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Selfadjoint operators and symmetric operators

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Abstract. Our study is in the set $\mathcal{S}(H)$ of all semiclosed operators in a Hilbert space H. We show that the set $\mathcal{S}_{sa}(H)$ of all selfadjoint operators is relatively open in the set $\mathcal{S}_{sym}(H)$ of all semiclosed symmetric operators. We calculate the value of a radius of minus-Laplacian $-\Delta$. As a topological approach, we show the selfadjointness of the Schrödinger operator with a Kato–Rellich potential.

1. Introduction

Let H be an infinite dimensional, complex Hilbert space. An operator means a linear mapping from a domain in H into H, and the notation $\mathcal{B}(H)$ stands for the set of all bounded operators with the domain H. An operator is said to be semiclosed if its graph is a semiclosed subspace in the product Hilbert space $H \times H$. Characterizations for semiclosed operators in a Hilbert space are accomplished in [8]. According to it, an operator is semiclosed if and only if it is a quotient of bounded operators. Quotients of bounded operators are treated in [5], [7]. Moreover, topological structures for them are studied in [3] and [4]. Especially, in the latter, we introduced the metric which is called the q-metric on the set $\mathcal{S}(H)$ of all semiclosed operators. The q-metric restricted to $\mathcal{B}(H)$ coincides with the metric induced from the operator norm, and the q-metric is stronger than the gap metric on the set $\mathcal{CD}(H)$ of all closed and densely defined operators. Moreover, we showed that $\mathcal{CD}(H)$ is open in the metric space $\mathcal{S}(H)$.

A reason we study semiclosed operators is the following. The set $\mathcal{S}(H)$ which contains the set of all closed operators is closed under sums, products, adjoints and

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closures if they exist. Hence sums of closed operators are necessarily semiclosed. When we encounter sums of selfadjoint operators like Schrödinger type operators, it is not so easy to decide that sums are selfadjoint or not, even closed or not. But we can immediately deduce that their sums are symmetric and semiclosed. So we believe that the set $\mathcal{S}(H)$ is one of the most suitable ones as a universal set when we handle unbounded operators.

In this paper, we consider the set of selfadjoint operators in the set of semiclosed symmetric operators from a topological stance. Our interest is a problem that 'when are semiclosed and symmetric operators selfadjoint?'. For this, we obtain the following main theorem which is proved in Section 3.

Theorem 1.1. *Let $(\mathcal{S}(H), q)$ be the metric space. The set $\mathcal{S}_{sa}(H)$ of all selfadjoint operators is relatively open in the set $\mathcal{S}_{sym}(H)$ of all semiclosed and symmetric operators: For $s \in \mathcal{S}_{sa}(H)$, there exists a positive constant $\delta > 0$ such that*

$$q(s,t) < \delta \text{ and } t \in \mathcal{S}_{sym}(H) \text{ imply } t \in \mathcal{S}_{sa}(H). \tag{1.1}$$

In Section 4, we calculate the value of a positive constant $\delta > 0$ as above theorem for a selfadjoint operator $s = -\Delta$. That is,

$$q(-\Delta, t) < \frac{\sqrt{2}}{4} \text{ and } t \in \mathcal{S}_{sym}(H) \text{ imply } t \in \mathcal{S}_{sa}(H).$$

As an application, the selfadjointness of the Schrödinger operator $-\Delta + V$ with a Kato–Rellich potential V is shown in Section 5, which gives another proof for the Kato–Rellich Theorem.

2. Preliminaries

The notation $(\cdot,\cdot)$ denotes the inner product equipped with H and put $\|\cdot\| := (\cdot,\cdot)^{\frac{1}{2}}$. A subspace M in H is said to be semiclosed if there exists an inner product $(\cdot,\cdot)_M$ on M such that M is a complete inner product space (i.e. a Hilbert space) and the inclusion mapping $J\colon (M, \|\cdot\|_M) \to H$ is continuous. When the inclusion mapping J is continuous, that is, there exists a constant $c > 0$ such that $\|u\| \le c\|u\|_M$ for $u \in M$, we write $(M, \|\cdot\|_M) \hookrightarrow H$. In this case, we call $\|\cdot\|_M$ a Hilbert norm on M. It is known that a subspace M in H is semiclosed if and only if M is an operator range in H, that is, $M = YH$ for some $Y \in \mathcal{B}(H)$.

An operator $s\colon \mathrm{dom}(s) \to H$ with a domain $\mathrm{dom}(s) \subseteq H$ is said to be semiclosed if its graph $\{(u, su) : u \in \mathrm{dom}(s)\}$ is a semiclosed subspace in the product Hilbert space $H \times H$. Characterizations for semiclosed operators are given as follows.

Theorem 2.1. ([8]) *Let $s\colon \operatorname{dom}(s) \to H$. Then the following conditions are equivalent.*

(1) *The operator s is a semiclosed operator in H.*
(2) *The domain $\operatorname{dom}(s)$ is a semiclosed subspace in H, so that*

$$\tilde{s}\colon (\operatorname{dom}(s), \|\cdot\|_{\operatorname{dom}(s)}) \to H$$

is bounded with respect to some (equivalently, any) Hilbert norm $\|\cdot\|_{\operatorname{dom}(s)}$ on $\operatorname{dom}(s)$. Here, $\tilde{s}u = su$ for $u \in \operatorname{dom}(s)$.

(3) *The operator s is represented by a quotient of bounded operators, namely there exist $A, B \in \mathcal{B}(H)$ such that*

$$\ker A \subseteq \ker B,\ \operatorname{dom}(s) = AH,\ \textit{and } sAu = Bu \textit{ for } u \in H.$$

In this case, we write $s = B/A$.

For $T \in \mathcal{B}(H)$, we define the inner product $(\cdot,\cdot)_T$ on the operator range TH by $(Tu, Tv)_T := (u, v)$ $u, v \in (\ker T)^\perp$. Then $(TH, (\cdot,\cdot)_T)$ is a complete inner product space and $(TH, (\cdot,\cdot)_T) \hookrightarrow H$. We call a Hilbert space $(TH, (\cdot,\cdot)_T)$ de Branges space induced by T and denote it by $\mathcal{M}(T)$. In case of considering de Branges space, positive operators have important roles as the following Lemma 2.2 shows. Here, an operator $T \in \mathcal{B}(H)$ is said to be positive, in short $T \geq 0$, if $(Tu, u) \geq 0$ for $u \in H$.

Lemma 2.2. ([1]) *Let M be a semiclosed subspace in H and let $\|\cdot\|_M$ be a Hilbert norm on M. Then there uniquely exists a positive operator $T \in \mathcal{B}(H)$ such that $(M, \|\cdot\|_M) = \mathcal{M}(T)$ (isometrically isomorphic). Here T is given by $(JJ^*)^{\frac{1}{2}} \geq 0$ satisfying $J : (M, \|\cdot\|_M) \hookrightarrow H$.*

From Lemma 2.2, there exists the bijective mapping from the set $\{\|\cdot\|_M : (M, \|\cdot\|_M) \hookrightarrow H\}$ of all Hilbert norms on M to the set $\{T \geq 0 : M = TH\}$ of all positive bounded operators whose ranges are M. Hence, we can choose a Hilbert norm $\|\cdot\|_M$ from the set of all Hilbert norms on M for each semiclosed subspace M, and let α be a corresponding $M \to \|\cdot\|_M$, equivalently $M \to T \geq 0$ such that $M = TH$ as above. Namely, a corresponding α is a choice function to choose a Hilbert norm for each semiclosed subspace.

Now, we lay down two rules concerning the choice. The first is to choose the original Hilbert norm $\|\cdot\|$ for H. Clearly the norm $\|\cdot\|$ corresponds to the identity operator I on H. The second is, for a closed subspace M, to choose the original Hilbert norm restricted on M, $\|\cdot\|_M (:= \|\cdot\| \text{ on } M)$. The norm $\|\cdot\|_M$ corresponds to the orthogonal projection P_M onto M.

Based on a corresponding α, we introduced in [4] the q_α-metric on the set $\mathcal{S}(H)$ of all semiclosed operators. Since, by Theorem 2.1 (2), a domain $\operatorname{dom}(s)$

of a semiclosed operator s is a semiclosed subspace, it is a Hilbert space with a Hilbert norm $\|\cdot\|_{\mathrm{dom}(s)}$ given by α. Hence, there uniquely exists a positive operator $A \in \mathcal{B}(H)$ such that a Hilbert space $\mathrm{dom}(s)$ is isometrically isomorphic to de Branges space $\mathcal{M}(A)$, that is, $\mathrm{dom}(s) = AH$, $\|\cdot\|_{\mathrm{dom}(s)} = \|\cdot\|_A$. Let $B := sA$. Since s and A are semiclosed, their product B is also semiclosed. Since clearly the domain of B is the whole space H, the operator B is in $\mathcal{B}(H)$ by the semiclosed graph theorem. This means that s is uniquely represented by a quotient B/A. When we emphasize the corresponding α, we write $s \overset{\alpha}{=} B/A$.

For $s, t \in \mathcal{S}(H)$, we set $s \overset{\alpha}{=} B/A$ and $t \overset{\alpha}{=} D/C$ as above. Then we define the metric depending on the corresponding α by

$$q(s,t) = q_\alpha(s,t) = \max\{\|A - C\|, \|B - D\|\}.$$

We simply denote $q(s,t)$ instead of $q_\alpha(s,t)$ without confusion. The term $\|A - C\|$ means the distance between semiclosed subspaces $\mathrm{dom}(s)$ and $\mathrm{dom}(t)$ under the corresponding α. In case of $S, T \in \mathcal{B}(H)$, it follows from $S \overset{\alpha}{=} S/I$ and $T \overset{\alpha}{=} T/I$ that $q(S,T) = \|S - T\|$.

For a quotient $F/E\colon Eu \to Fu$, $u \in H$ (E is not necessarily positive), a positive bounded operator $(E^*E + F^*F)^{\frac{1}{2}}$ has important roles in several situations. It is shown in [8] that a quotient F/E is closed if and only if $(E^*E + F^*F)^{\frac{1}{2}}$ has a closed range in H. The following lemma will be used later in our arguments.

Lemma 2.3. ([4]) *Let $s \in \mathcal{S}(H)$ and $s = B/A$ be a quotient such that $A \geq 0$. Then, s is in $\mathcal{CD}(H)$ if and only if $(A^2 + B^*B)^{\frac{1}{2}}$ has the inverse in $\mathcal{B}(H)$, that is, invertible.*

Remark 2.4. We call $(A^2 + B^*B)^{\frac{1}{2}}$ the attached positive operator for B/A.

For an integer $m \geq 1$, a subspace in $L^2(\mathbb{R})$

$$\mathrm{dom}(\mathcal{D}_1^m) = \{f \in L^2(\mathbb{R}) : \mathcal{D}_1 f, \mathcal{D}_1^2 f, \ldots, \mathcal{D}_1^m f \in L^2(\mathbb{R})\} \quad \left(\mathcal{D}_1 = \frac{1}{i}\frac{d}{dx}\right)$$

becomes a Hilbert space with a standard Hilbert structure

$$\|f\|_{W^{m,2}} := \left(\|f\|^2 + \|\mathcal{D}_1 f\|^2 + \cdots + \|\mathcal{D}_1^m f\|^2\right)^{1/2}.$$

($\|\cdot\|$ means L^2-norm and $\mathcal{D}_1\colon \mathrm{dom}(\mathcal{D}_1) \to L^2(\mathbb{R})$ is the differential operator in a weak sense.) Since $(\mathrm{dom}(\mathcal{D}_1^m), \|\cdot\|_{W^{m,2}}) \hookrightarrow L^2(\mathbb{R})$, a subspace $\mathrm{dom}(\mathcal{D}_1^m)$ is a semiclosed subspace in $L^2(\mathbb{R})$. We call the Hilbert space $(\mathrm{dom}(\mathcal{D}_1^m), \|\cdot\|_{W^{m,2}})$ the standard Sobolev space $W^{m,2}(\mathbb{R})$ with the order m. From $W^{m,2}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$,

$W^{m,2}(\mathbb{R})$ is represented by de Branges space $\mathcal{M}(A_m)$ for a unique positive operator A_m in isometrically isomorphic sense. It is shown in [4] that A_m is given by

$$A_m = (I + \mathcal{D}_1^2 + \cdots + \mathcal{D}_1^{2m})^{1/2}. \tag{2.1}$$

For $\sigma > 0$, a subspace $\{f \in L^2(\mathbb{R}^N) : (1 + |\xi|^2)^{\frac{\sigma}{2}} \widehat{f} \in L^2(\mathbb{R}^N)\}$ is semiclosed in $L^2(\mathbb{R}^N)$ $(N \geq 1)$. Because this is a continuously embedded Hilbert space in $L^2(\mathbb{R}^N)$ with a norm $\|f\|_{H^\sigma} := \|(1 + |\xi|^2)^{\sigma/2} \widehat{f}\|$, which is called the Fourier type Sobolev space $H^\sigma(\mathbb{R}^N)$ with the order $\sigma > 0$. ($\widehat{\cdot}$ means L^2-Fourier transform and $|\xi|^2 := \xi_1^2 + \cdots + \xi_N^2$ for $(\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$.) When $N = 1$, the standard Sobolev norm coincides with the Fourier type Sobolev norm, that is, $\| \cdot \|_{W^{1,2}} = \| \cdot \|_{H^1}$. The Sobolev space $H^\sigma(\mathbb{R}^N)$ is expressed by de Branges space $\mathcal{M}(\widetilde{A_\sigma})$ for a unique positive operator $\widetilde{A_\sigma}$. The operator $\widetilde{A_\sigma}$ is given ([4]) by

$$\widetilde{A_\sigma} = (I - \Delta)^{-\sigma/2}, \tag{2.2}$$

which is known as the Bessel potential of the order $\sigma > 0$.

3. A proof of the main theorem

In this section, we shall prove Theorem 1.1. A positive constant δ is said to be a radius of a selfadjoint operator s if it satisfies the condition (1.1) with respect to the q_α-metric. A radius does not mean the best possible constant.

To construct a radius of a given selfadjoint operator, we first prove a fundamental inequality as follows.

Lemma 3.1. *For $s, t \in \mathcal{S}(H)$, let $s \overset{\alpha}{=} B/A$ and $t \overset{\alpha}{=} D/C$. Then the following inequality holds:*

$$q(s - \zeta I, t - \zeta I) \leq (1 + |\zeta|)\, q(s, t) \quad \textit{for } \zeta \in \mathbb{C}. \tag{3.1}$$

Proof. Since $s - \zeta I = B/A - \zeta I \overset{\alpha}{=} (B - \zeta A)/A$ and $t - \zeta I \overset{\alpha}{=} (D - \zeta C)/C$,

$$\begin{aligned} q(s - \zeta I, t - \zeta I) &= \max\{\|A - C\|,\ \|(B - \zeta A) - (D - \zeta C)\|\} \\ &\leq \max\{\|A - C\|,\ \|B - D\| + |\zeta|\|A - C\|\} \\ &\leq \max\{q(s,t),\ q(s,t) + |\zeta| q(s,t)\} = (1 + |\zeta|)\, q(s,t). \end{aligned}$$

■

According to [4], since the set $\mathcal{CD}(H)$ of all closed and densely defined operators is open in $(\mathcal{S}(H), q)$, there exists a positive constant δ for $s \in \mathcal{CD}(H)$ such that $q(s,t) < \delta$ and $t \in \mathcal{S}(H)$ imply $t \in \mathcal{CD}(H)$. In the following, we explicitly give it under the hypothesis of nonempty resolvent set $\rho(s)$ of s.

Lemma 3.2. *Let $s \in \mathcal{CD}(H)$, the resolvent set $\rho(s)$ of s be nonempty and $s \overset{\alpha}{=} B/A$. For any nonempty compact set Φ such that $\rho(s) \supset \Phi$, we set*

$$\delta = \delta_\Phi = \min_{\zeta\in\Phi} \frac{\|(B-\zeta A)^{-1}\|^{-1}}{1+|\zeta|}. \tag{3.2}$$

Then, $q(s,t) < \delta$ and $t \in \mathcal{S}(H)$ imply $t \in \mathcal{CD}(H)$ and $\rho(t) \supset \Phi$.

Proof. We note that $A \geq 0$ and AH is dense in H, so that $\ker A = \{0\}$. For $\zeta \in \Phi$, we see that the operator $(s-\zeta I)^{-1} = ((B-\zeta A)/A)^{-1} = A/(B-\zeta A)$ belongs to $\mathcal{B}(H)$. Hence we have $\ker(B-\zeta A) \subseteq \ker A$ $(=\{0\})$ and $(B-\zeta A)H = H$. This means that there exists the inverse $(B-\zeta A)^{-1} \in \mathcal{B}(H)$.

If $q(s,t) < \delta$ and $t \overset{\alpha}{=} D/C$ in $\mathcal{S}(H)$, then, it follows from (3.2) that

$$(1+|\zeta|)\, q(s,t) < \|(B-\zeta A)^{-1}\|^{-1} \text{ for any } \zeta \in \Phi.$$

Since $\|(B-\zeta A)-(D-\zeta C)\| \leq (1+|\zeta|)\, q(s,t)$ for any $\zeta \in \Phi$ by the proof of Lemma 3.1, we have

$$\|(B-\zeta A)-(D-\zeta C)\| < \|(B-\zeta A)^{-1}\|^{-1}.$$

Hence, we see that $D-\zeta C$ is invertible. Then, by the relations $t-\zeta I \overset{\alpha}{=} (D-\zeta C)/C$ and $\ker C \subseteq \ker(D-\zeta C)$, we easily see $\ker C = \{0\}$. This means that $t-\zeta I$ has an inverse and $(t-\zeta I)^{-1} = C(D-\zeta C)^{-1} \in \mathcal{B}(H)$. It follows from this equation that $\zeta \in \rho(t)$, or $\rho(t) \supset \Phi$ and $t-\zeta I$ is closed, that is, t is closed. The denseness of $\mathrm{dom}(t)(=CH)$ follows from the conditions $\ker C = \{0\}$ and $C \geq 0$. Therefore $t \in \mathcal{CD}(H)$. ∎

We want to obtain a radius δ_Φ as large as possible for among a compact set Φ as in Lemma 3.2.

Hence, we deal with the compact set $\Phi = \{i, -i\}$ consisting of two elements.

Lemma 3.3. *Let $s \in \mathcal{S}_{sa}(H)$ and $s \overset{\alpha}{=} B/A$, and let $R = (A^2 + B^*B)^{1/2}$ be the attached positive operator for a quotient B/A as in Remark 2.4. Then, (3.2) for $\Phi := \{i, -i\}$ is given by*

$$\delta = \delta_\Phi = \frac{1}{2}\|(B \pm iA)^{-1}\|^{-1} = \frac{1}{2}\gamma(R). \tag{3.3}$$

Here $\gamma(R) = \inf\{\|Ru\| : u \in (\ker R)^\perp, \|u\| = 1\}$.

Proof. It easily follows from (3.2) that $\delta_\Phi = \|(B\pm iA)^{-1}\|^{-1}/2$. Thus it is sufficient to show the equation $\|(B\pm iA)^{-1}\|^{-1} = \gamma(R)$. Let ζ be i or $-i$. Then,

$$\|(B-\zeta A)^{-1}\|^2 = \sup_{u\neq 0} \frac{\|(B-\zeta A)^{-1}u\|^2}{\|u\|^2} = \sup_{v\neq 0} \frac{\|v\|^2}{\|(B-\zeta A)v\|^2}. \tag{3.4}$$

Note that (Av, Bv) is real for $v \in H$ from the selfadjointness of B/A and $\zeta + \bar{\zeta} = 0$. Hence we have

$$\begin{aligned}\|(B-\zeta A)v\|^2 &= ((B-\zeta A)v, (B-\zeta A)v) \\ &= \|Bv\|^2 + |\zeta|^2\|Av\|^2 - (\zeta+\bar{\zeta})(Av, Bv) = \|Av\|^2 + \|Bv\|^2.\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}\|(B \pm iA)^{-1}\|^2 &= \sup_{v\neq 0} \frac{\|v\|^2}{\|Av\|^2 + \|Bv\|^2} = \sup_{v\neq 0} \frac{1}{(\|Av\|/\|v\|)^2 + (\|Bv\|/\|v\|)^2} \\ &= \big(\inf_{\|v\|=1} \|Av\|^2 + \|Bv\|^2\big)^{-1} = \big(\inf_{\|v\|=1} \|(A^2 + B^*B)^{\frac{1}{2}} v\|^2\big)^{-1} \\ &= \big(\inf_{\|v\|=1} \|Rv\|\big)^{-2}.\end{aligned}$$

Hence we have $\|(B \pm iA)^{-1}\|^{-1} = \inf_{\|v\|=1} \|Rv\|$. Since $B/A \in \mathcal{CD}(H)$, $R = (A^2 + B^*B)^{\frac{1}{2}}$ has the inverse in $\mathcal{B}(H)$ by Lemma 2.3, so that $(\ker R)^{\perp} = H$. This means that

$$\inf_{\|v\|=1} \|Rv\| = \inf\{\|Rv\| : v \in (\ker R)^{\perp}, \|v\| = 1\} = \gamma(R) \ (= \|R^{-1}\|^{-1})$$

which completes the proof. ■

Proof of Theorem 1.1. For $s \in \mathcal{S}_{sa}(H)$, clearly $\rho(s) \supset \{i, -i\}$. We set

$$\delta = \delta_{\Phi} = \frac{1}{2}\gamma(R) \tag{3.6}$$

for $\Phi = \{i, -i\}$. Suppose that $q(s,t) < \delta$ and $t \in \mathcal{S}_{sym}(H)$. It follows from Lemma 3.2 that $t \in \mathcal{CD}(H)$ and $\rho(t) \supset \Phi$, so that t is closed symmetric and the range of $t \pm iI$ is H. Thus t is selfadjoint[1]. ■

4. Examples

From now on, we shall give some examples of a radius (3.6) of differential operators in the complex Hilbert space $L^2(\mathbb{R}^N)$ $(N \geq 1)$. Differential operators here are meant in a weak sense. The notation $\widehat{\cdot}$ stands for the L^2-Fourier transformation and $L^2_\xi(\mathbb{R}^N)$ means the range space of ξ-variable by $\widehat{\cdot}$.

[1] According to Theorem 4.2 in [10], a (not necessarily densely defined or closed) symmetric operator t satisfies that the range of $t \pm iI$ is H, then it is automatically selfadjoint.

Example 4.1. (A radius of $\frac{1}{i}\frac{d}{dx}$) Let $\mathcal{D}_1 := \frac{1}{i}\frac{d}{dx}$ be the selfadjoint operator with the maximal domain $\mathrm{dom}(\mathcal{D}_1)$ in $L^2(\mathbb{R})$. Let α be the choice function that we choose the standard Sobolev norm $\|\cdot\|_{W^{1,2}} (= \|\cdot\|_{H^1})$ for $\mathrm{dom}(\mathcal{D}_1)$, and we suitably choose a Hilbert norm for each semiclosed subspace except for $\mathrm{dom}(\mathcal{D}_1)$. Let $\mathcal{D}_1 \overset{\alpha}{=} B/A_1$ for $A_1 = (I + \mathcal{D}_1^2)^{-1/2}$ by (2.1), and let $R := (A_1^2 + B^*B)^{1/2}$ be the attached positive operator for a quotient B/A_1 in $\mathcal{CD}(H)$. Then $B = \mathcal{D}_1 A_1 = \mathcal{D}_1(I + \mathcal{D}_1^2)^{-1/2}$, and since $(\ker R)^\perp = L^2(\mathbb{R})$ by Lemma 2.3, we have that, for $f \in L^2(\mathbb{R})$,

$$\begin{aligned}\gamma(R)^2 &= \inf_{\|f\|=1} \|Rf\|^2 = \inf_{\|f\|=1} \{\|A_1 f\|^2 + \|Bf\|^2\} \\ &= \inf_{\|f\|=1} \{\|(I + \mathcal{D}_1^2)^{-\frac{1}{2}} f\|^2 + \|\mathcal{D}_1(I + \mathcal{D}_1^2)^{-\frac{1}{2}} f\|^2\} \\ &= \inf_{\|\widehat{f}\|=1} \{\|(1 + \xi^2)^{-\frac{1}{2}} \widehat{f}\|^2 + \|\xi(1 + \xi^2)^{-\frac{1}{2}} \widehat{f}\|^2\} = \inf_{\|\widehat{f}\|=1} \|(1 + \xi^2)^{\frac{1}{2}} (1 + \xi^2)^{-\frac{1}{2}} \widehat{f}\|^2 \\ &= 1.\end{aligned}$$

Hence we have $\gamma(R) = 1$, so that $\delta = \gamma(R)/2 = 1/2$ which is a radius of $\frac{1}{i}\frac{d}{dx}$ under the choice function α as above.

Example 4.2. (A radius of $-\frac{d^2}{dx^2}$) Let $\mathcal{D}_2 := -\frac{d^2}{dx^2} (= \mathcal{D}_1^2)$ be the selfadjoint operator with the maximal domain $\mathrm{dom}(\mathcal{D}_2)$ in $L^2(\mathbb{R})$. Let α be the choice function that we choose the standard Sobolev norm $\|\cdot\|_{W^{2,2}}$ for $\mathrm{dom}(\mathcal{D}_2)$, and we suitably choose a Hilbert norm for each semiclosed subspace except for $\mathrm{dom}(\mathcal{D}_2)$. Let $\mathcal{D}_2 \overset{\alpha}{=} B/A_2$ for $A_2 = (I + \mathcal{D}_1^2 + \mathcal{D}_1^4)^{-\frac{1}{2}}$ by (2.1), and let $R := (A_2^2 + B^*B)^{\frac{1}{2}}$ be the attached positive operator for a quotient B/A_2 in $\mathcal{CD}(H)$. Then $B = \mathcal{D}_2 A_2 = \mathcal{D}_2(I + \mathcal{D}_1^2 + \mathcal{D}_1^4)^{-\frac{1}{2}}$, and since $(\ker R)^\perp = L^2(\mathbb{R})$ by Lemma 2.3, we have that, for $f \in L^2(\mathbb{R})$,

$$\begin{aligned}\gamma(R)^2 &= \inf_{\|f\|=1} \|Rf\|^2 = \inf_{\|f\|=1} \{\|A_2 f\|^2 + \|Bf\|^2\} \\ &= \inf_{\|f\|=1} \{\|(I + \mathcal{D}_1^2 + \mathcal{D}_1^4)^{-1/2} f\|^2 + \|\mathcal{D}_2(I + \mathcal{D}_1^2 + \mathcal{D}_1^4)^{-1/2} f\|^2\} \\ &= \inf_{\|\widehat{f}\|=1} \{\|(1 + \xi^2 + \xi^4)^{-1/2} \widehat{f}\|^2 + \|\xi^2(1 + \xi^2 + \xi^4)^{-1/2} \widehat{f}\|^2\} \\ &= \inf_{\|\widehat{f}\|=1} \|(1 + \xi^4)^{\frac{1}{2}} (1 + \xi^2 + \xi^4)^{-1/2} \widehat{f}\|^2 = \inf_{\|\widehat{f}\|=1} \|M\widehat{f}\|^2 \\ &= \gamma(M)^2, \quad ((\ker M)^\perp = L^2_\xi(\mathbb{R})),\end{aligned}$$

where $M := (1 + \xi^4)^{1/2}(1 + \xi^2 + \xi^4)^{-1/2}$ is a bounded multiplication operator and invertible in $\mathcal{B}(L^2_\xi(\mathbb{R}))$. Hence, we have

$$\begin{aligned}\gamma(R) = \gamma(M) = \|M^{-1}\|^{-1} &= \Big\|\Big(\frac{1 + \xi^2 + \xi^4}{1 + \xi^4}\Big)^{1/2}\Big\|_\infty^{-1} = \Big\|\frac{1 + \xi^2 + \xi^4}{1 + \xi^4}\Big\|_\infty^{-1/2} \\ &= (3/2)^{-1/2} = \sqrt{6}/3,\end{aligned}$$

so that

$$\delta = \frac{1}{2}\gamma(R) = \frac{\sqrt{6}}{6} \ (= 0.408\ldots),$$

which is a radius of $-\frac{d^2}{dx^2}$ under the choice function α as above.

Example 4.3. (A radius of $-\frac{d^2}{dx^2}$) Let $\mathcal{D}_2 := -\frac{d^2}{dx^2}$ be the same operator as in Lemma 4.2. Let α be the choice function that we choose the Fourier type Sobolev norm $\|\cdot\|_{H^2}$ for $\mathrm{dom}(\mathcal{D}_2)$, and we suitably choose a Hilbert norm for each semiclosed subspace except for $\mathrm{dom}(\mathcal{D}_2)$. Let $\mathcal{D}_2 \overset{\alpha}{=} B/\widetilde{A_2}$ for $\widetilde{A_2} = (I+\mathcal{D}_2)^{-1}$ by (2.2), and let $R := (\widetilde{A_2}^2 + B^*B)^{\frac{1}{2}}$ be the attached positive operator for a quotient $B/\widetilde{A_2} = \mathcal{D}_2$ in $\mathcal{CD}(H)$. Then $B = \mathcal{D}_2\widetilde{A_2} = \mathcal{D}_2(I+\mathcal{D}_2)^{-1}$, and since $(\ker R)^\perp = L^2(\mathbb{R})$ by Lemma 2.3, we have that, for $f \in L^2(\mathbb{R})$,

$$\begin{aligned}
\gamma(R)^2 &= \inf_{\|f\|=1} \|Rf\|^2 = \inf_{\|f\|=1} \{\|\widetilde{A_2}f\|^2 + \|Bf\|^2\} \\
&= \inf_{\|f\|=1} \{\|(I+\mathcal{D}_2)^{-1}f\|^2 + \|\mathcal{D}_2(I+\mathcal{D}_2)^{-1}f\|^2\} \\
&= \inf_{\|\widehat{f}\|=1} \{\|(1+\xi^2)^{-1}\widehat{f}\|^2 + \|\xi^2(1+\xi^2)^{-1}\widehat{f}\|^2\} \\
&= \inf_{\|\widehat{f}\|=1} \|(1+\xi^4)^{\frac{1}{2}}(1+\xi^2)^{-1}\widehat{f}\|^2 = \inf_{\|\widehat{f}\|=1} \|M\widehat{f}\|^2 \\
&= \gamma(M)^2, \quad ((\ker M)^\perp = L^2_\xi(\mathbb{R})),
\end{aligned}$$

where $M := (1+\xi^4)^{\frac{1}{2}}(1+\xi^2)^{-1}$ is a bounded multiplication operator and invertible in $\mathcal{B}(L^2_\xi(\mathbb{R}))$. Hence, by calculations

$$\gamma(R) = \gamma(M) = \|M^{-1}\|^{-1} = \Big\|\Big(\frac{(1+\xi^2)^2}{1+\xi^4}\Big)^{1/2}\Big\|_\infty^{-1} = \Big\|\frac{(1+\xi^2)^2}{1+\xi^4}\Big\|_\infty^{-1/2}.$$

Since $\|\frac{(1+\xi^2)^2}{1+\xi^4}\|_\infty = 2$, we have

$$\delta = \frac{1}{2}\gamma(R) = \frac{1}{2}\cdot 2^{-1/2} = \frac{\sqrt{2}}{4} \ (= 0.353\ldots),$$

which is a radius of $-\frac{d^2}{dx^2}$ under the choice function α as above.

Example 4.4. (A radius of $k\Delta$) Let $k\Delta$ ($k \in \mathbb{R}\setminus\{0\}$) be the selfadjoint operator with the maximal domain $\mathrm{dom}(\Delta)$. Let α be the choice function that we choose the Fourier type Sobolev norm $\|\cdot\|_{H^2}$ for $\mathrm{dom}(\Delta)$, and we suitably choose a Hilbert norm for each semiclosed subspace except for $\mathrm{dom}(\Delta)$. Then we see that $\Delta \overset{\alpha}{=} B/\widetilde{A}_2$ for $\widetilde{A}_2 = (I-\Delta)^{-1}$ by (2.2). We also have $k\Delta \overset{\alpha}{=} kB/\widetilde{A}_2$, and let $R := (\widetilde{A_2}^2 +$

$k^2B^*B)^{\frac{1}{2}}$ be the attached positive operator for a quotient $kB/\widetilde{A_2}$ in $\mathcal{CD}(H)$. Then $B = \Delta(I-\Delta)^{-1}$, and since $(\ker R)^\perp = L^2(\mathbb{R}^N)$ by Lemma 2.3, we have that, for $f \in L^2(\mathbb{R}^N)$,

$$\begin{aligned}\gamma(R)^2 &= \inf_{\|f\|=1} \|Rf\|^2 = \inf_{\|f\|=1} \{\|\widetilde{A}_2 f\|^2 + \|kBf\|^2\} \\ &= \inf_{\|f\|=1} \{\|(I-\Delta)^{-1}f\|^2 + \|k\Delta(I-\Delta)^{-1}f\|^2\} \\ &= \inf_{\|\widehat{f}\|=1} \{\|(1+|\xi|^2)^{-1}\widehat{f}\|^2 + \|k|\xi|^2(1+|\xi|^2)^{-1}\widehat{f}\|^2\} \\ &= \inf_{\|\widehat{f}\|=1} \|(1+k^2|\xi|^4)^{\frac{1}{2}}(1+|\xi|^2)^{-1}\widehat{f}\|^2 = \inf_{\|\widehat{f}\|=1} \|M\widehat{f}\|^2 \\ &= \gamma(M)^2, \quad ((\ker M)^\perp = L^2_\xi(\mathbb{R}^N)),\end{aligned}$$

where $M := (1+k^2|\xi|^4)^{\frac{1}{2}}(1+|\xi|^2)^{-1}$ is a bounded multiplication operator and invertible in $\mathcal{B}(L^2_\xi(\mathbb{R}^N))$. Hence,

$$\gamma(R) = \gamma(M) = \left\|M^{-1}\right\|^{-1} = \left\|\left(\frac{(1+|\xi|^2)^2}{1+k^2|\xi|^4}\right)^{\frac{1}{2}}\right\|_\infty^{-1} = \left\|\frac{(1+|\xi|^2)^2}{1+k^2|\xi|^4}\right\|_\infty^{-1/2} = \frac{|k|}{\sqrt{1+k^2}}.$$

Hence, we have

$$\delta = \frac{1}{2}\gamma(R) = \frac{|k|}{2\sqrt{1+k^2}},$$

which is a radius of $k\Delta$ under the choice function α as above. This does not depend on the dimension N of $\mathbb{R}^N$. In paticular, $\delta = \sqrt{2}/4$ is a radius of $-\Delta$.

5. An application

In this section, we show the selfadjointness of the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^N)$ with a Kato–Rellich potential V. Related topics such as Schrödinger operators are found in [6] and [9].

A real-valued function $V(x)$ on $\mathbb{R}^N$ is said to be a Kato–Rellich potential if it is decomposed as $V = V_1 + V_2 \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. Here, $p = 2$ if $N = 1,2,3$ and p is some constant such that $p > \frac{N}{2}$ if $N \geq 4$.

A notation V has two meanings. One is a function and the other is a multiplication operator with $\operatorname{dom}(V) = \{f \in L^2(\mathbb{R}^N) : Vf \in L^2(\mathbb{R}^N)\}$. For a Kato-Rellich potential V, it is known that the relation $\operatorname{dom}(-\Delta) \subseteq \operatorname{dom}(V)$ holds. Hence $\operatorname{dom}(-\Delta) = \operatorname{dom}(-\Delta + V)$. The following theorem is well known as a breakthrough in the perturbation theory for operators.

Theorem 5.1. (cf. [6], [9]) *Let V be a Kato–Rellich potential on $\mathbb{R}^N$ $(N \geq 1)$. Then, $-\Delta + V$ is a selfadjoint operator with* $\operatorname{dom}(-\Delta)$ *in $L^2(\mathbb{R}^N)$.*

We shall prove the above theorem using topological arguments. The sketch for the proof is the following. For $V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, we can find sequences $\{V_{i,n}\}_{n=1}^\infty$ for $i = 1, 2$ such that $V = V_{1,n} + V_{2,n} \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ with $\|V_{1,n}\|_{L^p} \to 0$ as $n \to \infty$. Since $V_{2,n}$ is a bounded selfadjoint operator for each n, $-\Delta + V_{2,n}$ is the selfadjoint operator on $\mathrm{dom}(-\Delta)$. It follows from the boundedness for $V_{2,n}$ that a radius of $-\Delta + V_{2,n}$ can be taken by a radius $\sqrt{2}/4$ of $-\Delta$ (Lemma 5.3). We can see that $q(-\Delta + V, -\Delta + V_{2,n}) \to 0$ as $n \to \infty$. Thus, a semiclosed symmetric operator $-\Delta + V$ is very near to the selfadjoint operator $-\Delta + V_{2,n}$ with a radius $\sqrt{2}/4$ for sufficiently large n. It follows from Theorem 1.1 that $-\Delta + V$ is selfadjoint.

Lemma 5.2. *For $s, t \in \mathcal{S}(H)$ with $\mathrm{dom}(s) = \mathrm{dom}(t)$, let $s \overset{\alpha}{=} B/A$ and $t \overset{\alpha}{=} D/A$. Then, $q(s, t) = q(s + X, t + X)$ for any $X \in \mathcal{B}(H)$.*

Proof.

$$\begin{aligned} q(s+X, t+X) &= q(B/A + X/I, D/A + X/I) = q(B/A + XA/A, D/A + XA/A) \\ &= q((B+XA)/A, (D+XA)/A) = \|(B+XA) - (D+XA)\| \\ &= \|B - D\| = q(s, t). \end{aligned}$$

■

Lemma 5.3. *Let $s \in \mathcal{S}_{sa}(H)$ and let $\delta > 0$ be a radius of s. For any bounded selfadjoint operator $S \in \mathcal{B}(H)$, the sum $s + S$ is selfadjoint and δ is also a radius of $s + S$. That is, $q(s + S, t) < \delta$ and $t \in \mathcal{S}_{sym}(H)$ imply $t \in \mathcal{S}_{sa}(H)$.*

Proof. Clearly $s + S$ is selfadjoint. By Lemma 5.2, we see that $q(s + S, t) = q(s + S + (-S), t + (-S)) = q(s, t - S) < \delta$. Since $t - S$ is a semiclosed symmetric operator, we see that $t - S$ is selfadjoint. Therefore, t $(= (t - S) + S)$ is selfadjoint. ■

Proof of Theorem 5.1. Let $H = L^2(\mathbb{R}^N)$ and let α be the choice function that we choose the Fourier type Sobolev norm $\|\cdot\|_{H^2}$ for $\mathrm{dom}(-\Delta)$, and we suitably choose a Hilbert norm for each semiclosed subspace except for $\mathrm{dom}(-\Delta)$.

Let V be a Kato-Rellich potential such that $V = V_1 + V_2 \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. (If $V \in L^\infty$, then clearly $-\Delta + V$ is selfadjoint. In the sequel, it may be assumed that V is unbounded.) For sufficiently large n such that $\|V_2\|_\infty < n$, we define Z_n by $Z_n := \{x \in \mathbb{R}^N : |V(x)| > n\}$. Note that $V_2 = 0$ on Z_n (a.e.). Let $V_{1,n}(x) := V_1(x)\chi_{Z_n}(x) = V(x)\chi_{Z_n}(x)$, where $\chi_{Z_n}(x)$ is the characteristic function (the value is 1 if $x \in Z_n$, 0 otherwise), and let $V_{2,n}(x) := V(x) - V_{1,n}(x)$. Then $V = V_{1,n} + V_{2,n} \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ and $\|V_{1,n}\|_{L^p} \to 0$ as $n \to \infty$ by the Lebesgue convergence theorem.

Now, let $-\Delta \overset{\alpha}{=} B/\widetilde{A_2}$. De Branges space $\mathcal{M}(\widetilde{A_2})$ is isometrically isomorphic to the Sobolev space $H^2(\mathbb{R}^N)$ ($\|\cdot\|_{\widetilde{A_2}} = \|\cdot\|_{H^2}$). And let $V \overset{\alpha}{=} D/C$ be a multiplication operator. Since $\operatorname{dom}(-\Delta)(= \widetilde{A_2}H) \subseteq \operatorname{dom}(V)(= CH)$, there uniquely exists an operator $Y \in \mathcal{B}(H)$ such that $\widetilde{A_2} = CY$ (and $\ker Y^* \supseteq \ker C = \{0\}$) by Douglas's majorization theorem([2]). Now we have

$$\begin{aligned} q(-\Delta + V, -\Delta + V_{2,n}) &= q(B/\widetilde{A_2} + D/C, B/\widetilde{A_2} + V_{2,n}/I) \\ &= q(B/\widetilde{A_2} + DY/CY, B/\widetilde{A_2} + V_{2,n}\widetilde{A_2}/\widetilde{A_2}) \\ &= q((B + DY)/\widetilde{A_2}, (B + V_{2,n}\widetilde{A_2})/\widetilde{A_2}) \\ &= \|DY - V_{2,n}\widetilde{A_2}\| = \|VCY - V_{2,n}\widetilde{A_2}\| \quad (D = VC) \\ &= \|V\widetilde{A_2} - V_{2,n}\widetilde{A_2}\| = \|(V - V_{2,n})\widetilde{A_2}\| = \|V_{1,n}\widetilde{A_2}\|. \end{aligned}$$

Moreover,

$$\begin{aligned} \|V_{1,n}\widetilde{A_2}\| &= \sup_{\|g\|=1, g\in L^2} \|V_{1,n}\widetilde{A_2}g\| = \sup_{\|\widetilde{A_2}g\|_{\widetilde{A_2}}=1} \|V_{1,n}\widetilde{A_2}g\| \\ &= \sup_{\|f\|_{H^2}=1} \|V_{1,n}f\| \quad (f := \widetilde{A_2}g). \end{aligned}$$

The Sobolev embedding theorem says that $H^2(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ holds if $N = 1, 2, 3$ and $H^2(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ holds for any $0 < r < \infty$ if $N = 4$ and for any $0 < r < (\frac{1}{2} - \frac{2}{N})^{-1}$ if $N \geq 5$.

Now, let p be the positive constant in the definition of the Kato–Rellich potential and let p' be the positive constant such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$, where $p' = \infty$ if $p = 2$. If $N = 1, 2, 3$, then $p = 2$, so that $p' = \infty$. And if $N \geq 4$, then $p > \frac{N}{2}$, so that $p' = (\frac{1}{2} - \frac{1}{p})^{-1}$, or $p' < (\frac{1}{2} - \frac{2}{N})^{-1}$. Hence, it follows from the Sobolev embedding theorem that

$$H^2(\mathbb{R}^N) \hookrightarrow L^{p'}(\mathbb{R}^N), \text{ where } \begin{cases} p' = \infty, & (N = 1, 2, 3), \\ 0 < p' < (\frac{1}{2} - \frac{2}{N})^{-1}, & (N \geq 4). \end{cases} \tag{5.1}$$

Therefore, for $V_{1,n} \in L^p(\mathbb{R}^N)$ and $f \in H^2(\mathbb{R}^N)$, it follows from Hölder's inequality and (5.1) that

$$\begin{aligned} \sup_{\|f\|_{H^2}=1} \|V_{1,n}f\| &\leq \sup_{\|f\|_{H^2}=1} \|V_{1,n}\|_{L^p}\|f\|_{L^{p'}} \\ &\leq \sup_{\|f\|_{H^2}=1} \|V_{1,n}\|_{L^p} \cdot C\|f\|_{H^2} \quad \text{for some constant } C > 0 \\ &= C\|V_{1,n}\|_{L^p} \to 0 \quad (n \to \infty). \end{aligned}$$

This means that $q(-\Delta + V, -\Delta + V_{2,n}) \to 0$ $(n \to \infty)$.

On the other hand, since $V_{2,n}$ is a bounded selfadjoint operator, it follows from Lemma 5.3 that a radius of $-\Delta + V_{2,n}$ can be taken by a radius $\delta = \sqrt{2}/4$ (Example 4.4) of $-\Delta$. Therefore, the selfadjoint operator $-\Delta + V_{2,n}$ is sufficiently near to the semiclosed symmetric operator $-\Delta + V$ so that their distance is within $\sqrt{2}/4$ for large n. Hence we conclude that $-\Delta + V$ is selfadjoint. ∎

6. Concluding remarks

There is a situation that we handle operators on the constant domain as in the case of the Kato–Rellich theorem. Now, we shall consider a radius in such a situation. A positive constant δ is said to be a radius of a selfadjoint operator s on the constant domain if it satisfies the following conditions:

$$q(s,t) < \delta,\ t \in \mathcal{S}_{sym}(H) \text{ and } \mathrm{dom}(s) = \mathrm{dom}(t) \text{ imply } t \in \mathcal{S}_{sa}(H). \tag{6.1}$$

For $s, t \in \mathcal{S}(H)$ with $\mathrm{dom}(s) = \mathrm{dom}(t)$, we have representations of quotients $s \overset{\alpha}{=} B/A$ and $t \overset{\alpha}{=} D/A$. Then, since $s - \zeta I \overset{\alpha}{=} (B - \zeta A)/A$ and $t - \zeta I \overset{\alpha}{=} (D - \zeta A)/A$, we see that (3.1) corresponds to $q(s - \zeta I, t - \zeta I) = q(s,t)$. Hence, when $s \in \mathcal{CD}(H)$ and the resolvent $\rho(s)$ of s is nonempty, we see that (3.2) corresponds to

$$\delta_\Phi = \min_{\zeta \in \Phi} \|(B - \zeta A)^{-1}\|^{-1}.$$

Therefore, when $s \in \mathcal{S}_{sa}(H)$, we see that (3.3) corresponds to

$$\delta_\Phi = \|(B \pm iA)^{-1}\|^{-1} = \gamma(R),$$

where $R = (A^2 + B^*B)^{\frac{1}{2}}$. Now, we use the compact set $\Phi_c = \{ci, -ci\}$ $(c > 0)$ instead of $\Phi = \{i, -i\}$. Then, we have that

$$\delta_{\Phi_c} = \|(B \pm ciA)^{-1}\|^{-1} = \gamma(R_c),$$

where $R_c := (c^2A^2 + B^*B)^{1/2}$. A positive constant δ_{Φ_c} is a radius of s on the constant domain for each $c > 0$. However we want to get as large a radius as possible. Hence, taking the supremum among $c > 0$, we have

$$\delta = \sup_{c>0} \delta_{\Phi_c} = \sup_{c>0} \gamma(R_c),$$

which is a radius of s on the constant domain.

Example 6.1. Now we shall calculate a radius of $k\Delta$ ($k \in \mathbb{R} \setminus \{0\}$) on the constant domain. Let α be the choice function of Example 4.4. Then we see that $\Delta \overset{\alpha}{=} B/\widetilde{A}_2$ for $\widetilde{A}_2 = (I - \Delta)^{-1}$ by (2.2) and $k\Delta \overset{\alpha}{=} kB/\widetilde{A}_2$. Let $R_c = (c^2\widetilde{A_2}^2 + k^2 B^* B)^{\frac{1}{2}}$. Then, R_c is the attached positive operator for a quotient $kB/c\widetilde{A}_2 = \frac{k}{c}\Delta$ in $\mathcal{CD}(H)$. Thus, it follows from Lemma 2.3 that $(\ker R_c)^\perp = L^2(\mathbb{R}^N)$. So we have that, for $f \in L^2(\mathbb{R}^N)$,

$$\begin{aligned}
\gamma(R_c)^2 &= \inf_{\|f\|=1} \|R_c f\|^2 = \inf_{\|f\|=1} \{\|c\widetilde{A}_2 f\|^2 + \|kBf\|^2\} \ (B = \Delta(I-\Delta)^{-1}) \\
&= \inf_{\|f\|=1} \{\|c(I-\Delta)^{-1} f\|^2 + \|k\Delta(I-\Delta)^{-1} f\|^2\} \\
&= \inf_{\|\widehat{f}\|=1} \{\|c(1+|\xi|^2)^{-1}\widehat{f}\|^2 + \|k|\xi|^2(1+|\xi|^2)^{-1}\widehat{f}\|^2\} \\
&= \inf_{\|\widehat{f}\|=1} \|(c^2 + k^2|\xi|^4)^{1/2}(1+|\xi|^2)^{-1}\widehat{f}\|^2 = \inf_{\|\widehat{f}\|=1} \|M_c \widehat{f}\|^2 \\
&= \gamma(M_c)^2, \quad ((\ker M_c)^\perp = L^2_\xi(\mathbb{R}^N)),
\end{aligned}$$

where $M_c := (c^2 + k^2|\xi|^4)^{\frac{1}{2}}(1+|\xi|^2)^{-1}$ is a bounded multiplication operator which is invertible in $\mathcal{B}(L^2_\xi(\mathbb{R}^N))$. Hence,

$$\begin{aligned}
\gamma(R_c) = \gamma(M_c) = \|M_c^{-1}\|^{-1} &= \left\| \left(\frac{(1+|\xi|^2)^2}{c^2 + k^2|\xi|^4} \right)^{\frac{1}{2}} \right\|_\infty^{-1} = \left\| \frac{(1+|\xi|^2)^2}{c^2 + k^2|\xi|^4} \right\|_\infty^{-1/2} \\
&= |k| \cdot \left\| \frac{(1+|\xi|^2)^2}{(c/k)^2 + |\xi|^4} \right\|_\infty^{-1/2} = |k| \cdot \left(\frac{1 + (c/k)^2}{(c/k)^2} \right)^{-1/2} = |k| \cdot \frac{c}{\sqrt{c^2 + k^2}}.
\end{aligned}$$

Here, we used the fact that $\|g\|_\infty = \frac{1+\ell^2}{\ell^2}$ for $g(x) = \frac{(1+x^2)^2}{\ell^2 + x^4}$ ($x \geq 0, \ell$ is a constant). Therefore we have

$$\delta = \sup_{c>0} \gamma(R_c) = |k| \cdot \sup_{c>0} \frac{c}{\sqrt{c^2 + k^2}} = |k|,$$

which is a radius of $k\Delta$ on the constant domain. In paticular, a radius of $-\Delta$ on the constant domain is 1.

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