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Characterization of Euclidean geometry by existence of circumcenter or orthocenter

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Dedicated to László Leindler on his 80th birthday

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Abstract. A Minkowski geometry is Euclidean if and only if the altitudes of any trigon are concurrent. A Minkowski geometry is Euclidean if and only if the perpendicular bisectors of any trigon are concurrent.

1. Introduction

A Minkowski geometry is an affine geometry with distance function given by a centrally symmetric convex body, the indicatrix. It is widely investigated if presence of various geometric conditions in Minkowski geometry results in specific conditions regarding the indicatrix. For further reference see [1] and [5, 6].

Thorough articles were devoted to existence of certain points regarding triangles. See i.e. [10], [11] and [7, 8].

In this article we prove that the right-perpendicular bisectors are concurrent in every triangle in a Minkowski geometry if and only if the indicatrix is an ellipsoid. Then we prove that a Minkowski space, where triangles have concurrent right-altitudes, is Euclidean. An alternate proof for the plane was already given in [13].

In the last section the analogous results are proved for the left-perpendicularity.

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2. Preliminaries

Points of $\mathbb{R}^n$ are denoted as $\mathbf{a}, \mathbf{b}, \ldots$; the line through different points $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{ab}$, the open segment with endpoints $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\overline{\mathbf{ab}}$, the ray (open half-line) with starting point $\mathbf{a}$ passing through point $\mathbf{b}$ is denoted by $\overrightarrow{\mathbf{ab}}$. The arc of a curve $\mathcal{C}$ with endpoints $\mathbf{a}, \mathbf{b}$ will be denoted by $\mathcal{C}(\mathbf{ab})$[1]. Non-degenerate triangles are called *trigons*.

A Minkowski geometry $(\mathbb{A}^n; \mathcal{I}^n)$ is an affine space $\mathbb{A}^n$ of dimension n, provided with a centrally symmetric, strictly convex body $\mathcal{I}^n$, as the *indicatrix* of the geometry. In this article we assume that $\partial\mathcal{I}^n$ is twice differentiable.

Our considerations regard perpendicularity of lines. However, perpendicularity is not a symmetric relation as it is defined in Minkowski geometries (see [12, Definition 3.2.2]). We say that *line l is left-perpendicular to line m*, if $l' \parallel l$ is a line tangent to $\mathcal{I}$ at point $\mathbf{p}$ and m is parallel to line $\mathbf{cp}$, where $\mathbf{c}$ is the center of $\mathcal{I}$. In this case we say that *line m is right-perpendicular to the line l*. Note that by [13, Theorem 3.4.10] left- and right-perpendicularity coincide in dimensions higher than 2 if and only if $\mathcal{I}$ is an ellipsoid. In the plane this coincidence is satisfied if and only if $\partial\mathcal{I}$ is a Radon-curve (see [4]).

We are going to investigate the shape of $\mathcal{I}$ via considering triangles, that is why observations in two dimensions are beneficial for us, especially the following lemma which is the specific case of Theorem 1 [3], for $\delta = 0$. Here S^{d-1} denotes the unit sphere in the n-dimensional Euclidean space.

Lemma 2.1. *Let C be a convex body in $\mathbb{R}^d$ $(d \geq 3)$ and let us suppose that any hyperplane through the origin meets the interior of C and its intersection with C is a $(d-1)$-dimensional ellipsoid. Then C is an ellipsoid.*

In order to decide if a convex body is an ellipsoid or not, we shall compare it to its Lowner–John ellipsoid.

Lemma 2.2. ([8, Lemma 3.3]) *Let $\mathcal{H}$ be a convex body in the plane. Then*

(i) *there exists an ellipse $\mathcal{E}$ circumscribed around $\mathcal{H}$ with at least three different contact points $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ lying in $\partial\mathcal{H} \cap \partial\mathcal{E}$ such that the closed triangle $\triangle\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ contains the center $\mathbf{c}$ of $\mathcal{E}$, and*

(ii) *if $\mathcal{H} \not\equiv \mathcal{E}$, then these contact points can be chosen so that in every neighborhood of one of them $\partial\mathcal{H} \setminus \partial\mathcal{E} \neq \emptyset$.*

Let t_1, t_2, t_3 be the common support lines at $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, respectively. Then

(iii) *$\mathbf{c}$ is in the interior of $\triangle\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ if and only if t_1, t_2, t_3 form a trigon with vertices $\mathbf{m}_1 = t_2 \cap t_3$, $\mathbf{m}_2 = t_3 \cap t_1$ and $\mathbf{m}_3 = t_1 \cap t_2$;*

[1] Should this be not unique the context of its use will clarify what arc is thought about in the text.

(iv) $\mathbf{c}$ *is in one of the edges of* $\triangle\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$*, say* $\mathbf{c}\in\overline{\mathbf{e}_2\mathbf{e}_3}$*, if and only if* t_1,t_2,t_3 *form a half strip with vertices* $\mathbf{m}_2=t_1\cap t_3$*,* $\mathbf{m}_3=t_2\cap t_1$ *and the ideal point* $\mathbf{m}_1=t_2\cap t_3$*.*

If $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3$ *are the midpoints of the segments* $\overline{\mathbf{e}_2\mathbf{e}_3}$*,* $\overline{\mathbf{e}_3\mathbf{e}_1}$ *and* $\overline{\mathbf{e}_1\mathbf{e}_2}$*, respectively, then*

(v) *the straight lines* $\mathbf{m}_i\mathbf{b}_i$ $(i=1,2,3)$ *meet in* $\mathbf{c}$*.*

If $\mathcal{H}$ is centrally symmetric to a point $\mathbf{c}$ then the following complements Lemma 2.2.

Lemma 2.3. *The circumscribed Lowner–John-ellipse* $\mathcal{E}$ *of a convex body* $\mathcal{H}$ *centrally symmetric to point* $\mathbf{c}$ *has center* $\mathbf{c}$*.*

For the proof one only has to observe that the reflection $\tau_{\mathbf{c}}$ keeps $\mathcal{H}$ invariant and therefore, by its uniqueness, $\mathcal{E}=\tau_{\mathbf{c}}(\mathcal{E})$.

We need some more technical lemmas.

Lemma 2.4. *Let* $\mathbf{r},\mathbf{p}\colon[0;1]\to\mathbb{R}^2$ *be continuously differentiable curves with non-vanishing derivatives.*

(1) *If* (i) $\mathbf{r}\parallel\mathbf{p}$, (ii) $\dot{\mathbf{r}}\parallel\dot{\mathbf{p}}$*, and these curves intersect each other, then* $\mathbf{r}=\mathbf{p}$*.*

(2) *If* $\dot{\mathbf{r}}(0)\parallel\dot{\mathbf{p}}(0)$ *and* $\dot{\mathbf{r}}(1)\parallel\dot{\mathbf{p}}(1)$*, then there exists a* $t_0\in(0,1)$ *such that* $\dot{\mathbf{r}}(t_0)\parallel\dot{\mathbf{p}}(t_0)$*.*

Proof. (1) Conditions (i) and (ii) imply the existence of differentiable non-vanishing real functions λ and μ, respectively such that μ and $\mathbf{r}=\lambda\mathbf{p}$ and μ and $\dot{\mathbf{r}}=\mu\dot{\mathbf{p}}$ respectively. Substituting the latter equation into the derivative of the first one, results in $\mu\dot{\mathbf{p}}=\dot{\lambda}\mathbf{p}+\lambda\dot{\mathbf{p}}$, hence

$$\mathbf{0}=\dot{\lambda}\mathbf{p}+(\lambda-\mu)\dot{\mathbf{p}}.$$

If $\mathbf{p}\nparallel\dot{\mathbf{p}}$ then this gives $\dot{\lambda}=0$ and that implies the statement. If $\mathbf{p}\parallel\dot{\mathbf{p}}$ then $\mathbf{p}$ is a segment, hence we get the statement.

(2) It is enough to apply the mean value theorem to the function $\arccos(\langle\dot{\mathbf{r}}_1,\dot{\mathbf{r}}_2\rangle)$. ∎

Lemma 2.5. ([8, Lemma 3.4]) *For a small* $\varepsilon>0$ *let* $\mathbf{r},\mathbf{p}\colon(-\varepsilon,0]\to\mathbb{R}^2$ *be twice differentiable convex curves such that* $\mathbf{p}(\tau)=p(\tau)\mathbf{u}_\tau$ *and* $\mathbf{r}(\tau)=r(\tau)\mathbf{u}_\tau$*, where* $p,r\colon(-\varepsilon,0]\to\mathbb{R}_+$*,* $\lambda(\tau):=r(\tau)/p(\tau)$ *takes its minimum value* 1 *at* $\tau=0$*, and* $\max_{(-\delta,0]}\lambda>1$ *for every* $\delta\in(0,\varepsilon)$*.*

Let τ_n *be a sequence in* $(-\varepsilon,0]$ *tending to* 0 *such that* $\lambda(\tau_n)>1$ *for every* $n\in\mathbb{N}$*. Then the tangent lines of* $\mathbf{r}$ *and* $\mathbf{p}$ *at* $\mathbf{r}(\tau_n)$ *and* $\mathbf{p}(\tau_n)$*, respectively, intersect each other in a point* $\mathbf{m}(\tau_n)$ *that tends to* $\mathbf{p}(0)$ *as* $\tau_n\to 0$ *so that it is on the same side of the line* $\mathbf{0}\mathbf{p}(\tau_n)$ *as* $\mathbf{p}(0)$ *is.*

3. Characterization by right-circumcenter and right-orthocenter

Three right-perpendicular bisectors of a trigon pairwise have point of intersection. If all of them pass through one point we can call it the *right-circumcenter* of the trigon. In this section we prove that the existence of right-circumcenter for every trigon implies that the geometry is Euclidean. We use ideas taken from [7,8].

Theorem 3.1. *If in a Minkowski geometry the right-perpendicular bisectors of every trigon are concurrent, then the geometry is Euclidean.*

Proof. For easier references later on, we number the main steps of our proofs.

(Step a1) In virtue of Lemma 2.1, it is enough to show that every two-dimensional sections of the border of the indicatrix containing its center is an ellipse.

Take a plane through the origin $\mathbf{o}$, and let $\mathcal{I}'$ be its intersection with the border of the indicatrix. Furthermore take the circumscribed Lowner–John ellipse $\mathcal{E}$ of $\mathcal{I}'$.

We shall conduct an indirect proof. Assuming that $\mathcal{I}' \neq \mathcal{E}$, we want to arrive at a contradiction.

Let us take an affinity of the plane, keeping the origin $\mathbf{o}$ fixed, and mapping the ellipse $\mathcal{E}$ into a circle $\mathcal{C}$ with center $\mathbf{o}$, $\mathcal{I}'$ into a curve $\mathcal{I}$, centrally symmetric in the origin, and inscribed into the circle. Since any affinity of the plane keeps the incidence of points and lines, it keeps perpendicularity of lines, as well. So we can deal with a centrally symmetric curve $\mathcal{I}$ and its circumscribed circle $\mathcal{C}$, concentric with $\mathcal{I}$.

Let us notice that under these circumstances, a tangent to the circle is right-perpendicular to another tangent to the circle if and only if they are perpendicular in Euclidean terms.

(Step a2) According to Lemma 2.2, there exist three points in $\mathcal{I} \cap \mathcal{C}$. Because of central symmetry, common points occur in pairs symmetric with respect to $\mathbf{o}$, so we have at least four points in common. We shall investigate basically two cases: (1) the number of points of the set $\mathcal{I} \cap \mathcal{C}$ is not less than 5; (2) the number of points of the set $\mathcal{I} \cap \mathcal{C}$ is 4.

These two cases will be related to non-perpendicularity respectively perpendicularity of certain sides of the trigon we construct to gain contradiction.

Case 1. Curves $\mathcal{I}$ and $\mathcal{E}$ have at least five common points.

(Step a3) Common points of the symmetric curves are in antipodal pairs, so we can chose and number five common points $\mathbf{e}_i$ $i = 1, \dots, 5$ such that $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_3, \mathbf{e}_4)$ are antipodal pairs. We shall denote the common tangents at these points by t_i respectively. Then the tangents t_3 and t_5 meet t_1 and t_2, moreover, either t_3

or t_5 is not right-perpendicular to them. Let us change indexes, if necessary, in such a way that t_3 and t_2 are not perpendicular.

So we can suppose that we have four points $\mathbf{e}_1 \prec \mathbf{e}_4 \prec \mathbf{e}_2 \prec \mathbf{e}_3$ in a circuit of $\mathcal{C}$ such that $t_1 \parallel t_2$ and t_3 is not right-perpendicular to t_2.

(Step a4) As a consequence of the central symmetry of the two different curves, we know that equalities $\mathcal{C}(\mathbf{e}_1\mathbf{e}_4) = \mathcal{I}(\mathbf{e}_1\mathbf{e}_4)$ and $\mathcal{C}(\mathbf{e}_4\mathbf{e}_2) = \mathcal{I}(\mathbf{e}_4\mathbf{e}_2)$ can not hold simultaneously, otherwise $\mathcal{C} = \mathcal{I}$ would follow. Without loss of generality we may suppose that $\mathcal{C}(\mathbf{e}_1\mathbf{e}_4) \neq \mathcal{I}(\mathbf{e}_1\mathbf{e}_4)$.[2]

(Step a5) Tangents t_2 and t_3 are not parallel, because $\mathbf{e}_2$ and $\mathbf{e}_3$ are not antipodal points. Furthermore, all tangents to the open arc $\mathcal{C}(\mathbf{e}_1\mathbf{e}_4)$ meet both t_2 and t_3. Applying Lemma 2.4 to the arcs $\mathcal{C}(\mathbf{e}_1\mathbf{e}_4)$ and $\mathcal{I}(\mathbf{e}_1\mathbf{e}_4)$ one obtains that there must be points $\mathbf{p}_1^{\mathcal{C}} \in \mathcal{C}(\mathbf{e}_1\mathbf{e}_4)$ and $\mathbf{p}_1^{\mathcal{I}} \in \mathcal{I}(\mathbf{e}_1\mathbf{e}_4)$ such that the following properties hold true:

(i) $\mathbf{p}_1^{\mathcal{C}} \nparallel \mathbf{p}_1^{\mathcal{I}}$;
(ii) the respective tangents $t_1^{\mathcal{C}}, t_1^{\mathcal{I}}$ at these points are parallel.

Further

(iii) $t_1^{\mathcal{C}}$ and $t_1^{\mathcal{I}}$ meet both t_2 and t_3,
(iv) $\mathbf{p}_1^{\mathcal{C}}, \mathbf{p}_1^{\mathcal{I}}$ are not perpendicular to t_3, or to t_2,

otherwise a point, sufficiently close to $\mathbf{p}_1^{\mathcal{C}}$, could be used instead of $\mathbf{p}_1^{\mathcal{C}}$.

(Step a6) Let $\mathbf{m}_1 = t_2 \cap t_3$, $\mathbf{m}_2^{\mathcal{C}} = t_1^{\mathcal{C}} \cap t_3$, $\mathbf{m}_3^{\mathcal{C}} = t_1^{\mathcal{C}} \cap t_2$, $\mathbf{m}_2^{\mathcal{I}} = t_1^{\mathcal{I}} \cap t_3$, $\mathbf{m}_3^{\mathcal{I}} = t_1^{\mathcal{I}} \cap t_2$. (See Figure 3.1.)

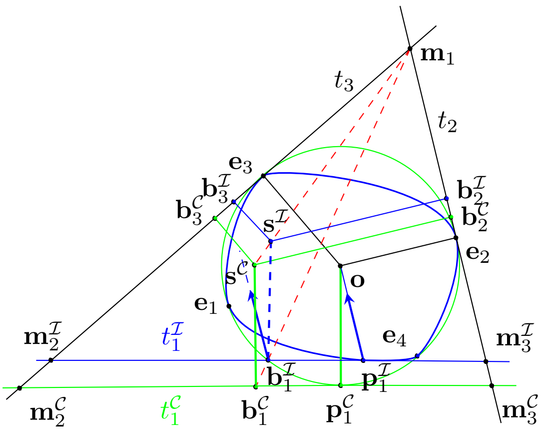

Figure 3.1. Left-perpendicular bisectors if $\mathbf{s}^{\mathcal{C}} \neq \mathbf{p}_1^{\mathcal{C}}$

[2]If $\mathcal{C}(\mathbf{e}_1\mathbf{e}_4) \neq \mathcal{I}(\mathbf{e}_1\mathbf{e}_4)$, we can exchange indexes 1 and 2, without disimproving the non-perpendicularity of t_3 and t_2.

Furthermore denote the midpoints of the edges of trigon $\triangle \mathbf{m}_1\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$ by $\mathbf{b}_3^{\mathcal{I}}, \mathbf{b}_1^{\mathcal{I}}, \mathbf{b}_2^{\mathcal{I}}$; the midpoints of the edges of trigon $\triangle \mathbf{m}_1\mathbf{m}_2^{\mathcal{C}}\mathbf{m}_3^{\mathcal{C}}$ by $\mathbf{b}_3^{\mathcal{C}}, \mathbf{b}_1^{\mathcal{C}}, \mathbf{b}_2^{\mathcal{C}}$ respectively. Perpendicular bisectors of the trigon $\triangle \mathbf{m}_1\mathbf{m}_2^{\mathcal{C}}\mathbf{m}_3^{\mathcal{C}}$, by Euclidean geometry, have a common point $\mathbf{s}^{\mathcal{C}}$. It is different from $\mathbf{b}_i^{\mathcal{C}}$, since this trigon is not a right one. Furthermore, $\mathbf{b}_i^{\mathcal{C}}\mathbf{s}^{\mathcal{C}} \parallel \mathbf{e}_i\mathbf{o}$ for $i = 2, 3$, and $\mathbf{b}_1^{\mathcal{C}}\mathbf{s}^{\mathcal{C}} \parallel \mathbf{p}_1^{\mathcal{C}}\mathbf{o}$.

By assumption, the intersection of the right-perpendicular bisectors of trigon $\triangle \mathbf{m}_1\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$ through the midpoints $\mathbf{b}_3^{\mathcal{I}}$ and $\mathbf{b}_2^{\mathcal{I}}$ is $\mathbf{s}^{\mathcal{I}}$, the right-circumcenter of $\triangle \mathbf{m}_1\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$. Trigons $\triangle \mathbf{m}_1\mathbf{m}_2^{\mathcal{C}}\mathbf{m}_3^{\mathcal{C}}$ and $\triangle \mathbf{m}_1\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$ are homothetic with center $\mathbf{m}_1$ since $t_1^{\mathcal{C}} \parallel t_1^{\mathcal{I}}$. This homothety χ sends $\mathbf{b}_i^{\mathcal{C}}$ to $\mathbf{b}_i^{\mathcal{I}}$ ($i = 1, 2, 3$), hence it sends lines $\mathbf{b}_i^{\mathcal{C}}\mathbf{s}^{\mathcal{C}}$ to the lines going through $\mathbf{b}_i^{\mathcal{I}}$, parallel with them ($i = 2, 3$). Therefore, χ maps point of intersection $\mathbf{s}^{\mathcal{C}}$ into point of intersection $\mathbf{s}^{\mathcal{I}}$, with $\mathbf{s}^{\mathcal{C}}$ different from $\mathbf{b}_i^{\mathcal{I}}$. Consequently, we have $\mathbf{p}_1^{\mathcal{C}}\mathbf{o} \parallel \mathbf{b}_1^{\mathcal{C}}\mathbf{s}^{\mathcal{C}} \parallel \mathbf{b}_1^{\mathcal{I}}\mathbf{s}^{\mathcal{I}}$. The right-perpendicular bisector, through midpoint $\mathbf{b}_1^{\mathcal{I}}$ must go through $\mathbf{s}^{\mathcal{I}}$, the point of intersection of the other two right-perpendicular bisectors, therefore is parallel to $\mathbf{p}_1^{\mathcal{C}}\mathbf{o}$ on one hand, and parallel to $\mathbf{p}_1^{\mathcal{I}}\mathbf{o}$, by the definition of right-perpendicularity, on the other hand. This contradiction proves the theorem in this case.

Case 2. Curves $\mathcal{I}$ and $\mathcal{C}$ have exactly four common points.

(Step a7) We can number the common points so that one has the order $\mathbf{e}_1 \prec \mathbf{e}_4 \prec \mathbf{e}_2 \prec \mathbf{e}_3$ in a circuit of $\mathcal{C}$.

If $\mathbf{e}_2$ is not perpendicular to $\mathbf{e}_3$ then for the tangents at these points $t_1 \parallel t_2$ and t_3 is not right-perpendicular to t_2.

Furthermore, the following open arcs have no common points: $\mathcal{C}(\mathbf{e}_1\mathbf{e}_4) \cap \mathcal{I}(\mathbf{e}_1\mathbf{e}_4) = \emptyset$ and $\mathcal{C}(\mathbf{e}_4\mathbf{e}_2) \cap \mathcal{I}(\mathbf{e}_4\mathbf{e}_2) = \emptyset$.

So one can go back to the end of (Step a4), continue with (Step a5) and finish with (Step a6), completing the proof.

(Step a8) If $\mathbf{e}_2$ is perpendicular to $\mathbf{e}_3$, we have two antipodal pairs of common points such that $\mathbf{e}_1 \prec \mathbf{e}_4 \prec \mathbf{e}_2 \prec \mathbf{e}_3$ in a circuit of $\mathcal{C}$ with corresponding tangents t_3 perpendicular to t_2.

Applying Lemma 2.4 to the open arcs $\mathcal{C}(\mathbf{e}_3\mathbf{e}_1)$ and $\mathcal{I}(\mathbf{e}_3\mathbf{e}_1)$ we can find points $\mathbf{p}_3^{\mathcal{C}}$, $\mathbf{p}_3^{\mathcal{I}}$ collinear with $\mathbf{o}$ such that their respective tangents $t_3^{\mathcal{C}}, t_3^{\mathcal{I}}$ to $\mathcal{C}$ and $\mathcal{I}$, respectively, are parallel. Moreover, these tangents are not perpendicular to t_2. Let us denote the magnitude of the angle $\angle(\mathbf{e}_3\mathbf{o}\mathbf{p}_3^{\mathcal{C}})$ by α_3.

(Step a9) We can apply Lemma 2.5 to the open arcs $\mathcal{C}(\mathbf{e}_4\mathbf{e}_1)$ and $\mathcal{I}(\mathbf{e}_4\mathbf{e}_1)$ to find points $\mathbf{p}_1^{\mathcal{C}}$, $\mathbf{p}_1^{\mathcal{I}}$ arbitrarily close to $\mathbf{e}_4$ on the respective arcs such that

(i) $\mathbf{p}_1^{\mathcal{C}} \parallel \mathbf{p}_1^{\mathcal{I}}$,

(ii) the respective tangents $t_1^{\mathcal{C}}, t_1^{\mathcal{I}}$ to $\mathcal{C}$ and $\mathcal{I}$, respectively, are not parallel,

(iii) $t_1^{\mathcal{C}} \cap t_1^{\mathcal{I}}$ is close to $\mathbf{e}_4$, and is on the same side of $\mathbf{o}\mathbf{p}_1^{\mathcal{C}}$ where $\mathbf{e}_4$ is.

We can chose $\mathbf{p}_1^{\mathcal{C}}$ so much close to $\mathbf{e}_4$ that the magnitude α_1 of angle $\angle(\mathbf{p}_1^{\mathcal{C}}\mathbf{o}\mathbf{e}_4)$ is smaller than $\min(\alpha_3, \pi/2 - \alpha_3)$.

(Step a10) Tangent $t_1^{\mathcal{C}}$ meets tangent t_2 in a point $\mathbf{m}_3^{\mathcal{C}}$ since t_4 meets it and directions of $t_1^{\mathcal{C}}$ and t_4 are close to each other. Tangent $t_1^{\mathcal{I}}$ meets tangent t_2 in a point $\mathbf{m}_3^{\mathcal{I}}$ for the same reason. Angles $\angle \mathbf{p}_1^{\mathcal{C}}\mathbf{m}_3^{\mathcal{C}}\mathbf{e}_2$, $\angle \mathbf{p}_1^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}\mathbf{e}_2$ are both acute.

Tangent $t_3^{\mathcal{C}}$ meets tangent t_2 in a point $\mathbf{m}_1^{\mathcal{C}}$, tangent $t_3^{\mathcal{I}}$ meets tangent t_2 in a point $\mathbf{m}_1^{\mathcal{I}}$ since $0 < \alpha_3 < \pi/2$, furthermore, angles $\angle \mathbf{p}_3^{\mathcal{C}}\mathbf{m}_1^{\mathcal{C}}\mathbf{e}_2$, $\angle \mathbf{p}_3^{\mathcal{I}}\mathbf{m}_1^{\mathcal{I}}\mathbf{e}_2$ are acute, and the angle $\angle \mathbf{p}_3^{\mathcal{C}}\mathbf{o}\mathbf{p}_1^{\mathcal{C}}$ is obtuse. It follows that tangent $t_1^{\mathcal{C}}$ meets tangent $t_3^{\mathcal{C}}$ in a point $\mathbf{m}_2^{\mathcal{C}}$, and the angle $\angle \mathbf{p}_3^{\mathcal{C}}\mathbf{m}_2^{\mathcal{C}}\mathbf{p}_1^{\mathcal{C}}$ is acute. Since tangent $t_1^{\mathcal{C}}$ and $t_1^{\mathcal{I}}$ differ just a little in their direction, $t_1^{\mathcal{I}}$ and $t_3^{\mathcal{I}}$ meet in a point $\mathbf{m}_2^{\mathcal{I}}$. (See Figure 3.2.)

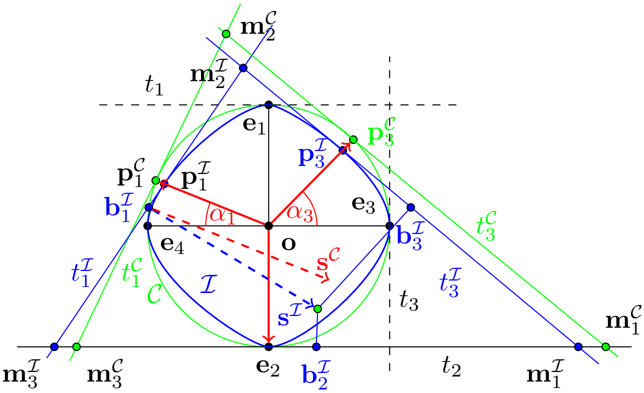

Figure 3.2. Case 2: if tangents t_3 and t_2 are orthogonal

When all comes to all we obtain that both triangles $\triangle \mathbf{m}_1^{\mathcal{C}}\mathbf{m}_2^{\mathcal{C}}\mathbf{m}_3^{\mathcal{C}}$, $\triangle \mathbf{m}_1^{\mathcal{I}}\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$ do exist and all angles are acute.

(Step a11) Let us denote midpoints of the edges of the trigon opposite to the vertex $\mathbf{m}_i^{\mathcal{I}}$ by $\mathbf{b}_i^{\mathcal{I}}$ $(i = 1, 2, 3)$. Right-perpendicular bisectors through $\mathbf{b}_2^{\mathcal{I}}$ and $\mathbf{b}_3^{\mathcal{I}}$ coincide with the Euclidean perpendicular bisectors, and have point of intersection $\mathbf{s}^{\mathcal{I}}$. This point is the Euclidean circumcenter of $\triangle \mathbf{m}_1^{\mathcal{I}}\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$, and it also the right-perpendicular circumcenter $\mathbf{s}^{\mathcal{I}}$ which is different from each midpoints $\mathbf{b}_i^{\mathcal{I}}$ as the triangle has no right angle. Hence $\mathbf{b}_1^{\mathcal{I}}\mathbf{s}^{\mathcal{I}}$ is the third Euclidean perpendicular bisector which is perpendicular to tangent $t_1^{\mathcal{I}}$. At the same time, $\mathbf{b}_1^{\mathcal{I}}\mathbf{s}^{\mathcal{I}}$ must be the third right-perpendicular bisector which is parallel to $\mathbf{p}_1^{\mathcal{I}}\mathbf{o}$ and consequently, perpendicular to $t_1^{\mathcal{C}}$. Since the two tangents $t_1^{\mathcal{I}}$ and $t_1^{\mathcal{C}}$ are not parallel, we have arrived at a contradiction.

The proof is complete. ∎

A line passing through a vertex of a trigon, right-perpendicular to its opposite side is called the *right-altitude* through that point. We say that a trigon has a right-orthocenter if the three right-altitudes are concurrent. Weiss proved [13, Theorem 1] that if every trigon has a right-orthocenter in a two-dimensional Minkowski space, then its indicatrix is an ellipse. We present here an alternate proof (valid in higher dimensions, as well).

Theorem 3.2. *If the right-altitudes of any trigon in a Minkowski geometry are concurrent, then its geometry is Euclidean.*

Proof. (Step b1) We can start with (Step a1) and (Step a2) of the proof of Theorem 3.1. So we need to consider two cases.

Case 1. $|\mathcal{C} \cap \mathcal{I}| \geq 5$.

(Step b2) Exactly the same way as in the proof of Theorem 3.1, via (Step a3)–(Step a5) one can conclude to a configuration where tangents at $\mathbf{p}_1^{\mathcal{C}} \in \mathcal{C}$ and $\mathbf{p}_1^{\mathcal{I}} \in \mathcal{I}$ are parallel, and $\mathbf{p}_1^{\mathcal{I}}$ and $\mathbf{p}_1^{\mathcal{C}}$ are not parallel at the same time (see Figure 3.3). The intersections of the tangent lines are denoted the same way as before.

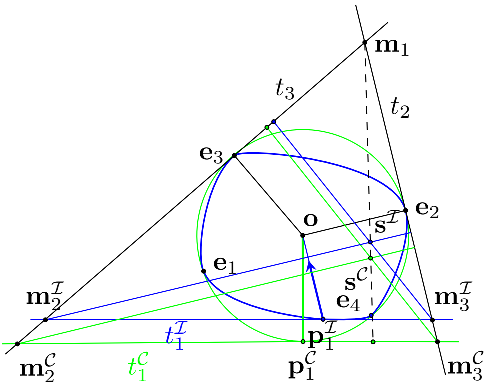

Figure 3.3. Altitudes in the first case

(Step b3) By Euclidean geometry, right-altitudes of $\triangle \mathbf{m}_1 \mathbf{m}_2^{\mathcal{C}} \mathbf{m}_3^{\mathcal{C}}$, clearly have a common point $\mathbf{s}^{\mathcal{C}}$, furthermore, $\mathbf{m}_i^{\mathcal{C}} \mathbf{s}^{\mathcal{C}} \parallel \mathbf{e}_i \mathbf{o}$ for $i = 2, 3$, and $\mathbf{m}_1^{\mathcal{C}} \mathbf{s}^{\mathcal{C}} \parallel \mathbf{o}\mathbf{p}_1^{\mathcal{C}}$.

Let us denote the intersection of the right-altitudes of trigon $\triangle \mathbf{m}_1 \mathbf{m}_2^{\mathcal{I}} \mathbf{m}_3^{\mathcal{I}}$ through the vertices $\mathbf{m}_3^{\mathcal{I}}$ and $\mathbf{m}_2^{\mathcal{I}}$ by $\mathbf{s}^{\mathcal{I}}$. Since trigons $\triangle \mathbf{m}_1 \mathbf{m}_3^{\mathcal{C}} \mathbf{m}_2^{\mathcal{C}}$ and $\triangle \mathbf{m}_1 \mathbf{m}_3^{\mathcal{I}} \mathbf{m}_2^{\mathcal{I}}$ are evidently homothetic with center $\mathbf{m}_1$, points $\mathbf{m}_1, \mathbf{s}^{\mathcal{I}}, \mathbf{s}^{\mathcal{C}}$ are collinear and clearly different. Thus, $\mathbf{m}_1 \mathbf{s}^{\mathcal{I}}$ is parallel to $\mathbf{p}_1^{\mathcal{C}} \mathbf{o}$. On the other hand, the right-altitude is parallel to $\mathbf{p}_1^{\mathcal{I}} \mathbf{o}$ (because of the definition of right-perpendicularity). This is a contradiction that proves the statement in the first case.

Case 2. $|\mathcal{C} \cap \mathcal{I}| = 4$.

(Step b4) We can continue with (Step a7) and finish the proof when $\mathbf{e}_2$ and $\mathbf{e}_3$ are not perpendicular.

(Step b5) It remains to consider the case where t_3 is perpendicular (right-perpendicular) to t_2.

Let us choose points $\mathbf{p}_1^{\mathcal{C}}$ and $\mathbf{p}_3^{\mathcal{C}}$ on $\mathcal{C}$ close to $\mathbf{e}_1$ and $\mathbf{e}_3$ such that $|\angle(\mathbf{e}_1\mathbf{o}\mathbf{p}_1^{\mathcal{C}})| = |\angle(\mathbf{e}_3\mathbf{o}\mathbf{p}_3^{\mathcal{C}})| = \alpha$, and the angles are directed in the same way (see Figure 3.4). Let the tangents to $\mathcal{C}$ at these points be denoted by $t_i^{\mathcal{C}}$ $(i = 1, 3)$.

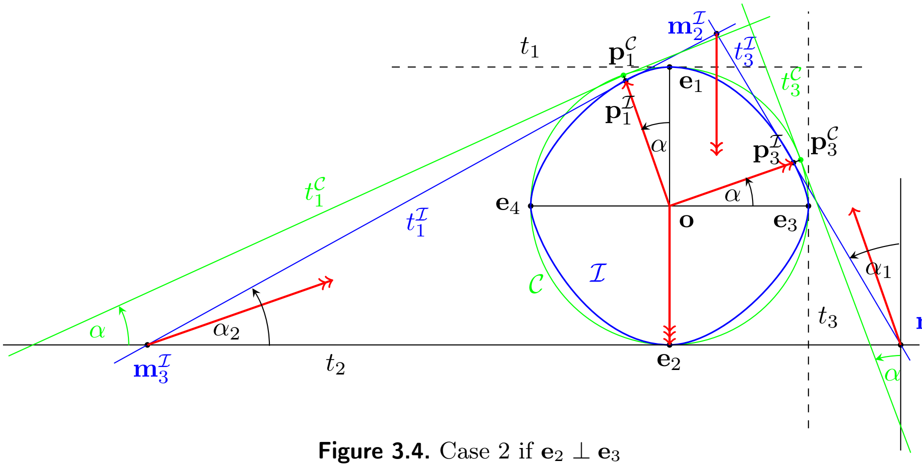

Figure 3.4. Case 2 if $\mathbf{e}_2 \perp \mathbf{e}_3$

Let us denote the point of the ray $\overrightarrow{\mathbf{op}_i^{\mathcal{C}}}$ on the curve $\mathcal{I}$ by $\mathbf{p}_i^{\mathcal{I}}$ $(i = 1, 3)$. The tangents to $\mathcal{I}$ through these points are denoted by $t_i^{\mathcal{I}}$ $(i = 1, 3)$. Clearly, the tangents $t_1^{\mathcal{C}}$, $t_3^{\mathcal{C}}$ and t_2 form a trigon, as well as the tangents $t_1^{\mathcal{I}}$, $t_3^{\mathcal{I}}$ and t_2 are the side lines of the trigon $\triangle\mathbf{m}_1^{\mathcal{I}}\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$.

Obviously, the magnitude of the angle between $t_1^{\mathcal{C}}$ and t_2 is α, as well as that of the angle between $t_3^{\mathcal{C}}$ and the direction of $\mathbf{e}_1$. As a consequence of Lemma 2.5, we obtain that $t_1^{\mathcal{I}}$ and t_2 close angle α_2 bigger than α, and similarly, the angle α_1 between $t_3^{\mathcal{I}}$ and $\mathbf{oe}_1$ is bigger than α.

According to the right-perpendicularity, altitudes of the trigon $\triangle\mathbf{m}_1^{\mathcal{I}}\mathbf{m}_2^{\mathcal{I}}\mathbf{m}_3^{\mathcal{I}}$ are parallel to $\mathbf{p}_1^{\mathcal{I}}, \mathbf{p}_3^{\mathcal{I}}, \mathbf{e}_2$, hence the altitude through vertex $\mathbf{m}_1^{\mathcal{I}}$ meets the altitude through $\mathbf{m}_2^{\mathcal{I}}$ outside the triangle, while the altitude through vertex $\mathbf{m}_3^{\mathcal{I}}$ meets the altitude through $\mathbf{m}_2^{\mathcal{I}}$ inside the triangle. It follows that the three altitudes can not form a pencil, which proves the theorem in this final case. ■

4. Characterizations by left-circumcenter and left-orthocenter

Concepts of left-circumcenter and left-orthocenter use the notion of left-perpendicularity in Minkowsky geometries. We say that a point is the *left-circumcenter* of a trigon if the three left-perpendicular bisectors of the edges of the trigon pass through that point. Similarly, we say that a point is the *left-orthocenter* of a trigon if the three lines through the vertexes of the trigon, left-perpendicular to the respective opposite edges, pass through that point.

Theorem 4.1. *If the left-perpendicular bisectors of every trigon in a Minkowski geometry are concurrent, then its geometry is Euclidean.*

Proof. (Step c1) We can repeat the first step of the proof of Theorem 3.1, obtain a centrally symmetric twice differentiable curve $\mathcal{I}$ and its circumscribed concentric circle $\mathcal{C}$ (its Lowner–John ellipse), and our aim is to construct a trigon, the three left-orthogonal lines in question of which can not pass through one point. Existence of such a configuration contradicts the assumption of the theorem.

(Step c2) From Lemma 2.2 we know that there exist at least three points in $\mathcal{I} \cap \mathcal{C}$. Furthermore, according to part (ii) of Lemma 2.2, we also know that in arbitrarily small neighborhood of (at least) one of the common points these curves are different. Since common points occur in antipodal pairs we can find four different common points $\mathbf{e}_i$ such that

(i) $\mathbf{e}_2$ and $\mathbf{e}_1$ are antipodal points;
(ii) $\mathbf{e}_3$ and $\mathbf{e}_4$ are antipodal points;
(iii) $\mathbf{e}_1$ is a point such that there are different points $\mathbf{p}_1^{\mathcal{C}} \in \mathcal{C}$ and $\mathbf{p}_1^{\mathcal{I}} \in \mathcal{I}$ sufficiently close to $\mathbf{e}_1$, being on the same ray starting from the origin $\mathbf{o}$ and having tangents $t_1^{\mathcal{C}}$ of $\mathcal{C}$ and $t_1^{\mathcal{I}}$ of $\mathcal{I}$, respectively, intersecting each other on that side of line $\mathbf{p}_1^{\mathcal{C}}\mathbf{p}_1^{\mathcal{I}}$ where $\mathbf{e}_1$ is (see Lemma 2.5);
(iv) in a circuit of $\mathcal{C}$, one has the points in order $\mathbf{e}_1 \prec \mathbf{p}_1^{\mathcal{C}} \prec \mathbf{e}_4 \prec \mathbf{e}_2 \prec \mathbf{e}_3$.

Let us denote the common tangent of the curves at $\mathbf{e}_i$ by t_i, $i = 2, 3$. The tangents $t_1^{\mathcal{C}}, t_2, t_3$ of the circle are perpendicular in Euclidean terms to $\mathbf{p}_1^{\mathcal{C}}, \mathbf{e}_2$ and $\mathbf{e}_3$, respectively, hence neither two of them are parallel. Since the directions of $t_1^{\mathcal{C}}$ and $t_1^{\mathcal{I}}$ are close to each other, chosing $\mathbf{p}_1^{\mathcal{C}}$ properly close to $\mathbf{e}_1$, tangent $t_1^{\mathcal{I}}$ meets t_2 and t_3, as well (see Figure 4.1, left).

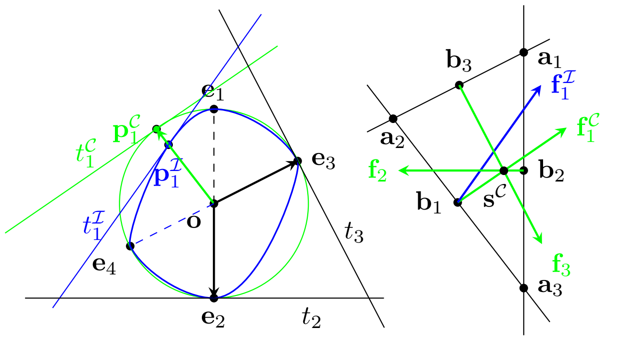

Figure 4.1. Left-perpendicularity and bisectors if $\mathbf{s}^{\mathcal{C}} \neq \mathbf{b}_1$

According to the definition of left-perpendicularity, line $t_1^{\mathcal{I}}$ is left-perpendicular to $\mathbf{p}_1^{\mathcal{I}}$.

Clearly, vectors $\mathbf{p}_1^{\mathcal{C}}$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent, so one can choose a triangle $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ such that $\mathbf{a}_3\mathbf{a}_2 \parallel \mathbf{p}_1^{\mathcal{C}}$, $\mathbf{a}_2\mathbf{a}_1 \parallel \mathbf{e}_3$, $\mathbf{a}_1\mathbf{a}_3 \parallel \mathbf{e}_2$ and the circuit of the vertexes $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is the same as that of points $\mathbf{e}_3, \mathbf{p}^{\mathcal{C}}, \mathbf{e}_2$ (see Figure 4.1, right).

Case 1. Vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are not perpendicular.

(Step c3) Let us denote the midpoints of the edges $\overline{\mathbf{a}_i\mathbf{a}_j}$ of $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ by $\mathbf{b}_k$, $\{i,j,k\} = \{1,2,3\}$ (see Figure 4.1, right).

Let the lines $\mathbf{b}_1\mathbf{f}_1^{\mathcal{C}}, \mathbf{b}_2\mathbf{f}_2, \mathbf{b}_3\mathbf{f}_3$ be taken so that $\mathbf{b}_1\mathbf{f}_1^{\mathcal{C}} \parallel t_1^{\mathcal{C}}$, $\mathbf{b}_2\mathbf{f}_2 \parallel t_2$, $\mathbf{b}_3\mathbf{f}_3 \parallel t_3$, respectively. Since these lines are (in Euclidean terms) perpendicular to their respective edges of the trigon, they have a common point, $\mathbf{s}^{\mathcal{C}}$, the Euclidean circumcenter. Since the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are not perpendicular, lines $\mathbf{a}_2\mathbf{a}_1$ and $\mathbf{a}_3\mathbf{a}_1$ are also not perpendicular, therefore $\mathbf{s}^{\mathcal{C}} \neq \mathbf{b}_1$.

Consider now the line $\mathbf{b}_1\mathbf{f}_1^{\mathcal{I}} \parallel t_1^{\mathcal{I}}$. It should pass through the left-circummcenter $\mathbf{s}^{\mathcal{I}} = \mathbf{b}_2\mathbf{f}_2 \cap \mathbf{b}_3\mathbf{f}_3$ of the trigon $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, by assumption. Since $\mathbf{s}^{\mathcal{C}} = \mathbf{b}_2\mathbf{f}_2 \cap \mathbf{b}_3\mathbf{f}_3$ and $\mathbf{s}^{\mathcal{C}} \neq \mathbf{b}_1$, one gets $\mathbf{b}_1\mathbf{f}_1^{\mathcal{I}} = \mathbf{b}_1\mathbf{f}_1^{\mathcal{C}}$. However, this contradicts $t_1^{\mathcal{I}} \nparallel t_1^{\mathcal{C}}$, so the theorem is proved in this case.

Case 2. Vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are perpendicular.

(Step c4) If there were more antipodal pairs of common points in $\mathcal{C} \cap \mathcal{I}$, than just $\mathbf{e}_1, \mathbf{e}_3$ and $\mathbf{e}_2, \mathbf{e}_4$, say $\mathbf{e}_6 \in \mathcal{C}(\mathbf{e}_1, \mathbf{e}_2)$, then exchanging indexes 6 and 3, one gains a configuration where t_3 and t_2 are not perpendicular, and the proof follows from (Step a3).

(Step c5) If $\mathcal{C} \cap \mathcal{I} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, then the proof of Theorem 3.2 in Case 2 (Step b4) is to follow. Consider the configuration on Figure 4.2, which is a modified version of Figure 3.4.

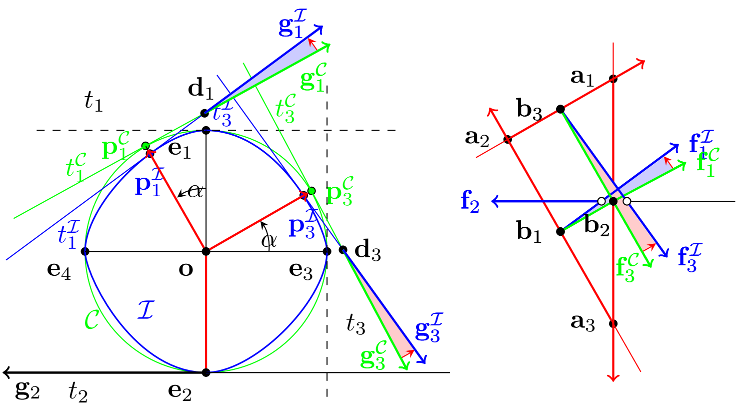

Figure 4.2. Left-perpendicularity and bisectors if $\mathbf{s}^{\mathcal{C}} = \mathbf{b}_1$

Let $\mathbf{d}_i = t_i^{\mathcal{C}} \cap t_i^{\mathcal{I}}$ $(i = 1, 3)$, and introduce the notations $\overrightarrow{\mathbf{d}_i\mathbf{g}_i^{\mathcal{C}}}$ and $\overrightarrow{\mathbf{d}_i\mathbf{g}_i^{\mathcal{I}}}$ $(i = 1, 3)$ for the rays starting from $\mathbf{d}_i$ in the respective tangents, not containing the touching points. It is clear that directed angles $\angle(\mathbf{g}_i^{\mathcal{C}}\mathbf{d}_1\mathbf{g}_i^{\mathcal{I}})$ have the same orientation as the circuit of the curves $\mathcal{C}$ and $\mathcal{I}$.

(Step c6) Take the rays $\overrightarrow{\mathbf{b}_1\mathbf{f}_1^{\mathcal{C}}} \parallel \overrightarrow{\mathbf{d}_1\mathbf{g}_1^{\mathcal{C}}}$, $\overrightarrow{\mathbf{b}_2\mathbf{f}_2} \parallel \overrightarrow{\mathbf{e}_2\mathbf{g}_2}$ and $\overrightarrow{\mathbf{b}_3\mathbf{f}_3^{\mathcal{C}}} \parallel \overrightarrow{\mathbf{d}_3\mathbf{g}_3^{\mathcal{C}}}$. The lines of these rays are perpendicular bisectors of the respective edges of the right trigon $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, hence, by Euclidean geometry, they pass trough the circumcenter $\mathbf{b}_2$.

Consider now the rays $\overrightarrow{\mathbf{b}_1\mathbf{f}_1^{\mathcal{I}}} \parallel \overrightarrow{\mathbf{d}_1\mathbf{g}_1^{\mathcal{I}}}$ and $\overrightarrow{\mathbf{b}_3\mathbf{f}_3^{\mathcal{I}}} \parallel \overrightarrow{\mathbf{d}_3\mathbf{g}_3^{\mathcal{I}}}$. The lines of these rays are left-perpendicular bisectors of the respective edges of the trigon $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, hence by assumption, they pass trough one point, the left-circumcenter of the trigon.

Investigating the orientation of the directed angles $\angle(\mathbf{f}_1^{\mathcal{C}}\mathbf{b}_1\mathbf{f}_1^{\mathcal{I}})$ and $\angle(\mathbf{f}_3^{\mathcal{C}}\mathbf{b}_3\mathbf{f}_3^{\mathcal{I}})$, one finds that rays $\overrightarrow{\mathbf{b}_1\mathbf{f}_1^{\mathcal{I}}}$ and $\overrightarrow{\mathbf{b}_3\mathbf{f}_3^{\mathcal{I}}}$ meet line $\mathbf{b}_2\mathbf{f}_2$ in opposite sides of $\mathbf{b}_2$, hence the three left-perpendicular bisectors of this trigon can not go through one point. This contradiction proves our theorem in this case, and completes the proof. ■

Theorem 4.2. *If the left-altitudes of every trigon in a Minkowski geometry are concurrent, then its geometry is Euclidean.*

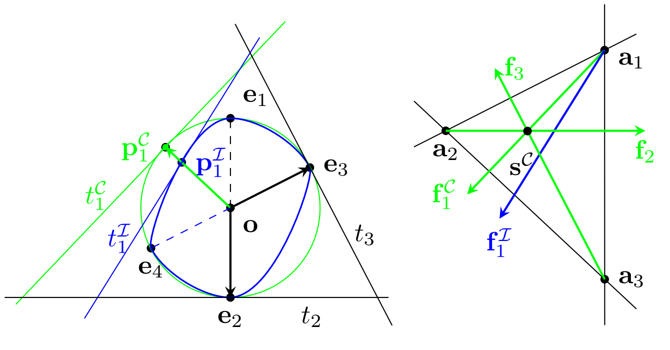

Figure 4.3. Left-perpendicularity and altitudes if $\mathbf{s}^{\mathcal{C}} \neq \mathbf{a}_2$

Proof. (Step d1) We can repeat (Step c1)–(Step c3) of the previous proof.

Then, as before, we can construct a trigon $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, such that $\mathbf{a}_1\mathbf{a}_2 \parallel \mathbf{oe}_3$, $\mathbf{a}_2\mathbf{a}_3 \parallel \mathbf{op}_1^{\mathcal{C}}$, $\mathbf{a}_3\mathbf{a}_1 \parallel \mathbf{oe}_2$, furthermore the circuit of points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is the same as that of points $\mathbf{e}_3, \mathbf{p}_1^{\mathcal{C}}, \mathbf{e}_2$. We investigate two cases (see Figure 4.3).

Case 1. Vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are not perpendicular.

(Step d2) In this step we apply arguments similar to that of (Step c3).

Let the lines $\mathbf{a}_1\mathbf{f}_1^{\mathcal{C}}, \mathbf{a}_2\mathbf{f}_2, \mathbf{a}_3\mathbf{f}_3$ be chosen so that $\mathbf{a}_1\mathbf{f}_1^{\mathcal{C}} \parallel t_1^{\mathcal{C}}$, $\mathbf{a}_2\mathbf{f}_2 \parallel t_2$ and $\mathbf{a}_3\mathbf{f}_3 \parallel t_3$. Since these lines are (in Euclidean terms) perpendicular to the respective opposite edges of the trigon, they all pass through the Euclidean orthocenter $\mathbf{s}^{\mathcal{C}}$.

Consider now the line $\mathbf{a}_1\mathbf{f}_1^{\mathcal{I}} \parallel t_1^{\mathcal{I}}$. The lines $\mathbf{a}_1\mathbf{f}_1^{\mathcal{I}}, \mathbf{a}_2\mathbf{f}_2$ and $\mathbf{a}_3\mathbf{f}_3$ are the left-perpendicular altitudes of trigon $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, hence, by assumption, they pass through the left-orthocenter $\mathbf{s}^{\mathcal{I}}$.

Since $t_1^{\mathcal{C}} \nparallel t_1^{\mathcal{I}}$, we have $\mathbf{a}_1\mathbf{f}_1^{\mathcal{C}} \neq \mathbf{a}_1\mathbf{f}_1^{\mathcal{I}}$, but this contradicts $\mathbf{s}^{\mathcal{C}} = \mathbf{a}_2\mathbf{f}_2 \cap \mathbf{a}_3\mathbf{f}_3 = \mathbf{s}^{\mathcal{I}}$, because $\mathbf{a}_1 \neq \mathbf{s}^{\mathcal{C}} \in \mathbf{a}_1\mathbf{f}_1^{\mathcal{C}} \cap \mathbf{a}_1\mathbf{f}_1^{\mathcal{I}} = \{\mathbf{a}_1\}$. This contradiction proves Case 1.

Case 2. Vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are perpendicular.

(Step d3) We can proceed with (Step c4)–(Step c5), and finish with a modified version of (Step c6), as follows (see Figure 4.4).

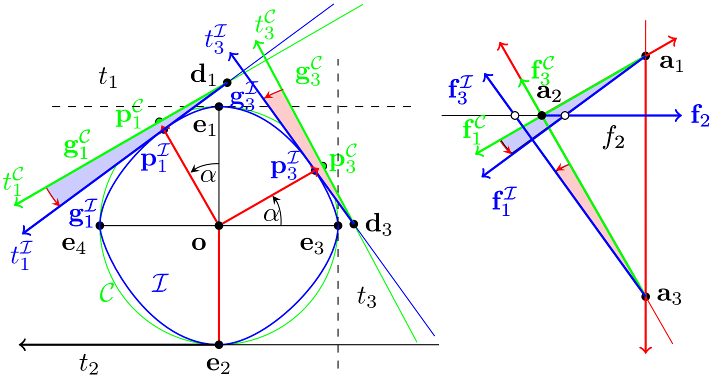

Figure 4.4. Left-perpendicularaty and altitudes if $\mathbf{s}^{\mathcal{C}} = \mathbf{a}_2$

Take the rays $\overrightarrow{\mathbf{a}_1\mathbf{f}_1^{\mathcal{C}}} \parallel \overrightarrow{\mathbf{d}_1\mathbf{g}_1^{\mathcal{C}}}, \overrightarrow{\mathbf{a}_2\mathbf{f}_2} \parallel t_2$ and $\overrightarrow{\mathbf{a}_3\mathbf{f}_3^{\mathcal{C}}} \parallel \overrightarrow{\mathbf{d}_3\mathbf{g}_3^{\mathcal{C}}}$. The lines $\mathbf{a}_i\mathbf{f}_i^{\mathcal{C}}$ $(i = 1, 3)$ and $\mathbf{a}_2\mathbf{f}_2$ are (Euclidean) altitudes of the right trigon $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, hence they pass trough the orthocenter $\mathbf{a}_2$ of the trigon.

Consider now the rays $\overrightarrow{\mathbf{a}_1\mathbf{f}_1^{\mathcal{I}}} \parallel \overrightarrow{\mathbf{d}_1\mathbf{g}_1^{\mathcal{I}}}$ and $\overrightarrow{\mathbf{a}_3\mathbf{f}_3^{\mathcal{I}}} \parallel \overrightarrow{\mathbf{d}_3\mathbf{g}_3^{\mathcal{I}}}$. The lines $\mathbf{a}_i\mathbf{f}_i^{\mathcal{I}}$ $(i = 1, 3)$ and $\mathbf{a}_2\mathbf{f}_2$ are left-perpendicular altitudes of the right trigon $\triangle\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, hence by assumption, they should pass trough the left-orthocenter of the trigon.

However, considering the directed angles $\angle(\mathbf{f}_1^{\mathcal{C}}\mathbf{a}_1\mathbf{f}_1^{\mathcal{I}})$ and $\angle(\mathbf{f}_3^{\mathcal{C}}\mathbf{a}_3\mathbf{f}_3^{\mathcal{I}})$, one finds that rays $\overrightarrow{\mathbf{a}_1\mathbf{f}_1^{\mathcal{I}}}$ and $\overrightarrow{\mathbf{a}_3\mathbf{f}_3^{\mathcal{I}}}$ meet $\mathbf{a}_2\mathbf{f}_2$ in the opposite sides of $\mathbf{a}_2$, hence $\mathbf{a}_1\mathbf{f}_1^{\mathcal{I}}, \mathbf{a}_2\mathbf{f}_2$ and $\mathbf{a}_3\mathbf{f}_3^{\mathcal{I}}$ can not go through one point. This contradiction proves our theorem in this final case.

■

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