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One more refinement of Tandori's fundamental divergence theorem

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Dedicated to the memory of my honoured Masters Professors G. Alexits and K. Tandori

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Abstract. The aim of the paper is to generalize two fundamental theorems of K. Tandori. The coefficient sequences in his theorems are classical monotone decreasing. We moderate the classical monotonicity to locally almost monotonicity assumption.

1 Introduction

K. Tandori proved several very interesting theorems pertaining to the convergence and divergence of orthogonal series. We refer to the monograph of G. Alexits [1], where you find his twelve papers. The highest originality and worth of his results, in my view, appears in those theorems in which he refines D. Menšov's fundamental results. Tandori proved — roughly speaking — that if a standard sufficient condition for certain monotone coefficients does not hold, then there exists a special orthonormal system $\{\Phi_n(x)\}$ such that the orthogonal series with these monotone coefficients and functions $\Phi_n(x)$ has divergence phenomenon. In other words, he proved that the condition in question for monotone coefficients is a necessary and sufficient condition.

One of his most celebrated theorems proved in [4], a remarkable refinement of Menšov's theorem ([3]), reads as follows:

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Theorem T. *If $\{a_n\}$ is a positive monotone non-increasing sequence of numbers for which*

$$\sum_{n=2}^{\infty} a_n^2 \log^2 n = \infty, \tag{1.1}$$

then there exists a uniformly bounded orthonormal function system $\{\Phi_n(x)\}$ dependent on $\{a_n\}$ such that the orthogonal series

$$\sum_{n=0}^{\infty} a_n \Phi_n(x) \tag{1.2}$$

diverges everywhere in the interval (a, b) of orthogonality.

Remarks. 1. In [4], among others, the following natural generalization of Theorem T is proved: *If* (1.1) *holds and $a_n^* \geqq \eta a_n$, $\eta > 0$, then every series $\sum a_n^* \Phi_n(x)$ is also divergent everywhere.*

2. Tandori himself emphasized that the monotonicity assumption is crucial just if $\{a_n\}$ belongs to ℓ^2, i.e. $\sum a_n^2 < \infty$, namely if $\sum a_n^2 = \infty$ then the Rademacher series $\sum a_n r_n(x)$ diverges almost everywhere.

In a previous paper [2] we established two theorems in which the normal monotonicity hypothesis is moderated to locally almost monotonicity assumption. Unfortunately, in the theorem of that time stating divergence we could prove only a weaker statement, namely our result asserts only that the series (1.2) with (1.1) is not everywhere convergent, instead of that (1.2) diverges everywhere, as in Theorem T. This is an essential difference.

The aim of this paper is to eliminate this shortcoming.

To perform this we received a great encouragement from the reconstructed proof given by Alexits in his monograph for Theorem T.

2 Results

Our real aim is to extend Theorem T and one of its consequences in such a way that the ordinary monotonicity assumption on the sequence $\{a_n\}$ will be replaced by locally almost monotone condition.

Now we recall some definitions.

A positive sequence $\mathbf{c} := \{c_n\}$ is called *locally almost monotone non-increasing* if there exists a constant $K(\mathbf{c})$, depending on the sequence $\mathbf{c}$ only, such that for all m and $m \leqq n \leqq 2m$

$$c_n \leqq K(\mathbf{c}) c_m \tag{2.1}$$

holds. Such a sequence will be denoted by $\mathbf{c} \in$ LAMS.

Hence throughout we assume that $(a,b) \equiv (0,1)$, furthermore that the coefficient sequences belong to ℓ^2.

Theorem 1. *If* $\{a_n\} \in \mathrm{LAMS}$ *and satisfies* (1.1) *then there exists a uniformly bounded orthonormal system* $\{\Phi_n(x)\}$ *such that the series* (1.2) *is divergent everywhere in* $(0,1)$.

Combining this theorem with the classical Rademacher–Menšov theorem we obtain immediately Theorem 2, see below.

The Rademacher–Menšov theorem ([1, Theorem 2.3.2]) reads as follows:

Theorem. *If* $\{\varphi_n(x)\}$ *is an orthonormal system then the series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{2.2}$$

is convergent almost everywhere if

$$\sum_{n=2}^{\infty} c_n^2 \log^2 n < \infty. \tag{2.3}$$

This classical theorem and Theorem 1 clearly imply:

Theorem 2. *If* $\{c_n\} \in \mathrm{LAMS}$ *then condition* (2.3) *is a necessary and sufficient condition that the series* (2.2) *for every orthonormal system* $\{\varphi_n(x)\}$ *should be convergent almost everywhere.*

The proofs of Theorem 1 and the lemmas, specially (4.2) show that a slight generalization of Theorem 1 as in the case of Theorem T, clearly holds. This reads as follows:

Theorem 1*. *If* $a_n^* \geqq \eta a_n$, $\eta > 0$, $\mathbf{a} \in \mathrm{LAMS}$ *and* (1.1) *hold, then every series* $\sum a_n^* \Phi_n(x)$ *is also divergent everywhere.*

3 Lemmas

To the proof of Theorem 1 we shall use the following three lemmas.

Lemma 1. ([1, Theorem 2.4.4]). *Let* $C \geqq 1$, $p \geqq 2$ *be integers and* $I = [u,v]$ *an arbitrary interval. Then there exists in* $[u,v]$ *a system* $\{f_\ell(C,p,I;x)\}_1^{2p}$ *of step-functions* (*i.e. the interval* $[u,v]$ *can be divided into finite subintervals such that on these the functions are constant*) *orthogonal to one another with the following properties:*

1° *We have*

$$\int_u^v f_\ell^2(C,p,I;x)dx = v-u.$$

2° *For every point* x *of the half-open interval*

$$F(C,I) := \Big[u + \frac{v-u}{5}\cdot\frac{2}{C}, u+\frac{v-u}{5}\cdot\frac{3}{C}\Big)$$

there exists an index $m(x) < p$ *dependent on* x *such that the functions* $f_\ell(C,p,I;x)$, $\ell = 1,2,\dots,p+m(x)$, *are positive and the inequality*

$$\sum_{\ell=1}^{p+m(x)} f_\ell(C,p,I;x) \geqq A(Cp)^{1/2}\log p$$

holds, where A *is a positive constant, independent of* C, p, I *and* x.

Lemma 2. *Let* $\{c_n\}$ *be a positive sequence of numbers and let* $N_m = 2^{m+2}-4$ *for all natural* m. *For every value of* m *it is possible to construct in* $[0,1]$ *a measurable set* F_m *of measure*

$$|F_m| \geqq \frac{1}{10}\min\big(1, N_{m+1}c^2_{N_{m+1}}\log^2 N_{m+1}\big) \tag{3.1}$$

and a system $\{\Phi_n(x)\}$ *of orthonormal step-functions (uniformly bounded) with following properties*:

a) *The sets* $F_0, F_1, \dots$ *are stochastically independent, i.e. for every sequence of indices* $k_1 < k_2 < \cdots < k_s$ *the relation*

$$|F_{k_1}\cap F_{k_2}\cap\cdots\cap F_{k_s}| = |F_{k_1}|\cdot|F_{k_2}|\cdots|F_{k_s}|$$

holds.

b) *For every* $x \in F_m$ *there exists an index* $n_{m(x)} < 2^{m+2}-1$ *such that* $\Phi_{N_m}(x),\dots,\Phi_{N_m+n_m(x)}(x)$ *have equal signs and*

$$\big|\Phi_{N_m}(x)+\cdots+\Phi_{N_m+n_m(x)}(x)\big| \geqq \frac{B}{c_{N_{m+1}}}, \tag{3.2}$$

where B *is a positive constant, independent of* x *and* m.

Lemma 2 is proved in the monograph of Alexits as Theorem 2.4.5, namely the assumption that $\{c_n\}$ is monotone decreasing was nowhere utilized in the proof.

In the proof of Tandori's Hilfsatz III in [4] Lemma 2 is also verified implicitly, but it appears in a more hidden form.

Alexits proficiently separated the statements of Hilfssatz III pertaining to the construction of the functions and sets, thus it becomes clear that only in his Theorem 2.4.6 (here the next Lemma 3) is used the monotonicity assumption combining with condition (1.1). The other parts of the proofs do not utilize the monotonicity of the coefficients.

We shall prove that the assumption $\{c_n\} \in \mathrm{LAMS}$ with (1.1) is also sufficient to the assertion of Alexits's Theorem 2.4.6, herewith we arrive at Lemma 3.

Lemma 3. *If $\{c_n\} \in \mathrm{LAMS}$ satisfies the condition $\sum_{n=2}^{\infty} c_n^2 \log^2 n = \infty$, then the set*

$$F := \bigcap_{k=1}^{\infty} \Big(\bigcup_{\ell=k}^{\infty} F_\ell \Big) = \overline{\lim_{m\to\infty}} F_m$$

is of measure $|F| = 1$, where F_k are from Lemma 2.

Proof of Lemma 3. Denote by CG the complement of an arbitrary set $G \subset [0,1]$ to $[0,1]$. Then we can represent the union set $F_k \cup F_{k+1} \cup \cdots \cup F_s$ as follows:

$$\bigcup_{\ell=k}^{s} F_\ell = [0,1] - C \bigcup_{\ell=k}^{s} F_\ell = [0,1] - \bigcap_{\ell=k}^{s} CF_\ell.$$

In view of the stochastic independence of the sets F_ℓ we obtain that

$$\Big| \bigcup_{\ell=k}^{s} F_\ell \Big| = 1 - \prod_{\ell=k}^{s} (1 - |F_\ell|). \tag{3.3}$$

Next we use that $\{c_n\} \in \mathrm{LAMS}$. Then for every m and $m \leqq n \leqq 4m$ holds that

$$c_n \leqq K(\mathbf{c})^2 c_m,$$

using this and the fact that $N_{k+1} \leqq 4N_k$, we obtain that

$$\begin{aligned} \sum_{n=N_1}^{N_m} c_n^2 \log^2 n &\leqq \sum_{k=1}^{m} \sum_{n=N_k}^{N_{k+1}-1} c_n^2 \log^2 n \leqq K(\mathbf{c})^2 \sum_{k=1}^{m} (N_{k+1} - N_k) c_{N_k}^2 \log^2 N_{k+1} \\ &\leqq 2K(\mathbf{c})^2 \sum_{k=1}^{m} 2^{k+2} c_{N_k}^2 \log^2 N_k \leqq 4K(\mathbf{c})^2 \sum_{k=1}^{m} N_k c_{N_k}^2 \log^2 N_k. \end{aligned} \tag{3.4}$$

Since it is assumed that $\sum\limits_{n=1}^{\infty} c_n^2 \log^2 n = \infty$, (3.4) implies that

$$\sum_{k=1}^{\infty} N_k c_{N_k}^2 \log^2 N_k = \infty,$$

whence, by (3.1),

$$\sum_{m=1}^{\infty} |F_m| = \infty \tag{3.5}$$

also holds. Next, following Alexits's reasoning, by an elementary theorem concerning infinite products and (3.5), we get that

$$\prod_{\ell=k}^{\infty} (1 - |F_\ell|) = 0,$$

thus (3.3) implies that

$$\Big| \bigcup_{\ell=k}^{\infty} F_\ell \Big| = 1 \qquad (k = 0, 1, \ldots),$$

and so

$$|F| = 1 \tag{3.6}$$

also follows.

This completes the proof. ■

4 Proof of Theorem 1

By Lemma 2 with a_n in place of c_n we have an orthonormal system $\{\Phi_n(x)\}$ and sets F_m for which at every point $x \in F_m$ there exists an index $n_m(x) < 2^{m+2}$ such that (3.2) holds. By the definition of the set F if $x \in F$ then $x \in F_m$ also holds for infinitely many m, consequently (3.2) is also satisfied for infinitely many m. Since the functions being in (3.2) have equal signs, thus

$$\begin{aligned} &|a_{N_m} \Phi_{N_m}(x) + \cdots + a_{N_m+n_m(x)} \Phi_{N_m+n_m(x)}(x)| \\ &\qquad \geqq \min_{N_m \leqq n \leqq N_m+n_m(x)} a_n |\Phi_{N_m}(x) + \cdots + \Phi_{N_m+n_m(x)}(x)| \\ &\qquad \geqq \min_{N_m \leqq n \leqq N_{m+1}} a_n \frac{B}{a_{N_{m+1}}}. \end{aligned} \tag{4.1}$$

Since $\{a_n\} \in$ LAMS implies that $a_n \geqq K^{-2}(\mathbf{a}) a_{N_{m+1}}$ for any $N_m \leqq n \leqq N_{m+1}$, thus (4.1) verifies that

$$|a_{N_m} \Phi_{N_m}(x) + \cdots + a_{N_m+n_m(x)} \Phi_{N_m+n_m(x)}(x)| \geqq \frac{B}{K(\mathbf{a})^2} > 0 \tag{4.2}$$

holds at every $x \in F$ for infinitely many m; whence, on account of Lemma 3, the series (1.2) is divergent almost everywhere.

In order to furnish that (1.2) should diverge everywhere, we only need to alter the values $\Phi(x)$ in a set of measure zero (putting $\Phi_n(x) = \infty$) where (1.2) is eventually convergent, thus the divergence is achieved everywhere, in accordance with our statement. ■

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