 3.15.

It is worth making a stop to comment on the first connection with a very recent contribution of M. Mori. In the preprint [34], M. Mori establishes a generalization of the above Theorem 3.21.

Theorem 3.22. ([34, Theorem 3.2]) *Let $\Delta\colon S(A) \to S(B)$ be a surjective isometry between the unit spheres of two unital C^*-algebras. Then Δ maps unitaries in A to unitaries in B.*

The proof presented by M. Mori in [34] is based on the following geometric result, which is a nice discovery by itself, and might be useful in some other contexts.

Lemma 3.23. *Let A be a unital C^*-algebra, and let x be an element in $\partial_e(\mathcal{B}_A)$. Then x is a unitary if and only if the set $A_x := \{y \in \partial_e(\mathcal{B}_A) : \|x \pm y\| = \sqrt{2}\}$ has an isolated point as a metric space.*

We finish this section with a couple of open problems.

Open Problem 1. (Tingley's problem for C^*-algebras) Let $\Delta\colon S(A)\to S(B)$ be a surjective isometry between the unit spheres of two C^*-algebras. Does Δ admit an extension to a surjective real linear isometry from A onto B?

When A is a unital C^*-algebra, M. Mori shows in [34, Proposition 3.4 and Problem 6.1] that a particular version of the above problem can be restated in the following terms:

Open Problem 2. Let A be a unital C^*-algebra and let $\Delta\colon S(A)\to S(A)$ be a surjective isometry. Suppose that $\Delta(x)=x$ for every invertible element in the unit sphere of A. Is Δ equal to the identity mapping on $S(A)$?

4. Tingley's problem on von Neumann algebra preduals

Let us begin this section with another result due to G.G. Ding. Let Γ be an index set, we denote by $\ell^1_{\mathbb{R}}(\Gamma)$ the Banach space of all absolutely summable families of real numbers equipped with the norm $\|(\xi_j)_j\|_1=\sum_{j\in\Gamma}|\xi_j|$.

Theorem 4.1. ([10, Theorem 1]) *Let $\Delta\colon S(\ell^1_{\mathbb{R}}(\Gamma_1))\to S(\ell^1_{\mathbb{R}}(\Gamma_2))$ be a surjective isometry. Then there exists a one-to-one bijection $\sigma\colon\Gamma_1\to\Gamma_2$ and a family of real numbers $\{\theta_j : j\in\Gamma_1\}\subseteq\mathbb{T}$ such that*

$$\Delta\Big(\sum_{j\in\Gamma_1}\xi_j e_j\Big)=\sum_{j\in\Gamma_2}\theta_j\xi_{\sigma(j)}\widehat{e}_j,$$

where $\{e_j : j\in\Gamma_1\}$ and $\{\widehat{e}_j : j\in\Gamma_2\}$ are the canonical basis of $\ell^1_{\mathbb{R}}(\Gamma_1)$ and $\ell^1_{\mathbb{R}}(\Gamma_2)$, respectively. In particular, there exists a surjective real linear isometry $T\colon\ell^1_{\mathbb{R}}(\Gamma_1)\to\ell^1_{\mathbb{R}}(\Gamma_2)$ whose restriction to $S(\ell^1_{\mathbb{R}}(\Gamma_1))$ coincides with Δ.

Given a σ-finite measure space (Ω,Σ,μ), the symbol $L^1_{\mathbb{R}}(\Omega,\Sigma,\mu)$ will denote the Banach space of real-valued measurable functions $f\colon\Omega\to\mathbb{R}$ satisfying $\int_\Omega|f|d\mu<\infty$, with norm $\|f\|_1=\int_\Omega|f|d\mu$. The Banach space of all essentially bounded real valued measurable functions on Ω will be denoted by $L^\infty_{\mathbb{R}}(\Omega,\Sigma,\mu)$.

The previous result of Ding is complemented by the following theorem due to D. Tan.

Theorem 4.2. ([49, Theorem 3.4]) *Let (Ω,Σ,μ) be a σ-finite measure space and let Y be a real Banach space. Then every surjective isometry $\Delta\colon S(L^1_{\mathbb{R}}(\Omega,\Sigma,\mu))\to S(Y)$ can be uniquely extended to a surjective real linear isometry from $L^1_{\mathbb{R}}((\Omega,\Sigma,\mu))$ onto Y.*

Regarding $\ell^1_{\mathbb{R}}(\Gamma_1)$ and $L^1_{\mathbb{R}}(\Omega, \Sigma, \mu)$ as predual spaces of the hermitian parts of the von Neumann algebras $\ell^\infty_{\mathbb{R}}(\Gamma_1)$ and $L^\infty_{\mathbb{R}}(\Omega, \Sigma, \mu)$, respectively, it seems natural to ask whether Theorems 4.1 and 4.2 admit non-commutative counterparts. The dualities $c_0^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$ admit a non-commutative alter ego in the form $K(H)^* = C_1(H)$ and $C_1(H)^* = B(H)$, where $C_1(H)$ is the space of trace class operators on a complex Hilbert space H. This will be treated in the next subsection.

4.1. Tingley's problem on trace class operators

Tingley's problem for surjective isometries between unit spheres of spaces of trace class operators has been approached by F.J. Fernández-Polo, J.J. Garcés, I. Villanueva and the author of this note in [21]. We shall review here the main achievements in this line.

When the space $C_1(H)$ is regarded as the predual of the von Neumann algebra $B(H)$, or as the dual space of the C^*-algebra $K(H)$, we can get back to Corollary 2.3 and subsequent comments whose consequences were already observed in [21].

Proposition 4.3. ([21, Proposition 2.6]) *Let $\Delta\colon S(C_1(H)) \to S(C_1(H'))$ be a surjective isometry, where H and H' are complex Hilbert spaces. Then the following statements hold:*

(a) *A subset $\mathcal{F} \subset S(C_1(H))$ is a proper norm-closed face of $\mathcal{B}_{C_1(H)}$ if and only if $\Delta(\mathcal{F})$ is.*
(b) *Δ maps $\partial_e(\mathcal{B}_{C_1(H)})$ into $\partial_e(\mathcal{B}_{C_1(H')})$.*
(c) $\dim(H) = \dim(H')$.
(d) *For each $e_0 \in \partial_e(\mathcal{B}_{C_1(H)})$ we have $\Delta(ie_0) = i\Delta(e_0)$ or $\Delta(ie_0) = -i\Delta(e_0)$.*
(e) *For each $e_0 \in \partial_e(\mathcal{B}_{C_1(H)})$ if $\Delta(ie_0) = i\Delta(e_0)$ (respectively, $\Delta(ie_0) = -i\Delta(e_0)$) then $\Delta(\lambda e_0) = \lambda\Delta(e_0)$ (respectively, $\Delta(\lambda e_0) = \overline{\lambda}\Delta(e_0)$) for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

The strategy to solve Tingley's problem on $C_1(H)$ is based on techniques of linear algebra and geometry to obtain first a solution in the case of finite-dimensional spaces.

Theorem 4.4. ([21, Theorem 3.7]) *Let $\Delta\colon S(C_1(H)) \to S(C_1(H))$ be a surjective isometry, where H is a finite-dimensional complex Hilbert space. Then there exists a surjective complex linear or conjugate linear isometry $T\colon C_1(H) \to C_1(H)$ satisfying $\Delta(x) = T(x)$ for every $x \in S(C_1(H))$. More concretely, there exist unitary elements $u, v \in M_n(\mathbb{C}) = B(H)$ such that one of the following statements holds:*

(a) *$\Delta(x) = uxv$, for every $x \in S(C_1(H))$;*
(b) *$\Delta(x) = ux^tv$, for every $x \in S(C_1(H))$;*

(c) $\Delta(x) = u\overline{x}v$, *for every* $x \in S(C_1(H))$;
(d) $\Delta(x) = ux^*v$, *for every* $x \in S(C_1(H))$,
where $\overline{(x_{ij})} = (\overline{x_{ij}})$.

Surprisingly, the solution in the finite-dimensional case is applied, in a very technical argument, to derive a solution to Tingley's problem for surjective isometries between the unit spheres of two spaces of trace class operators.

Theorem 4.5. ([21, Theorem 4.1]) *Let* $\Delta\colon S(C_1(H)) \to S(C_1(H))$ *be a surjective isometry, where* H *is an arbitrary complex Hilbert space. Then there exists a surjective complex linear or conjugate linear isometry* $T\colon C_1(H) \to C_1(H)$ *satisfying* $\Delta(x) = T(x)$, *for every* $x \in S(C_1(H))$.

4.2. Tingley's problem on von Neumann preduals

According to what is commented on in the introduction, a very recent contribution by M. Mori has changed the original plans and the structure of this survey. The preprint [34] contains, among other interesting results, a complete positive solution to Tingley's problem for surjective isometries between the unit spheres of von Neumann algebra preduals.

Theorem 4.6. ([34, Theorem 4.3]) *Let* M *and* N *be von Neumann algebras, and let* $\Delta\colon S(M_*) \to S(N_*)$ *be a surjective isometry. Then there exists a (unique) surjective real linear isometry* $T\colon M_* \to N_*$ *satisfying* $T(x) = \Delta(x)$, *for every* $x \in S(M_*)$.

It is perhaps interesting to take a brief look at the method applied by M. Mori to synthesize the surjective real linear isometry T. Let $\Delta\colon S(M_*) \to S(N_*)$ be a surjective isometry, where M and N are von Neumann algebras. When Corollary 2.3 and the subsequent comments are combined with the Akemann–Pedersen theorem (see Theorem 2.5), we can conclude that for each maximal partial isometry $u \in \partial_e(\mathcal{B}_M)$ there exists a unique maximal partial isometry $T_1(u) \in \partial_e(\mathcal{B}_N)$ satisfying $\Delta(\{u\}_\prime) = \{T_1(u)\}_\prime$. This gives a bijection $T_1\colon \partial_e(\mathcal{B}_M) \to \partial_e(\mathcal{B}_N)$.

Let (E,d) be a metric space. The *Hausdorff distance* between two sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq E$ is defined by

$$d_H(\mathcal{S}_1,\mathcal{S}_2) := \max\{\sup_{x\in\mathcal{S}_1}\inf_{y\in\mathcal{S}_2} d(x,y), \sup_{y\in\mathcal{S}_2}\inf_{x\in\mathcal{S}_1} d(x,y)\}.$$

The lattice of partial isometries can be equipped with a distance defined by

$$\delta_H(v,w) := d_H(\{v\}_\prime, \{w\}_\prime).$$

It is shown by M. Mori that this distance enjoys the following properties:

Proposition 4.7. ([34, Lemmas 4.1 and 4.2]) *Let M be a von Neumann algebra. Then the following statements hold:*

(a) $\delta_H(u,v) = \|u-v\|$, *for every $u\in\mathcal{U}(M)$ and every $v\in\partial_e(\mathcal{B}_M)$.*

(b) *An element $u\in\partial_e(\mathcal{B}_M)$ is a unitary if and only if the set*

$$\widehat{M_u} := \{e\in\partial_e(\mathcal{B}_M) : \delta_H(u,\pm e)\le\sqrt{2}\}$$

has an isolated point with respect to the metric δ_H.

Applying Proposition 2.4(a) and Proposition 4.7(b), M. Mori concludes that $T_1(\mathcal{U}(M)) = \mathcal{U}(N)$, and, by Proposition 4.7(a), $T_1|_{\mathcal{U}(M)}\colon\mathcal{U}(M)\to\mathcal{U}(N)$ is a surjective isometry. The mapping T_1 fulfills the hypothesis of the Hatori–Molnár theorem (see Theorem 3.3), and thus there exists a surjective real linear (weak*-continuous) isometry $\widetilde{T}_1\colon M\to N$ whose restriction to $\mathcal{U}(M)$ is T_1. The technical arguments developed by M. Mori in the proof of [34, Theorem 4.3] finally show that the mapping $T_2\colon N^*\to M^*$ defined by

$$T_2(\varphi)(x) := \operatorname{Re}\varphi(\widetilde{T}_1(x)) - i\operatorname{Re}\varphi(i\widetilde{T}_1(x)),\ \ \varphi\in N^*, x\in M,$$

is a real linear isometry whose restriction to N_* gives a surjective real linear isometry $T_2|_{N_*}\colon N_*\to M_*$ and $(T_2|_{N_*})^{-1}(\phi) = \Delta(\phi)$ for all ϕ in M_*.

5. Isometries between the spheres of hermitian operators

A second and interesting variant of Problem 1.1 is obtained when X and Y are von Neumann algebras or C^*-algebras and $\mathcal{S}_1$ and $\mathcal{S}_2$ are the unit spheres of their respective hermitian parts. In this section we consider two von Neumann algebras M, N and a surjective isometry $\Delta\colon S(M_{sa})\to S(N_{sa})$. Our goal will consist in showing that the same tools in [28] can be, almost literarily, applied to find a surjective complex linear isometry $T\colon M\to N$ satisfying $T(a^*) = T(a)^*$ for all $a\in M$ and $T(x) = \Delta(x)$ for all $x\in S(M_{sa})$.

Given a C^*-algebra A, its hermitian part, A_{sa}, is not, in general, a C^*-subalgebra of A. However, A_{sa} is a real closed subspace of A which satisfies the hypotheses of Corollary 2.3 (see the comments after this corollary). After applying this corollary, we find the necessity of describing the facial structure of $\mathcal{B}_{A_{sa}}$. Fortunately for us, the Akemann–Pedersen theorem (Theorem 2.5) has a forerunner in [16, Corollary 5.1] where C.M. Edwards and G.T. Rüttimann described the facial structure of the closed unit ball of the hermitian part of every C^*-algebra. We recall that partial isometries in A_{sa} are all elements of the form $e = p - q$, where p and q are orthogonal projections in A.

Theorem 5.1. ([16, Corollary 5.1]) *Let A be a C^*-algebra. Then for each norm-closed face F of $\mathcal{B}_{A_{sa}}$, there exists a unique pair of orthogonal compact projections p, q in A^{**} such that*

$$\begin{aligned} F &= \{x \in \mathcal{B}_{A_{sa}} : x(p-q) = p+q\} = \{p-q\}_{\prime\prime} \\ &= \{x \in \mathcal{B}_{A_{sa}} : x = (p-q) + (1-p-q)x(1-p-q)\}. \end{aligned}$$

Combining this theorem of Edwards and Rüttimann with the above Corollary 2.3 we easily get the following version of Theorem 3.16.

Theorem 5.2. *Let $\Delta\colon S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where A and B are C^*-algebras. Then the following statements hold:*

(a) *For each non-zero compact partial isometry $e \in A_{sa}^{**}$ there exists a unique (non-zero) compact partial isometry $\phi_\Delta^s(e) \in B_{sa}^{**}$ such that $\Delta(F_e) = F_{\phi_\Delta^s(e)}$, where $F_e = \big(e + (1-e^2)\mathcal{B}_{A_{sa}^{**}}(1-e^2)\big) \cap \mathcal{B}_{A_{sa}}$.*

(b) *The mapping $e \mapsto \phi_\Delta^s(e)$ defines an order preserving bijection between the set of non-zero compact partial isometries in A_{sa}^{**} and the set of non-zero compact partial isometries in B_{sa}^{**}.*

(c) *ϕ_Δ^s maps minimal partial isometries in A_{sa}^{**} to minimal partial isometries in B_{sa}^{**}.*

The arguments in the proofs of [28, Theorems 2.7, 2.8 and 2.11 and Proposition 2.12] literarily work to obtain the following four results.

Theorem 5.3. ([28, Theorem 2.7]) *Let $\Delta\colon S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where A and B are C^*-algebras. Let e be a non-zero partial isometry in A_{sa}. Then 1 is isolated in the spectrum of $|\Delta(e)|$.*

Theorem 5.4. ([28, Theorem 2.8]) *Let $\Delta\colon S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where A and B are C^*-algebras. Then Δ maps non-zero partial isometries in A_{sa} into non-zero partial isometries in B_{sa}. Moreover, for each non-zero partial isometry e in A_{sa}, we have $\phi_\Delta^s(e) = \Delta(e)$, where ϕ_Δ^s is the mapping given by Theorem 5.2, and there exists a surjective (real) linear isometry*

$$T_e\colon (1-e^2)A_{sa}(1-e^2) \to (1-\Delta(e)^2)B_{sa}(1-\Delta(e)^2)$$

such that $\Delta(e+x) = \Delta(e) + T_e(x)$, for all x in $\mathcal{B}_{(1-e^2)A_{sa}(1-e^2)}$. In particular the restriction of Δ to the face $F_e = e + (1-e^2)\mathcal{B}_{A_{sa}}(1-e^2)$ is a real affine function.

Theorem 5.5. ([28, Theorem 2.11]) *Let $\Delta\colon S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where A and B are C^*-algebras. Then, for each non-zero compact partial isometry e in A_{sa}^{**} we have $\phi_\Delta^s(-e) = -\phi_\Delta^s(e)$, where ϕ_Δ^s is the mapping given by*

Theorem 5.2. Consequently, for each non-zero partial isometry $e \in A_{sa}$ we have $\Delta(-e) = -\Delta(e)$.

Proposition 5.6. ([28, Proposition 2.12]) *Let $\Delta\colon S(A_{sa}) \to S(B_{sa})$ be a surjective isometry, where A and B are C^*-algebras. Then the following statements hold:*

(a) *For each non-zero partial isometry v in A_{sa}, the surjective real linear isometry*

$$T_v\colon (1-v^2)A_{sa}(1-v^2) \to (1-\Delta(v)^2)B_{sa}(1-\Delta(v)^2)$$

given by Theorem 5.4 satisfies $\Delta(e) = T_v(e)$, for every non-zero partial isometry $e \in (1-v^2)A_{sa}(1-v^2)$.

(b) *Let $w_1, \ldots, w_n$ be mutually orthogonal non-zero partial isometries in A_{sa}, and let $\lambda_1, \ldots, \lambda_n$ be real numbers with $1 = |\lambda_1| \geq \max\{|\lambda_j|\}$. Then*

$$\Delta\Big(\sum_{j=1}^{n} \lambda_j w_j\Big) = \sum_{j=1}^{n} \lambda_j \Delta(w_j).$$

(c) *Suppose v, w are mutually orthogonal non-zero partial isometries in A_{sa} then $T_v(x) = T_w(x)$ for every $x \in \{v\}^{\perp} \cap \{w\}^{\perp}$.*

(d) *If A is a von Neumann algebra, then for each non-zero partial isometry v in A_{sa} we have $\Delta(x) = T_v(x)$ for every $x \in S((1-vv^*)A_{sa}(1-v^*v))$.*

Back to our goal, we observe that the case of $M_2(\mathbb{C})$ of all 2×2 matrices with complex entries must be treated independently.

Proposition 5.7. *Let $A = M_2(\mathbb{C})$, B be a C^*-algebra, and let $\Delta\colon S(A_{sa}) \to S(B_{sa})$ be a surjective isometry. Then there exists a surjective complex linear isometry $T\colon A \to B$ satisfying $T(a^*) = T(a)^*$, for all $a \in A$, and $T(x) = \Delta(x)$, for all $x \in S(A_{sa})$.*

Proof. Since A is finite-dimensional, it follows from the hypotheses that $S(B_{sa})$ is compact, and hence B is finite-dimensional. Having in mind that the rank of a von Neumann algebra M is the cardinality of a maximal set of mutually orthogonal non-zero projections, Proposition 5.6 assures that B must have rank 2. Therefore $B = \mathbb{C} \oplus^{\infty} \mathbb{C}$ or $B = M_2(\mathbb{C})$. We shall show that the first case is impossible.

Suppose $B = \mathbb{C} \oplus^{\infty} \mathbb{C}$. We pick two orthogonal minimal projections $p_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $p_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and a symmetry $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in A.

By Theorem 5.2(b) and Proposition 5.6, $\Delta(p_1)$ and $\Delta(p_2)$ are orthogonal minimal partial isometries in B_{sa}, and $\Delta(s)$ is a symmetry in B. We can assume, without loss of generality, that $\Delta(p_1) = (\pm 1, 0)$, $\Delta(p_2) = (0, \pm 1)$, and $\Delta(s) =$

(σ_1, σ_2), where $\sigma_1, \sigma_2 \in \{\pm 1\}$. By hypotheses,

$$\frac{1+\sqrt{5}}{2} = \left\| \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \right\| = \|p_1 - s\| = \|\Delta(p_1) - \Delta(s)\|$$
$$= \|(\pm 1, 0) - (\sigma_1, \sigma_2)\| \in \{1, 2\},$$

which is impossible. Therefore, $B = M_2(\mathbb{C})$.

Let us take a surjective complex linear and symmetric isometry $T_1 \colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ mapping $\Delta(p_1)$ and $\Delta(p_2)$ to p_1 and p_2, respectively. We set $\Delta_1 = T_1 \circ \Delta$. Then $\Delta_1 \colon S(A_{sa}) \to S(B_{sa})$ is a surjective isometry with $\Delta_1(p_i) = p_i$ for $i = 1, 2$.

An arbitrary pair of orthogonal minimal projections in A_{sa} can be written in the form $q_1 = \begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}$ and $q_2 = \begin{pmatrix} 1-s_0 & -\lambda\sqrt{s_0(1-s_0)} \\ -\overline{\lambda}\sqrt{s_0(1-s_0)} & s_0 \end{pmatrix}$ for a unique $s_0 \in (0, 1)$ and a unique $\lambda \in \mathbb{T}$ (the cases $s_0 = 0, 1$ give p_1 and p_2). By Theorem 5.2(b) and Proposition 5.6, $\Delta_1(q_1)$ and $\Delta_1(q_2)$ are orthogonal minimal partial isometries in B_{sa}. It is well known that $\Delta_1(q_1) = \pm \begin{pmatrix} t_0 & \mu\sqrt{t_0(1-t_0)} \\ \overline{\mu}\sqrt{t_0(1-t_0)} & 1-t_0 \end{pmatrix}$ for a unique $t_0 \in [0, 1]$ and a unique $\mu \in \mathbb{T}$ (compare [45, Theorem 1.3] or [41, §3]).

If $\Delta_1(q_1) = -\begin{pmatrix} t_0 & \mu\sqrt{t_0(1-t_0)} \\ \overline{\mu}\sqrt{t_0(1-t_0)} & 1-t_0 \end{pmatrix}$, then by hypothesis,

$$1 + \sqrt{t_0} = \left\| \begin{pmatrix} t_0 + 1 & \mu\sqrt{t_0(1-t_0)} \\ \overline{\mu}\sqrt{t_0(1-t_0)} & 1 - t_0 \end{pmatrix} \right\|$$
$$= \| - \Delta_1(q_1) + \Delta_1(p_1)\| = \| - q_1 + p_1\| = \|q_1 - p_1\|$$
$$= \left\| \begin{pmatrix} s_0 - 1 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1 - s_0 \end{pmatrix} \right\| = \sqrt{(1 - s_0)},$$

which is impossible.

If $\Delta_1(q_1) = \begin{pmatrix} t_0 & \mu\sqrt{t_0(1-t_0)} \\ \overline{\mu}\sqrt{t_0(1-t_0)} & 1-t_0 \end{pmatrix}$, then by hypothesis,

$$\sqrt{(1 - t_0)} = \left\| \begin{pmatrix} t_0 - 1 & \mu\sqrt{t_0(1-t_0)} \\ \overline{\mu}\sqrt{t_0(1-t_0)} & 1 - t_0 \end{pmatrix} \right\|$$
$$= \|\Delta_1(q_1) - \Delta_1(p_1)\| = \|q_1 - p_1\|$$
$$= \left\| \begin{pmatrix} s_0 - 1 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1 - s_0 \end{pmatrix} \right\| = \sqrt{(1 - s_0)},$$

which implies that $t_0 = s_0$. That is, for each $s_0 \in [0, 1]$ and $\lambda \in \mathbb{T}$, there exists a unique $\mu \in \mathbb{T}$ such that

$$\Delta_1\left(\begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1 - s_0 \end{pmatrix}\right) = \begin{pmatrix} s_0 & \mu\sqrt{s_0(1-s_0)} \\ \overline{\mu}\sqrt{s_0(1-s_0)} & 1 - s_0 \end{pmatrix}. \tag{6}$$

In particular, $\Delta_1\left(\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}\right) = \begin{pmatrix} \frac12 & \mu_0\frac12 \\ \overline{\mu_0}\frac12 & \frac12 \end{pmatrix}$, for a certain $\mu_0 \in \mathbb{T}$.

Let us take a surjective complex linear symmetric isometry $T_2\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ satisfying $T_2(p_j) = p_j$ for every $j = 1, 2$ and $T_2\Delta_1\left(\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}\right) = \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$. We set $\Delta_2 = T_2 \circ \Delta_1\colon S(A_{sa}) \to S(B_{sa})$. Proposition 5.6(b) applied to Δ_2 gives

$$1 = \Delta_2(p_1) + \Delta_2(p_2) = \Delta_2(1) = \Delta_2\left(\begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}\right) + \Delta_2\left(\begin{pmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{pmatrix}\right),$$

which assures that $\Delta_2\left(\begin{pmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{pmatrix}\right) = \begin{pmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{pmatrix}$. Let us denote $r_1 = \begin{pmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{pmatrix}$, and $r_2 = 1 - r_1$. A new application of Proposition 5.6(b) gives

$$\Delta_2(r_1 - r_2) = \Delta_2(r_1) - \Delta_2(r_2) = r_1 - r_2.$$

Take an arbitrary projection $q_1 = \begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}$ with $s_0 \in (0,1)$ and $\lambda \in \mathbb{T}$. We deduce from the hypothesis and (6) (applied to Δ_2) that

$$\begin{aligned} &\frac{1 + \sqrt{5 - 8\operatorname{Re}(\lambda)\sqrt{s_0(1-s_0)}}}{2} \\ &= \left\| \begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| \\ &= \left\| \Delta_2\begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} - \Delta_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} s_0 & \mu\sqrt{s_0(1-s_0)} \\ \overline{\mu}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| \\ &= \frac{1 + \sqrt{5 - 8\operatorname{Re}(\mu)\sqrt{s_0(1-s_0)}}}{2}, \end{aligned}$$

which assures that the scalar μ in (6) for Δ_2 must satisfy $\mu = \lambda$ or $\mu = \overline{\lambda}$. Consequently, for each $s_0 \in (0,1)$ and $\lambda \in \mathbb{T}$, we have

$$\Delta_2\left(\begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) = \begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} \tag{7}$$

or

$$\Delta_2\left(\begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) = \begin{pmatrix} s_0 & \overline{\lambda}\sqrt{s_0(1-s_0)} \\ \lambda\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}.$$

We can also deduce from the above identities and Proposition 5.6(b) that

$$\Delta_2\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \text{or} \quad \Delta_2\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Suppose first that $\Delta_2\left(\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$. Given $s_0 \in (0,1)$ and $\lambda \in \mathbb{T}$, we have

$$\frac{1+\sqrt{5+8\,\mathrm{Im}(\lambda)\sqrt{s_0(1-s_0)}}}{2} = \left\| \begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\|,$$

$$\frac{1+\sqrt{5-8\,\mathrm{Im}(\lambda)\sqrt{s_0(1-s_0)}}}{2} = \left\| \begin{pmatrix} s_0 & \overline{\lambda}\sqrt{s_0(1-s_0)} \\ \lambda\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\|,$$

and thus, (7) and the hypothesis prove that

$$\begin{aligned} \Delta_2(q_1) &= \Delta_2\left(\begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) \\ &= \begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} = q_1, \end{aligned}$$

for every q_1 as above. Let $q_2 = 1 - q_1$. By Proposition 5.6(b) we also have

$$1 = \Delta_2(p_1) + \Delta_2(p_2) = \Delta_2(1) = \Delta_2(q_1) + \Delta_2(q_2),$$

which assures that $\Delta_2(q_2) = q_2$. We have therefore proved that $\Delta_2(q_i) = q_i$, for every pair of orthogonal minimal projections q_1, q_2 in A_{sa}. Since every element x in $S(A_{sa})$ can be written as a linear combination of the form $x = \sum_{j=1}^{2} \mu_j q_j$, where q_1 and q_2 are orthogonal minimal projections in A_{sa}, $\mu_j \in \mathbb{R}$ and $\max\{|\mu_j|\} = 1$, a new application of Proposition 5.6(b) gives

$$\Delta_2(x) = \Delta_2\Big(\sum_{j=1}^{2} \mu_j q_j\Big) = \sum_{j=1}^{2} \mu_j \Delta_2(q_j) = \sum_{j=1}^{2} \mu_j q_j = x.$$

This shows that $\Delta_2(x) = x$, for every x in $S(A_{sa})$, and hence $\Delta(x) = T_1^{-1}T_2^{-1}(x)$, for every x in $S(A_{sa})$.

Assume now that $\Delta_2\left(\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Similar arguments to those given above show that, in this case, we have

$$\begin{aligned} \Delta_2(q_1) &= \Delta_2\left(\begin{pmatrix} s_0 & \lambda\sqrt{s_0(1-s_0)} \\ \overline{\lambda}\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix}\right) \\ &= \begin{pmatrix} s_0 & \overline{\lambda}\sqrt{s_0(1-s_0)} \\ \lambda\sqrt{s_0(1-s_0)} & 1-s_0 \end{pmatrix} = \overline{q_1}, \end{aligned}$$

for every minimal projection q_1 as above, where $\overline{(x_{ij})} = (\overline{x_{i,j}})$, and $\Delta_2(x) = \overline{x}$, for every x in $S(A_{sa})$. Therefore, $\Delta(x) = T_1^{-1}T_2^{-1}(\overline{x})$, for every x in $S(A_{sa})$. Denoting $S = T_1^{-1}T_2^{-1}$ we have a complex linear and symmetric isometry $S\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$. We define $T\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$T(h+ik) := S(\overline{h}) + iS(\overline{k}) = S(\overline{h}+i\overline{k}) = S(\overline{h-ik}) = S(\overline{(h+ik)^*}) = S((h+ik)^t),$$

for every $h, k \in A_{sa}$, which provides the mapping T in the statement of the proposition.

■

We can state now the desired result and its proof, where we show that the synthesis of a surjective isometry is even easier in this setting.

Theorem 5.8. *Let $\Delta\colon S(M_{sa}) \to S(N_{sa})$ be a surjective isometry, where M and N are von Neumann algebras. Then there exists a surjective complex linear isometry $T\colon M \to N$ satisfying $T(a^*) = T(a)^*$, for all $a \in M$, and $T(x) = \Delta(x)$, for all $x \in S(M_{sa})$.*

Proof. We shall distinguish the following three cases:

(1) M contains no type I_2 von Neumann factors;
(2) M contains a type I_2 von Neumann factor but M is not a type I_2 von Neumann factor;
(3) M is a type I_2 von Neumann factor.

Case (3) is solved by Proposition 5.7.

Case (2). We can assume that $M = J_1 \oplus J_2$, where J_1 and J_2 are non-zero orthogonal weak* closed ideals of M and $J_1 = M_2(\mathbb{C})$. We can now mimic the arguments we gave in the solution to Tingley's problem for compact operators on page 96. Let us take two non-zero projections p_1 in J_1 and $p_2 \in J_2$, and define the mapping $T\colon M \to N$ given by $T(x) = T_{p_1}(\pi_2(x)) + T_{p_2}(\pi_1(x))$, where π_1 and π_2 stand for the canonical projections of M onto J_1 and J_2, respectively, and T_{p_1} and T_{p_2} are the surjective weak* continuous complex linear and symmetric isometries given by Theorem 5.4. The mapping T is complex linear and weak* continuous because T_{p_1} and T_{p_2} are. Any projection p in M can be written in the form $p = p_1 + p_2$ where p_j is a projection in J_j. Let us pick an algebraic element x in $S(M_{sa})$ which can be written in the form $x = \sum_{j=1} \alpha_j p_j + \sum_{k=1} \beta_k q_k$, where p_j, q_k are mutually orthogonal non-zero projections in M_{sa}, α_j, $\beta_k \in \mathbb{R}\backslash\{0\}$, $\max\{|\alpha_j|, |\beta_k|\} = 1$, $p_j \in J_1$ and $q_k \in J_2$ for all j, k. By the definition of T and

Proposition 5.6(b) we have

$$\Delta(x) = \sum_{j=1} \alpha_j \Delta(p_j) + \sum_{k=1} \beta_k \Delta(q_k) = \sum_{j=1} \alpha_j T_{p_2}(p_j) + \sum_{k=1} \beta_k T_{p_1}(q_k)$$
$$= \sum_{j=1} \alpha_j T(p_j) + \sum_{k=1} \beta_k T(q_k) = T(x).$$

The norm density of this kind of algebraic elements x in $S(M_{sa})$ together with the norm continuity of T and Δ prove that $T(x) = \Delta(x)$ for all $x \in S(M)$.

Case (1). M contains no type I_2 von Neumann factors. Let us define a vector measure on the lattice $\operatorname{Proj}(M)$ of all projections of M defined by $\mu\colon \operatorname{Proj}(M) \to N$, $\mu(p) = \Delta(p)$ if $p \in S(M)$ and $\mu(0) = 0$. Proposition 5.6(b) assures that μ is finitely additive, that is $\mu\big(\sum_{j=1}^m p_j\big) = \sum_{j=1}^m \mu(p_j)$, whenever $p_1, \ldots, p_m$ are mutually orthogonal projections in M. We further have $\|\mu(p)\| \leq 1$ for every $p \in \operatorname{Proj}(M)$. By the Bunce–Wright–Mackey–Gleason theorem (see [3, Theorem A] or [4, Theorem A]) there exists a bounded (complex) linear operator $T\colon M \to N$ satisfying $T(p) = \mu(p) = \Delta(p)$, for every $p \in \operatorname{Proj}(M)\backslash\{0\}$. By definition $T(p) \in N_{sa}$ for every projection p in M. Therefore T is a symmetric map, that is, $T(a^*) = T(a)^*$ for all $a \in M$.

Finally, Proposition 5.6(b) also guarantees that Δ and T coincide on algebraic elements in $S(M_{sa})$ which can be written as finite real linear combinations of mutually orthogonal projections. Since this kind of algebraic elements are norm dense in $S(M_{sa})$, we deduce from the norm continuity of Δ and T that $T(x) = \Delta(x)$ for all $x \in S(M_{sa})$.

■

Remark 5.9. After completing the writing of this chapter, the preprint by M. Mori [34] became available in arxiv. Section 5 in the just quoted paper is devoted to study Theorem 5.8 with a different proof based on a theorem of Dye on orthoisomorphisms (see [34, §5] and [13]). So, Theorem 5.8 should be also credited to M. Mori. It is surprising that the arguments developed by Mori find a similar obstacle with type I_2 von Neumann factors when applying Dye's theorem. To solve the difficulties Mori built an analogue to our Proposition 5.7 in [34, Proposition 5.2 and its proof]. The proof of Proposition 5.7 is a bit simpler with pure geometry-linear algebra arguments.

Open Problem 3. Let $\Delta\colon S(A_{sa}) \to S(B_{sa})$ be a surjective isometry between the unit spheres of the hermitian parts of two C^*-algebras. Does Δ admit an extension to a surjective complex linear isometry from A onto B?

6. Isometries between the spheres of positive operators

Contrary to the results revised in the previous sections, in the third variant of Problem 1.1 treated in this survey the theory on the facial structure of a C^*-algebra revised in Section 2 will not play any role. Let us establish the concrete statement. Given a subset B of a Banach space X, the symbol $S(B)$ will stand for the intersection of B and $S(X)$. Given a C^*-algebra A, the symbol A^+ will denote the cone of positive elements in A, while $S(A^+)$ will stand for the sphere of positive norm-one operators. The concrete variant of Problem 1.1 reads as follows.

Problem 6.1. Let $\Delta\colon S(X^+) \to S(Y^+)$ be a surjective isometry, where X and Y are Banach spaces which can be regarded as linear subspaces of two C^*-algebras A and B, $S(X^+) = S(X) \cap A^+$ and $S(Y^+) = S(Y) \cap B^+$. Does Δ admit an extension to a surjective complex linear isometry $T\colon X \to Y$?

Problem 6.1 is too general. We can easily find non-isomorphic Banach spaces X and Y which are linear subspaces of two C^*-algebras A and B, for which $S(X^+)$ and $S(Y^+)$ reduce to a single point.

Before dealing with the historical background and forerunners, we shall make some observations. If we have a surjective isometry $\Delta\colon S(A^+) \to S(B^+)$ between the spheres of positive elements in two arbitrary C^*-algebras, the application of Theorems 2.1 and 2.2 is non-viable because A^+ and B^+ are not Banach spaces.

Another comment: the hypotheses in Problem 6.1 are strictly weaker than those in Theorems 3.4, 3.8, 3.10, 3.15, 4.5, 4.6, and 5.8. However, the conclusion is also weaker because we need to find a surjective isometry $T\colon A \to B$ whose restriction to $S(A^+)$ coincides with Δ, we do not have to show that T and Δ coincide on the whole $S(A)$ nor on $S(A_{sa})$. That is, the synthesis of the mapping T is, a priori, easier at the cost of losing the main geometric tools.

We can now survey the main achievements in this line. Let us recall some terminology. According to the notation in previous sections, we shall denote by $(C_p(H), \|\cdot\|_p)$ the Banach space of all p-Schatten–von Neumann operators on a complex Hilbert space H, where $1 \leq p \leq \infty$. For $p = 1$ we find the space of trace class operators. By a standard abuse of notation we identify $C_\infty(H)$ with $B(H)$. Let the symbol $C_p(H)^+$ denote the set of all positive operators in $C_p(H)$. The elements in the set $S(C_1(H)^+) = S(C_1(H)) \cap C_1(H)^+$ are usually called *density operators*.

Our first result, which was obtained by L. Molnár and W. Timmermann in [36], provides a complete positive solution to Problem 6.1 for the space $C_1(H)$ of trace class operators on an arbitrary complex Hilbert space H.

Theorem 6.2. ([36, Theorem 4]) *Let H be an arbitrary complex Hilbert space. Then every surjective isometry $\Delta\colon S(C_1(H)^+) \to S(C_1(H)^+)$ admits a unique extension to a surjective complex linear isometry on $C_1(H)$.*

In 2012, G. Nagy and L. Molnár studied a problem connected to our Problem 6.1 in the finite-dimensional case for every $1 \leq p$.

Theorem 6.3. ([35, Theorem 1]) *Let H be a finite-dimensional complex Hilbert space, and let $\infty > p \geq 1$. Then every isometry $\Delta\colon (S(C_1(H)^+), \|.\|_p) \to (S(C_1(H)^+), \|.\|_p)$ admits a unique extension to a surjective complex linear isometry on $C_1(H)$, where $(S(C_1(H)^+), \|.\|_p)$ denotes the unit sphere of $C_1(H)^+)$ equipped with the norm $\|.\|_p$.*

Let us observe that the mapping Δ in the above theorem is not assumed to be surjective a priori. However, as a consequence of the result, Δ is surjective.

Theorem 6.3 has been extended by G. Nagy in [37].

Theorem 6.4. ([37, Theorem 1]) *Let H be an arbitrary complex Hilbert space, and let $p \in (1, \infty)$. Then every surjective isometry $\Delta\colon S(C_p(H)^+) \to S(C_p(H)^+)$ admits a unique extension to a surjective complex linear isometry on $C_p(H)$.*

Problem 6.1 has been explored, in a very recent paper due to G. Nagy, for surjective isometries $\Delta\colon S(B(H)^+) \to S(B(H)^+)$ under the hypothesis of H being finite-dimensional. In the paper [38] we can find the following result.

Theorem 6.5. ([38, Theorem]) *Let H be a finite-dimensional complex Hilbert space, and let $\Delta\colon S(B(H)^+) \to S(B(H)^+)$ be an isometry. Then Δ is surjective and there exists a (unique) surjective complex linear isometry $T\colon B(H) \to B(H)$ satisfying $T(x) = \Delta(x)$, for all $x \in S(B(H)^+)$.*

The arguments developed by Nagy in the paper [38] develop some interesting tools and results in the finite-dimensional setting. Some of them have been successfully extended to arbitrary dimensions. Let E and P be subsets of a Banach space X. Following the notation employed in the recent paper [43], the *unit sphere around E in P* is defined as the set

$$\mathrm{Sph}(E; P) := \{x \in P : \|x - b\| = 1 \text{ for all } b \in E\}.$$

To simplify the notation, given a C^*-algebra A and a subset $E \subset A$, we shall write $\mathrm{Sph}^+(E)$ or $\mathrm{Sph}^+_A(E)$ for the set $\mathrm{Sph}(E; S(A^+))$.

In [38, Proof of Claim 1] G. Nagy proves that if H is a finite-dimensional complex Hilbert space, and a is a positive norm-one element in $B(H) = M_n(\mathbb{C})$, then

$$a \text{ is a projection if, and only if, } \mathrm{Sph}^+_{M_n(\mathbb{C})}(\mathrm{Sph}^+_{M_n(\mathbb{C})}(a)) = \{a\}.$$

We have recently generalized Nagy's result to the setting of atomic von Neumann algebras. We recall that a von Neumann algebra M is called *atomic* if it coincides with the weak* closure of the linear span of its minimal projections. It is known that every atomic von Neumann algebra M can be written in the form $M = \bigoplus_j^{\ell_\infty} B(H_j)$, where each H_j is a complex Hilbert space (compare [47, §V.1] or [46, §2.2]).

Theorem 6.6. ([43, Theorem 2.3]) *Let M be an atomic von Neumann algebra, and let a be a positive norm-one element in M. Then the following statements are equivalent:*

(a) *a is a projection;*
(b) $\mathrm{Sph}^+_M\left(\mathrm{Sph}^+_M(a)\right) = \{a\}$.

Actually, if a is a positive norm-one element in an arbitrary C^*-algebra A satisfying $\mathrm{Sph}^+_A\left(\mathrm{Sph}^+_A(a)\right) = \{a\}$, then a is a projection (see [43, Proposition 2.2]).

Open Problem 4. Does the equivalence in Theorem 6.6 hold when M is a general von Neumann algebra or a C^*-algebra?

For a separable infinite-dimensional complex Hilbert space H_3 and the C^*-algebra $K(H_3)$ of compact operators on H_3, we have actually established a more general result, whose finite-dimensional version was given by G. Nagy in [38, Proof of Claim 1].

Theorem 6.7. ([43, Theorem 3.3]) *Let H_3 be a separable infinite-dimensional complex Hilbert space. Then the identity*

$$\mathrm{Sph}^+_{K(H_3)}\left(\mathrm{Sph}^+_{K(H_3)}(a)\right) = \left\{ b \in S(K(H_3)^+) : \begin{array}{c} s_{K(H_3)}(a) \leq s_{K(H_3)}(b), \text{ and} \\ \mathbf{1} - r_{B(H_3)}(a) \leq \mathbf{1} - r_{B(H_3)}(b) \end{array} \right\},$$

holds for every a in the unit sphere of $K(H_3)^+$.

A consequence of the above theorem gives an appropriate version of Theorem 6.6 for $K(H_3)$.

Theorem 6.8. ([43, Theorem 2.5]) *Let a be a positive norm-one element in $K(H_3)$, where H_3 is a separable complex Hilbert space. Then the following statements are equivalent:*

(a) *a is a projection;*
(b) $\mathrm{Sph}^+_{K(H_3)}\left(\mathrm{Sph}^+_{K(H_3)}(a)\right) = \{a\}$.

Thanks to Theorems 6.6 and 6.8 it can be concluded that given two atomic von Neumann algebras M and N (respectively, separable complex Hilbert spaces H_3 and H_4), and a surjective isometry $\Delta\colon S(M^+) \to S(N^+)$ (respectively, $\Delta\colon S(K(H_3)^+) \to S(K(H_4)^+)$), then Δ maps $\operatorname{Proj}(M)\backslash\{0\}$ onto $\operatorname{Proj}(N)\backslash\{0\}$ (respectively, $\operatorname{Proj}(K(H_3))\backslash\{0\}$ onto $\operatorname{Proj}(K(H_4))\backslash\{0\}$), and the restriction

$$\Delta|_{\operatorname{Proj}(M)\backslash\{0\}}\colon \operatorname{Proj}(M)\backslash\{0\} \to \operatorname{Proj}(N)\backslash\{0\}$$

(respectively, $\Delta|_{\operatorname{Proj}(K(H_3))\backslash\{0\}}\colon \operatorname{Proj}(K(H_3))\backslash\{0\} \to \operatorname{Proj}(K(H_4))\backslash\{0\}$) is a surjective isometry.

These are some of the tools that, combined with many other technical arguments, are applied to give a partial solution to Problem 6.1 in the setting of compact operators.

Theorem 6.9. ([43, Theorem 3.7]) *Let H_3 and H_4 be separable complex Hilbert spaces. Let us assume that H_3 is infinite-dimensional. We suppose that $\Delta\colon S(K(H_3)^+) \to S(K(H_4)^+)$ is a surjective isometry. Then there exists a surjective complex linear isometry $T\colon K(H_3) \to K(H_4)$ satisfying $T(x) = \Delta(x)$, for all $x \in S(K(H_3)^+)$. We can further conclude that T is a *-isomorphism or a *-anti-isomorphism.*

Additional technical results are given in [43, §4] to give a complete solution to Problem 6.1 in the setting of atomic von Neumann algebras. For brevity we shall not comment on some of the deep technical results required to establish this solution. The final statement reads as follows:

Theorem 6.10. ([43, Theorem 4.5]) *Let $\Delta\colon S(B(H_1)^+) \to S(B(H_2)^+)$ be a surjective isometry, where H_1 and H_2 are complex Hilbert spaces. Then there exists a surjective complex linear isometry (actually, a *-isomorphism or a *-anti-automorphism) $T\colon B(H_1) \to B(H_2)$ satisfying $\Delta(x) = T(x)$, for all $x \in S(B(H_1)^+)$.*

Open Problem 5. Let $\Delta\colon S(A^+) \to S(B^+)$ be a surjective isometry, where A and B are C^*-algebras. Does Δ admit an extension to a surjective complex linear isometry from A onto B?

Open Problem 6. Let H be an arbitrary complex Hilbert space, and let $p \in (1,\infty)$. Suppose $\Delta\colon S(C_p(H)) \to S(C_p(H))$ is a surjective isometry. Does Δ admit a unique extension to a surjective real linear isometry on $C_p(H)$?

A more general version has been also posed by M. Mori in [34, Problem 6.3].

Open Problem 7. Let $1 < p < \infty$, $p \neq 2$, let M, N be von Neumann algebras and $\Delta\colon S(L^p(M)) \to S(L^p(N))$ be a surjective isometry between the unit spheres of two noncommutative L^p-spaces (with respect to fixed normal semifinite faithful weights). Does Δ admit an extension to a real linear surjective isometry $T\colon L^p(M) \to L^p(N)$?

Note added in proof. Problem 6 has been recently solved by F. J. Fernández-Polo, E. Jordá and the author of this note in [22].

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