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On the essential minimum modulus of linear operators in Banach spaces

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Abstract. We shall show in this paper a quite useful formula connecting the essential minimum modulus and minimum modulus of any linear bounded operator defined on a separable Hilbert space. Since this formula does not depend on the structure of Hilbert space, this result enables us to define the essential minimum modulus of linear operators in the more general context of Banach spaces. The connection between our definition and that given by Zemánek [Geometric interpretation of the essential minimum modulus, Operator Theory: Adv. Appl., **6**, (1982), 225-227] is discussed. Moreover, the notion of the essential surjectivity modulus and left (resp. right) essential minimum modulus on Banach spaces are also defined and will be studied in this paper. The asymptotic formula for the essential spectrum of a semi-Fredholm operator with index zero in terms of the left and right essential minimum moduli is proved.

1. Terminology and introduction

Let $\mathscr{B}(\mathsf{X})$ be the algebra of bounded linear operators acting on an infinite-dimensional complex Banach space X. We will denote by X' the dual space of X and the conjugate of $T \in \mathscr{B}(\mathsf{X})$ by T'. Let $\mathscr{K}(\mathsf{X})$ be the set of all compact operators and $\mathscr{F}(\mathsf{X})$ be the set of all finite rank operators in $\mathscr{B}(\mathsf{X})$. Denote by $\mathcal{C}(\mathsf{X}) = \mathscr{B}(\mathsf{X})/\mathscr{K}(\mathsf{X})$ the Calkin algebra and by $\pi\colon \mathscr{B}(\mathsf{X}) \longrightarrow \mathcal{C}(\mathsf{X})$ the canonical

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projection. Endowed with the essential norm $\|T\|_e = \|\pi(T)\|$, $\mathcal{C}(\mathsf{X})$ is a Banach algebra. The essential spectrum of $T \in \mathscr{B}(\mathsf{X})$ is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \pi(T - \lambda I) \text{ is not invertible in } \mathcal{C}(\mathsf{X})\}.$$

The *minimum modulus* (also called injectivity modulus) of an operator $T \in \mathscr{B}(\mathsf{X})$ is defined by

$$\mathsf{m}(T) = \inf\{\|T(x)\| : x \in \mathsf{X} \text{ and } \|x\| = 1\}, \tag{1.1}$$

and the *surjectivity modulus* of T is defined by

$$\mathsf{q}(T) = \sup\{r \geq 0 : r\mathsf{B}(0, 1) \subset T(\mathsf{B}(0, 1))\}, \tag{1.2}$$

where, as usual $\mathsf{B}(0, 1)$ denotes the closed unit ball of X.

We refer the reader to [4, 6, 7] for the properties of these quantities. It is well known that for every $T \in \mathscr{B}(\mathsf{X})$,

$$\mathsf{m}(T) = \mathsf{q}(T') \tag{1.3}$$

and

$$\mathsf{m}(T') = \mathsf{q}(T). \tag{1.4}$$

A bounded operator T is called *bounded below* if $\mathsf{m}(T) > 0$. Recall that T is bounded below if and only if it is injective and has closed range and T is onto if and only if $\mathsf{q}(T) > 0$, see, for instance, [6, Theorem 9.4].

We define the *left essential minimum modulus* of $T \in \mathscr{B}(\mathsf{X})$ by

$$\mathsf{n_e}(T) = \sup\{\mathsf{m}(T + K) : K \in \mathscr{K}(\mathsf{X})\}, \tag{1.5}$$

and the *right essential minimum modulus* of T by

$$\mathsf{p_e}(T) = \sup\{\mathsf{q}(T + K) : K \in \mathscr{K}(\mathsf{X})\}. \tag{1.6}$$

We note that the existence of the supremum in (1.5) and (1.6) will be proved in Section 3.

For an operator $T \in \mathscr{B}(\mathsf{X})$ we shall denote by $\alpha(T)$ the dimension of the kernel $\mathsf{ker}(T)$, and by $\beta(T)$ the codimension of the range $\mathsf{R}(T)$. We recall that an operator $T \in \mathscr{B}(\mathsf{X})$ is called *upper semi-Fredholm* if $\alpha(T) < +\infty$ and $\mathsf{R}(T)$ is closed, while $T \in \mathscr{B}(\mathsf{X})$ is called *lower semi-Fredholm* if $\beta(T) < +\infty$. Let $\Phi_+(\mathsf{X})$ and $\Phi_-(\mathsf{X})$ denote the class of all upper semi-Fredholm operators and the class of all lower semi-Fredholm operators, respectively. The class of all semi-Fredholm operators is defined by $\Phi_\pm(\mathsf{X}) = \Phi_+(\mathsf{X}) \cup \Phi_-(\mathsf{X})$, while the class of all Fredholm operators is defined by $\Phi(\mathsf{X}) = \Phi_+(\mathsf{X}) \cap \Phi_-(\mathsf{X})$. Recall that T is Fredholm if and only if $\pi(T)$ is invertible in $\mathcal{C}(\mathsf{X})$. If $T \in \Phi_\pm(\mathsf{X})$, the *index* of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. It is well known that the index is a continuous function on the set of semi-Fredholm operators.

The paper is organized as follows. In Section 2, we prove, for a separable Hilbert space H, a useful formula for calculating the essential minimum modulus of any linear bounded operator in terms of the minimum modulus. More precisely, we prove in Theorem 2.1, that the essential minimum modulus $\mathsf{m_e}(T) = \sup\{\mathsf{m}(T+K) : K \in \mathscr{K}(\mathsf{H})\}$, for every $T \in \mathscr{B}(\mathsf{H})$. Furthermore, we show that this supremum is always attained. In Section 3, we introduce the concepts of the essential minimum modulus and the essential surjectivity modulus for any linear bounded operator defined on a Banach space. We prove for these moduli some analogous properties to the Hilbert space case. We also give stability perturbation results involving the essential (resp. left essential, right essential) minimum modulus and the surjectivity modulus in Banach spaces. In Section 4, we study the continuity of the function $\mathsf{M_e}(\cdot)$, which denotes any one of the quantities $\mathsf{m_e}(\cdot)$, $\mathsf{q_e}(\cdot)$ $\mathsf{n_e}(\cdot)$ and $\mathsf{p_e}(\cdot)$ given respectively by relations (3.9), (3.10), (1.5) and (1.6). We also prove an asymptotic formula for the essential spectrum in terms of $\mathsf{M_e}(\cdot)$:

$$\operatorname{dist}(0, \sigma_e(T)) = \sup\left\{\mathsf{M_e}(T^k)^{1/k} : k \in \mathbb{N}\backslash\{0\}\right\} = \lim_{k\to+\infty} \mathsf{M_e}(T^k)^{1/k},$$

for every semi-Fredholm operator T with index zero. At the end of this paper, we discuss the connection between our definitions of the essential minimum modulus and the surjectivity modulus in Banach spaces and Zemánek's definitions [11, 12].

2. The essential minimum modulus of linear operators in Hilbert space

Let H be a separable complex Hilbert space. For $T \in \mathscr{B}(\mathsf{H})$, we will denote by T^* the adjoint of T and by $|T| = (T^*T)^{1/2}$ the square root of T^*T. The *essential minimum modulus* of an operator $T \in \mathscr{B}(\mathsf{H})$ is defined by (see [2])

$$\mathsf{m_e}(T) = \inf\{\lambda : \lambda \in \sigma_e(|T|)\}. \tag{2.1}$$

The following basic properties of $\mathsf{m_e}(T)$ were proved for Hilbert spaces in [2, Theorem 2]:

$$\mathsf{m_e}(T) > 0 \Longleftrightarrow T \in \Phi_+(\mathsf{H}), \tag{2.2}$$

$$\mathsf{m_e}(T) > 0 \text{ and } \mathsf{m_e}(T^*) > 0 \Longrightarrow \mathsf{m_e}(T) = \mathsf{m_e}(T^*). \tag{2.3}$$

We note that in Hilbert space the minimum modulus of $T \in \mathscr{B}(\mathsf{H})$ is also equal to (see [2, Theorem 1])

$$\mathsf{m}(T) = \inf\{\lambda \,:\, \lambda \in \sigma(|T|)\}, \tag{2.4}$$

where, for each $T \in \mathscr{B}(\mathsf{H})$, $\sigma(T)$ stands for the spectrum of the operator T.

Here there is a quite useful formula connecting the essential minimum modulus to the minimum modulus.

Theorem 2.1. *Let $T \in \mathscr{B}(\mathsf{H})$.*

(i) *If $\alpha(T) < +\infty$ and $\alpha(T) \leq \alpha(T^*)$, then*

$$\mathsf{m}_\mathsf{e}(T) = \sup\{\mathsf{m}(T+K) \,:\, K \in \mathscr{K}(\mathsf{H})\}; \tag{2.5}$$
$$= \sup\{\mathsf{m}(T+K) \,:\, K \in \mathscr{F}(\mathsf{H})\}. \tag{2.6}$$

(ii) *If $\alpha(T^*) < +\infty$ and $\alpha(T^*) \leq \alpha(T)$, then*

$$\mathsf{m}_\mathsf{e}(T^*) = \sup\{\mathsf{q}(T+K) \,:\, K \in \mathscr{K}(\mathsf{H})\}; \tag{2.7}$$
$$= \sup\{\mathsf{q}(T+K) \,:\, K \in \mathscr{F}(\mathsf{H})\}. \tag{2.8}$$

Proof. (i) First, from (2.1) and (2.4), it follows that

$$\mathsf{m}_\mathsf{e}(T) = \inf \sigma_e(|T|) \geq \inf \sigma(|T|) = \mathsf{m}(T).$$

Now by [2, Theorem 2], we see that $\mathsf{m}_\mathsf{e}(T) = \mathsf{m}_\mathsf{e}(T+K)$, for every $K \in \mathscr{K}(\mathsf{H})$; this implies that

$$\mathsf{m}_\mathsf{e}(T) \geq \sup\{\mathsf{m}(T+K) \,:\, K \in \mathscr{K}(\mathsf{H})\} \geq \sup\{\mathsf{m}(T+K) \,:\, K \in \mathscr{F}(\mathsf{H})\}. \tag{a}$$

For the reverse inequalities, if $\mathsf{m}_\mathsf{e}(T) = 0$, there is nothing to prove. Thus, one may assume that $\mathsf{m}_\mathsf{e}(T) > 0$. Let $T = V|T|$ with an isometry operator V and let $\mathsf{E}(\cdot)$ be the spectral measure for $|T|$. For $\varepsilon > 0$, define

$$K_\varepsilon = \mathsf{m}_\mathsf{e}(T)\mathsf{E}([0, \mathsf{m}_\mathsf{e}(T) - \varepsilon[) - V|T|\mathsf{E}(]0, \mathsf{m}_\mathsf{e}(T) - \varepsilon[).$$

In view of [2, Theorem 2] and [3, Proposition XI.4.6], we see that

$$\dim \mathsf{E}([0, \mathsf{m}_\mathsf{e}(T) - \varepsilon[)\mathsf{H} < +\infty.$$

From this, we infer that $K_\varepsilon \in \mathscr{F}(\mathsf{H})$. On the other hand, since

$$T + K_\varepsilon = \mathsf{m}_\mathsf{e}(T)\mathsf{E}([0, \mathsf{m}_\mathsf{e}(T) - \varepsilon[) + V|T|\mathsf{E}([\mathsf{m}_\mathsf{e}(T) - \varepsilon, +\infty[),$$

it follows that

$$\mathsf{m}(T + K_\varepsilon) \geq \mathsf{m_e}(T) - \varepsilon.$$

Therefore,

$$\sup_{K\in\mathscr{K}(\mathsf{H})} \mathsf{m}(T+K) \geq \sup_{K\in\mathscr{F}(\mathsf{H})} \mathsf{m}(T+K) \geq \sup_{\varepsilon>0} \mathsf{m}(T+K_\varepsilon) \geq \mathsf{m_e}(T). \tag{b}$$

Thus (2.5) and (2.6) follows from (a) and (b).

(ii) Equalities (2.7) and (2.8) can be derived analogously, or by passing to adjoint operators in (2.5) and (2.6), respectively. This completes the proof of Theorem 2.1. ∎

As an immediate consequence of the previous theorem and the relation (2.3), we have the following result:

Corollary 2.2. *Let $T \in \mathscr{B}(\mathsf{H})$. Then*

$$\mathsf{m_e}(T) = \begin{cases} \max\{\mathsf{n_e}(T),\ \mathsf{p_e}(T)\}, & \text{if } \alpha(T) < +\infty, \\ 0, & \text{if } \alpha(T) = +\infty. \end{cases}$$

Now we show that the suprema in (2.5) and (2.7) are always attained.

Theorem 2.3. *Let $T \in \mathscr{B}(\mathsf{H})$.*

(i) *If $\alpha(T) \leq \alpha(T^*)$, then there exists $K \in \mathscr{K}(\mathsf{H})$ such that $\mathsf{m_e}(T) = \mathsf{m}(T+K)$.*

(ii) *If $\alpha(T^*) \leq \alpha(T)$, then there exists $K \in \mathscr{K}(\mathsf{H})$ such that $\mathsf{m_e}(T^*) = \mathsf{q}(T+K)$.*

Proof. (i) If $\mathsf{m_e}(T) = 0$, Theorem 2.1 shows that $\mathsf{m}(T+K) = 0$ for every $K \in \mathscr{K}(\mathsf{H})$. Thus we can choose $K = 0$. Now assume that $\mathsf{m_e}(T) > 0$, then $T \in \Phi_+(\mathsf{H})$. By Theorem 2.1, we deduce that there exists $K_0 \in \mathscr{K}(\mathsf{H})$ such that $\mathsf{m}(T + K_0) > 0$. Put $L = T + K_0$, then $\mathsf{m_e}(L) = \mathsf{m_e}(T) > 0$. Let $\mathsf{E}_L(\cdot)$ be the spectral mesure for $|L|$. Define $K_1 = \mathsf{E}_L([0, \mathsf{m_e}(T)[)(-|L| + \mathsf{m_e}(T)I)$, then $K_1 \in \mathscr{K}(\mathsf{H})$. Indeed, let $\varepsilon > 0$ be given, and define

$$K_\varepsilon = (-|L| + \mathsf{m_e}(T)I)\ \mathsf{E}_L([0, \mathsf{m_e}(T) - \varepsilon[).$$

From [2, Theorem 2] and [3, Proposition XI.4.6], we see that $K_\varepsilon \in \mathscr{K}(\mathsf{H})$. Furthermore, it is clear that $\|K_1 - K_\varepsilon\| \leq \varepsilon$. From this, it follows that $K_1 \in \mathscr{K}(\mathsf{H})$ because $\mathscr{K}(\mathsf{H})$ is a closed ideal in $\mathscr{B}(\mathsf{H})$.

On the other hand, from

$$|L| + K_1 = \mathsf{m_e}(T)\mathsf{E}_L([0, \mathsf{m_e}(T)[) + |L|\mathsf{E}_L([\mathsf{m_e}(T), +\infty[),$$

we infer that

$$\mathsf{m}(|L| + K_1) = \mathsf{m_e}(T).$$

Since $\operatorname{ind}(L) = \operatorname{ind}(T) \leq 0$, from [5, Solution 135], it follows that there exists an isometry operator W such that $L = W|L|$. Put $K = K_0 + WK_1$, then $K \in \mathscr{K}(\mathsf{H})$. From this, we obtain

$$\mathsf{m}(T + K) = \mathsf{m}(L + WK_1) = \mathsf{m}(|L| + K_1) = \mathsf{m_e}(T).$$

This proves the first assertion.

(ii) By making use of the symmetry between T and T^* and using (i) we have the result.

The proof is complete. ∎

3. The essential minimum modulus of linear operators in Banach spaces

In order to define the concept of essential minimum modulus of every operator on Banach spaces we need to prove the following lemma.

Lemma 3.1. *Let $T \in \mathscr{B}(\mathsf{X})$. Then*

(i) $\mathsf{m}(T + K) \leq \|T\|$ *for all* $K \in \mathscr{K}(\mathsf{X})$;

(ii) $\mathsf{q}(T + K) \leq \|T\|$ *for all* $K \in \mathscr{K}(\mathsf{X})$.

Proof. (i) Assume, to the contrary, that there is a compact operator $K_0 \in \mathscr{K}(\mathsf{X})$ such that $\|T\| < \mathsf{m}(T + K_0)$. Since $\|T + K_0 - K_0\| < \mathsf{m}(T + K_0)$, from [4, Lemma 2.3], we deduce that K_0 is bounded below, which is a contradiction.

(ii) follows from (1.4) and assertion (i). This completes the proof. ∎

The next two corollaries follow easily from the preceding lemma.

Corollary 3.2. *Let $T \in \mathscr{B}(\mathsf{X})$. Then*

(i) $\mathsf{m}(T) \leq \|T\|_e \leq \|T\|$;

(ii) $\mathsf{q}(T) \leq \|T\|_e \leq \|T\|$.

Corollary 3.3. *Let $T \in \mathscr{B}(\mathsf{X})$. Then*

(i) $\sup\{\mathsf{m}(T + K) : K \in \mathscr{K}(\mathsf{X})\} \leq \|T\|_e \leq \|T\|$;

(ii) $\sup\{\mathsf{q}(T + K) : K \in \mathscr{K}(\mathsf{X})\} \leq \|T\|_e \leq \|T\|$.

Note that Corollary 3.3 ensures the finiteness of $\mathsf{n_e}(T)$ and $\mathsf{p_e}(T)$ for every $T \in \mathscr{B}(\mathsf{X})$.

We define also

$$\widetilde{\mathsf{n_e}}(T') = \sup\{\mathsf{m}(T' + K') : K \in \mathscr{K}(\mathsf{X})\}, \tag{3.1}$$

$$\widetilde{\mathsf{p_e}}(T') = \sup\{\mathsf{q}(T' + K') : K \in \mathscr{K}(\mathsf{X})\}. \tag{3.2}$$

From (1.3) and (1.4), we can only deduce that

$$\mathsf{n_e}(T) = \widetilde{\mathsf{p_e}}(T') \leq \mathsf{p_e}(T'), \quad \text{for all } T \in \mathscr{B}(\mathsf{X}), \tag{3.3}$$

$$\mathsf{p_e}(T) = \widetilde{\mathsf{n_e}}(T') \leq \mathsf{n_e}(T'), \quad \text{for all } T \in \mathscr{B}(\mathsf{X}). \tag{3.4}$$

If, in addition, X is reflexive, then

$$\mathsf{n_e}(T) = \mathsf{p_e}(T'), \quad \text{for all } T \in \mathscr{B}(\mathsf{X}), \tag{3.5}$$

$$\mathsf{n_e}(T') = \mathsf{p_e}(T), \quad \text{for all } T \in \mathscr{B}(\mathsf{X}). \tag{3.6}$$

Let us notice that, since $\mathsf{m}(\cdot)$ and $\mathsf{q}(\cdot)$ are super-multiplicative, it is easy to see that $\mathsf{n_e}(\cdot)$ and $\mathsf{p_e}(\cdot)$ are also super-multiplicative, that is

$$\mathsf{n_e}(TL) \geq \mathsf{n_e}(T)\mathsf{n_e}(L), \quad \text{for all } T,\, L \in \mathscr{B}(\mathsf{X}), \tag{3.7}$$

$$\mathsf{p_e}(TL) \geq \mathsf{p_e}(T)\mathsf{p_e}(L), \quad \text{for all } T,\, L \in \mathscr{B}(\mathsf{X}). \tag{3.8}$$

We have the following proposition, which will be needed in the sequel.

Proposition 3.4. *Let $T \in \mathscr{B}(\mathsf{X})$.*
(i) *If $\alpha(T) \leq \alpha(T')$ then $\mathsf{p_e}(T) \leq \mathsf{n_e}(T)$.*
(ii) *If $\alpha(T') \leq \alpha(T)$ then $\mathsf{n_e}(T) \leq \mathsf{p_e}(T)$.*

Proof. (i) We will argue by contradiction, and so assume that $\mathsf{n_e}(T) < \mathsf{p_e}(T)$. Then there is $K_0 \in \mathscr{K}(\mathsf{X})$, such that

$$\mathsf{m}(T + K) < \mathsf{q}(T + K_0), \quad \text{for all } K \in \mathscr{K}(\mathsf{X}). \tag{a}$$

From this, we infer that $T + K_0$ is surjective. In particular, $T + K_0$ is semi-Fredholm with non-negative index. Now, since $K_0 \in \mathscr{K}(\mathsf{X})$, it follows that T is also semi-Fredholm with non-negative index. Hence, in view of hypothesis, we have $\operatorname{ind}(T) = 0$. Therefore $T + K_0 \in \Phi_\pm(\mathsf{X})$ with $\operatorname{ind}(T + K_0) = 0$. Moreover, since $\alpha(T' + K_0') = 0$, it follows that $T + K_0$ is an invertible operator. Applying [6, Theorem 9.7], we obtain

$$\mathsf{q}(T + K_0) = \frac{1}{\|(T + K_0)^{-1}\|} = \mathsf{m}(T + K_0).$$

But this is in contradiction with (a).

(ii) Note that in general, we cannot deduce this assertion by passing to conjugate because inequalities (3.3) and (3.4) are not always equalities. But we can prove it in the same way as (i). This completes the proof. ∎

Since relations (2.5) and (2.7) do not depend on the structure of Hilbert space, Corollary 2.2 suggests the following definition of the essential minimum modulus and the essential surjectivity modulus for any Banach space operator.

The *essential minimum modulus* of an operator $T \in \mathscr{B}(\mathsf{X})$ is defined by

$$\mathsf{m_e}(T) = \begin{cases} \max\{\mathsf{n_e}(T),\, \mathsf{p_e}(T)\}, & \text{if } \alpha(T) < +\infty \\ 0, & \text{if } \alpha(T) = +\infty, \end{cases} \tag{3.9}$$

and the *essential surjectivity modulus* of T is defined by

$$\mathsf{q_e}(T) = \begin{cases} \max\{\mathsf{n_e}(T),\, \mathsf{p_e}(T)\}, & \text{if } \alpha(T') < +\infty \\ 0, & \text{if } \alpha(T') = +\infty. \end{cases} \tag{3.10}$$

Note that in [11, 12] J. Zemánek has given another definitions of the essential minimum modulus and the essential surjectivity modulus in Banach spaces without using the notions of the minimum modulus and the surjectivity modulus. He has just used the upper and lower semi-Fredholm classes. But our definitions are more natural because these are based on the minimum and the surjectivity moduli. Clearly, from Corollary 3.3, we have

$$\mathsf{m_e}(T) \le \|T\|_e \le \|T\|, \quad \text{for all } T \in \mathscr{B}(\mathsf{X}), \tag{3.11}$$

$$\mathsf{q_e}(T) \le \mathsf{m_e}(T') \le \|T'\|_e \le \|T\|_e \le \|T\|, \quad \text{for all } T \in \mathscr{B}(\mathsf{X}). \tag{3.12}$$

If, in addition, X is reflexive, then

$$\mathsf{q_e}(T) = \mathsf{m_e}(T'), \quad \text{for all } T \in \mathscr{B}(\mathsf{X}). \tag{3.13}$$

Let $T \in \mathscr{B}(\mathsf{X})$ and $K \in \mathscr{K}(\mathsf{X})$. Assume that $\alpha(T) < +\infty$, then $\mathsf{m_e}(T) = \max\{\mathsf{n_e}(T), \mathsf{p_e}(T)\}$, and so, if $\alpha(T+K) < +\infty$, then $\mathsf{m_e}(T) = \mathsf{m_e}(T+K)$. Now, if $\alpha(T+K) = +\infty$, then $\mathsf{m_e}(T+K) = 0$ and $\mathsf{R}(T)$ is not closed. From this, it is clear that $\mathsf{n_e}(T) = \mathsf{p_e}(T) = 0$, and hence $\mathsf{m_e}(T) = 0$.

Assume now that $\alpha(T) = +\infty$, then $\mathsf{m_e}(T) = 0$. If $\alpha(T+K) < +\infty$, then necessarily $\mathsf{R}(T+K)$ cannot be closed, and hence $\mathsf{m_e}(T+K) = 0$. Consequently,

$$\mathsf{m_e}(T) = \mathsf{m_e}(T+K), \quad \text{for all } T \in \mathscr{B}(\mathsf{X}), \text{ for all } K \in \mathscr{K}(\mathsf{X}). \tag{3.14}$$

Exactly in the same way as above, we can prove that

$$\mathsf{q_e}(T) = \mathsf{q_e}(T+K), \quad \text{for all } T \in \mathscr{B}(\mathsf{X}), \text{ for all } K \in \mathscr{K}(\mathsf{X}). \tag{3.15}$$

We introduce the following notations for two special subsets of semi-Fredholm operators:

- $\Phi_+^-(\mathsf{X}) = \{T \in \Phi_+(\mathsf{X}) : \operatorname{ind}(T) \le 0\}$,
- $\Phi_-^+(\mathsf{X}) = \{T \in \Phi_-(\mathsf{X}) : \operatorname{ind}(T) \ge 0\}$.

As it is known in Hilbert space, an upper (resp. lower) semi-Fredholm operator can be characterized by the essential minimum (resp. surjectivity) modulus. Our definitions enable us to characterize upper and lower semi-Fredholm operators even in a Banach space and to retrieve the case of Hilbert space, which is the subject of Theorem 3.5. This theorem also gives a characterization of a lower (resp. upper) semi-Fredholm operator with non-negative (resp. non-positive) index in terms of the right (resp. left) essential minimum modulus.

Theorem 3.5. *Let $T \in \mathscr{B}(\mathsf{X})$.*

(i) $\mathsf{n_e}(T) > 0 \iff T \in \Phi_+^-(\mathsf{X})$.
(ii) $\mathsf{p_e}(T) > 0 \iff T \in \Phi_-^+(\mathsf{X})$.
(iii) $\mathsf{m_e}(T) > 0 \iff T \in \Phi_+(\mathsf{X})$.
(iv) $\mathsf{q_e}(T) > 0 \iff T \in \Phi_-(\mathsf{X})$.

Proof. (i) Let $T \in \mathscr{B}(\mathsf{X})$ and assume that $\mathsf{n_e}(T) > 0$. Then there exists $K \in \mathscr{K}(\mathsf{X})$ such that $\mathsf{m}(T+K) > 0$, which implies that $T + K \in \Phi_+(\mathsf{X})$ and $\operatorname{ind}(T) = \operatorname{ind}(T+K) \le 0$. This implies that $T \in \Phi_+^-(\mathsf{X})$.

For the converse, by [8, Theorem 1.2], we see that there exists a compact operator $K \in \mathscr{K}(\mathsf{X})$ such that $T + K$ is bounded below, and hence $\mathsf{m}(T+K) > 0$. Finally, from (3.14), it follows that $\mathsf{n_e}(T) > 0$.

Assertion (ii) can be proved exactly in the same way as (i).

(iii) If $\mathsf{m_e}(T) = \mathsf{n_e}(T) > 0$, the result follows from (i), and so we may assume that $\mathsf{m_e}(T) = \mathsf{p_e}(T) > 0$. So, it follows from (ii) that $T \in \Phi_-(\mathsf{X})$ and $\operatorname{ind}(T) \ge 0$. Since $\alpha(T) < +\infty$, T has necessarily finite index, and consequently $T \in \Phi_+(\mathsf{X})$.

For the converse, let $T \in \Phi_+(\mathsf{X})$ so $\alpha(T) < +\infty$. If $\operatorname{ind}(T) \le 0$, then $T \in \Phi_+^-(\mathsf{X})$ and from (i), we deduce that $\mathsf{m_e}(T) \ge \mathsf{n_e}(T) > 0$. Now if $\operatorname{ind}(T) > 0$, first we remark that $\operatorname{ind}(T)$ cannot be $+\infty$. Therefore, T is Fredholm, and consequently $T \in \Phi_-^+(\mathsf{X})$. Finally, from (ii) it follows that $\mathsf{m_e}(T) \ge \mathsf{p_e}(T) > 0$.

Assertion (iv) can be proved exactly in the same way as (iii). This completes the proof of Theorem 3.5. ∎

Remark. Other characterizations of upper semi-Fredholm and lower semi-Fredholm operators can be found in [13, Theorem 4.1].

It is well known that an operator T belongs to $\Phi_+(\mathsf{X})$ if and only if its conjugate T' belongs to $\Phi_-(\mathsf{X}')$. Also, an operator T belongs to $\Phi_-(\mathsf{X})$ if and only if its conjugate T' belongs to $\Phi_+(\mathsf{X}')$. Now, by using Proposition 3.4, Theorem 3.5 and formulas (3.3)–(3.6), one can prove the following result.

Corollary 3.6. *Let $T \in \mathscr{B}(\mathsf{X})$. Then*

(i) $T \in \Phi_-(\mathsf{X}) \Longleftrightarrow \mathsf{m_e}(T') > 0$.

(ii) $T \in \Phi(\mathsf{X}) \Longleftrightarrow \mathsf{m_e}(T) > 0$ *and* $\mathsf{q_e}(T) > 0$. *In this case,* $\mathsf{m_e}(T) = \mathsf{q_e}(T)$.

(iii) $T \in \Phi(\mathsf{X}) \Longleftrightarrow \mathsf{m_e}(T) > 0$ *and* $\mathsf{m_e}(T') > 0$. *In this case, if in addition* X *is reflexive then* $\mathsf{m_e}(T) = \mathsf{m_e}(T')$.

(iv) *If* $\alpha(T) < +\infty$ *and* $\alpha(T') < +\infty$, *then* $\mathsf{m_e}(T) > 0 \Longleftrightarrow \mathsf{m}_e(T') > 0$, *whence* $T \in \Phi_+(\mathsf{X}) \Longleftrightarrow T \in \Phi(\mathsf{X}) \Longleftrightarrow T \in \Phi_-(\mathsf{X})$.

(v) *If* $\alpha(T') < +\infty$ *then* $\mathsf{m_e}(T') \geq \mathsf{q_e}(T) \geq \mathsf{m}_e(T)$.

Now we prove the following result.

Theorem 3.7. *Let T, $L \in \mathscr{B}(\mathsf{X})$. If $\|T - L\| < \mathsf{m_e}(T)$ then*

(i) $\mathsf{m_e}(L) > 0$;

(ii) $L \in \Phi_+(\mathsf{X})$;

(iii) $\operatorname{ind}(T) = \operatorname{ind}(L)$.

Analogous assertions hold when replacing $\mathsf{m_e}(\cdot)$ *with* $\mathsf{n_e}(\cdot)$ (*resp.* $\mathsf{p_e}(\cdot)$, $\mathsf{q_e}(\cdot)$) *and* $\Phi_+(\mathsf{X})$ *with* $\Phi_+^-(\mathsf{X})$ (*resp.* $\Phi_-^+(\mathsf{X})$, $\Phi_-(\mathsf{X})$).

Proof. With our hypothesis we necessarily have $\mathsf{m_e}(T) > 0$. We first assume that $\mathsf{m_e}(T) = \mathsf{n_e}(T) > 0$. From the definition of $\mathsf{n_e}(T)$, we can find $K \in \mathscr{K}(\mathsf{X})$ such that $\|T - L\| < \mathsf{m}(T + K)$. Since $\|T + K - (L + K)\| = \|T - L\|$, from [4, Lemma 2.3], it follows that $L + K$ is bounded below. Therefore $\mathsf{m}_e(L) \geq \mathsf{m}(L + K) > 0$. Moreover, from Theorem 3.5, it follows that $L \in \Phi_+(\mathsf{X})$. By the continuity of the index on $\Phi_+(\mathsf{X})$, we deduce that $\operatorname{ind}(T) = \operatorname{ind}(L) \leq 0$.

Suppose now that $\mathsf{m_e}(T) = \mathsf{p_e}(T)$, by taking (1.4) into account we deduce, as above, that there is $K \in \mathscr{K}(\mathsf{X})$ such that $L' + K'$ is bounded below. In particular L is semi-Fredholm, and the continuity of the index implies that $\operatorname{ind}(T) = \operatorname{ind}(L) \geq 0$. Therefore $\alpha(L') < +\infty$ and hence $L \in \Phi_-(\mathsf{X})$. Finally, from Theorem 3.5, we obtain $\mathsf{p_e}(L) > 0$. A similar proof follows for the other functions. This proves the theorem. ■

The following corollary is an immediate consequence of Theorem 3.7.

Corollary 3.8. *Let* $T \in \mathscr{B}(\mathsf{X})$.

(i) $\mathsf{n_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_+^-(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\| < r\}$.

(ii) $\mathsf{p_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_-^+(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\| < r\}$.

(iii) $\mathsf{m_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_+(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\| < r\}$.

(iv) $\mathsf{q_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_-(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\| < r\}$.

Corollary 3.9. *Let* $T, L \in \mathscr{B}(\mathsf{X})$ *be such that* $\|T - L\|_e < \mathsf{m_e}(T)$. *Then*

(i) $\mathsf{m_e}(L) > 0$;

(ii) $L \in \Phi_+(\mathsf{X})$;

(iii) $\mathrm{ind}(T) = \mathrm{ind}(L)$.

Analogous assertions hold when replacing $\mathsf{m_e}(\cdot)$ *with* $\mathsf{n_e}(\cdot)$ (*resp.* $\mathsf{p_e}(\cdot)$, $\mathsf{q_e}(\cdot)$) *and* $\Phi_+(\mathsf{X})$ *with* $\Phi_+^-(\mathsf{X})$ (*resp.* $\Phi_-^+(\mathsf{X})$, $\Phi_-(\mathsf{X})$).

Proof. Let $K \in \mathscr{K}(\mathsf{X})$ be such that $\|T - L + K\| < \mathsf{m_e}(T)$. Clearly, the corollary follows from equality (3.14) and Theorem 3.7. ∎

As consequence of Corollary 3.9, we have the following result.

Corollary 3.10. *Let* $T \in \mathscr{B}(\mathsf{X})$.

(i) $\mathsf{n_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_+^-(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\|_e < r\}$.

(ii) $\mathsf{p_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_-^+(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\|_e < r\}$.

(iii) $\mathsf{m_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_+(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\|_e < r\}$.

(iv) $\mathsf{q_e}(T) \leq \sup\{r > 0 \,:\, T - S \in \Phi_-(\mathsf{X}),$ *for all* $S \in \mathscr{B}(\mathsf{X}),\ \|S\|_e < r\}$.

For $T \in \mathscr{B}(\mathsf{X})$, we denote by $\mathsf{B}_e(T, r) = \{S \in \mathscr{B}(\mathsf{X}) \,:\, \|T - S\|_e < r\}$.

Corollary 3.11. *Let* $T, L \in \mathscr{B}(\mathsf{X})$. *If* $\|T - L\|_e < \mathsf{m_e}(T) + \mathsf{m_e}(L)$ *then*

(i) $T, L \in \Phi_+(\mathsf{X})$,

(ii) $\mathrm{ind}(T) = \mathrm{ind}(L)$.

Analogous assertions hold when replacing $\mathsf{m_e}(\cdot)$ *with* $\mathsf{n_e}(\cdot)$ (*resp.* $\mathsf{p_e}(\cdot)$, $\mathsf{q_e}(\cdot)$) *and* $\Phi_+(\mathsf{X})$ *with* $\Phi_+^-(\mathsf{X})$ (*resp.* $\Phi_-^+(\mathsf{X})$, $\Phi_-(\mathsf{X})$).

Proof. If we assume that $\mathsf{m_e}(T) = 0$, then $\mathsf{m_e}(L) > 0$. So it follows from Corollary 3.9, that $\mathsf{m_e}(T) > 0$, which is a contradiction. Therefore $\mathsf{m_e}(T) > 0$ and $\mathsf{m_e}(L) > 0$ hold, and consequently $T, L \in \Phi_+(\mathsf{X})$ by Theorem 3.5. Finally, if $S \in \mathsf{B}_e(T, \mathsf{m_e}(T)) \cap \mathsf{B}_e(L, \mathsf{m_e}(L))$, then from Corollary 3.9, we have $\mathrm{ind}(T) = \mathrm{ind}(S) = \mathrm{ind}(L)$. ∎

Remark. Analogous result of Corollary 3.11 can be found in [12, Theorem 4], for the essential minimum modulus given by Zemánek.

From the last corollary we can further deduce the following.

Corollary 3.12. *Let T, $L \in \mathscr{B}(\mathsf{X})$. If $\|T-L\| < \mathsf{m_e}(T) + \mathsf{m_e}(L)$, then*

(i) $T, L \in \Phi_+(\mathsf{X})$,

(ii) $\operatorname{ind}(T) = \operatorname{ind}(L)$.

Analogous assertions hold when replacing $\mathsf{m_e}(\cdot)$ with $\mathsf{n_e}(\cdot)$ (resp. $\mathsf{p_e}(\cdot)$, $\mathsf{q_e}(\cdot)$) and $\Phi_+(\mathsf{X})$ with $\Phi_+^-(\mathsf{X})$ (resp. $\Phi_-^+(\mathsf{X})$, $\Phi_-(\mathsf{X})$).

As an immediate consequence of Corollary 3.11 we obtain

Corollary 3.13. *Let T, $L \in \Phi_\pm(\mathsf{X})$. If $\operatorname{ind}(T) \neq \operatorname{ind}(L)$ then*

$$\|T-L\|_e \geq \mathsf{m_e}(T) + \mathsf{m_e}(L).$$

Analogous inequality holds for the function $\mathsf{q_e}(\cdot)$.

Corollary 3.14. *Let T, $L \in \Phi_\pm(\mathsf{X})$. If $\operatorname{ind}(T) \neq \operatorname{ind}(L)$ then*

$$\|T-L\|_e \geq \max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\} + \max\{\mathsf{m_e}(L), \mathsf{q_e}(L)\}.$$

Proof. We will distinguish four cases.

Case 1. If $\max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\} = \mathsf{m_e}(T) > 0$ and $\max\{\mathsf{m_e}(L), \mathsf{q_e}(L)\} = \mathsf{m_e}(L) > 0$, the result follows immediately from Corollary 3.13.

Case 2. If $\max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\} = \mathsf{q_e}(T) > 0$ and $\max\{\mathsf{m_e}(L), \mathsf{q_e}(L)\} = \mathsf{q_e}(L) > 0$, the result again follows from Corollary 3.13.

Case 3. If $\max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\} = \mathsf{m_e}(T) > 0$ and $\max\{\mathsf{m_e}(L), \mathsf{q_e}(L)\} = \mathsf{q_e}(L) > 0$, assume that $\|T-L\|_e < \mathsf{m_e}(T) + \mathsf{q_e}(L)$, then $\mathsf{B}_e(T, \mathsf{m_e}(T)) \cap \mathsf{B}_e(L, \mathsf{q_e}(L)) \neq \emptyset$. Let $S \in \mathsf{B}_e(T, \mathsf{m_e}(T)) \cap \mathsf{B}_e(L, \mathsf{q_e}(L))$, from Corollary 3.9, we deduce that $\operatorname{ind}(T) = \operatorname{ind}(S) = \operatorname{ind}(L)$ and we get the contradiction, since $\operatorname{ind}(T) \neq \operatorname{ind}(L)$.

Case 4. If $\max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\} = \mathsf{q_e}(T) > 0$ and $\max\{\mathsf{m_e}(L), \mathsf{q_e}(L)\} = \mathsf{m_e}(L) > 0$; this follows exactly in the same way as the previous case. ∎

As a consequence, we have the following result.

Corollary 3.15. *Let T, $L \in \mathscr{B}(\mathsf{X})$ be such that T is bounded below, L is onto and $\operatorname{ind}(T) \neq \operatorname{ind}(L)$. Then*

$$\|T-L\|_e \geq \mathsf{m_e}(T) + \mathsf{q_e}(L).$$

Now define the following sets:

- $\mathsf{S}(\mathsf{X}) = \{T \in \mathscr{B}(\mathsf{X}) : \|T(x)\| = \|x\|$, for all $x \in \mathsf{X}\}$, the set of all isometries;
- $\mathsf{U}(\mathsf{X}) = \{T \in \mathsf{S}(\mathsf{X}) : T$ is bijective$\}$, the set of all bijective isometries.

It is clear that for all $S \in \mathsf{S}(\mathsf{X})$,

$$\mathsf{m}(S) = \mathsf{m_e}(S) = \|S\|_e = \|S\| = 1. \tag{3.16}$$

As a consequence of Corollary 3.11, we obtain the following result.

Corollary 3.16. *Let T, $S \in \mathsf{S}(\mathsf{X})$. If $\|T - S\|_e < 2$ then $\alpha(T') = \alpha(S') < +\infty$ or $\alpha(T') = \alpha(S') = +\infty$.*

Corollary 3.17. *Let T, $S \in \mathscr{B}(\mathsf{X})$. If one of the conditions*
(i) *$\alpha(T') < \alpha(T)$ and $S \in \mathsf{S}(\mathsf{X})$, and*
(ii) *$\alpha(T) < \alpha(T')$ or $\alpha(T') < \alpha(T)$ and $S \in \mathsf{U}(\mathsf{X})$*
is satisfied, then

$$\|T - S\|_e \geq 1 + \max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\}.$$

Proof. Note that if $\mathsf{m_e}(T) = \mathsf{q_e}(T) = 0$, the result follows from Corollary 3.11. Now assume that $\max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\} > 0$, then from Theorem 3.5, we deduce that $T \in \Phi_{\pm}(\mathsf{X})$. Therefore by Corollary 3.14, we must have $\|T - S\|_e \geq 1 + \max\{\mathsf{m_e}(T), \mathsf{q_e}(T)\}$. ∎

4. Continuity of the essential minimum modulus and an asymptotic formula

Let us now present some useful lemmas to be used in the sequel.

Lemma 4.1. *Let T, $L \in \mathscr{B}(\mathsf{X})$. Then*

$$|\mathsf{n_e}(T) - \mathsf{n_e}(L)| \leq \|T - L\|_e, \tag{4.1}$$

$$|\mathsf{p_e}(T) - \mathsf{p_e}(L)| \leq \|T - L\|_e. \tag{4.2}$$

Proof. First, by [4, Lemma 2.2], we obtain

$$\mathsf{m}(T + K_1) - \mathsf{m}(L + K_2) \leq \|T + K_1 - L - K_2\|, \text{ for all } K_1, K_2 \in \mathscr{K}(\mathsf{X}).$$

From this, we infer that

$$\mathsf{m}(T + K_1) - \mathsf{n_e}(L) \leq \|T + K_1 - L - K_2\|, \quad \text{for all } K_1, K_2 \in \mathscr{K}(\mathsf{X}).$$

Now, taking the infimum of the right-hand side over all $K_2 \in \mathscr{K}(\mathsf{X})$, we get

$$\mathsf{m}(T + K_1) - \mathsf{n_e}(L) \leq \|T - L\|_e, \quad \text{for all } K_1 \in \mathscr{K}(\mathsf{X}),$$

and by taking the supremum of the left-hand side over all $K_1 \in \mathscr{K}(\mathsf{X})$, we obtain

$$\mathsf{n_e}(T) - \mathsf{n_e}(L) \leq \|T - L\|_e. \tag{a}$$

By interchanging T and L in (a), we deduce $\mathsf{n_e}(L) - \mathsf{n_e}(T) \leq \|T - L\|_e$. Therefore

$$|\mathsf{n_e}(T) - \mathsf{n_e}(L)| \leq \|T - L\|_e.$$

This proves (4.1). Inequality (4.2) can be proved exactly in the same way as (4.1). This completes the proof. ∎

As a consequence of Proposition 3.4 and Lemma 4.1, we have the following result.

Lemma 4.2. *Let T, $L \in \mathscr{B}(\mathsf{X})$. If one of the following conditions*
(i) $\alpha(T) \le \alpha(T')$ *and* $\alpha(L) \le \alpha(L')$,
(ii) $\alpha(T') \le \alpha(T)$ *and* $\alpha(L') \le \alpha(L)$
is satisfied, then

$$|\mathsf{m_e}(T) - \mathsf{m_e}(L)| \le \|T - L\|_e, \tag{4.3}$$

$$|\mathsf{q_e}(T) - \mathsf{q_e}(L)| \le \|T - L\|_e. \tag{4.4}$$

By [12, Proposition 1], we know that the essential minimum modulus and the essential surjectivity modulus for any Banach space operator given by Zemánek are both continuous functions. The following theorem shows that the same is true for the functions $\mathsf{m_e}(\cdot)$ and $\mathsf{q_e}(\cdot)$.

Theorem 4.3. *Let T, $L \in \mathscr{B}(\mathsf{X})$.*
(i) *If* $\|T - L\|_e < \mathsf{m_e}(T)$ *then* $|\mathsf{m_e}(T) - \mathsf{m_e}(L)| \le \|T - L\|_e$.
(ii) *If* $\|T - L\|_e < \mathsf{q_e}(T)$ *then* $|\mathsf{q_e}(T) - \mathsf{q_e}(L)| \le \|T - L\|_e$.

Proof. (i) From Corollary 3.9, we know that $\mathsf{m_e}(L) > 0$ and $\mathrm{ind}(T) = \mathrm{ind}(L)$. Thus, by Lemma 4.2, $|\mathsf{m_e}(T) - \mathsf{m_e}(L)| \le \|T - L\|_e$.

(ii) can be proved exactly in the same way as (i). ∎

We prove now the following proposition.

Proposition 4.4. *Let $T \in \Phi_\pm(\mathsf{X})$ be such that* $\mathrm{ind}(T) = 0$ *and let $L \in \mathscr{B}(\mathsf{X})$ with $\alpha(L) \le \alpha(L')$ or $\alpha(L) = +\infty$. Then*

$$\mathsf{m_e}(TL) \le \|T\|_e \, \mathsf{m_e}(L).$$

Analogous inequality holds for the function $\mathsf{n_e}(\cdot)$.

Proof. Note that the inequality is clear if $\alpha(L) = +\infty$. Assume that $\alpha(L) < +\infty$. Clearly, if $\mathsf{m_e}(TL) = 0$, there is nothing to prove, so we can assume that $\mathsf{m_e}(TL) > 0$. In view of Theorem 3.5, we see that $TL \in \Phi_+(\mathsf{X})$. Now from [6, Theorem 16.6], it follows that $L \in \Phi_+(\mathsf{X})$. Consequently,

$$\mathrm{ind}(TL) = \mathrm{ind}(T) + \mathrm{ind}(L) = \mathrm{ind}(L) \le 0$$

which implies that $\alpha(TL) \le \alpha(L'T')$. Thus, by Proposition 3.4, $\mathsf{m_e}(TL) = \mathsf{n_e}(TL)$.

First we prove this result when T is invertible. Let $\varepsilon > 0$ and $K_1 \in \mathscr{K}(\mathsf{X})$ be such that $\|T + K_1\| < \|T\|_e + \varepsilon$. By [8, Theorem 1.2], it is easy to see that we can find $K_2 \in \mathscr{K}(\mathsf{X})$ such that $T + K_2$ is invertible and $\|T + K_2\| < \|T\|_e + \varepsilon$. Since

$$\mathsf{m}\big((T + K_2)(L + K)\big) \le \|T + K_2\| \, \mathsf{m}(L + K), \quad \text{for all } K \in \mathscr{K}(\mathsf{X}),$$

it follows that

$$\mathsf{m}\big((T + K_2)(L + K)\big) \le \|T + K_2\| \, \mathsf{m_e}(L), \quad \text{for all } K \in \mathscr{K}(\mathsf{X}). \tag{a}$$

On the other hand, we have

$$(T + K_2)(L + K) = TL + K_2 L + (T + K_2)K \tag{b}$$

and

$$\mathscr{K}(\mathsf{X}) = \{K_2 L + (T + K_2)K \,:\, K \in \mathscr{K}(\mathsf{X})\}. \tag{c}$$

Indeed, let $S \in \mathscr{K}(\mathsf{X})$ and put $K_0 = (T + K_2)^{-1}(S - K_2 L)$. It is clear that $K_0 \in \mathscr{K}(\mathsf{X})$ and $S = K_2 L + (T + K_2)K_0$. The reverse inclusion is trivial.

Now, by combining (a), (b) and (c), we get

$$\mathsf{m_e}(TL) = \mathsf{n_e}(TL) \le \|T + K_2\| \, \mathsf{m_e}(L).$$

From this, we infer that

$$\mathsf{m_e}(TL) \le (\|T\|_e + \varepsilon) \, \mathsf{m_e}(L).$$

Since ε is arbitrary, we deduce that $\mathsf{m_e}(TL) \le \|T\|_e \, \mathsf{m_e}(L)$.

Assume now that $T \in \Phi_\pm(\mathsf{X})$ and $\mathrm{ind}(T) = 0$. From [8, Theorem 1.2], we know that there exists $K \in \mathscr{K}(\mathsf{X})$ such that $T + K$ is invertible. Since $\mathsf{m_e}(TL) = \mathsf{m_e}((T + K)L)$, from the previous step, it follows that

$$\mathsf{m_e}(TL) = \mathsf{m_e}((T + K)L) \le \|T + K\|_e \, \mathsf{m_e}(L) = \|T\|_e \, \mathsf{m_e}(L).$$

This completes the proof. ∎

Using the facts, that for $T, L \in \mathscr{B}(\mathsf{X})$,

$$\mathsf{q}(LT) \le \mathsf{q}(L) \, \|T\| \quad \text{and} \quad LT \in \Phi_-(\mathsf{X}) \Longrightarrow L \in \Phi_-(\mathsf{X}),$$

the following result can be proved in the same way as the previous proposition.

Proposition 4.5. *Let $T \in \Phi_\pm(\mathsf{X})$ be such that* $\operatorname{ind}(T) = 0$ *and let $L \in \mathscr{B}(\mathsf{X})$ with $\alpha(L') \leq \alpha(L)$ or $\alpha(L') = +\infty$. Then*

$$\mathsf{q_e}(LT) \leq \mathsf{q_e}(L)\, \|T\|_e.$$

Analogous inequality holds for the function $\mathsf{p_e}(\cdot)$.

In the following we prove an asymptotic formula for the essential spectrum in terms of the essential minimum modulus and the essential surjectivity modulus in Banach spaces for every semi-Fredholm operator with index zero. In [12, Theorem 1] there is a similar asymptotic formula for an arbitrary operator $T \in \mathscr{B}(\mathsf{X})$ using Zemánek's definitions.

Theorem 4.6. *Let $T \in \Phi_\pm(\mathsf{X})$ be such that* $\operatorname{ind}(T) = 0$. *Then*

$$\operatorname{dist}(0, \sigma_e(T)) = \sup\left\{\mathsf{m_e}(T^k)^{1/k} : k \in \mathbb{N}\backslash\{0\}\right\} = \lim_{k\to+\infty} \mathsf{m_e}(T^k)^{1/k}.$$

Analogous equalities hold when replacing $\mathsf{m_e}(\cdot)$ with $\mathsf{n_e}(\cdot)$ (resp. $\mathsf{p_e}(\cdot)$, $\mathsf{q_e}(\cdot)$).

Proof. Let $L \in \mathscr{B}(\mathsf{X})$ be the inverse of T modulo the set of compact operators. By Theorem 3.7 and Proposition 4.4, we deduce that

$$\frac{1}{\|L\|_e} \leq \mathsf{m_e}(T) \leq \operatorname{dist}(0, \sigma_e(T)). \tag{a}$$

On the other hand, let $\rho_e(L) = \sup\{|\lambda| \in \mathbb{C} : \lambda \in \sigma_e(L)\}$. We have

$$\frac{1}{\rho_e(L)} = \frac{1}{\inf\limits_{1\leq k} \|L^k\|_e^{1/k}} = \sup_{1\leq k} \frac{1}{\|L^k\|_e^{1/k}} \leq \sup_{1\leq k} \mathsf{m_e}(T^k)^{1/k}. \tag{b}$$

Using the fact that $\operatorname{dist}(0, \sigma_e(T)) = \frac{1}{\rho_e(L)}$ (see [1, Theorem 3.3.5]) together with $\rho_e((L)^k) = \rho_e(L)^k$, for all $k \geq 1$ and by taking account of (a), we get

$$\sup_{1\leq k} \mathsf{m_e}(T^k)^{1/k} \leq \sup_{1\leq k} \frac{1}{\rho_e(L^k)^{1/k}} = \frac{1}{\rho_e(L)}. \tag{c}$$

By combining (b) and (c), we get $\operatorname{dist}(0, \sigma_e(T)) = \sup\left\{\mathsf{m_e}(T^k)^{1/k} : k \in \mathbb{N}\backslash\{0\}\right\}$ and

$$\overline{\lim_{k\to+\infty}}\, \mathsf{m_e}(T^k)^{1/k} \leq \frac{1}{\rho_e(L)}. \tag{d}$$

Furthermore, it is clear that

$$\frac{1}{\rho_e(L)} = \lim_{k\to+\infty} \frac{1}{\|L^k\|_e^{1/k}} \leq \underline{\lim_{k\to+\infty}}\, \mathsf{m_e}(T^k)^{1/k}. \tag{e}$$

Therefore, from (d) and (e), we conclude $\operatorname{dist}(0, \sigma_e(T)) = \lim_{k\to+\infty} \mathsf{m_e}(T^k)^{1/k}$. A similar proof follows for the other functions. This completes the proof. ∎

We finish this paper with some open questions.

Question 4.7. In [11], Zemánek for Hilbert space (see also [9, Theorem 3] for the separable case) proved that

$$\mathsf{m_e}(T) = \operatorname{dist}(T, \mathscr{B}(\mathsf{H})\backslash\Phi_+(\mathsf{H})), \quad \text{for all } T \in \mathscr{B}(\mathsf{H}); \tag{4.5}$$

$$\mathsf{q_e}(T) = \operatorname{dist}(T, \mathscr{B}(\mathsf{H})\backslash\Phi_-(\mathsf{H})), \quad \text{for all } T \in \mathscr{B}(\mathsf{H}). \tag{4.6}$$

We note that from Corollary 3.8, it follows that

$$\mathsf{m_e}(T) \le \operatorname{dist}(T, \mathscr{B}(\mathsf{X})\backslash\Phi_+(\mathsf{X})) \text{ and } \mathsf{q_e}(T) \le \operatorname{dist}(T, \mathscr{B}(\mathsf{H})\backslash\Phi_-(\mathsf{H})).$$

Can we prove (4.5) and (4.6) for Banach spaces?

If the answer to Question 4.7 is positive, then our definitions of the essential minimum modulus and the surjectivity modulus are equivalent to Zemánek's definitions [11, 12], so in Banach spaces, we have a new characterization of the essential minimum modulus and the surjectivity modulus in terms of the minimum and the surjectivity moduli.

Question 4.8. In Hilbert space, we know from [10, Theorem 2.2, Corollary 2.3], that

$$T \in \mathsf{S}(\mathsf{H}) + \mathscr{K}(\mathsf{H}) \iff 1 = \mathsf{m_e}(T) = \|T\|_e \text{ and } \operatorname{ind}(T) \le 0. \tag{4.7}$$

From relation (3.16) it is clear that if $T \in \mathsf{S}(\mathsf{X}) + \mathscr{K}(\mathsf{X})$, then $\mathsf{m_e}(T) = \|T\|_e = 1$ and $\operatorname{ind}(T) \le 0$. It is a natural question whether (4.7) can be extended to the case of Banach spaces.

Question 4.9. Can we prove Proposition 4.4 and Proposition 4.5, for all $T \in \mathscr{B}(\mathsf{X})$?

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