doi: 10.14232/actasm-016-012-2

Acta Sci. Math. (*Szeged*)
85 (2019), 59–68

z-ideals in lattices

Vinayak Joshi and Shubhangi Kavishwar

Communicated by G. Czédli

Abstract. In this paper, we define z-ideals in bounded lattices. A separation theorem for the existence of prime z-ideals is proved in distributive lattices. As a consequence, we prove that every z-ideal is the intersection of some prime z-ideals. Lastly, we prove a characterization of dually semi-complemented lattices.

1. Introduction

The concept of z-ideals, which are both algebraic and topological objects, were first introduced by Kohls [9] and played a fundamental role in studying the ideal theory of $C(X)$, the ring of continuous real-valued functions on a completely regular Hausdorff space X; see Gillman and Jerison [3]. An ideal I of a commutative ring R with unity is a *z-ideal* if whenever any two elements of R are contained in the same set of maximal ideals and I contains one of them, then it also contains the other (see Gillman and Jerison [3] for an equivalent definition). Mason [10] studied z-ideals in general commutative rings. He proved that maximal ideals, minimal prime ideals and some other important ideals in commutative rings are z-ideals (see [10, p. 281]). As a generalization of z-ideals, the concept of z^0-ideals is introduced and studied in $C(X)$. Note that in [5], Huijsmans and de Pagter studied z^0-ideals under the name of d-ideals in Riesz spaces. Speed [14] introduced and studied the concept of Baer ideals in a commutative Baer ring, which are essentially z^0-ideals (equivalently, d-ideals) and characterized regular rings and quasi-regular rings. Jayaram [6], Anderson, Jayaram and Phiri [1] defined this concept (Baer ideals) for lattices and multiplicative lattices respectively. Recently, Joshi and Mundlik [8] extended the concept of Baer ideals to posets. Since z-ideal and z^0-ideals (Baer

Received February 22, 2016 and in final form September 3, 2018.
AMS Subject Classification: 06B10, 06D75.
Key words and phrases: z-ideals, Baer ideal, 0-ideal, closed ideal, minimal prime ideal, maximal ideal, dense ideal, dually semi-complemented lattice.

ideals or d-ideals) are closely related in commutative rings, hence it is natural to study the analogous concept of z-ideals in lattices.

In this paper, we define and study z-ideals in bounded lattices. A separation theorem for the existence of prime z-ideals is proved in distributive lattices. As a consequence, we prove that every z-ideal is the intersection of some prime z-ideals. Lastly, we prove a characterization of dually semi-complemented lattices.

2. Preliminaries

We use the standard terminology of Lattice Theory; see, for example, Burris and Sankappanavar [2], Grätzer [4], and Nation [11]. In particular, a nonempty subset I of a lattice L is a *down-set* if $(\forall x \in I)$ $(\forall y \in L)(y \leq x \Rightarrow y \in I)$; up-sets are defined dually. A subset of L is *proper* if it is distinct from L. By a *maximal* ideal or a *maximal* down-set, etc., we mean a maximal proper ideal, maximal proper down-set, etc., respectively. A down-set or an ideal I is *prime* if it is proper and its complement, $L \setminus I$, is meet-closed. A prime ideal P of L is a *minimal prime ideal* if for any prime ideal Q of L, $Q \subseteq P$ implies $P = Q$. The set of all minimal prime ideals is denoted by $\mathcal{I}^{\mathrm{p}}_{\min}(L)$.

An ideal I is *semiprime*, if for all $x, y, z \in L$, the inclusion $\{x \wedge y, x \wedge z\} \subseteq I$ implies $x \wedge (y \vee z) \in I$; see Rav [13]. A lattice L is 0-*distributive* if $0 \in L$ and $\{0\}$ is a semiprime ideal. If both L and its dual are 0-distributive then L is 0-1-*distributive*.

A lattice L with 0 is *semi-complemented* if for any element $a \in L$ (with $a \neq 1$, if 1 exists) there exists a nonzero element $b \in L$ such that $a \wedge b = 0$. Dually, we can define *dually semi-complemented* lattices. A lattice L with 0 is *sectionally semi-complemented* (in brief SSC), if every interval $[0, a]$ is semi-complemented for $a > 0$; see Janowitz [7]. By Janowitz [7, Remark 2.2], a lattice L is SSC if and only if it satisfies the condition $(*)$: For $a, b \in L$, if $b < a$, then there is a nonzero $c \in L$ such that $c \leq a$ and $c \wedge b = 0$. We observe that condition $(*)$ is equivalent to the following condition $(**)$: For $a, b \in L$, if $a \nleq b$, then there exists $c \in L$ such that $0 < c \leq a$ and $c \wedge b = 0$. For this, if $a \nleq b$, then we can apply the condition $(*)$ to $a \wedge b < a$. Hence there is a nonzero c such that $c \leq a$ and $c \wedge b = c \wedge (a \wedge b) = 0$. The other implication $(**) \Rightarrow (*)$ is trivial. Note that in [7], sectionally semi-complemented lattices are known as section semi-complemented lattices.

For an ideal I and a prime ideal P of a lattice L, we define the set $I[P]$ as follows:

$I[P] = \{x \in L \mid x \wedge y \in I$ for some $y \in L \setminus P\}$. If $I = \{0\}$, then $I[P]$ is denoted by $O[P]$. Let A be a nonempty subset and I be an ideal of a lattice L. We define the set $(I : A)$ as follows: $(I : A) = \{x \in L \mid x \wedge a \in I$ for all $a \in A\}$. In particular, if $A = \{a\}$, we write $(I : a)$ instead of $(I : \{a\})$. If $I = \{0\}$, then $(I : A)$ is denoted by $A^{\perp}$ and is called the *annihilator* of A. For $a \in L$, we use the notation $a^{\perp} = \{x \in L | x \wedge a = 0\}$. Note that $A^{\perp}$ is not an ideal in general, but if L is a 0-distributive lattice, then $A^{\perp}$ is an ideal.

An ideal I of a lattice L with 0 is a *Baer ideal* if $a \in I$ implies $a^{\perp\perp} \subseteq I$. An ideal I of a lattice L with 0 is a *dense ideal* if $I^{\perp} = (0]$ and is a *non-dense ideal* if $I^{\perp} \neq (0]$. An ideal I of a lattice L with 0 is a 0-*ideal* if there exists a proper filter F such that $I = F^0$, where $F^0 = \{x \in L | x \wedge y = 0 \text{ for some } y \in F\}$. An ideal I of a lattice L with 0 is a *closed ideal* if $I = I^{\perp\perp}$.

Let $\mathcal{I}_{\max}(L)$ denote the set of all maximal ideals of a lattice L and let $\mathrm{Max}(a) = \{M \in \mathcal{I}_{\max}(L) \mid a \in M\}$ for $a \in L$. Further, the intersection of all maximal ideals in L containing an element a of L is denoted by M_a, that is, $M_a = \bigcap \mathrm{Max}(a)$. Note that $M_1 = L$, if $1 \in L$.

In the rest of the paper, L *will denote a non-singleton lattice with the greatest element* 1.

3. z-ideals

Now, we define the concept of a z-ideal. Lemma 3.7 will provide a good connection to the ring-theoretical concepts mentioned in the introduction.

Definition 3.1. Let L be a lattice. A proper ideal I of L is a z-*ideal*, if $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$ and $b \in I$ imply $a \in I$.

Since the assignment $a \mapsto \mathrm{Max}(a)$ is antitone, it is easy to see that the z-ideals of a lattice L are exactly the join-closed nonempty subsets I with the property that $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$ and $b \in I$ imply $a \in I$.

Since $1 \in L$, Zorn's Lemma implies that every proper ideal of L is a subset of a maximal ideal. Hence, in particular, L has at least one maximal ideal.

Lemma 3.2. *Every maximal ideal of a lattice L is a z-ideal. Moreover, if L has only one maximal ideal, then no other ideal of L is a z-ideal.*

Proof. Clearly, every maximal ideal is a z-ideal. To prove the moreover part, assume that M is the unique maximal ideal of L and I is a proper ideal of L distinct from M. Since every proper ideal is included in a maximal ideal, $I \subsetneq M$. Hence there exists $x \in M$ such that $x \notin I$. Let $i \in I$. Since M is the unique maximal ideal, we have $\mathrm{Max}(i) = \mathrm{Max}(x)$. Thus I is not a z-ideal. ∎

Lemma 3.3. (Pawar and Thakare [12]) *Every maximal ideal of a* 1-*distributive* (*in particular, a distributive*) *lattice is a prime ideal.*

Lemma 3.4. *Let L be a lattice and $a, b \in L$. Then $a \in M_b$ if and only if $M_a \subseteq M_b$ if and only if* $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$.

Proof. If $M_a \subseteq M_b$, then $a \in M_b$, since $a \in M_a$. If $a \in M_b$, then all maximal ideals containing b also contain a, that is, $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$. Finally, larger sets have smaller intersections, whereby $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$ implies that $M_a \subseteq M_b$. ■

The following lemma is frequently used in the sequel.

Lemma 3.5. *Let L be a 1-distributive lattice and $a, b \in L$. Then the following statements hold.*

(1) $M_{a\wedge b} = M_a \cap M_b$.

(2) *If* $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$, *then* $\mathrm{Max}(b \wedge c) \subseteq \mathrm{Max}(a \wedge c)$ *for any* $c \in L$.

Proof. 1) Since the assignment $x \mapsto \mathrm{Max}(x)$ is antitone, Lemma 3.4 gives that $x \leq y$ implies $M_x \subseteq M_y$. We conclude from this fact that $M_{a\wedge b} \subseteq M_a \cap M_b$.

Now, let $x \in M_a \cap M_b$ and $x \notin M_{a\wedge b}$. Then there is a maximal ideal, say M', such that $a \wedge b \in M'$ but $x \notin M'$. Since L is 1-distributive, Lemma 3.3 gives that M' is prime. This gives that $a \in M'$ or $b \in M'$. Without loss of generality, assume that $a \in M'$. But then $x \in M_a \subseteq M'$, a contradiction. Hence $M_a \cap M_b \subseteq M_{a\wedge b}$. Thus $M_{a\wedge b} = M_a \cap M_b$.

2) Let M be a maximal ideal containing $b \wedge c$, i.e., $M \in \mathrm{Max}(b \wedge c)$. By Lemma 3.3, M is a prime ideal. Therefore $b \in M$ or $c \in M$. If $c \in M$, then $a \wedge c \in M$, and we are through. Now, let $b \in M$. Then $M \in \mathrm{Max}(b) \subseteq \mathrm{Max}(a)$, so $a \in M$. This gives $a \wedge c \in M$. Thus $M \in \mathrm{Max}(a \wedge c)$. ■

Remark 3.6. Note that the assertion of Lemma 3.5 is not true, if we drop 1-distributivity. Consider the lattice L depicted in Figure 1. Clearly, L is not 1-distributive. In this lattice, $M_a = (d] = M_b$. Hence $M_a \cap M_b = (d]$ and $M_{a\wedge b} = (0]$. Thus $M_{a\wedge b} \subsetneqq M_a \cap M_b$. Also $\mathrm{Max}(a) = \mathrm{Max}(d) = (d]$ but $\mathrm{Max}(a \wedge b) \neq \mathrm{Max}(d \wedge b)$.

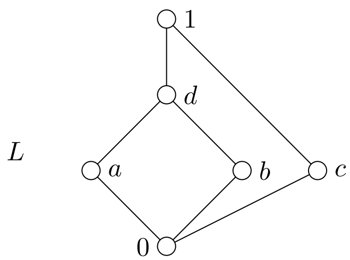

Figure 1. $M_{a\wedge b} \subsetneqq M_a \cap M_b$

In the following result, we characterize z-ideals in lattices.

Lemma 3.7. *Let I be an ideal of a 1-distributive lattice L. Then the following statements are equivalent.*

(1) *I is a z-ideal.*
(2) *If* $\mathrm{Max}(a) = \mathrm{Max}(b)$ *and $b \in I$, then $a \in I$.*
(3) *$M_a \subseteq I$ for all $a \in I$.*
(4) *If $M_b \subseteq M_a$ and $a \in I$, then $b \in I$.*

Proof. (1) $\Rightarrow$ (2): Obvious.
(2) $\Rightarrow$ (3): Let $x \in M_a$. Then by Lemma 3.4, $M_x \subseteq M_a$. Hence $M_x = M_x \cap M_a = M_{a \wedge x}$ by Lemma 3.5. This further gives $\mathrm{Max}(x) = \mathrm{Max}(x \wedge a)$, by Lemma 3.4. If $a \in I$, then $a \wedge x \in I$. Thus if $a \in I$, then by (2), $x \in I$ proving that $M_a \subseteq I$ for $a \in I$.
(3) $\Rightarrow$ (4): Let $a \in I$. Then by (3), $M_a \subseteq I$. Now, if $M_b \subseteq M_a$, then $b \in M_b \subseteq I$.
(4) $\Rightarrow$ (1): This follows from Lemma 3.4. ∎

Theorem 3.8. (Grätzer [4]) *Let L be a distributive lattice. If $I \cap F = \emptyset$ for an ideal I and for a filter F in L, then there exists a prime ideal P containing I and disjoint from F. Consequently, every ideal of L is the intersection of some prime ideals.*

Now, we prove a separation theorem for z-ideals. It is known that the behavior of ideals is influenced by that of prime ideals. The following result is an example of such type of behavior. In fact, we prove a stronger version mentioned below.

Theorem 3.9. *Let L be a distributive lattice. If $I \cap F = \emptyset$ for a z-ideal I and for a filter F in L, then there exists a prime z-ideal P containing I and disjoint from F. Consequently, every z-ideal of L is the intersection of some prime z-ideals.*

Proof. Consider $\mathcal{F} = \{J \mid J \text{ is a } z\text{-ideal including } I \text{ and } J \cap F = \emptyset\}$. Since $I \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Similarly to Grätzer [4], Zorn's Lemma yields a maximal element P of $\mathcal{F}$. We claim that P is a prime ideal. Let $a \wedge b \in P$ and $a, b \notin P$. Then $(P \vee (a]) \cap F \neq \emptyset$ and $(P \vee (b]) \cap F \neq \emptyset$. Let $x \in (P \vee (a]) \cap F$ and $y \in (P \vee (b]) \cap F$. Then $x \leq p_1 \vee a$ and $y \leq p_2 \vee b$ for some $p_1, p_2 \in P$. This gives $x \wedge y \leq (p_3 \vee a) \wedge (p_3 \vee b) = p_3 \vee (a \wedge b) \in P$, where $p_3 = p_1 \vee p_2$. Thus $x \wedge y \in P$. Also $x \wedge y \in F$ gives that $P \cap F \neq \emptyset$, a contradiction. Hence P is a prime z-ideal.

Clearly, the intersection of z-ideals is a z-ideal. Thus the second part of this theorem follows in the same way as in the case of Theorem 3.8. ∎

From the above result, it is clear that in a distributive lattice every prime ideal is a z-ideal if and only if every ideal is a z-ideal.

Now, we characterize dually semi-complemented lattices in terms of maximal ideals.

Lemma 3.10. *A lattice L with 0 is dually semi-complemented if and only if $\bigcap\{M|M \in \mathcal{I}_{\max}(L)\} = \{0\}$.*

Proof. Let L be a dually semi-complemented lattice and suppose $\bigcap\{M|M \in \mathcal{I}_{\max}(L)\} \neq \{0\}$. Let $a \in \bigcap\{M|M \in \mathcal{I}_{\max}(L)\}$ and $a \neq 0$. Since L is dually semi-complemented, there exists $b \neq 1$ such that $a \vee b = 1$. This implies that $b \notin \bigcap\{M|M \in \mathcal{I}_{\max}(L)\}$. Since $b \neq 1$, there exists a maximal ideal, say M', such that $b \in M'$. Since $a \in \bigcap\{M|M \in \mathcal{I}_{\max}(L)\}$, this implies $a \in M'$. Thus $1 = a \vee b \in M'$, a contradiction. Hence $\bigcap\{M|M \in \mathcal{I}_{\max}(L)\} = \{0\}$.

Conversely, suppose that $\bigcap\{M|M \in \mathcal{I}_{\max}(L)\} = \{0\}$. Let $(0 \neq)\ a \in L$. Clearly, $a \notin \bigcap\{M|M \in \mathcal{I}_{\max}(L)\} = \{0\}$. Then there exists a maximal ideal M' such that $a \notin M'$. Therefore $(a] \vee M' = L$ implies $1 \in (a] \vee M'$. Hence $1 = a \vee b$ for some $b \in M'$. Clearly, $b \neq 1$. Thus L is dually semi-complemented. ■

Proposition 3.11. *Let L be a dually semi-complemented lattice with 0. Then $(0]$ is a z-ideal. If, in addition, L is a 1-distributive lattice and $a \in L$, then $a^{\perp} = \bigcap\{M \in \mathcal{I}_{\max}(L)|\ a \notin M\}$ for $a \in L$.*

Proof. Let $\mathrm{Max}(a) \subseteq \mathrm{Max}(b)$ and $a \in I = (0]$. Then $\mathrm{Max}(b) = \mathcal{I}_{\max}(L)$ implies that $b \in \bigcap\{M|M \in \mathcal{I}_{\max}(L)\} = \{0\}$, by Lemma 3.10. Hence $b = 0$. Thus $(0]$ is a z-ideal.

To prove the second part, let $x \in a^{\perp}$ and $M \in \mathcal{I}_{\max}(L)$. By Lemma 3.3, M is prime. If $a \notin M$, then $x \in M$. Thus $a^{\perp} \subseteq \bigcap\{M \in \mathcal{I}_{\max}(L)|\ a \notin M\}$.

Conversely, with the notations $\mathcal{I}_1 = \{M \in \mathcal{I}_{\max}(L) : a \notin M\}$ and $\mathcal{I}_2 = \{M \in \mathcal{I}_{\max}(L) : a \in M\}$, assume that $x \in \bigcap \mathcal{I}_1$. For $M \in \mathcal{I}_1$, $a \wedge x \leq x$ gives that $a \wedge x \in M$. Similarly, for $M \in \mathcal{I}_2$, we obtain $a \wedge x \in M$, since $a \wedge x \leq a$. Thus $a \wedge x$ belongs to $\bigcap(\mathcal{I}_1 \cup \mathcal{I}_2) = \bigcap \mathcal{I}_{\max}(L)$, which is $(0]$ by Lemma 3.10. Thus, $x \in a^{\perp}$, as required. ■

Remark 3.12. Note that, by Lemma 3.2, every finite chain with at least three elements exemplify that dual semi-complementedness cannot be omitted from Proposition 3.11.

Let I be an ideal of a lattice L. Define the set $I_z = \bigcap\{J \supseteq I|\ J \text{ is a } z\text{-ideal}\}$. Clearly, I_z is the smallest z-ideal and the assignment $I \subseteq I_z$ is a closure operator.

Theorem 3.13. *Let L be a 1-distributive, dually semi-complemented lattice with 0 and I be an ideal of L. Then each of the following five conditions implies that I is a z-ideal:*
(1) *I is a non-dense prime ideal;*
(2) *$I = A^{\perp}$ for some subset A of L;*
(3) *I is a closed ideal;*
(4) *I is 0-ideal;*
(5) *$I = O[P]$ for some prime ideal P of L.*
Moreover, if L is SSC, then every principal ideal is a z-ideal.

Proof. (1) Let I be a non-dense prime ideal and $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$, $b \in I$. Hence there is a nonzero element x such that $x \in I^{\perp}$. Hence $x \wedge b = 0$, as $b \in I$. Since $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$, by Lemma 3.5, we have $\mathcal{I}_{\max}(L) = \mathrm{Max}(b \wedge x) \subseteq \mathrm{Max}(a \wedge x)$. Thus $a \wedge x \in M$ for all $M \in \mathcal{I}_{\max}(L)$. Thus $a \wedge x = 0$, as $\bigcap\{M | M \in \mathcal{I}_{\max}(L)\} = \{0\}$, by Lemma 3.10. This yields $a \wedge x \in I$. Since I is a prime ideal, $a \in I$ or $x \in I$. If $x \in I$, then $x \in I \cap I^{\perp} = \{0\}$, a contradiction. Thus $a \in I$. Hence I is a z-ideal.

(2) Let $I = A^{\perp} = \{x \in L | x \wedge a = 0 \text{ for all } a \in A\}$ and $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$ with $b \in I$. Now, $b \in I = A^{\perp}$ implies $b \wedge c = 0$ for all $c \in A$. Since $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$, we have $\mathcal{I}_{\max}(L) = \mathrm{Max}(b \wedge c) \subseteq \mathrm{Max}(a \wedge c)$, by Lemma 3.5. Hence $a \wedge c \in \bigcap\{M | M \in \mathcal{I}_{\max}(L)\} = \{0\}$, by Lemma 3.10. Therefore $a \wedge c = 0$ for all $c \in A$. Thus $a \in A^{\perp} = I$. Hence I is a z-ideal.

(3) follows from (2) by substituting $A = I^{\perp}$.

(4) Let I be a 0-ideal. Then $I = F^0 = \{x \in L | x \wedge y = 0 \text{ for some } y \in F\}$ for some proper filter F of L. Let $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$ and $b \in I$. Since $b \in I = F^0$, we have $b \wedge y = 0$ for some $y \in F$. Now, $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$ and Lemma 3.5 imply that $\mathcal{I}_{\max}(L) = \mathrm{Max}(b \wedge y) \subseteq \mathrm{Max}(a \wedge y)$. Hence $a \wedge y \in \bigcap\{M | M \in \mathcal{I}_{\max}(L)\} = \{0\}$, by Lemma 3.10. Thus $a \wedge y = 0$ for some $y \in F$, that is, $a \in F^0 = I$. Hence I is a z-ideal.

(5) Let $I = O[P] = \{x \in L |\ x \wedge y = 0 \text{ for some } y \notin P\}$ for some prime ideal P of L. Then $F = L \setminus P$ is a proper filter. This gives that $I = F^0$. Hence the result follows from (4).

In order to prove the last statement, assume that L is an SSC lattice. Let $I = (x]$ be a principal ideal of L. Let $\mathrm{Max}(b) \subseteq \mathrm{Max}(a)$ and $b \in I$. Now, suppose $a \notin I$. Then there exists a nonzero c such that $c \leq a$ and $c \wedge x = 0$. This gives $b \wedge c = 0$. Thus $\mathcal{I}_{\max}(L) = \mathrm{Max}(b \wedge c) \subseteq \mathrm{Max}(a \wedge c)$, by Lemma 3.5. Then $c = a \wedge c \in \bigcap\{M | M \in \mathcal{I}_{\max}(L)\} = \{0\}$, by Lemma 3.10. Therefore $c = 0$, a contradiction. Thus $a \in I$. ■

As mentioned in the introduction, the concept of Baer ideals (equivalently

z^0-ideals) and z-ideals are related in commutative rings with unity. In the following remark, we show that in general lattices they are not related.

Remark 3.14. In view of Lemma 3.10 and Theorem 3.13, it is clear that in a 1-distributive, dually semi-complemented lattice with 0 every closed ideal is a z-ideal. However, the assertion is not true, if we drop the condition of dual semi-complementedness. Consider the three-element chain C_3. Then the ideal $(0]$ is a closed ideal and hence a Baer ideal but not a z-ideal. Now, if we consider the nonzero proper ideal in C_3, then it is a z-ideal but not a Baer ideal and hence not a closed ideal.

However, in a 0-1-distributive lattice with the additional condition $\bigcap\{M|M \in \mathcal{I}_{\max}(L)\} = \{0\}$, we prove that every Baer ideal is a z-ideal. For this purpose, we need the following three results.

Lemma 3.15. (Thakare and Pawar [15, Theorem 4]) *A prime ideal M is a minimal prime ideal in a 0-distributive lattice L if and only if it contains precisely one of $(x]$ or $x^\perp$ for every $x \in L$.*

Lemma 3.16. (Thakare and Pawar [15, Theorem 7]) *In a 0-distributive lattice L, the pseudocomplement of any ideal I is the intersection of all minimal prime ideals not containing I.*

Lemma 3.17. *In a 0-distributive lattice L, $a^{\perp\perp} = \bigcap\{P \in \mathcal{I}^{\mathrm{p}}_{\min}(L)|a \in P\}$ for $a \in L$.*

Proof. Let $I = a^\perp$. By Lemma 3.16, we have $a^{\perp\perp} = \bigcap\{P \in \mathcal{I}^{\mathrm{p}}_{\min}(L)|a^\perp \nsubseteq P\}$. Further, by Lemma 3.15, it is clear that $\{P \in \mathcal{I}^{\mathrm{p}}_{\min}(L)|a^\perp \nsubseteq P\} = \{P \in \mathcal{I}^{\mathrm{p}}_{\min}(L)|a \in P\}$. Hence $a^{\perp\perp} = \bigcap\{P \in \mathcal{I}^{\mathrm{p}}_{\min}(L)|a \in P\}$ for $a \in L$. ■

With this preparation, we now prove that in a 0-1-distributive, dually semi-complemented lattice, every Baer ideal is a z-ideal.

Lemma 3.18. *Let L be a 0-1-distributive, dually semi-complemented lattice. Then every Baer ideal is a z-ideal.*

Proof. Suppose I is a Baer ideal and $\mathrm{Max}(a) = \mathrm{Max}(b)$, $a \in I$ but $b \notin I$. By Lemma 3.17, we have $a^{\perp\perp} = \bigcap\{P \in \mathcal{I}^{\mathrm{p}}_{\min}(L)|a \in P\} = P_a$ (say). Since I is Baer and $a \in I$, we have $a^{\perp\perp} = P_a \subseteq I$. Note that $(b] \cap a^\perp \neq \{0\}$, otherwise $b \in a^{\perp\perp} \subseteq I$, a contradiction. Since L is dually semi-complemented, by Lemma 3.10, $\bigcap\{M|M \in \mathcal{I}_{\max}(L)\} = \{0\}$. Hence there exists a maximal ideal M such that $(b] \cap a^\perp \nsubseteq M$. Clearly, $a^\perp \nsubseteq M$ and $b \notin M$. By 1-distributivity, M is a prime ideal. Since $(a] \cap a^{\perp} = \{0\} \subseteq M$ and $a^{\perp} \nsubseteq M$, we have $a \in M$. Thus there exists a maximal ideal M such that $a \in M$ and $b \notin M$. Hence $\mathrm{Max}(a) \neq \mathrm{Max}(b)$, a contradiction to $\mathrm{Max}(a) = \mathrm{Max}(b)$. So $b \in I$. Thus I is a z-ideal. ■

Remark 3.19. Note that in a 0-1-distributive, dually semi-complemented lattice L a z-ideal is not a Baer ideal in general. For this, consider the set $L = \{X \subseteq \mathbb{N} | X$ is an infinite set $\} \cup \{\emptyset\}$, where $\mathbb{N}$ is the set of natural numbers. Clearly, L is a 0-distributive lattice under the set inclusion and $\bigcap\{M | M \in \mathcal{I}_{\max}(L)\} = \{0\}$. Let $I = (\mathbb{N} - \{1\}]$, the principal ideal generated by $\mathbb{N} - \{1\}$. Since every maximal ideal is a z-ideal, we have $(\mathbb{N} - \{1\}]$ is a z-ideal. Clearly, $(\mathbb{N} - \{1\}]^{\perp} = \{0\}$ and hence $(\mathbb{N} - \{1\}]^{\perp\perp} = \{0\}^{\perp} = L$. This gives $(\mathbb{N} - \{1\}]^{\perp\perp} \nsubseteq (\mathbb{N} - \{1\}]$. Therefore $(\mathbb{N} - \{1\}]$ is not a Baer ideal.

Lemma 3.20. *Let L be a* 0-1-*distributive lattice such that every Baer ideal is a z-ideal. Then L is dually semi-complemented.*

Proof. Suppose that every Baer ideal is a z-ideal. On the contrary, assume that L is not dually semi-complemented. Hence by Lemma 3.10, $\bigcap\{M | M \in \mathcal{I}_{\max}(L)\} \neq \{0\}$. Then for a nonzero element $a \in \bigcap\{M | M \in \mathcal{I}_{\max}(L)\}$, we have $\mathrm{Max}(a) = \mathrm{Max}(0)$. Since $(0]$ is a Baer ideal, by the hypothesis, $(0]$ is a z-ideal. Then $\mathrm{Max}(0) = \mathrm{Max}(a)$ and $0 \in (0]$ imply $a \in (0]$, a contradiction. ■

We conclude the paper by combining Lemma 3.10, Lemma 3.18 and Lemma 3.20 to obtain the following result.

Theorem 3.21. *Let L be a* 0-1-*distributive lattice. Then the following statements are equivalent.*

(1) *Every Baer ideal is a z-ideal.*
(2) *L is dually semi-complemented.*
(3) $\bigcap\{M | M \in \mathcal{I}_{\max}(L)\} = \{0\}$.

Acknowledgment. The authors are grateful to the referee for many fruitful suggestions which improved the quality of the paper.

References

[1] D. D. Anderson, C. Jayaram and P. A. Phiri, Baer lattices, *Acta Sci. Math. (Szeged)*, **59** (1994), 61–74.

[2] S. Burris and H. P. Sankappanavar,, *A Course in Universal Algebra*, Graduate Texts in Mathematics **78**, Springer, 1981.

[3] L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N.J.– Toronto – London – New York, 1960.
[4] G. Grätzer, *General Lattice Theory*, Second Edition, Birkhäuser Verlag, 1998.
[5] C. B. Huijsmans and de Pagter, On z-ideals and *d*-ideals in Riesz spaces I., *Nederl. Akad. Wetensch. Indag. Math.*, **42** (1980), 183–195.
[6] C. Jayaram, Baer ideals in commutative semiprime rings, *Indian J. Pure Appl. Math.*, **15** (1984), 855–864.
[7] M. F. Janowitz, Section semi-complemented lattices, *Math. Z.*, **108** (1968), 63–76.
[8] V. Joshi and N. Mundlik, Bear ideals in 0-distributive posets, *Asian- Euro. J. Math.*, **9** (2016), 1650055 (16 pages).
[9] C. W. Kohls, Ideals in rings of continuous functions, *Fund. Math.*, **45** (1957), 28–50.
[10] G. Mason, *z*-Ideals and prime ideals, *J. Algebra*, **26** (1973), 280–297.
[11] J. B. Nation, *Notes on Lattice Theory*, http://people.math.sc.edu/mcnulty/alglatvar/lat0.pdf.
[12] Y. S. Pawar and N. K. Thakare, 0-distributive semilattices, *Canad. Math. Bull.*, **21** (1978), 469–481.
[13] Y. Rav, Semiprime ideals in general lattices, *J. Pure Appl. Algebra*, **56** (1989), 105–118.
[14] T. P. Speed, A note on commutative Baer rings, *J. Austral. Math. Soc.*, **14** (1972), 257–263.
[15] N. K. Thakare and Y. S. Pawar, Minimal prime ideals in 0-distributive semilattices, *Period. Math. Hungar.*, **3** (1982), 237–246.
[16] J. C. Varlet, A generalization of the notion of pseudo-complementedness, *Bull. Soc. Roy. Sci. Liêge*, **37** (1968), 149–158.

V. Joshi, Department of Mathematics, Savitribai Phule Pune University, Pune-411007, India; *e-mail*: vinayakjoshi111@yahoo.com; vvjoshi@unipune.ac.in

S. Kavishwar, Department of Mathematics, College of Engineering, Pune-411005, India; *e-mail*: spk.maths@coep.ac.in