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Linear maps that preserve semi-Fredholm operators acting on Banach spaces

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Abstract. We consider the linear maps $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ that preserve the semi-Fredholm operators in both directions or the essential spectrum of an operator, where $\mathcal{B}(X)$ is the algebra of all bounded linear operators on an infinite-dimensional Banach space X. We describe some known results in the Hilbert space case, provide some basic results and examples in the general case, and state several open problems.

1. Introduction

There is a considerable interest in the so-called linear preserver problems (see [3], [5], [6], [7], [16], [18], [26], [31]) which study the linear maps between two Banach algebras that preserve a given class of elements of the algebras (e.g. the invertible elements, the nilpotents, the idempotents), or a property of the elements (e.g. the spectrum or the spectral radius). In this direction, *Kaplansky's problem* [19] can be stated as follows: Let φ be a surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\sigma(\varphi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$. Is it true that φ is a Jordan automorphism?

This problem was first solved in the finite-dimensional case [9], [21] by showing that every unital invertibility preserving linear map on a complex matrix algebra is an inner automorphism or an inner anti-automorphism, and a similar result was

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later obtained for the algebra of bounded linear operators on a Banach space [32], and for von Neumann algebras [4].

Here we consider the linear maps $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ that preserve the Fredholm or the semi-Fredholm operators in both directions, or the essential spectrum of an operator. We survey some relevant known results, give some basic properties, and describe several open questions.

In Section 2 we describe some known results [22], [25], [17] for $X = Y$ a Hilbert space, which form the model we have in mind later when we consider operators on Banach spaces.

Section 3 gives basic results for maps preserving Atkinson operators acting on Banach spaces, and gives some examples involving Banach spaces with finite dimensional Calkin algebra. Section 4 considers maps preserving semi-Fredholm operators acting on some classes of Banach spaces, the subprojective and the superprojective spaces, and shows that in this case semi-Fredholm operators admit characterizations in terms of their image in the Calkin algebra. For a survey on linear preserver problems involving a wider choice of topics we refer to [23].

Given a Banach space X, $\mathcal{B}(X)$ is the algebra of all bounded operators on X, and $\mathcal{K}(X)$ is the closed ideal of all compact operators. Moreover X^* is the dual space of X, and T^* is the conjugate operator of $T \in \mathcal{B}(X)$, even when X is a Hilbert space. Note that the conjugate operator is slightly different from the Hilbert space adjoint. Also $\mathrm{N}(T)$ and $\mathrm{R}(T)$ are the kernel and the range of T. An operator $T \in \mathcal{B}(X)$ is *upper semi-Fredholm* if $\mathrm{R}(T)$ is closed and $\dim \mathrm{N}(T)$ is finite; T is *lower semi-Fredholm* if $\mathrm{R}(T)$ is closed and $\operatorname{codim} \mathrm{R}(T)$ is finite, and T is *Fredholm* if it is upper and lower semi-Fredholm. The *index* of a semi-Fredholm operator $T \in \mathcal{B}(X)$ is defined as $\operatorname{ind}(T) = \dim \mathrm{N}(T) - \dim X/\mathrm{R}(T) \in \mathbb{Z} \cup \{\pm\infty\}$.

Let $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi(X)$ denote the subsets of all upper semi-Fredholm, lower semi-Fredholm and Fredholm operators acting on X. We also denote by $\Phi_l(X)$ and $\Phi_r(X)$ the *left-Atkinson* and the *right-Atkinson* operators, which are respectively the operators in $\Phi_+(X)$ and $\Phi_-(X)$ with complemented kernel and range. The operator $T \in \mathcal{B}(X)$ is called *inessential* if $I - TB \in \Phi(X)$ for every $B \in \mathcal{B}(X)$. We denote by $\mathcal{I}(X)$ the subset of all inessential operators. When X is a Hilbert space, $\Phi_+(X) = \Phi_l(X)$, $\Phi_-(X) = \Phi_r(X)$ and $\mathcal{I}(H) = \mathcal{K}(H)$.

Let $\mathcal{S}$ denote one of the classes of operators $\Phi, \Phi_+, \Phi_-, \Phi_r, \Phi_s$. The *perturbation class $P\mathcal{S}$ of $\mathcal{S}$* is defined by its components:

$$P\mathcal{S}(X) = \{A \in \mathcal{B}(X) : \text{ for every } T \in \mathcal{S}(X),\, T + A \in \mathcal{S}(X)\}.$$

We denote the Calkin algebra $\mathcal{B}(X)/\mathcal{I}(X)$ by $\mathcal{C}(X)$ and $\pi\colon \mathcal{B}(X) \to \mathcal{C}(X)$ is the quotient map. Observe that $T \in \Phi(X)$ if and only if $\pi(T)$ is invertible in $\mathcal{C}(X)$.

The *essential spectrum* of $T \in \mathcal{B}(X)$ is the set $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(X)\}$. It coincides with the spectrum of $\pi(T)$ in $\mathcal{C}(X)$. For information on Fredholm theory we refer to [8], [10], [20] and [27].

We say that a map $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ *preserves the set of Fredholm operators in both directions* if $T \in \mathcal{B}(X)$ is in $\Phi(X)$ if and only if $\varphi(T) \in \Phi(Y)$. Also φ is *surjective up to compact operators* when $\varphi\big(\mathcal{B}(X)\big) + \mathcal{K}(Y) = \mathcal{B}(Y)$. Similarly we can define these two concepts for other classes of operators.

Let $\mathcal{A}$ and $\mathcal{B}$ be complex unital Banach algebras. A bijective linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a *Jordan isomorphism* if $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$. It is an anti-isomorphism if $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$.

An automorphism $\varphi\colon \mathcal{A} \to \mathcal{A}$ is *inner* if there is an invertible $u \in \mathcal{A}$ such that $\varphi(x) = u^{-1}xu$ for every $x \in \mathcal{A}$.

2. The case of Hilbert space operators

In this section H denotes an infinite-dimensional, complex Hilbert space. We consider operators acting on a Hilbert space, and we describe known results and a few questions. The results we give in this section will be the model we will have in mind when we consider operators on Banach spaces.

Our first result deals with maps that preserve the set of Fredholm operators in both directions.

Theorem 2.1. ([22, Theorem 2.1]) *Let $\varphi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map which is surjective up to compact operators. Then the following conditions are equivalent:*

(1) *φ preserves the set of Fredholm operators in both directions.*
(2) *$\varphi\big(\mathcal{K}(H)\big) \subset \mathcal{K}(H)$ and the induced map $\widehat{\varphi}\colon \mathcal{C}(H) \to \mathcal{C}(H)$ is given by $\widehat{\varphi}(\hat{x}) = \hat{a} \cdot \tau(\hat{x})$ ($\hat{x} \in \mathcal{C}(H)$), where $\hat{a}$ is invertible in $\mathcal{C}(H)$ and τ is an automorphism or an anti-automorphism of $\mathcal{C}(H)$.*

In this case the map φ is continuous.

In Theorem 2.1 the converse implication is trivial, and the direct one is a consequence of [16, Theorem 4.1] and the following facts:

(1) An operator $A \in \mathcal{B}(H)$ is compact if and only if $T + A \in \Phi(H)$ for every $T \in \Phi(H)$; i.e., $P\Phi(H) = \mathcal{K}(H)$.
(2) Denoting by e the identity in $\mathcal{C}(H)$ and defining $\Psi(\hat{x}) = \widehat{\varphi}(\hat{e})^{-1}\widehat{\varphi}(\hat{x})$, the map Ψ preserves the orthogonal idempotents in $\mathcal{C}(H)$.
(3) The linear span of the set of orthogonal projections is dense in the set of selfadjoint elements of $\mathcal{C}(H)$.

From Theorem 2.1 we can derive a description of the maps that preserve the essential spectrum.

Corollary 2.2. ([22, Theorems 3.2 and 3.3]) *Let $\varphi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map which is surjective up to compact operators. Then the following conditions are equivalent:*

(1) *φ preserves the set of Fredholm operators in both directions and it satisfies $\varphi(I) = I - K$ for some $K \in \mathcal{K}(H)$.*
(2) *$\sigma_e\big(\varphi(T)\big) = \sigma_e(T)$ for every $T \in \mathcal{B}(H)$.*
(3) *$\varphi\big(\mathcal{K}(H)\big) \subset \mathcal{K}(H)$ and the induced map $\widehat{\varphi}\colon \mathcal{C}(H) \to \mathcal{C}(H)$ is an automorphism or an anti-automorphism.*

In this case $\widehat{\varphi}$ is continuous and unital, and $\operatorname{ind}(\varphi(T)) = \operatorname{ind}(T)$ *for every* $T \in \Phi(H)$, *or* $\operatorname{ind}(\varphi(T)) = -\operatorname{ind}(T)$ *for every* $T \in \Phi(H)$.

The following question was raised in [22, Conjecture 3.4].

Question 1. Suppose that $\varphi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies the hypothesis of Corollary 2.2. Is it possible to find a linear map $\chi\colon \mathcal{B}(H) \to \mathcal{K}(H)$ and Fredholm operators $A, B \in \Phi(H)$ such that $\varphi(T) = ATB + \chi(T)$ for all $T \in \mathcal{B}(H)$, or $\varphi(T) = AT^*B + \chi(T)$ for all $T \in \mathcal{B}(H)$?

What happens if we additionally assume that φ is continuous (hence χ is also continuous)?

In Question 1 we are asking if given a map φ satisfying the conditions of Corollary 2.2, the associated automorphism or anti-automorphism τ of $\mathcal{C}(H)$ is inner: if there is an invertible $u \in \mathcal{C}(H)$ such that $\tau(x) = uxu^{-1}$ for every $x \in \mathcal{C}(H)$ or $\tau(x) = ux^*u^{-1}$ for every $x \in \mathcal{C}(H)$.

We observe that (for H separable) Phillips and Weaver [30] proved that, if one assumes the continuum hypothesis, then there exist non-inner automorphisms of $\mathcal{C}(H)$; and Farah [11] showed that, if one assumes the open coloring axiom, then every automorphism of $\mathcal{C}(H)$ is inner. In the latter case the answer to Question 1 is obviously positive.

Next we consider the maps φ that preserve the set of semi-Fredholm operators in both directions. In this case it is important to be able to assure that $\varphi(I)$ is Fredholm not just semi-Fredholm. For this reason we assume that H is separable.

Theorem 2.3. ([25, Theorems 1.2 and 1.3]) *Suppose that H is separable and let $\varphi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ be a linear map which is surjective up to compact operators. Then the following conditions are equivalent:*

(1) *φ preserves the set of semi-Fredholm operators in both directions.*

(2) *$\varphi\big(\mathcal{K}(H)\big) \subseteq \mathcal{K}(H)$ and the induced map $\widehat{\varphi}\colon \mathcal{C}(H) \to \mathcal{C}(H)$ is given by $\widehat{\varphi}(\hat{x}) = \hat{a} \cdot \tau(\hat{x})$ ($\hat{x} \in \mathcal{C}(H)$), where $\hat{a}$ is invertible in $\mathcal{C}(H)$ and τ is an automorphism or an anti-automorphism of $\mathcal{C}(H)$.*

In this case φ preserves the Fredholm operators in both directions and there exists $n \in \mathbb{Z}$ such that $\operatorname{ind}(\varphi(T)) = n + \operatorname{ind}(T)$ for every $T \in \Phi(H)$, or $\operatorname{ind}(\varphi(T)) = n - \operatorname{ind}(T)$ for every $T \in \Phi(H)$.

In the case of an anti-automorphism in Theorem 2.3, upper (lower) semi-Fredholm operators are taken into lower (upper) semi-Fredholm operators. So we get the following result.

Corollary 2.4. *Under the hypothesis of Theorem 2.3, the following conditions are equivalent:*

(1) *φ preserves the set $\Phi_+(H)$ in both directions.*
(2) *φ preserves the set $\Phi_-(H)$ in both directions.*
(3) *$\varphi\big(\mathcal{K}(H)\big) \subseteq \mathcal{K}(H)$ and the induced map $\widehat{\varphi}\colon \mathcal{C}(H) \to \mathcal{C}(H)$ is given by $\widehat{\varphi}(\hat{x}) = \hat{a} \cdot \tau(\hat{x})$ ($\hat{x} \in \mathcal{C}(H)$), where $\hat{a}$ is invertible in $\mathcal{C}(H)$ and τ is an automorphism of $\mathcal{C}(H)$.*

3. Preservation of Atkinson operators

In this section X denotes an infinite-dimensional, complex Banach space. We replace $\mathcal{K}(X)$ by $\mathcal{I}(X)$ in the definition of $\mathcal{C}(X)$ because $\mathcal{I}(X)$ is the perturbation class for the Fredholm and the Atkinson operators in X, and the quotient algebra $\mathcal{B}(X)/\mathcal{I}(X)$ is semi-simple.

Again we begin with maps that preserve the set of Fredholm operators in both directions or the essential spectrum. Recall that $\pi\colon \mathcal{B}(X) \to \mathcal{C}(X)$ denotes the quotient map. The arguments in the proofs of the following two results are similar to those of the proofs of [22, Theorems 2.1 and 3.2].

Lemma 3.1. *Let $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map which is surjective up to inessential operators and preserves the set of Fredholm operators in both directions. Then $\varphi\big(\mathcal{I}(X)\big) \subseteq \mathcal{I}(X)$ and the kernel of φ is contained in $\mathcal{I}(X)$. Hence the induced map $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$, determined by the equality $\widehat{\varphi} \circ \pi = \pi \circ \varphi$, is bijective.*

Proof. The key fact is that $\mathcal{I}(X)$ coincides with $P\Phi(X)$, the perturbation class of $\Phi(X)$.

Given $B \in \mathcal{I}(X)$, we have to show that $T + \varphi(B) \in \Phi(X)$ for each $T \in \Phi(X)$. By hypothesis, there exist $S \in \Phi(X)$ and $A \in \mathcal{I}(X)$ so that $T = \varphi(S) + A$. Hence $T + \varphi(B) = \varphi(S + B) + A \in \Phi(X)$, and we conclude that $\varphi(B) \in \mathcal{I}(X)$.

Suppose that $A \in \mathcal{B}(X)$ is in the kernel of φ and $T \in \Phi(X)$. Then $\varphi(T+A) = \varphi(T) \in \Phi(X)$, hence $T + A \in \Phi(X)$ and we conclude that $A \in \mathcal{I}(X)$. ■

Theorem 3.2. *Let $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map which is surjective up to inessential operators. Then the following assertions hold:*

(1) *φ preserves the set of Fredholm operators in both directions if and only if $\varphi\big(\mathcal{I}(X)\big) \subseteq \mathcal{I}(X)$ and the induced map $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$ preserves the invertible elements of $\mathcal{C}(X)$ in both directions.*

(2) *φ preserves the set of Fredholm operators in both directions and $\varphi(I) = I - K$ for some $K \in \mathcal{I}(X)$ if and only if $\sigma_e\big(\varphi(T)\big) = \sigma_e(T)$ for every $T \in \mathcal{B}(H)$; or equivalently, the induced map $\widehat{\varphi}$ preserves the spectrum.*

Proof. (1) This is a consequence of the equivalence $T \in \mathcal{B}(X)$ is Fredholm if and only if $\pi(T)$ is invertible in $\mathcal{C}(X)$.

(2) Observe that $\varphi(I) = I - K$ with $K \in \mathcal{I}(X)$ is equivalent to $\pi(\varphi(I))$ being the identity in $\mathcal{C}(X)$. So the result follows from the fact that, given $T \in \mathcal{B}(X)$, $\lambda \in \sigma_e(T)$ if and only if $\lambda I - T \notin \Phi(X)$, and $\sigma_e(T) = \sigma(\pi(T))$. ■

We showed in Section 2 that the answer to the following question is positive when X is a Hilbert space.

Question 2. Under the hypothesis of part (2) in Theorem 3.2, does the induced map $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$ is an automorphism or an anti-automorphism of $\mathcal{C}(X)$?

In the previous question $\mathcal{C}(X)$ is a semi-simple Banach algebra and the map $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$ is bijective and preserves the spectrum. So we are asking if Kaplansky's problem has a positive answer for some special maps acting on a Calkin algebra.

Recall that every complex finite-dimensional semi-simple Banach algebra A has the form $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.

Example 1. (See the proof of [13, Theorem 2.18].) For every complex finite-dimensional semi-simple Banach algebra A there exists a Banach space X_A such that $\mathcal{C}(X_A) = A$.

Question 2 admits a positive answer for the spaces X_A in Example 1.

Remark 3.3. Let $X_{1,1}$ denote the Banach space in Example 1 corresponding to the algebra $\mathbb{C} \oplus \mathbb{C}$. We can see in the proof of [13, Theorem 2.18] that $X_{1,1} = Y \times Z$, where the Calkin algebras of Y and Z are $\mathbb{C}$, and the operators from Y to Z, and from Z to Y, are inessential. So every operator $T \in \mathcal{B}(X_{1,1})$ can be represented as a matrix $T = \begin{pmatrix} \lambda I+A & B \\ C & \mu I+D \end{pmatrix}$ where $A\colon Y \to Y$, $B\colon Z \to Y$, $C\colon Y \to Z$ and $D\colon Z \to Z$ are inessential operators. Moreover $\sigma_e(T) = \{\lambda, \mu\}$.

Example 2. Let $\varphi_1\colon \mathcal{B}(X_{1,1}) \to \mathcal{B}(X_{1,1})$ be the map that permutes λ and μ in the matrix of T and leaves the rest unchanged.

The map φ_1 is linear, bijective, and preserves the essential spectrum, but it does not admit a representation like the ones considered in Question 1.

Indeed, suppose that there exists $A, B \in \Phi(X_{1,1})$ such that $\varphi(T) = ATB + \chi(T)$ for all $T \in \mathcal{B}(X_{1,1})$ with $\chi(T)$ inessential. Denoting by λ_a, μ_a and λ_b, μ_b the scalar terms in the matrices of A and B, it is easy to check that $\lambda_a \lambda \lambda_b = \mu$, which is not possible when $\lambda = 0 \neq \mu$.

Similarly we can show that $\varphi(T) = AT^*B + \chi(T)$ leads to a contradiction.

Example 3. Let $\varphi_2\colon \mathcal{B}(X_{1,1}) \to \mathcal{B}(X_{1,1})$ be the map that replaces λ by μ and μ by $-\lambda$, and leaves the rest unchanged. Then φ_2 preserves the Fredholm operators in both directions, but $\varphi_2(I)$ does not have the form $kI + C$ with k a constant and C an inessential operator.

Next we consider the maps φ that preserve the set of Atkinson operators $\mathcal{A}(X) = \Phi_l(X) \cup \Phi_r(X)$ in both directions. A similar argument to that in the proof of Theorem 3.2 provides the following result.

Proposition 3.4. *Let $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map which is surjective up to inessential operators. Then φ preserves the set $\Phi_l(X)$ (respectively $\Phi_r(X)$) in both directions if and only if $\varphi\big(\mathcal{I}(X)\big) \subseteq \mathcal{I}(X)$ and the induced map $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$ preserves the left invertible (respectively right invertible) elements of $\mathcal{C}(X)$ in both directions.*

The next result relates Theorem 3.2 and Proposition 3.4.

For a generalization to Banach algebras of the following theorem, see [5].

Theorem 3.5. *Let $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map which is surjective up to inessential operators. If φ preserves one of the sets $\Phi_l(X)$, $\Phi_r(X)$ or $\mathcal{A}(X)$ in both directions, then φ preserves the set of Fredholm operators in both directions.*

Proof. First observe that

$$\begin{aligned} T \in \mathcal{I}(X) &\iff T + S \in \Phi(X),\ \forall S \in \Phi(X) \iff T + S \in \Phi_l(X),\ \forall S \in \Phi_l(X) \\ &\iff T + S \in \Phi_r(X),\ \forall S \in \Phi_r(X) \iff T + S \in \mathcal{A}(X),\ \forall S \in \mathcal{A}(X). \end{aligned}$$

Consequently, if φ preserves one of the sets $\Phi_l(X)$, $\Phi_r(X)$ or $\mathcal{A}(X)$ in both directions, then $\varphi\big(\mathcal{I}(X)\big) \subseteq \mathcal{I}(X)$ and the induced map $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$ preserves the left invertible, right invertible or semi-invertible (that is left or right invertible) elements of $\mathcal{C}(X)$ in both directions.

We assume that $\widehat{\varphi}$ preserves the left invertible elements in both directions, since the proofs in the other cases are similar.

We denote by e the unit of $\mathcal{C}(X)$, select $s \in \mathcal{C}(X)$ such that $s\widehat{\varphi}(e) = e$, and define $\Psi\colon \mathcal{C}(X) \to \mathcal{C}(X)$ by $\Psi(x) = s\widehat{\varphi}(x)$. Then $\Psi(e) = e$ and

$$\Psi(x) \text{ is left invertible} \Rightarrow \widehat{\varphi}(x) \text{ is left invertible} \Rightarrow x \text{ is left invertible.}$$

Hence the left spectrum in $\mathcal{C}(X)$ satisfies $\sigma_l(x) \subset \sigma_l(\Psi(x))$. Moreover, since

$$\partial\sigma(x) \subset \sigma_l(x) \subset \sigma_l(\Psi(x)) \subset \sigma(\Psi(x)),$$

the spectral radius satisfies $r(x) \leq r(\Psi(x))$ for every $x \in \mathcal{C}(X)$.

Let us see that Ψ and $\widehat{\varphi}$ are injective: If $x_0 \in \mathcal{C}(X)$ and $\Psi(x_0) = 0$ then

$$r(\lambda x_0 + x) \leq r(\Psi(\lambda x_0 + x)) = r(\Psi(x)) \quad \text{for every } \lambda \in \mathbb{C} \text{ and } x \in \mathcal{C}(X).$$

Since the map $\lambda \in \mathbb{C} \to r(\lambda x_0 + x)$ is subharmonic [2, Theorem 3.4.7], it follows from Liouville's theorem for subharmonic functions [2, Theorem 3.4.14] that $r(x_0 + x) = r(x)$ for all $x \in \mathcal{C}(X)$. Hence [2, Theorem 5.3.1] allows us to conclude that x_0 is in the Jacobson radical of $\mathcal{C}(X)$, which means $x_0 = 0$ because $\mathcal{C}(X)$ is semi-simple.

Let us see that $\widehat{\varphi}(e)$ is invertible: Since $\widehat{\varphi}$ is surjective, we can take $y_0 \in \mathcal{C}(X)$ such that $\widehat{\varphi}(y_0) = e - \widehat{\varphi}(e)s$. Then $\Psi(y_0) = s\widehat{\varphi}(y_0) = s(e - \widehat{\varphi}(e)s) = s - s = 0$. Since Ψ is injective $\widehat{\varphi}(e)s = e$, hence $\widehat{\varphi}(e)$ is invertible.

Let us see that $\widehat{\varphi}$ preserves invertibility in both directions: If $a \in \mathcal{C}(X)$ is invertible, then by the previous arguments $F(x) := \widehat{\varphi}(ax)$ defines a bijective map $F\colon \mathcal{C}(X) \to \mathcal{C}(X)$ that preserves left invertibility in both directions and satisfies that $F(e) = \widehat{\varphi}(a)$ is invertible. And applying the argument to $\widehat{\varphi}^{-1}$ we obtain that $\widehat{\varphi}(a)$ invertible implies a is invertible.

Therefore φ preserves the set of Fredholm operators in both directions. ■

Remark 3.6. In Theorem 2.3 and Corollary 2.4 it was assumed that the Hilbert space H is separable. Theorem 3.5 shows that Corollary 2.4 is valid in the non-separable case.

Remark 3.7. The converse of Theorem 3.5 is not valid. There are maps $\varphi\colon \mathcal{C}(X) \to \mathcal{C}(X)$ that preserve the invertible elements of $\mathcal{C}(X)$ in both directions, but take left invertible elements to right invertible ones, as in the anti-automorphism case of Theorem 2.1.

By Corollary 2.2 and Theorem 3.5, the answer to the following question is positive when X is a Hilbert space.

Question 3. Let $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map which is surjective up to inessential operators and $\varphi(I) = I - K$ for some $K \in \mathcal{I}(X)$. We consider the following conditions:

(1) φ preserves the set $\Phi_l(X)$ in both directions.

(2) φ preserves the set $\Phi_r(X)$ in both directions.

(3) the induced map $\varphi\colon \mathcal{C}(X) \to \mathcal{C}(X)$ is an automorphism.

It is clear that (3) implies (2) and (1). However, we do not know if (1) or (2) implies (3).

4. Preservation of semi-Fredholm operators

Now we consider the semi-Fredholm operators on a complex Banach space X. In this case the perturbation classes $P\Phi_+(X)$ and $P\Phi_-(X)$ satisfy $\mathcal{SS}(X) \subset P\Phi_+(X) \subset \mathcal{I}(X)$ and $\mathcal{SC}(X) \subset P\Phi_-(X) \subset \mathcal{I}(X)$, where $\mathcal{SS}(X)$ and $\mathcal{SC}(X)$ denote the strictly singular and the strictly cosingular operators, and there are examples of Banach spaces showing that all the inclusions can be proper [12].

In order to apply the arguments considered in Section 3, we need that $P\Phi_+(X)$ or $P\Phi_-(X)$ coincide with $\mathcal{I}(X)$, and we also need characterizations of the semi-Fredholm operators in terms of their image in the Calkin algebra. To get that we will restrict ourselves to some special classes of Banach spaces.

Definition 4.1. Let X be a Banach space.

- X is called *subprojective* if every infinite-dimensional closed subspace of X contains an infinite-dimensional complemented subspace.
- X is called *superprojective* if every infinite-codimensional closed subspace of X is contained in an infinite-codimensional complemented subspace.

These two classes of Banach spaces were introduced in [33] to study the operators whose conjugate operator is strictly singular, and they were useful to provide positive answers to the perturbation classes problem for semi-Fredholm operators in [14].

The spaces ℓ_p $(1 \leq p < \infty)$ and c_0 are subprojective and superprojective, $L_p(0,1)$ is subprojective but not superprojective for $2 < p < \infty$, and it is superprojective but not subprojective for $1 < p < 2$. For the stability properties of subprojective and superprojective spaces, and plenty of additional examples we refer to [28] and [15].

Note that $\mathcal{SS}(X) = \mathcal{I}(X)$ when X is subprojective and $\mathcal{SC}(X) = \mathcal{I}(X)$ when X is subprojective [1]. Moreover we have the following descriptions of $\Phi_+(X)$ and $\Phi_-(X)$.

Proposition 4.2. *Let X be a Banach space and $T \in \mathcal{B}(X)$.*

(1) *Suppose that X is subprojective. Then $T \in \Phi_+(X)$ if and only if $\pi(T)$ is not a left divisor of 0 in $\mathcal{C}(X)$.*

(2) *Suppose that X is superprojective. Then $T \in \Phi_-(X)$ if and only if $\pi(T)$ is not a right divisor of 0 in $\mathcal{C}(X)$.*

Proof. (1) Suppose that X is subprojective.

Let $T \in \Phi_+(X)$ and suppose that $\pi(T)\pi(S) = 0$. Then $TS \in \mathcal{I}(X) = \mathcal{SS}(X)$, hence $S \in \mathcal{SS}(X)$ and $\pi(S) = 0$. Thus $\pi(T)$ is not a left divisor of 0 in $\mathcal{C}(X)$.

Conversely, if $T \notin \Phi_+(X)$, then there exists a compact operator $K \in \mathcal{B}(X)$ so that $\dim N(T + K) = \infty$. We can find a projection $P \in \mathcal{B}(X)$ with infinite dimensional range contained in $N(T + K)$. Thus $\pi(P) \neq 0$ and $\pi(T)\pi(P) = 0$.

(2) Suppose that X is superprojective.

Let $T \in \Phi_-(X)$ and suppose that $\pi(S)\pi(T) = 0$. Then $ST \in \mathcal{I}(X) = \mathcal{SS}(X)$, hence $S \in \mathcal{SC}(X)$ and $\pi(S) = 0$. Thus $\pi(T)$ is not a right divisor of 0 in $\mathcal{C}(X)$.

Conversely, if $T \notin \Phi_-(X)$, then there exists a compact operator $K \in \mathcal{B}(X)$ so that $\dim X/\overline{R(T + K)} = \infty$. We can find a projection $P \in \mathcal{B}(X)$ with infinite-dimensional range such that $R(T + K) \subset N(P)$. Thus $\pi(P) \neq 0$ and $\pi(P)\pi(T) = 0$. ■

A space X is *uniformly subprojective* if there exists a constant $C > 0$ such that the projections P associated to the complemented subspaces that appear in the definition of subprojective space can be chosen with $\|P\| \leq C$. Similarly we can define the *uniformly superprojective* spaces.

In [28] and [15], the proof that some concrete spaces are subprojective or superprojective shows that they satisfy the corresponding uniform property.

Remark 4.3. It is not difficult to check that, if we ask the spaces in Proposition 4.2 to be uniformly subprojective (superprojective) then we can replace left (right) divisor of 0 by left (right) topological divisor of 0 in the statement, as it is done in [20] under different conditions.

Now we can obtain some results for the maps preserving semi-Fredholm operators.

Proposition 4.4. *Let $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map which is surjective up to inessential operators.*

(1) *Suppose that X is subprojective. Then φ preserves the set of upper semi-Fredholm operators in both directions if and only if $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$ preserves the set of left divisors of 0 in both directions.*

(2) *Suppose that X is superprojective. Then φ preserves the set of lower semi-Fredholm operators in both directions if and only if $\widehat{\varphi}\colon \mathcal{C}(X) \to \mathcal{C}(X)$ preserves the set of right divisors of* 0 *in both directions.*

We can consider the following questions.

Question 4. Suppose that $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies the hypothesis of (1) or (2) in Proposition 4.4.

Is it possible to find a linear map $\chi\colon \mathcal{B}(H) \to \mathcal{I}(X)$ and upper (lower) semi-Fredholm operators $A, B \in \Phi(X)$ such that $\varphi(T) = ATB + \chi(T)$ for all $T \in \mathcal{B}(X)$?

What happens if we additionally assume that φ is continuous?

Question 5. Suppose that the space X is subprojective and superprojective. Let $\varphi\colon \mathcal{B}(X) \to \mathcal{B}(X)$ be a linear map which is surjective up to inessential operators.

Is it true that φ preserves the set of upper semi-Fredholm operators in both directions if and only if φ preserves the set of lower semi-Fredholm operators in both directions?

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