Stochastic Gravity: Theory and Applications

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Abstract

Whereas semiclassical gravity is based on the semiclassical Einstein equation with sources given by the expectation value of the stress-energy tensor of quantum fields, stochastic semiclassical gravity is based on the Einstein–Langevin equation, which has in addition sources due to the noise kernel. The noise kernel is the vacuum expectation value of the (operator-valued) stress-energy bi-tensor which describes the fluctuations of quantum matter fields in curved spacetimes. In the first part, we describe the fundamentals of this new theory via two approaches: the axiomatic and the functional. The axiomatic approach is useful to see the structure of the theory from the framework of semiclassical gravity, showing the link from the mean value of the stress-energy tensor to their correlation functions. The functional approach uses the Feynman–Vernon influence functional and the Schwinger–Keldysh closed-time-path effective action methods which are convenient for computations. It also brings out the open systems concepts and the statistical and stochastic contents of the theory such as dissipation, fluctuations, noise, and decoherence. We then focus on the properties of the stress-energy bi-tensor. We obtain a general expression for the noise kernel of a quantum field defined at two distinct points in an arbitrary curved spacetime as products of covariant derivatives of the quantum field's Green function. In the second part, we describe three applications of stochastic gravity theory. First, we consider metric perturbations in a Minkowski spacetime. We offer an analytical solution of the Einstein–Langevin equation and compute the two-point correlation functions for the linearized Einstein tensor and for the metric perturbations. Second, we discuss structure formation from the stochastic gravity viewpoint, which can go beyond the standard treatment by incorporating the full quantum effect of the inflaton fluctuations. Third, we discuss the backreaction of Hawking radiation in the gravitational background of a quasi-static black hole (enclosed in a box). We derive a fluctuation-dissipation relation between the fluctuations in the radiation and the dissipative dynamics of metric fluctuations.

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Contents

1 Overview

Stochastic semiclassical gravity[1] is a theory developed in the 1990s using semiclassical gravity (quantum fields in classical spacetimes, solved self-consistently) as the starting point and aiming at a theory of quantum gravity as the goal. While semiclassical gravity is based on the semiclassical Einstein equation with the source given by the expectation value of the stress-energy tensor of quantum fields, stochastic gravity includes also its fluctuations in a new stochastic semiclassical or the Einstein–Langevin equation. If the centerpiece in semiclassical gravity theory is the vacuum expectation value of the stress-energy tensor of a quantum field, and the central issues being how well the vacuum is defined and how the divergences can be controlled by regularization and renormalization, the centerpiece in stochastic semiclassical gravity theory is the stress-energy bi-tensor and its expectation value known as the noise kernel. The mathematical properties of this quantity and its physical content in relation to the behavior of fluctuations of quantum fields in curved spacetimes are the central issues of this new theory. How they induce metric fluctuations and seed the structures of the universe, how they affect the black hole horizons and the backreaction of Hawking radiance in black hole dynamics, including implications on trans-Planckian physics, are new horizons to explore. On the theoretical issues, stochastic gravity is the necessary foundation to investigate the validity of semiclassical gravity and the viability of inflationary cosmology based on the appearance and sustenance of a vacuum energy-dominated phase. It is also a useful beachhead supported by well-established low energy (sub-Planckian) physics to explore the connection with high energy (Planckian) physics in the realm of quantum gravity.

In this review we summarize major work on and results of this theory since 1998. It is in the nature of a progress report rather than a review. In fact we will have room only to discuss a handful of topics of basic importance. A review of ideas leading to stochastic gravity and further developments originating from it can be found in [149, 154]; a set of lectures which include a discussion of the issue of the validity of semiclassical gravity in [168]; a pedagogical introduction of stochastic gravity theory with a more complete treatment of backreaction problems in cosmology and black holes in [169]. A comprehensive formal description of the fundamentals is given in [207, 208] while that of the noise kernel in arbitrary spacetimes in [208, 244, 245]. Here we will try to mention all related work so the reader can at least trace out the parallel and sequential developments. The references at the end of each topic below are representative work where one can seek out further treatments.

Stochastic gravity theory is built on three pillars: general relativity, quantum fields, and nonequilibrium statistical mechanics. The first two uphold semiclassical gravity, the last two span statistical field theory. Strictly speaking one can understand a great deal without appealing to statistical mechanics, and we will try to do so here. But concepts such as quantum open systems [71, 200, 291] and techniques such as the influence functional [89, 88] (which is related to the closed-time-path effective action [257, 11, 184, 66, 272, 41, 70, 76, 181, 39, 182, 236]) were a great help in our understanding of the physical meaning of issues involved toward the construction of this new theory, foremost because quantum fluctuations and correlation have become the focus. Quantum statistical field theory and the statistical mechanics of quantum field theory [40, 42, 44, 46] also aided us in searching for the connection with quantum gravity through the retrieval of correlations and coherence. We show the scope of stochastic gravity as follows:

[1] We will often use the shortened term *stochastic gravity* as there is no confusion as to the nature and source of stochasticity in gravity being induced from the quantum fields and not a priori from the classical spacetime.

1 Ingredients:

(a) From general relativity [215, 285] to semiclassical gravity.

(b) Quantum field theory in curved spacetimes [25, 100, 286, 113]:

i. Stress-energy tensor: Regularization and renormalization.

ii. Self-consistent solution: Backreaction problems [203, 115, 158, 159, 124, 3, 4].

iii. Effective action: Closed time path, initial value formulation [257, 11, 184, 66, 272, 41, 70, 76, 181, 39, 182, 236].

iv. Equation of motion: Real and causal.

(c) Nonequilibrium statistical mechanics:

i. Open quantum systems [71, 200, 291].

ii. Influence functional: Stochastic equations [89].

iii. Noise and decoherence: Quantum to classical transition [303, 304, 305, 306, 180, 33, 279, 307, 109, 114, 221, 222, 223, 224, 225, 226, 105, 125, 83, 120, 122, 30, 239, 278, 170, 171, 172, 121, 81, 82, 185, 186, 187, 173].

(d) Decoherence in quantum cosmology and emergence of classical spacetimes [188, 119, 228, 150, 36, 37, 160].

2 Theory:

(a) Dissipation from particle creation [76, 181, 39, 182, 236, 57];
backreaction as fluctuation-dissipation relation (FDR) [167].

(b) Noise from fluctuations of quantum fields [149, 151, 43].

(c) Einstein–Langevin equations [43, 157, 167, 58, 59, 38, 202, 207, 208, 206].

(d) Metric fluctuations in Minkowski spacetime [209].

3 Issues:

(a) Validity of semiclassical gravity [163, 243].

(b) Viability of vacuum dominance and inflationary cosmology.

(c) Stress-energy bi-tensor and noise kernel: Regularization reassessed [244, 245].

4 Applications: Early universe and black holes:

(a) Wave propagation in stochastic geometry [166].

(b) Black hole horizon fluctuations: Spontaneous/active versus induced/passive [94, 294, 267, 268, 14, 15, 211, 232, 245].

(c) Noise induced inflation [50].

(d) Structure formation [45, 213, 212, 51, 254];
trace anomaly-driven inflation [269, 280, 132].

(e) Black hole backreaction as FDR [60, 258, 259, 217, 164, 54, 55, 264].

5 Related Topics:

(a) Metric fluctuations and trans-Planckian problem [14, 15, 211, 232, 219].

(b) Spacetime fam [62, 63, 101, 102, 103].

(c) Universal 'metric conductance' fluctuations [261].

6 Ideas:

(a) General relativity as geometro-hydrodynamics [146].

(b) Semiclassical gravity as mesoscopic physics [153].

(c) From stochastic to quantum gravity:

i. Via correlation hierarchy of interacting quantum fields [154, 42, 46, 155].

ii. Possible relation to string theory and matrix theory.

We list only the latest work in the respective topics above describing ongoing research. The reader should consult the references therein for earlier work and the background material. We do not seek a complete coverage here, but will discuss only the selected topics in theory, issues, and applications. We use the $(+,+,+)$ sign conventions of [215, 285], and units in which $c = \hbar = 1$.

2 From Semiclassical to Stochastic Gravity

There are three main steps that lead to the recent development of stochastic gravity. The first step begins with *quantum field theory in curved spacetime* [75, 25, 100, 286, 113], which describes the behavior of quantum matter fields propagating in a specified (not dynamically determined by the quantum matter field as source) background gravitational field. In this theory the gravitational field is given by the classical spacetime metric determined from classical sources by the classical Einstein equations, and the quantum fields propagate as test fields in such a spacetime. An important process described by quantum field theory in curved spacetime is indeed particle creation from the vacuum, and effects of vacuum fluctuations and polarizations, in the early universe [234, 260, 300, 301, 147, 21, 22, 23, 75, 96, 65], and Hawking radiation in black holes [130, 131, 174, 235, 282].

The second step in the description of the interaction of gravity with quantum fields is backreaction, i.e., the effect of the quantum fields on the spacetime geometry. The source here is the expectation value of the stress-energy operator for the matter fields in some quantum state in the spacetime, a classical observable. However, since this object is quadratic in the field operators, which are only well defined as distributions on the spacetime, it involves ill defined quantities. It contains ultraviolet divergences, the removal of which requires a renormalization procedure [75, 67, 68]. The final expectation value of the stress-energy operator using a reasonable regularization technique is essentially unique, modulo some terms which depend on the spacetime curvature and which are independent of the quantum state. This uniqueness was proved by Wald [283, 284] who investigated the criteria that a physically meaningful expectation value of the stress-energy tensor ought to satisfy.

The theory obtained from a self-consistent solution of the geometry of the spacetime and the quantum field is known as *semiclassical gravity*. Incorporating the backreaction of the quantum matter field on the spacetime is thus the central task in semiclassical gravity. One assumes a general class of spacetime where the quantum fields live in and act on, and seek a solution which satisfies simultaneously the Einstein equation for the spacetime and the field equations for the quantum fields. The Einstein equation which has the expectation value of the stress-energy operator of the quantum matter field as the source is known as the *semiclassical Einstein equation*. Semiclassical gravity was first investigated in cosmological backreaction problems [203, 115, 158, 159, 124, 3, 4, 123, 90, 129]; an example is the damping of anisotropy in Bianchi universes by the backreaction of vacuum particle creation. Using the effect of quantum field processes such as particle creation to explain why the universe is so isotropic at the present was investigated in the context of chaotic cosmology [214, 19, 20] in the late 1970s prior to the inflationary cosmology proposal of the 1980s [117, 2, 197, 198], which assumes the vacuum expectation value of an inflaton field as the source, another, perhaps more well-known, example of semiclassical gravity.

2.1 The importance of quantum fluctuations

For a free quantum field, semiclassical gravity is fairly well understood. The theory is in some sense unique, since the only reasonable c-number stress-energy tensor that one may construct [283, 284] with the stress-energy operator is a renormalized expectation value. However, the scope and limitations of the theory are not so well understood. It is expected that the semiclassical theory would break down at the Planck scale. One can conceivably assume that it would also break down when the fluctuations of the stress-energy operator are large [92, 194]. Calculations of the fluctuations of the energy density for Minkowski, Casimir and hot flat spaces as well as Einstein and de Sitter universes are available [194, 242, 163, 243, 244, 245, 241, 208, 209, 251, 254, 227, 69]. It is less clear, however, how to quantify what a large fluctuation is, and different criteria have been proposed [194, 93, 95, 163, 243, 9, 10]. The issue of the validity of the semiclassical gravity viewed in the light of quantum fluctuations is summarized in our Erice lectures [168]. One can see the essence of the problem by the following example inspired by Ford [92].

Let us assume a quantum state formed by an isolated system which consists of a superposition with equal amplitude of one configuration of mass M with the center of mass at X_1, and another configuration of the same mass with the center of mass at X_2. The semiclassical theory as described by the semiclassical Einstein equation predicts that the center of mass of the gravitational field of the system is centered at $\frac{1}{2}(X_1 + X_2)$. However, one would expect that if we send a succession of test particles to probe the gravitational field of the above system, half of the time they would react to a gravitational field of mass M centered at X_1 and half of the time to the field centered at X_2. The two predictions are clearly different; note that the fluctuation in the position of the center of masses is of the order of $(X_1 - X_2)^2$. Although this example raises the issue of how to place the importance of fluctuations to the mean, a word of caution should be added to the effect that it should not be taken too literally. In fact, if the previous masses are macroscopic, the quantum system decoheres very quickly [306, 307] and instead of being described by a pure quantum state it is described by a density matrix which diagonalizes in a certain pointer basis. For observables associated to such a pointer basis, the density matrix description is equivalent to that provided by a statistical ensemble. The results will differ, in any case, from the semiclassical prediction.

In other words, one would expect that a stochastic source that describes the quantum fluctuations should enter into the semiclassical equations. A significant step in this direction was made in [149], where it was proposed to view the backreaction problem in the framework of an open quantum system: the quantum fields seen as the "environment" and the gravitational field as the "system". Following this proposal a systematic study of the connection between semiclassical gravity and open quantum systems resulted in the development of a new conceptual and technical framework where (semiclassical) Einstein–Langevin equations were derived [43, 157, 167, 58, 59, 38, 202]. The key technical factor to most of these results was the use of the influence functional method of Feynman and Vernon [89], when only the coarse-grained effect of the environment on the system is of interest. Note that the word semiclassical put in parentheses refers to the fact that the noise source in the Einstein–Langevin equation arises from the quantum field, while the background spacetime is classical; generally we will not carry this word since there is no confusion that the source which contributes to the stochastic features of this theory comes from quantum fields.

In the language of the consistent histories formulation of quantum mechanics [114, 221, 222, 223, 224, 225, 226, 105, 125, 83, 120, 122, 30, 239, 278, 170, 171, 172, 121, 81, 82, 185, 186, 187, 173] for the existence of a semiclassical regime for the dynamics of the system, one needs two requirements: The first is decoherence, which guarantees that probabilities can be consistently assigned to histories describing the evolution of the system, and the second is that these probabilities should peak near histories which correspond to solutions of classical equations of motion. The effect of the environment is crucial, on the one hand, to provide decoherence and, on the other hand, to produce both dissipation and noise to the system through backreaction, thus inducing a semiclassical stochastic dynamics on the system. As shown by different authors [106, 303, 304, 305, 306, 180, 33, 279, 307, 109], indeed over a long history predating the current revival of decoherence, stochastic semiclassical equations are obtained in an open quantum system after a coarse graining of the environmental degrees of freedom and a further coarse graining in the system variables. It is expected but has not yet been shown that this mechanism could also work for decoherence and classicalization of the metric field. Thus far, the analogy could be made formally [206] or under certain assumptions, such as adopting the Born–Oppenheimer approximation in quantum cosmology [237, 238].

An alternative axiomatic approach to the Einstein–Langevin equation without invoking the open system paradigm was later suggested, based on the formulation of a self-consistent dynamical equation for a perturbative extension of semiclassical gravity able to account for the lowest order stress-energy fluctuations of matter fields [207]. It was shown that the same equation could be derived, in this general case, from the influence functional of Feynman and Vernon [208]. The field equation is deduced via an effective action which is computed assuming that the gravitational field is a c-number. The important new element in the derivation of the Einstein–Langevin equation, and of the stochastic gravity theory, is the physical observable that measures the stress-energy fluctuations, namely, the expectation value of the symmetrized bi-tensor constructed with the stress-energy tensor operator: the *noise kernel*. It is interesting to note that the Einstein–Langevin equation can also be understood as a useful intermediary tool to compute symmetrized two-point correlations of the quantum metric perturbations on the semiclassical background, independent of a suitable classicalization mechanism [255].

3 The Einstein–Langevin Equation: Axiomatic Approach

In this section we introduce *stochastic semiclassical gravity*, or *stochastic gravity* for short, in an axiomatic way. It is introduced as an extension of semiclassical gravity motivated by the search of self-consistent equations which describe the backreaction of the quantum stress-energy fluctuations on the gravitational field [207].

3.1 Semiclassical gravity

Semiclassical gravity describes the interaction of a classical gravitational field with quantum matter fields. This theory can be formally derived as the leading $1/N$ approximation of quantum gravity interacting with N independent and identical free quantum fields [142, 143, 128, 277] which interact with gravity only. By keeping the value of NG finite, where G is Newton's gravitational constant, one arrives at a theory in which formally the gravitational field can be treated as a c-number field (i.e. quantized at tree level) while matter fields are fully quantized. The semiclassical theory may be summarized as follows.

Let $(\mathcal{M}, g_{ab})$ be a globally hyperbolic four-dimensional spacetime manifold $\mathcal{M}$ with metric g_{ab}, and consider a real scalar quantum field ϕ of mass m propagating on that manifold; we just assume a scalar field for simplicity. The classical action $S_{\rm m}$ for this matter field is given by the functional

$$S_{\rm m}[g,\phi] = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{ab}\nabla_a\phi\nabla_b\phi + \left(m^2+\xi R\right)\phi^2\right], \tag{1}$$

where ∇_a is the covariant derivative associated to the metric g_{ab}, ξ is a coupling parameter between the field and the scalar curvature of the underlying spacetime R, and $g = \det g_{ab}$.

The field may be quantized in the manifold using the standard canonical quantization formalism [25, 100, 286]. The field operator in the Heisenberg representation $\hat{\phi}$ is an operator-valued distribution solution of the Klein–Gordon equation, the field equation derived from Equation (1),

$$(\Box - m^2 - \xi R)\hat{\phi} = 0. \tag{2}$$

We may write the field operator as $\hat{\phi}[g;x)$ to indicate that it is a functional of the metric g_{ab} and a function of the spacetime point x. This notation will be used also for other operators and tensors.

The classical stress-energy tensor is obtained by functional derivation of this action in the usual way, $T^{ab}(x) = (2/\sqrt{-g})\,\delta S_{\rm m}/\delta g_{ab}$, leading to

$$T^{ab}[g,\phi] = \nabla^a\phi\nabla^b\phi - \frac{1}{2}g^{ab}\left(\nabla^c\phi\nabla_c\phi + m^2\phi^2\right) + \xi\left(g^{ab}\Box - \nabla^a\nabla^b + G^{ab}\right)\phi^2, \tag{3}$$

where $\Box = \nabla_a\nabla^a$, and G_{ab} is the Einstein tensor. With the notation $T^{ab}[g,\phi]$ we explicitly indicate that the stress-energy tensor is a functional of the metric g_{ab} and the field ϕ.

The next step is to define a stress-energy tensor operator $\hat{T}^{ab}[g;x)$. Naively one would replace the classical field $\phi[g;x)$ in the above functional by the quantum operator $\hat{\phi}[g;x)$, but this procedure involves taking the product of two distributions at the same spacetime point. This is ill-defined and we need a regularization procedure. There are several regularization methods which one may use; one is the point-splitting or point-separation regularization method [67, 68], in which one introduces a point y in a neighborhood of the point x and then uses as the regulator the vector tangent at the point x of the geodesic joining x and y; this method is discussed for instance in [243, 244, 245] and in Section 5. Another well known method is dimensional regularization in which one works in arbitrary n dimensions, where n is not necessarily an integer, and then uses as the regulator the parameter $\epsilon = n - 4$; this method is implicitly used in this section. The regularized stress-energy operator using the Weyl ordering prescription, i.e. symmetrical ordering, can be written as

$$\hat{T}^{ab}[g] = \frac{1}{2}\{\nabla^a\hat{\phi}[g], \nabla^b\hat{\phi}[g]\} + \mathcal{D}^{ab}[g]\,\hat{\phi}^2[g], \tag{4}$$

where $\mathcal{D}^{ab}[g]$ is the differential operator

$$\mathcal{D}^{ab} \equiv (\xi - 1/4)\, g^{ab}\Box + \xi \left(R^{ab} - \nabla^a\nabla^b\right). \tag{5}$$

Note that if dimensional regularization is used, the field operator $\hat{\phi}[g;x)$ propagates in an n-dimensional spacetime. Once the regularization prescription has been introduced, a regularized and renormalized stress-energy operator $\hat{T}^{\rm R}_{ab}[g;x)$ may be defined as

$$\hat{T}^{\rm R}_{ab}[g;x) = \hat{T}_{ab}[g;x) + F^{C}_{ab}[g;x)\hat{I}, \tag{6}$$

which differs from the regularized $\hat{T}_{ab}[g;x)$ by the identity operator times some tensor counterterms $F^{C}_{ab}[g;x)$, which depend on the regulator and are local functionals of the metric (see [208] for details). The field states can be chosen in such a way that for any pair of physically acceptable states (i.e., Hadamard states in the sense of [286]), $|\psi\rangle$ and $|\varphi\rangle$, the matrix element $\langle\psi|T^{\rm R}_{ab}|\varphi\rangle$, defined as the limit when the regulator takes the physical value is finite and satisfies Wald's axioms [100, 283]. These counterterms can be extracted from the singular part of a Schwinger–DeWitt series [100, 67, 68, 31]. The choice of these counterterms is not unique, but this ambiguity can be absorbed into the renormalized coupling constants which appear in the equations of motion for the gravitational field.

The *semiclassical Einstein equation* for the metric g_{ab} can then be written as

$$G_{ab}[g] + \Lambda g_{ab} - 2(\alpha A_{ab} + \beta B_{ab})[g] = 8\pi G\langle\hat{T}^{\rm R}_{ab}[g]\rangle, \tag{7}$$

where $\langle\hat{T}^{\rm R}_{ab}[g]\rangle$ is the expectation value of the operator $\hat{T}^{\rm R}_{ab}[g,x)$ after the regulator takes the physical value in some physically acceptable state of the field on $(\mathcal{M}, g_{ab})$. Note that both the stress tensor and the quantum state are functionals of the metric, hence the notation. The parameters G, Λ, α, and β are respectively the renormalized coupling constants, the gravitational constant, the cosmological constant, and two dimensionless coupling constants which are zero in the classical Einstein equation. These constants must be understood as the result of "dressing" the bare constants which appear in the classical action before renormalization. The values of these constants must be determined by experiment. The left-hand side of Equation (7) may be derived from the gravitational action

$$S_{\rm g}[g] = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\left[\frac{1}{2}R - \Lambda + \alpha C_{abcd}C^{abcd} + \beta R^2\right], \tag{8}$$

where C_{abcd} is the Weyl tensor. The tensors A_{ab} and B_{ab} come from the functional derivatives with respect to the metric of the terms quadratic in the curvature in Equation (8); they are explicitly given by

$$\begin{aligned} A^{ab} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}\int d^4\sqrt{-g}C_{cdef}C^{cdef} \\ &= \frac{1}{2}g^{ab}C_{cdef}C^{cdef} - 2R^{acde}R^{b}{}_{cde} + 4R^{ac}R_{c}{}^{b} - \frac{2}{3}RR^{ab} - 2\Box R^{ab} + \frac{2}{3}\nabla^a\nabla^b R + \frac{1}{3}g^{ab}\Box R, \end{aligned} \tag{9}$$

$$\begin{aligned} B^{ab} &= \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}\int d^4\sqrt{-g}R^2 \\ &= \frac{1}{2}g^{ab}R^2 - 2RR^{ab} + 2\nabla^a\nabla^b R - 2g^{ab}\Box R, \end{aligned} \tag{10}$$

where R_{abcd} and R_{ab} are the Riemann and Ricci tensors, respectively. These two tensors are, like the Einstein and metric tensors, symmetric and divergenceless: $\nabla^a A_{ab} = 0 = \nabla^a B_{ab}$.

A solution of semiclassical gravity consists of a spacetime $(\mathcal{M}, g_{ab})$, a quantum field operator $\hat{\phi}[g]$ which satisfies the evolution equation (2), and a physically acceptable state $|\psi[g]\rangle$ for this field, such that Equation (7) is satisfied when the expectation value of the renormalized stress-energy operator is evaluated in this state.

For a free quantum field this theory is robust in the sense that it is self-consistent and fairly well understood. As long as the gravitational field is assumed to be described by a classical metric, the above semiclassical Einstein equations seems to be the only plausible dynamical equation for this metric: The metric couples to matter fields via the stress-energy tensor, and for a given quantum state the only physically observable c-number stress-energy tensor that one can construct is the above renormalized expectation value. However, lacking a full quantum gravity theory, the scope and limits of the theory are not so well understood. It is assumed that the semiclassical theory should break down at Planck scales, which is when simple order of magnitude estimates suggest that the quantum effects of gravity should not be ignored, because the energy of a quantum fluctuation in a Planck size region, as determined by the Heisenberg uncertainty principle, is comparable to the gravitational energy of that fluctuation.

The theory is expected to break down when the fluctuations of the stress-energy operator are large [92]. A criterion based on the ratio of the fluctuations to the mean was proposed by Kuo and Ford [194] (see also work via zeta-function methods [242, 69]). This proposal was questioned by Phillips and Hu [163, 243, 244] because it does not contain a scale at which the theory is probed or how accurately the theory can be resolved. They suggested the use of a smearing scale or point-separation distance for integrating over the bi-tensor quantities, equivalent to a stipulation of the resolution level of measurements; see also the response by Ford [93, 95]. A different criterion is recently suggested by Anderson et al. [9, 10] based on linear response theory. A partial summary of this issue can be found in our Erice Lectures [168].

3.2 Stochastic gravity

The purpose of stochastic gravity is to extend the semiclassical theory to account for these fluctuations in a self-consistent way. A physical observable that describes these fluctuations to lowest order is the *noise kernel* bi-tensor, which is defined through the two point correlation of the stress-energy operator as

$$N_{abcd}[g;x,y]=\frac{1}{2}\langle\{\hat{t}_{ab}[g;x),\hat{t}_{cd}[g;y)\}\rangle, \tag{11}$$

where the curly brackets mean anticommutator, and where

$$\hat{t}_{ab}[g;x)\equiv\hat{T}_{ab}[g;x)-\langle\hat{T}_{ab}[g;x)\rangle. \tag{12}$$

This bi-tensor can also be written as $N_{ab,c'd'}[g;x,y)$, or $N_{ab,c'd'}(x,y)$ as we do in Section 5, to emphasize that it is a tensor with respect to the first two indices at the point x and a tensor with respect to the last two indices at the point y, but we shall not follow this notation here. The noise kernel is defined in terms of the unrenormalized stress-tensor operator $\hat{T}_{ab}[g;x)$ on a given background metric g_{ab}, thus a regulator is implicitly assumed on the right-hand side of Equation (11). However, for a linear quantum field the above kernel – the expectation function of a bi-tensor – is free of ultraviolet divergences because the regularized $T_{ab}[g;x)$ differs from the renormalized $T^{\rm R}_{ab}[g;x)$ by the identity operator times some tensor counterterms (see Equation (6)), so that in the subtraction (12) the counterterms cancel. Consequently the ultraviolet behavior of $\langle\hat{T}_{ab}(x)\hat{T}_{cd}(y)\rangle$ is the same as that of $\langle\hat{T}_{ab}(x)\rangle\langle\hat{T}_{cd}(y)\rangle$, and $\hat{T}_{ab}$ can be replaced by the renormalized operator $\hat{T}^{\rm R}_{ab}$ in Equation (11); an alternative proof of this result is given in [244, 245]. The noise kernel should be thought of as a distribution function; the limit of coincidence points has meaning only in the sense of distributions. The bi-tensor $N_{abcd}[g;x,y)$, or $N_{abcd}(x,y)$ for short, is real and positive semi-definite, as a consequence of $\hat{T}^{\rm R}_{ab}$ being self-adjoint. A simple proof is given in [169].

Once the fluctuations of the stress-energy operator have been characterized, we can perturbatively extend the semiclassical theory to account for such fluctuations. Thus we will assume that the background spacetime metric g_{ab} is a solution of the semiclassical Einstein Equations (7), and we will write the new metric for the extended theory as $g_{ab}+h_{ab}$, where we will assume that h_{ab} is a perturbation to the background solution. The renormalized stress-energy operator and the state of the quantum field may now be denoted by $\hat{T}^{\mathrm{R}}_{ab}[g+h]$ and $|\psi[g+h]\rangle$, respectively, and $\langle \hat{T}^{\mathrm{R}}_{ab}[g+h]\rangle$ will be the corresponding expectation value.

Let us now introduce a Gaussian stochastic tensor field $\xi_{ab}[g;x)$ defined by the following correlators:

$$\langle \xi_{ab}[g;x)\rangle_{\mathrm{s}} = 0, \qquad \langle \xi_{ab}[g;x)\xi_{cd}[g;y)\rangle_{\mathrm{s}} = N_{abcd}[g;x,y), \tag{13}$$

where $\langle \dots \rangle_{\mathrm{s}}$ means statistical average. The symmetry and positive semi-definite property of the noise kernel guarantees that the stochastic field tensor $\xi_{ab}[g,x)$, or $\xi_{ab}(x)$ for short, just introduced is well defined. Note that this stochastic tensor captures only partially the quantum nature of the fluctuations of the stress-energy operator since it assumes that cumulants of higher order are zero.

An important property of this stochastic tensor is that it is covariantly conserved in the background spacetime, $\nabla^a \xi_{ab}[g;x) = 0$. In fact, as a consequence of the conservation of $\hat{T}^{\mathrm{R}}_{ab}[g]$ one can see that $\nabla^a_x N_{abcd}(x,y) = 0$. Taking the divergence in Equation (13) one can then show that $\langle \nabla^a \xi_{ab}\rangle_{\mathrm{s}} = 0$ and $\langle \nabla^a_x \xi_{ab}(x)\xi_{cd}(y)\rangle_{\mathrm{s}} = 0$, so that $\nabla^a \xi_{ab}$ is deterministic and represents with certainty the zero vector field in $\mathcal{M}$.

For a conformal field, i.e., a field whose classical action is conformally invariant, ξ_{ab} is traceless: $g^{ab}\xi_{ab}[g;x) = 0$; thus, for a conformal matter field the stochastic source gives no correction to the trace anomaly. In fact, from the trace anomaly result which states that $g^{ab}\hat{T}^{\mathrm{R}}_{ab}[g]$ is, in this case, a local c-number functional of g_{ab} times the identity operator, we have that $g^{ab}(x)N_{abcd}[g;x,y) = 0$. It then follows from Equation (13) that $\langle g^{ab}\xi_{ab}\rangle_{\mathrm{s}} = 0$ and $\langle g^{ab}(x)\xi_{ab}(x)\xi_{cd}(y)\rangle_{\mathrm{s}} = 0$; an alternative proof based on the point-separation method is given in [244, 245] (see also Section 5).

All these properties make it quite natural to incorporate into the Einstein equations the stress-energy fluctuations by using the stochastic tensor $\xi_{ab}[g;x)$ as the source of the metric perturbations. Thus we will write the following equation:

$$G_{ab}[g+h] + \Lambda(g_{ab}+h_{ab}) - 2(\alpha A_{ab} + \beta B_{ab})[g+h] = 8\pi G \left(\langle \hat{T}^{\mathrm{R}}_{ab}[g+h]\rangle + \xi_{ab}[g] \right). \tag{14}$$

This equation is in the form of a *(semiclassical) Einstein–Langevin equation*; it is a dynamical equation for the metric perturbation h_{ab} to linear order. It describes the backreaction of the metric to the quantum fluctuations of the stress-energy tensor of matter fields, and gives a first order extension to semiclassical gravity as described by the semiclassical Einstein equation (7).

Note that we refer to the Einstein–Langevin equation as a first order extension to the semiclassical Einstein equation of semiclassical gravity and the lowest level representation of stochastic gravity. However, stochastic gravity has a much broader meaning, as it refers to the range of theories based on second and higher order correlation functions. Noise can be defined in effectively open systems (e.g., correlation noise [46] in the Schwinger–Dyson equation hierarchy) to some degree, but one should not expect the Langevin form to prevail. In this sense we say that stochastic gravity is the intermediate theory between semiclassical gravity (a mean field theory based on the expectation values of the energy-momentum tensor of quantum fields) and quantum gravity (the full hierarchy of correlation functions retaining complete quantum coherence [154, 155]).

The renormalization of the operator $\hat{T}_{ab}[g+h]$ is carried out exactly as in the previous case, now in the perturbed metric $g_{ab}+h_{ab}$. Note that the stochastic source $\xi_{ab}[g;x)$ is not dynamical; it is independent of h_{ab} since it describes the fluctuations of the stress tensor on the semiclassical background g_{ab}.

An important property of the Einstein–Langevin equation is that it is gauge invariant under the change of h_{ab} by $h'_{ab} = h_{ab} + \nabla_a \zeta_b + \nabla_b \zeta_a$, where ζ^a is a stochastic vector field on the background manifold $\mathcal{M}$. Note that a tensor such as $R_{ab}[g+h]$ transforms as $R_{ab}[g+h'] = R_{ab}[g+h] + \mathcal{L}_\zeta R_{ab}[g]$ to linear order in the perturbations, where $\mathcal{L}_\zeta$ is the Lie derivative with respect to ζ^a. Now, let us write the source tensors in Equations (14) and (7) to the left-hand sides of these equations. If we substitute h by h' in this new version of Equation (14), we get the same expression, with h instead of h', plus the Lie derivative of the combination of tensors which appear on the left-hand side of the new Equation (7). This last combination vanishes when Equation (7) is satisfied, i.e., when the background metric g_{ab} is a solution of semiclassical gravity.

A solution of Equation (14) can be formally written as $h_{ab}[\xi]$. This solution is characterized by the whole family of its correlation functions. From the statistical average of this equation we have that $g_{ab} + \langle h_{ab} \rangle_{\mathrm{s}}$ must be a solution of the semiclassical Einstein equation linearized around the background g_{ab}; this solution has been proposed as a test for the validity of the semiclassical approximation [9, 10]. The fluctuations of the metric around this average are described by the moments of the stochastic field $h^{\mathrm{s}}_{ab}[\xi] = h_{ab}[\xi] - \langle h_{ab} \rangle_{\mathrm{s}}$. Thus the solutions of the Einstein–Langevin equation will provide the two-point metric correlation functions $\langle h^{\mathrm{s}}_{ab}(x) h^{\mathrm{s}}_{cd}(y) \rangle_{\mathrm{s}}$.

We see that whereas the semiclassical theory depends on the expectation value of the point-defined value of the stress-energy operator, the stochastic theory carries information also on the two point correlation of the stress-energy operator. We should also emphasize that, even if the metric fluctuations appears classical and stochastic, their origin is quantum not only because they are induced by the fluctuations of quantum matter, but also because they are the suitably coarse-grained variables left over from the quantum gravity fluctuations after some mechanism for decoherence and classicalization of the metric field [106, 126, 83, 120, 122, 293]. One may, in fact, derive the stochastic semiclassical theory from a full quantum theory. This was done via the world-line influence functional method for a moving charged particle in an electromagnetic field in quantum electrodynamics [178]. From another viewpoint, quite independent of whether a classicalization mechanism is mandatory or implementable, the Einstein–Langevin equation proves to be a useful tool to compute the symmetrized two point correlations of the quantum metric perturbations [255]. This is illustrated in the linear toy model discussed in [169], which has features of some quantum Brownian models [49, 47, 48].

4 The Einstein–Langevin Equation: Functional Approach

The Einstein–Langevin equation (14) may also be derived by a method based on functional techniques [208]. Here we will summarize these techniques starting with semiclassical gravity.

In semiclassical gravity functional methods were used to study the backreaction of quantum fields in cosmological models [123, 90, 129]. The primary advantage of the effective action approach is, in addition to the well-known fact that it is easy to introduce perturbation schemes like loop expansion and nPI formalisms, that it yields a *fully* self-consistent solution. For a general discussion in the semiclassical context of these two approaches, equation of motion versus effective action, see, e.g., the work of Hu and Parker (1978) versus Hartle and Hu (1979) in [203, 115, 158, 159, 124, 3, 4]. See also comments in Sec. 5.6 of [169] on the black hole backreaction problem comparing the approach by York et al. [297, 298, 299] versus that of Sinha, Raval, and Hu [264].

The well known in-out effective action method treated in textbooks, however, led to equations of motion which were not real because they were tailored to compute transition elements of quantum operators rather than expectation values. The correct technique to use for the backreaction problem is the Schwinger–Keldysh closed-time-path (CTP) or 'in-in' effective action [257, 11, 184, 66, 272, 41, 70]. These techniques were adapted to the gravitational context [76, 181, 39, 182, 236, 57] and applied to different problems in cosmology. One could deduce the semiclassical Einstein equation from the CTP effective action for the gravitational field (at tree level) with quantum matter fields.

Furthermore, in this case the CTP functional formalism turns out to be related [272, 43, 58, 201, 112, 54, 55, 216, 196, 208, 206] to the influence functional formalism of Feynman and Vernon [89], since the full quantum system may be understood as consisting of a distinguished subsystem (the "system" of interest) interacting with the remaining degrees of freedom (the environment). Integrating out the environment variables in a CTP path integral yields the influence functional, from which one can define an effective action for the dynamics of the system [43, 167, 156, 112]. This approach to semiclassical gravity is motivated by the observation [149] that in some open quantum systems classicalization and decoherence [303, 304, 305, 306, 180, 33, 279, 307, 109] on the system may be brought about by interaction with an environment, the environment being in this case the matter fields and some "high-momentum" gravitational modes [188, 119, 228, 150, 36, 37, 160, 293]. Unfortunately, since the form of a complete quantum theory of gravity interacting with matter is unknown, we do not know what these "high-momentum" gravitational modes are. Such a fundamental quantum theory might not even be a field theory, in which case the metric and scalar fields would not be fundamental objects [154]. Thus, in this case, we cannot attempt to evaluate the influence action of Feynman and Vernon starting from the fundamental quantum theory and performing the path integrations in the environment variables. Instead, we introduce the influence action for an effective quantum field theory of gravity and matter [79, 78, 77, 80, 263, 237, 238], in which such "high-momentum" gravitational modes are assumed to have already been "integrated out."

4.1 Influence action for semiclassical gravity

Let us formulate semiclassical gravity in this functional framework. Adopting the usual procedure of effective field theories [289, 290, 79, 78, 77, 80, 52], one has to take the effective action for the metric and the scalar field of the most general local form compatible with general covariance: $S[g, \phi] \equiv S_{\rm g}[g] + S_{\rm m}[g, \phi] + \dots$, where $S_{\rm g}[g]$ and $S_{\rm m}[g, \phi]$ are given by Equations (8) and (1), respectively, and the dots stand for terms of order higher than two in the curvature and in the number of derivatives of the scalar field. Here, we shall neglect the higher order terms as well as self-interaction terms for the scalar field. The second order terms are necessary to renormalize one-loop ultraviolet divergences of the scalar field stress-energy tensor, as we have already seen. Since $\mathcal{M}$ is a globally hyperbolic manifold, we can foliate it by a family of $t =$ const. Cauchy hypersurfaces Σ_t, and we will indicate the initial and final times by t_i and t_f, respectively.

The *influence functional* corresponding to the action (1) describing a scalar field in a spacetime (coupled to a metric field) may be introduced as a functional of two copies of the metric, denoted by g^+_{ab} and g^-_{ab}, which coincide at some final time $t = t_\mathrm{f}$. Let us assume that, in the quantum effective theory, the state of the full system (the scalar and the metric fields) in the Schrödinger picture at the initial time $t = t_\mathrm{i}$ can be described by a density operator which can be written as the tensor product of two operators on the Hilbert spaces of the metric and of the scalar field. Let $\rho_\mathrm{i}(t_\mathrm{i}) \equiv \rho_\mathrm{i}[\phi_+(t_\mathrm{i}), \phi_-(t_\mathrm{i})]$ be the matrix element of the density operator $\hat{\rho}^\mathrm{S}(t_\mathrm{i})$ describing the initial state of the scalar field. The Feynman–Vernon influence functional is defined as the following path integral over the two copies of the scalar field:

$$\mathcal{F}_\mathrm{IF}[g^\pm] \equiv \int \mathcal{D}\phi_+ \,\mathcal{D}\phi_- \;\rho_\mathrm{i}(t_\mathrm{i})\delta\left[\phi_+(t_\mathrm{f}) - \phi_-(t_\mathrm{f})\right] e^{i\left(S_\mathrm{m}[g^+,\phi_+]-S_\mathrm{m}[g^-,\phi_-]\right)}. \tag{15}$$

Alternatively, the above double path integral can be rewritten as a CTP integral, namely, as a single path integral in a complex time contour with two different time branches, one going forward in time from t_i to t_f, and the other going backward in time from t_f to t_i (in practice one usually takes $t_\mathrm{i} \to -\infty$). From this influence functional, the *influence action* $S_\mathrm{IF}[g^+, g^-]$, or $S_\mathrm{IF}[g^\pm]$ for short, defined by

$$\mathcal{F}_\mathrm{IF}[g^\pm] \equiv e^{iS_\mathrm{IF}[g^\pm]}, \tag{16}$$

carries all the information about the environment (the matter fields) relevant to the system (the gravitational field). Then we can define the CTP *effective action* for the gravitational field, $S_\mathrm{eff}[g^\pm]$, as

$$S_\mathrm{eff}[g^\pm] \equiv S_\mathrm{g}[g^+] - S_\mathrm{g}[g^-] + S_\mathrm{IF}[g^\pm]. \tag{17}$$

This is the effective action for the classical gravitational field in the CTP formalism. However, since the gravitational field is treated only at the tree level, this is also the effective classical action from which the classical equations of motion can be derived.

Expression (15) contains ultraviolet divergences and must be regularized. We shall assume that dimensional regularization can be applied, that is, it makes sense to dimensionally continue all the quantities that appear in Equation (15). For this we need to work with the n-dimensional actions corresponding to S_m in Equation (15) and S_g in Equation (8). For example, the parameters G, Λ, α, and β of Equation (8) are the bare parameters G_B, Λ_B, α_B, and β_B, and in $S_\mathrm{g}[g]$, instead of the square of the Weyl tensor in Equation (8), one must use $\frac{2}{3}(R_{abcd}R^{abcd} - R_{ab}R^{ab})$, which by the Gauss–Bonnet theorem leads to the same equations of motion as the action (8) when $n = 4$. The form of S_g in n dimensions is suggested by the Schwinger–DeWitt analysis of the ultraviolet divergences in the matter stress-energy tensor using dimensional regularization. One can then write the Feynman–Vernon effective action $S_\mathrm{eff}[g^\pm]$ in Equation (17) in a form suitable for dimensional regularization. Since both S_m and S_g contain second order derivatives of the metric, one should also add some boundary terms [285, 167]. The effect of these terms is to cancel out the boundary terms which appear when taking variations of $S_\mathrm{eff}[g^\pm]$ keeping the value of g^+_{ab} and g^-_{ab} fixed at Σ_{t_i} and Σ_{t_f}. Alternatively, in order to obtain the equations of motion for the metric in the semiclassical regime, we can work with the action terms without boundary terms and neglect all boundary terms when taking variations with respect to $g^\pm_{ab}$. From now on, all the functional derivatives with respect to the metric will be understood in this sense.

The semiclassical Einstein equation (7) can now be derived. Using the definition of the stress-energy tensor $T^{ab}(x) = (2/\sqrt{-g})\delta S_\mathrm{m}/\delta g_{ab}$ and the definition of the influence functional, Equations (15) and (16), we see that

$$\langle \hat{T}^{ab}[g;x]\rangle = \frac{2}{\sqrt{-g(x)}}\left.\frac{\delta S_\mathrm{IF}[g^\pm]}{\delta g^+_{ab}(x)}\right|_{g^\pm=g}, \tag{18}$$

where the expectation value is taken in the n-dimensional spacetime generalization of the state described by $\hat{\rho}^{\rm S}(t_{\rm i})$. Therefore, differentiating $S_{\rm eff}[g^{\pm}]$ in Equation (17) with respect to g^{+}_{ab}, and then setting $g^{+}_{ab} = g^{-}_{ab} = g_{ab}$, we get the semiclassical Einstein equation in n dimensions. This equation is then renormalized by absorbing the divergences in the regularized $\langle \hat{T}^{ab}[g]\rangle$ into the bare parameters. Taking the limit $n \to 4$ we obtain the physical semiclassical Einstein equation (7).

4.2 Influence action for stochastic gravity

In the spirit of the previous derivation of the Einstein–Langevin equation, we now seek a dynamical equation for a linear perturbation h_{ab} to the semiclassical metric g_{ab}, solution of Equation (7). Strictly speaking, if we use dimensional regularization we must consider the n-dimensional version of that equation. From the results just described, if such an equation were simply a linearized semiclassical Einstein equation, it could be obtained from an expansion of the effective action $S_{\rm eff}[g + h^{\pm}]$. In particular, since, from Equation (18), we have that

$$\langle \hat{T}^{ab}[g+h;x]\rangle = \frac{2}{\sqrt{-\det{(g+h)(x)}}} \left. \frac{\delta S_{\rm IF}[g+h^{\pm}]}{\delta h^{+}_{ab}(x)} \right|_{h^{\pm}=h}, \qquad (19)$$

the expansion of $\langle \hat{T}^{ab}[g + h]\rangle$ to linear order in h_{ab} can be obtained from an expansion of the influence action $S_{\rm IF}[g + h^{\pm}]$ up to second order in $h^{\pm}_{ab}$.

To perform the expansion of the influence action, we have to compute the first and second order functional derivatives of $S_{\rm IF}[g + h^{\pm}]$ and then set $h^{+}_{ab} = h^{-}_{ab} = h_{ab}$. If we do so using the path integral representation (15), we can interpret these derivatives as expectation values of operators. The relevant second order derivatives are

$$\begin{aligned} \frac{4}{\sqrt{-g(x)}\sqrt{-g(y)}} \left. \frac{\delta^2 S_{\rm IF}[g+h^{\pm}]}{\delta h^{+}_{ab}(x)\delta h^{+}_{cd}(y)} \right|_{h^{\pm}=h} &= -H^{abcd}_{\rm S}[g;x,y] - K^{abcd}[g;x,y] + iN^{abcd}[g;x,y], \\ \frac{4}{\sqrt{-g(x)}\sqrt{-g(y)}} \left. \frac{\delta^2 S_{\rm IF}[g^{\pm}]}{\delta h^{+}_{ab}(x)\delta h^{-}_{cd}(y)} \right|_{h^{\pm}=h} &= -H^{abcd}_{\rm A}[g;x,y] - iN^{abcd}[g;x,y], \end{aligned} \qquad (20)$$

where

$$\begin{aligned} N^{abcd}[g;x,y] &\equiv \frac{1}{2}\left\langle \left\{ \hat{t}^{ab}[g;x], \hat{t}^{cd}[g;y] \right\} \right\rangle, \\ H^{abcd}_{\rm S}[g;x,y] &\equiv {\rm Im} \left\langle {\rm T}^{*} \left(\hat{T}^{ab}[g;x]\hat{T}^{cd}[g;y] \right) \right\rangle, \\ H^{abcd}_{\rm A}[g;x,y] &\equiv -\frac{i}{2} \left\langle \left[\hat{T}^{ab}[g;x],\, \hat{T}^{cd}[g;y] \right] \right\rangle, \\ K^{abcd}[g;x,y] &\equiv \frac{-4}{\sqrt{-g(x)}\sqrt{-g(y)}} \left\langle \left. \frac{\delta^2 S_{\rm m}[g+h,\phi]}{\delta h_{ab}(x)\delta h_{cd}(y)} \right|_{\phi=\hat{\phi}} \right\rangle, \end{aligned}$$

with $\hat{t}^{ab}$ defined in Equation (12); [,] denotes the commutator and { , } the anti-commutator. Here we use a Weyl ordering prescription for the operators. The symbol ${\rm T}^{*}$ denotes the following ordered operations: First, time order the field operators $\hat{\phi}$ and then apply the derivative operators which appear in each term of the product $T^{ab}(x)T^{cd}(y)$, where T^{ab} is the functional (3). This ${\rm T}^{*}$ "time ordering" arises because we have path integrals containing products of derivatives of the field, which can be expressed as derivatives of the path integrals which do not contain such derivatives. Notice, from their definitions, that all the kernels which appear in expressions (20) are real and also $H^{abcd}_{\rm A}$ is free of ultraviolet divergences in the limit $n \to 4$.

From Equation (18) and (20), since $S_{\rm IF}[g,g]=0$ and $S_{\rm IF}[g^-,g^+]=-S^*_{\rm IF}[g^+,g^-]$, we can write the expansion for the influence action $S_{\rm IF}[g+h^\pm]$ around a background metric g_{ab} in terms of the previous kernels. Taking into account that these kernels satisfy the symmetry relations

$$H_{\rm S}^{abcd}(x,y)=H_{\rm S}^{cdab}(y,x), \qquad H_{\rm A}^{abcd}(x,y)=-H_{\rm A}^{cdab}(y,x), \qquad K^{abcd}(x,y)=K^{cdab}(y,x), \quad (21)$$

and introducing the new kernel

$$H^{abcd}(x,y)\equiv H_{\rm S}^{abcd}(x,y)+H_{\rm A}^{abcd}(x,y), \quad (22)$$

the expansion of $S_{\rm IF}$ can be finally written as

$$\begin{aligned} S_{\rm IF}[g+h^\pm] = & \frac{1}{2}\int d^4x\,\sqrt{-g(x)}\langle \hat{T}^{ab}[g;x]\rangle\,[h_{ab}(x)] \\ & -\frac{1}{8}\int d^4x\,d^4y\,\sqrt{-g(x)}\sqrt{-g(y)}\,[h_{ab}(x)]\left(H^{abcd}[g;x,y]+K^{abcd}[g;x,y]\right)\{h_{cd}(y)\} \\ & +\frac{i}{8}\int d^4x\,d^4y\,\sqrt{-g(x)}\sqrt{-g(y)}\,[h_{ab}(x)]\,N^{abcd}[g;x,y]\,[h_{cd}(y)]+\mathcal{O}(h^3), \end{aligned} \quad (23)$$

where we have used the notation

$$[h_{ab}]\equiv h^+_{ab}-h^-_{ab}, \qquad \{h_{ab}\}\equiv h^+_{ab}+h^-_{ab}. \quad (24)$$

From Equations (23) and (19) it is clear that the imaginary part of the influence action does not contribute to the perturbed semiclassical Einstein equation (the expectation value of the stress-energy tensor is real), however, as it depends on the noise kernel, it contains information on the fluctuations of the operator $\hat{T}^{ab}[g]$.

We are now in a position to carry out the derivation of the semiclassical Einstein–Langevin equation. The procedure is well known [43, 167, 58, 110, 26, 296, 246]: It consists of deriving a new "stochastic" effective action from the observation that the effect of the imaginary part of the influence action (23) on the corresponding influence functional is equivalent to the averaged effect of the stochastic source ξ^{ab} coupled linearly to the perturbations $h^\pm_{ab}$. This observation follows from the identity first invoked by Feynman and Vernon for such purpose:

$$\begin{aligned} \exp\left(-\frac{1}{8}\int d^4x\,d^4y\,\sqrt{-g(x)}\sqrt{-g(y)}\,[h_{ab}(x)]\,N^{abcd}(x,y)\,[h_{cd}(y)]\right) = \\ \int \mathcal{D}\xi\mathcal{P}[\xi]\exp\left(\frac{i}{2}\int d^4x\sqrt{-g(x)}\xi^{ab}(x)\,[h_{ab}(x)]\right), \end{aligned} \quad (25)$$

where $\mathcal{P}[\xi]$ is the probability distribution functional of a Gaussian stochastic tensor ξ^{ab} characterized by the correlators (13) with N^{abcd} given by Equation (11), and where the path integration measure is assumed to be a scalar under diffeomorphisms of $(\mathcal{M},g_{ab})$. The above identity follows from the identification of the right-hand side of Equation (25) with the characteristic functional for the stochastic field ξ^{ab}. The probability distribution functional for ξ^{ab} is explicitly given by

$$\mathcal{P}[\xi]=\det(2\pi N)^{-1/2}\exp\left[-\frac{1}{2}\int d^4x d^4y\sqrt{-g(x)}\sqrt{-g(y)}\,\xi^{ab}(x)N^{-1}_{abcd}(x,y)\xi^{cd}(y)\right]. \quad (26)$$

We may now introduce the *stochastic effective action* as

$$S^{\rm s}_{\rm eff}[g+h^\pm,\xi]\equiv S_{\rm g}[g+h^+]-S_{\rm g}[g+h^-]+S^{\rm s}_{\rm IF}[g+h^\pm,\xi], \quad (27)$$

where the "stochastic" influence action is defined as

$$S^{\rm s}_{\rm IF}[g+h^\pm,\xi]\equiv {\rm Re}\,S_{\rm IF}[g+h^\pm]+\frac{1}{2}\int d^4x\,\sqrt{-g(x)}\,\xi^{ab}(x)\,[h_{ab}(x)]+\mathcal{O}(h^3). \quad (28)$$

Note that, in fact, the influence functional can now be written as a statistical average over ξ^{ab}:

$$\mathcal{F}_{\rm IF}[g+h^\pm] = \left\langle \exp\left(iS^{\rm s}_{\rm IF}[g+h^\pm,\xi]\right)\right\rangle_{\rm s}.$$

The stochastic equation of motion for h_{ab} reads

$$\left.\frac{\delta S^{\rm s}_{\rm eff}[g+h^\pm,\xi]}{\delta h^+_{ab}(x)}\right|_{h^\pm=h} = 0, \tag{29}$$

which is the Einstein–Langevin equation (14); notice that only the real part of $S_{\rm IF}$ contributes to the expectation value (19). To be precise, we get first the regularized n-dimensional equations with the bare parameters, with the tensor A^{ab} replaced by $\frac{2}{3}D^{ab}$, where

$$\begin{aligned} D^{ab} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}}\int d^nx\,\sqrt{-g}\left(R_{cdef}R^{cdef} - R_{cd}R^{cd}\right) \\ &= \frac{g^{ab}}{2}\left(R_{cdef}R^{cdef} - R_{cd}R^{cd} + \Box R\right) - 2R^{acde}{R^b}_{cde} - 2R^{acbd}R_{cd} + 4R^{ac}{R_c}^b - 3\Box R^{ab} + \nabla^a\nabla^b R. \end{aligned} \tag{30}$$

Of course, when $n=4$ these tensors are related, $A^{ab} = \frac{2}{3}D^{ab}$. After that we renormalize and take the limit $n\to 4$ to obtain the Einstein–Langevin equations in the physical spacetime.

4.3 Explicit form of the Einstein–Langevin equation

We can write the Einstein–Langevin equation in a more explicit form by working out the expansion of $\langle \hat{T}^{ab}[g+h]\rangle$ up to linear order in the perturbation h_{ab}. From Equation (19), we see that this expansion can be easily obtained from Equation (23). The result is

$$\langle \hat{T}^{ab}_n[g+h;x]\rangle = \langle \hat{T}^{ab}_n[g,x]\rangle + \langle \hat{T}^{(1)ab}_n[g,h;x]\rangle - \frac{1}{2}\int d^ny\,\sqrt{-g(y)}H^{abcd}_n[g;x,y]h_{cd}(y) + \mathcal{O}(h^2). \tag{31}$$

Here we use a subscript n on a given tensor to indicate that we are explicitly working in n dimensions, as we use dimensional regularization, and we also use the superindex $^{(1)}$ to generally indicate that the tensor is the first order correction, linear in h_{ab}, in a perturbative expansion around the background g_{ab}.

Using the Klein–Gordon equation (2), and expressions (3) for the stress-energy tensor and the corresponding operator, we can write

$$\hat{T}^{(1)ab}_n[g,h] = \left(\frac{1}{2}g^{ab}h_{cd} - \delta^a_c h^b_d - \delta^b_c h^a_d\right)\hat{T}^{cd}_n[g] + \mathcal{F}^{ab}[g,h]\,\hat{\phi}^2_n[g], \tag{32}$$

where $\mathcal{F}^{ab}[g;h]$ is the differential operator

$$\begin{aligned} \mathcal{F}^{ab} \equiv &\left(\xi - \frac{1}{4}\right)\left(h^{ab} - \frac{1}{2}g^{ab}h^c_c\right)\Box \\ &+\frac{\xi}{2}\left[\nabla^c\nabla^a h^b_c + \nabla^c\nabla^b h^a_c - \Box h^{ab} - \nabla^a\nabla^b h^c_c - g^{ab}\nabla^c\nabla^d h_{cd} + g^{ab}\Box h^c_c\right. \\ &\qquad \left. + \left(\nabla^a h^b_c + \nabla^b h^a_c - \nabla_c h^{ab} - 2g^{ab}\nabla^d h_{cd} + g^{ab}\nabla_c h^d_d\right)\nabla^c - g^{ab}h_{cd}\nabla^c\nabla^d\right]. \end{aligned} \tag{33}$$

It is understood that indices are raised with the background inverse metric g^{ab}, and that all the covariant derivatives are associated to the metric g_{ab}.

Substituting Equation (31) into the n-dimensional version of the Einstein–Langevin Equation (14), taking into account that g_{ab} satisfies the semiclassical Einstein equation (7), and substituting expression (32), we can write the Einstein–Langevin equation in dimensional regularization as

$$\begin{aligned} &\frac{1}{8\pi G_{\rm B}}\left[G^{(1)ab}-\frac{1}{2}\,g^{ab}G^{cd}h_{cd}+G^{ac}h^{b}_{c}+G^{bc}h^{a}_{c}+\Lambda_{\rm B}\left(h^{ab}-\frac{1}{2}g^{ab}h^{c}_{c}\right)\right]\\ &-\frac{4\alpha_{\rm B}}{3}\left(D^{(1)ab}-\frac{1}{2}g^{ab}D^{cd}h_{cd}+D^{ac}h^{b}_{c}+D^{bc}h^{a}_{c}\right)\\ &-2\beta_{\rm B}\left(B^{(1)ab}-\frac{1}{2}g^{ab}B^{cd}h_{cd}+B^{ac}h^{b}_{c}+B^{bc}h^{a}_{c}\right)\\ &-\mu^{-(n-4)}\mathcal{F}^{ab}_{x}\langle\hat{\phi}^{2}_{n}[g;x)\rangle+\frac{1}{2}\int d^{n}y\sqrt{-g(y)}\,\mu^{-(n-4)}H^{abcd}_{n}[g;x,y)h_{cd}(y)\\ =\;&\mu^{-(n-4)}\xi^{ab}_{n}, \end{aligned} \tag{34}$$

where the tensors G^{ab}, D^{ab}, and B^{ab} are computed from the semiclassical metric g_{ab}, and where we have omitted the functional dependence on g_{ab} and h_{ab} in $G^{(1)ab}$, $D^{(1)ab}$, $B^{(1)ab}$, and $\mathcal{F}^{ab}$ to simplify the notation. The parameter μ is a mass scale which relates the dimensions of the physical field ϕ with the dimensions of the corresponding field in n dimensions, $\phi_n = \mu^{(n-4)/2}\phi$. Notice that, in Equation (34), all the ultraviolet divergences in the limit $n \to 4$, which must be removed by renormalization of the coupling constants, are in $\langle\hat{\phi}^2_n(x)\rangle$ and the symmetric part $H^{abcd}_{{\rm S}_n}(x,y)$ of the kernel $H^{abcd}_n(x,y)$, whereas the kernels $N^{abcd}_n(x,y)$ and $H^{abcd}_{{\rm A}_n}(x,y)$ are free of ultraviolet divergences. If we introduce the bi-tensor $F^{abcd}_n[g;x,y)$ defined by

$$F^{abcd}_{n}[g;x,y)\equiv\left\langle\hat{t}^{ab}_{n}[g;x)\,\hat{t}^{\rho\sigma}_{n}[g;y)\right\rangle, \tag{35}$$

where $\hat{t}^{ab}$ is defined by Equation (12), then the kernels N and $H_{\rm A}$ can be written as

$$N^{abcd}_{n}[g;x,y)={\rm Re}\,F^{abcd}_{n}[g;x,y),\qquad H^{abcd}_{{\rm A}_n}[g;x,y)={\rm Im}\,F^{abcd}_{n}[g;x,y), \tag{36}$$

where we have used that

$$2\langle\hat{t}^{ab}(x)\,\hat{t}^{cd}(y)\rangle=\langle\{\hat{t}^{ab}(x),\hat{t}^{cd}(y)\}\rangle+\langle[\hat{t}^{ab}(x),\hat{t}^{cd}(y)]\rangle,$$

and the fact that the first term on the right-hand side of this identity is real, whereas the second one is pure imaginary. Once we perform the renormalization procedure in Equation (34), setting $n = 4$ will yield the physical Einstein–Langevin equation. Due to the presence of the kernel $H^{abcd}_n(x,y)$, this equation will be usually non-local in the metric perturbation. In Section 6 we will carry out an explicit evaluation of the physical Einstein–Langevin equation which will illustrate the procedure.

4.3.1 The kernels for the vacuum state

When the expectation values in the Einstein–Langevin equation are taken in a vacuum state $|0\rangle$, such as, for instance, an "in" vacuum, we can be more explicit, since we can write the expectation values in terms of the Wightman and Feynman functions, defined as

$$G^{+}_{n}[g;x,y)\equiv\langle 0|\hat{\phi}_{n}[g;x)\hat{\phi}_{n}[g;y)|0\rangle,\qquad iG_{F_n}[g;x,y)\equiv\langle 0|{\rm T}\left(\hat{\phi}_{n}[g;x)\hat{\phi}_{n}[g;y)\right)|0\rangle. \tag{37}$$

These expressions for the kernels in the Einstein–Langevin equation will be very useful for explicit calculations. To simplify the notation, we omit the functional dependence on the semiclassical metric g_{ab}, which will be understood in all the expressions below.

From Equations (36), we see that the kernels $N_n^{abcd}(x,y)$ and $H_{\mathrm{A}_n}^{abcd}(x,y)$ are the real and imaginary parts, respectively, of the bi-tensor $F_n^{abcd}(x,y)$. From the expression (4) we see that the stress-energy operator $\hat{T}_n^{ab}$ can be written as a sum of terms of the form $\left\{\mathcal{A}_x\hat{\phi}_n(x),\,\mathcal{B}_x\hat{\phi}_n(x)\right\}$, where $\mathcal{A}_x$ and $\mathcal{B}_x$ are some differential operators. It then follows that we can express the bi-tensor $F_n^{abcd}(x,y)$ in terms of the Wightman function as

$$
\begin{aligned}
F_n^{abcd}(x,y) = {} & \nabla_x^a\nabla_y^c G_n^+(x,y)\nabla_x^b\nabla_y^d G_n^+(x,y) + \nabla_x^a\nabla_y^d G_n^+(x,y)\nabla_x^b\nabla_y^c G_n^+(x,y) \\
& + 2\mathcal{D}_x^{ab}\left(\nabla_y^c G_n^+(x,y)\nabla_y^d G_n^+(x,y)\right) + 2\mathcal{D}_y^{cd}\left(\nabla_x^a G_n^+(x,y)\nabla_x^b G_n^+(x,y)\right) \\
& + 2\mathcal{D}_x^{ab}\mathcal{D}_y^{cd}\left(G_n^{+2}(x,y)\right),
\end{aligned}
\tag{38}
$$

where $\mathcal{D}_x^{ab}$ is the differential operator (5). From this expression and the relations (36), we get expressions for the kernels N_n and H_{A_n} in terms of the Wightman function $G_n^+(x,y)$.

Similarly the kernel $H_{\mathrm{S}_n}^{abcd}(x,y)$ can be written in terms of the Feynman function as

$$
\begin{aligned}
H_{\mathrm{S}_n}^{abcd}(x,y) = -\mathrm{Im}\Big[& \nabla_x^a\nabla_y^c G_{F_n}(x,y)\nabla_x^b\nabla_y^d G_{F_n}(x,y) + \nabla_x^a\nabla_y^d G_{F_n}(x,y)\nabla_x^b\nabla_y^c G_{F_n}(x,y) \\
& - g^{ab}(x)\nabla_x^e\nabla_y^c G_{F_n}(x,y)\nabla_e^x\nabla_y^d G_{F_n}(x,y) \\
& - g^{cd}(y)\nabla_x^a\nabla_y^e G_{F_n}(x,y)\nabla_x^b\nabla_e^y G_{F_n}(x,y) \\
& + \frac{1}{2}g^{ab}(x)g^{cd}(y)\nabla_x^e\nabla_y^f G_{F_n}(x,y)\nabla_e^x\nabla_f^y G_{F_n}(x,y) \\
& + \mathcal{K}_x^{ab}\left(2\nabla_y^c G_{F_n}(x,y)\nabla_y^d G_{F_n}(x,y) - g^{cd}(y)\nabla_y^e G_{F_n}(x,y)\nabla_e^y G_{F_n}(x,y)\right) \\
& + \mathcal{K}_y^{cd}\left(2\nabla_x^a G_{F_n}(x,y)\nabla_x^b G_{F_n}(x,y) - g^{ab}(x)\nabla_x^e G_{F_n}(x,y)\nabla_e^x G_{F_n}(x,y)\right) \\
& + 2\mathcal{K}_x^{ab}\mathcal{K}_y^{cd}\left(G_{F_n}^2(x,y)\right)\Big],
\end{aligned}
\tag{39}
$$

where $\mathcal{K}_x^{ab}$ is the differential operator

$$
\mathcal{K}_x^{ab} \equiv \xi\left(g^{ab}(x)\Box_x - \nabla_x^a\nabla_x^b + G^{ab}(x)\right) - \frac{1}{2}m^2 g^{ab}(x). \tag{40}
$$

Note that, in the vacuum state $|0\rangle$, the term $\langle\hat{\phi}_n^2(x)\rangle$ in Equation (34) can also be written as $\langle\hat{\phi}_n^2(x)\rangle = iG_{F_n}(x,x) = G_n^+(x,x)$.

Finally, the causality of the Einstein–Langevin equation (34) can be explicitly seen as follows. The non-local terms in that equation are due to the kernel $H(x,y)$ which is defined in Equation (22) as the sum of $H_{\mathrm{S}}(x,y)$ and $H_{\mathrm{A}}(x,y)$. Now, when the points x and y are spacelike separated, $\hat{\phi}_n(x)$ and $\hat{\phi}_n(y)$ commute and, thus, $G_n^+(x,y) = iG_{F_n}(x,y) = \frac{1}{2}\langle 0|\{\hat{\phi}_n(x),\hat{\phi}_n(y)\}|0\rangle$, which is real. Hence, from the above expressions, we have that $H_{\mathrm{A}_n}^{abcd}(x,y) = H_{\mathrm{S}_n}^{abcd}(x,y) = 0$, and thus $H_n^{abcd}(x,y) = 0$. This fact is expected since, from the causality of the expectation value of the stress-energy operator [283], we know that the non-local dependence on the metric perturbation in the Einstein–Langevin equation, see Equation (14), must be causal. See [169] for an alternative proof of the causal nature of the Einstein–Langevin equation.

5 Noise Kernel and Point-Separation

In this section we explore further the properties of the noise kernel and the stress-energy bi-tensor. Similar to what was done for the stress-energy tensor it is desirable to relate the noise kernel defined at separated points to the Green function of a quantum field. We pointed out earlier [154] that field quantities defined at two separated points may possess important information which could be the starting point for probes into possible extended structures of spacetime. Of more practical concern is how one can define a finite quantity at one point or in some small region around it from the noise kernel defined at two separated points. When we refer to, say, the fluctuations of energy density in ordinary (point-wise) quantum field theory, we are in actuality asking such a question. This is essential for addressing fundamental issues like

- the validity of semiclassical gravity [194] – e.g., whether the fluctuations to mean ratio is a correct criterion [163, 243, 93, 95, 9, 10];
- whether the fluctuations in the vacuum energy density which drives some models of inflationary cosmology violates the positive energy condition;
- physical effects of black hole horizon fluctuations and Hawking radiation backreaction – to begin with, is the fluctuations finite or infinite?
- general relativity as a low energy effective theory in the geometro-hydrodynamic limit towards a kinetic theory approach to quantum gravity [146, 154, 155].

Thus, for comparison with ordinary phenomena at low energy we need to find a reasonable prescription for obtaining a finite quantity of the noise kernel in the limit of ordinary (point-defined) quantum field theory. Regularization schemes used in obtaining a finite expression for the stress-energy tensor have been applied to the noise kernel[2]. This includes the simple normal ordering [194, 295] and smeared field operator [243] methods applied to the Minkowski and Casimir spaces, zeta-function [87, 189, 53] for spacetimes with an Euclidean section, applied to the Casimir effect [69] and the Einstein Universe [242], or the covariant point-separation methods applied to the Minkowski [243], hot flat space and the Schwarzschild spacetime [245]. There are differences and deliberations on whether it is meaningful to seek a point-wise expression for the noise kernel, and if so what is the correct way to proceed – e.g., regularization by a subtraction scheme or by integrating over a test-field. Intuitively the smear field method [243] may better preserve the integrity of the noise kernel as it provides a sampling of the two point function rather than using a subtraction scheme which alters its innate properties by forcing a nonlocal quantity into a local one. More investigation is needed to clarify these points, which bear on important issues like the validity of semiclassical gravity. We shall set a more modest goal here, to derive a general expression for the noise kernel for quantum fields in an arbitrary curved spacetime in terms of Green functions and leave the discussion of point-wise limit to a later date. For this purpose the covariant point-separation method which highlights the bi-tensor features, when used not as a regularization scheme, is perhaps closest to the spirit of stochastic gravity.

The task of finding a general expression of the noise-kernel for quantum fields in curved spacetimes was carried out by Phillips and Hu in two papers using the “modified” point separation

[2] It is well-known that several regularization methods can work equally well for the removal of ultraviolet divergences in the stress-energy tensor of quantum fields in curved spacetime. Their mutual relations are known, and discrepancies explained. This formal structure of regularization schemes for quantum fields in curved spacetime should remain intact when applied to the regularization of the noise kernel in general curved spacetimes; it is the meaning and relevance of regularization of the noise kernel which is more of a concern (see comments below). Specific considerations will of course enter for each method. But for the methods employed so far, such as zeta-function, point separation, dimensional, smeared-field, applied to simple cases (Casimir, Einstein, thermal fields) there is no new inconsistency or discrepancy.

scheme [282, 1, 284]. Their first paper [244] begins with a discussion of the procedures for dealing with the quantum stress tensor bi-operator at two separated points, and ends with a general expression of the noise kernel defined at separated points expressed as products of covariant derivatives up to the fourth order of the quantum field's Green function. (The stress tensor involves up to two covariant derivatives.) This result holds for $x \neq y$ without recourse to renormalization of the Green function, showing that $N_{abc'd'}(x,y)$ is always finite for $x \neq y$ (and off the light cone for massless theories). In particular, for a massless conformally coupled free scalar field on a four dimensional manifold, they computed the trace of the noise kernel at both points and found this double trace vanishes identically. This implies that there is no stochastic correction to the trace anomaly for massless conformal fields, in agreement with results arrived at in [43, 58, 208] (see also Section 3). In their second paper [245] a Gaussian approximation for the Green function (which is what limits the accuracy of the results) is used to derive finite expressions for two specific classes of spacetimes, ultrastatic spacetimes, such as the hot flat space, and the conformally- ultrastatic spacetimes, such as the Schwarzschild spacetime. Again, the validity of these results may depend on how we view the relevance and meaning of regularization. We will only report the result of their first paper here.

5.1 Point separation

The point separation scheme introduced in the 1960s by DeWitt [74] was brought to more popular use in the 1970s in the context of quantum field theory in curved spacetimes [75, 67, 68] as a means for obtaining a finite quantum stress tensor. Since the stress-energy tensor is built from the product of a pair of field operators evaluated at a single point, it is not well-defined. In this scheme, one introduces an artificial separation of the single point x to a pair of closely separated points x and x'. The problematic terms involving field products such as $\hat{\phi}(x)^2$ becomes $\hat{\phi}(x)\hat{\phi}(x')$, whose expectation value is well defined. If one is interested in the low energy behavior captured by the point-defined quantum field theory – as the effort in the 1970s was directed – one takes the coincidence limit. Once the divergences present are identified, they may be removed (regularization) or moved (by renormalizing the coupling constants), to produce a well-defined, finite stress tensor at a single point.

Thus the first order of business is the construction of the stress tensor and then to derive the symmetric stress-energy tensor two point function, the noise kernel, in terms of the Wightman Green function. In this section we will use the traditional notation for index tensors in the point-separation context.

5.1.1 *n*-tensors and end-point expansions

An object like the Green function $G(x,y)$ is an example of a *bi-scalar*: It transforms as scalar at both points x and y. We can also define a *bi-tensor* $T_{a_1\dots a_n\, b'_1\dots b'_m}(x,y)$: Upon a coordinate transformation, this transforms as a rank n tensor at x and a rank m tensor at y. We will extend this up to a *quad-tensor* $T_{a_1\dots a_{n_1}\, b'_1\dots b'_{n_2}\, c''_1\dots c''_{n_3}\, d'''_1\dots d'''_{n_4}}$ which has support at four points x, y, x', y', transforming as rank n_1, n_2, n_3, n_4 tensors at each of the four points. This also sets the notation we will use: unprimed indices referring to the tangent space constructed above x, single primed indices to y, double primed to x' and triple primed to y'. For each point, there is the covariant derivative ∇_a at that point. Covariant derivatives at different points commute, and the covariant derivative at, say, point x' does not act on a bi-tensor defined at, say, x and y:

$$T_{ab';c;d'} = T_{ab';d';c} \qquad \text{and} \qquad T_{ab';c''} = 0. \tag{41}$$

To simplify notation, henceforth we will eliminate the semicolons after the first one for multiple covariant derivatives at multiple points.

Having objects defined at different points, the *coincident limit* is defined as evaluation "on the diagonal", in the sense of the spacetime support of the function or tensor, and the usual shorthand $[G(x,y)] \equiv G(x,x)$ is used. This extends to n-tensors as

$$\left[T_{a_1\dots a_{n_1}\, b'_1\dots b'_{n_2}\, c''_1\dots c''_{n_3}\, d'''_1\dots d'''_{n_4}}\right] = T_{a_1\dots a_{n_1}\, b_1\dots b_{n_2}\, c_1\dots c_{n_3}\, d_1\dots d_{n_4}}, \tag{42}$$

i.e., this becomes a rank $(n_1+n_2+n_3+n_4)$ tensor at x. The multi-variable chain rule relates covariant derivatives acting at different points, when we are interested in the coincident limit:

$$\left[T_{a_1\dots a_m\, b'_1\dots b'_n}\right]_{;c} = \left[T_{a_1\dots a_m\, b'_1\dots b'_n;c}\right] + \left[T_{a_1\dots a_m\, b'_1\dots b'_n;c'}\right]. \tag{43}$$

This result is referred to as *Synge's theorem* in this context; we follow Fulling's discussion [100].

The bi-tensor of *parallel transport* ${g_a}^{b'}$ is defined such that when it acts on a vector $v_{b'}$ at y, it parallel transports the vector along the geodesics connecting x and y. This allows us to add vectors and tensors defined at different points. We cannot directly add a vector v_a at x and vector $w_{a'}$ at y. But by using ${g_a}^{b'}$, we can construct the sum $v^a + {g_a}^{b'} w_{b'}$. We will also need the obvious property $\left[{g_a}^{b'}\right] = {g_a}^{b}$.

The main bi-scalar we need is the *world function* $\sigma(x,y)$. This is defined as a half of the square of the geodesic distance between the points x and y. It satisfies the equation

$$\sigma = \frac{1}{2}\sigma^{;p}\sigma_{;p}. \tag{44}$$

Often in the literature, a covariant derivative is implied when the world function appears with indices, $\sigma^a \equiv \sigma^{;a}$, i.e., taking the covariant derivative at x, while $\sigma^{a'}$ means the covariant derivative at y. This is done since the vector $-\sigma^a$ is the tangent vector to the geodesic with length equal to the distance between x and y. As σ^a records information about distance and direction for the two points, this makes it ideal for constructing a series expansion of a bi-scalar. The *end point* expansion of a bi-scalar $S(x,y)$ is of the form

$$S(x,y) = A^{(0)} + \sigma^p A^{(1)}_p + \sigma^p\sigma^q A^{(2)}_{pq} + \sigma^p\sigma^q\sigma^r A^{(3)}_{pqr} + \sigma^p\sigma^q\sigma^r\sigma^s A^{(4)}_{pqrs} + \dots, \tag{45}$$

where, following our convention, the expansion tensors $A^{(n)}_{a_1\dots a_n}$ with unprimed indices have support at x (hence the name end point expansion). Only the symmetric part of these tensors contribute to the expansion. For the purposes of multiplying series expansions it is convenient to separate the distance dependence from the direction dependence. This is done by introducing the unit vector $p^a = \sigma^a/\sqrt{2\sigma}$. Then the series expansion can be written

$$S(x,y) = A^{(0)} + \sigma^{\frac{1}{2}}A^{(1)} + \sigma A^{(2)} + \sigma^{\frac{3}{2}}A^{(3)} + \sigma^2 A^{(4)} + \dots. \tag{46}$$

The expansion scalars are related, via $A^{(n)} = 2^{n/2}A^{(n)}_{p_1\dots p_n}p^{p_1}\dots p^{p_n}$, to the expansion tensors.

The last object we need is the *Van Vleck–Morette* determinant $D(x,y)$, defined as $D(x,y) \equiv -\det\left(-\sigma_{;ab'}\right)$. The related bi-scalar

$$\Delta^{1/2} = \left(\frac{D(x,y)}{\sqrt{g(x)g(y)}}\right)^{\frac{1}{2}} \tag{47}$$

satisfies the equation

$$\Delta^{1/2}\left(4 - {\sigma_{;p}}^{p}\right) - 2{\Delta^{1/2}}_{;p}\sigma^{;p} = 0 \tag{48}$$

with the boundary condition $\left[\Delta^{1/2}\right] = 1$.

Further details on these objects and discussions of the definitions and properties are contained in [67, 68] and [240]. There it is shown how the defining equations for σ and $\Delta^{1/2}$ are used to determine the coincident limit expression for the various covariant derivatives of the world function ($[\sigma_{;a}]$, $[\sigma_{;ab}]$, etc.) and how the defining differential equation for $\Delta^{1/2}$ can be used to determine the series expansion of $\Delta^{1/2}$. We show how the expansion tensors $A^{(n)}_{a_1\ldots a_n}$ are determined in terms of the coincident limits of covariant derivatives of the bi-scalar $S(x,y)$. ([240] details how point separation can be implemented on the computer to provide easy access to a wider range of applications involving higher derivatives of the curvature tensors.)

5.2 Stress-energy bi-tensor operator and noise kernel

Even though we believe that the point-separated results are more basic in the sense that it reflects a deeper structure of the quantum theory of spacetime, we will nevertheless start with quantities defined at one point, because they are what enter in conventional quantum field theory. We will use point separation to introduce the bi-quantities. The key issue here is thus the distinction between point-defined (pt) and point-separated (bi) quantities.

For a free classical scalar field ϕ with the action $S_{\rm m}[g,\phi]$ defined in Equation (1), the classical stress-energy tensor is

$$T_{ab} = (1-2\xi)\phi_{;a}\phi_{;b} + \left(2\xi - \frac{1}{2}\right)\phi_{;p}\phi^{;p}\,g_{ab} + 2\xi\phi\left(\phi_{;p}{}^{p} - \phi_{;ab}\,g_{ab}\right)$$
$$+\phi^2\xi\left(R_{ab} - \frac{1}{2}R\,g_{ab}\right) - \frac{1}{2}m^2\phi^2 g_{ab}, \tag{49}$$

which is equivalent to the tensor of Equation (3), but written in a slightly different form for convenience. When we make the transition to quantum field theory, we promote the field $\phi(x)$ to a field operator $\hat\phi(x)$. The fundamental problem of defining a quantum operator for the stress tensor is immediately visible: The field operator appears quadratically. Since $\hat\phi(x)$ is an operator-valued distribution, products at a single point are not well-defined. But if the product is point separated, $\hat\phi^2(x) \to \hat\phi(x)\hat\phi(x')$, they are finite and well-defined.

Let us first seek a point-separated extension of these classical quantities and then consider the quantum field operators. Point separation is symmetrically extended to products of covariant derivatives of the field according to

$$(\phi_{;a})\,(\phi_{;b}) \to \frac{1}{2}\left(g_a{}^{p'}\nabla_{p'}\nabla_b + g_b{}^{p'}\nabla_a\nabla_{p'}\right)\phi(x)\phi(x'),$$
$$\phi\,(\phi_{;ab}) \to \frac{1}{2}\left(\nabla_a\nabla_b + g_a{}^{p'}g_b{}^{q'}\nabla_{p'}\nabla_{q'}\right)\phi(x)\phi(x').$$

The bi-vector of parallel displacement $g_a{}^{a'}(x,x')$ is included so that we may have objects that are rank 2 tensors at x and scalars at x'.

To carry out point separation on Equation (49), we first define the differential operator

$$\mathcal{T}_{ab} = \frac{1}{2}(1-2\xi)\left(g_a{}^{a'}\nabla_{a'}\nabla_b + g_b{}^{b'}\nabla_a\nabla_{b'}\right) + \left(2\xi - \frac{1}{2}\right)g_{ab}g^{cd'}\nabla_c\nabla_{d'}$$
$$-\xi\left(\nabla_a\nabla_b + g_a{}^{a'}g_b{}^{b'}\nabla_{a'}\nabla_{b'}\right) + \xi g_{ab}\left(\nabla_c\nabla^c + \nabla_{c'}\nabla^{c'}\right) + \xi\left(R_{ab} - \frac{1}{2}g_{ab}R\right) - \frac{1}{2}m^2 g_{ab}, \tag{50}$$

from which we obtain the classical stress tensor as

$$T_{ab}(x) = \lim_{x'\to x}\mathcal{T}_{ab}\phi(x)\phi(x'). \tag{51}$$

That the classical tensor field no longer appears as a product of scalar fields at a single point allows a smooth transition to the quantum tensor field. From the viewpoint of the stress tensor, the separation of points is an artificial construct, so when promoting the classical field to a quantum one, neither point should be favored. The product of field configurations is taken to be the symmetrized operator product, denoted by curly brackets:

$$\phi(x)\phi(y) \to \frac{1}{2}\left\{\hat{\phi}(x),\hat{\phi}(y)\right\} = \frac{1}{2}\left(\hat{\phi}(x)\hat{\phi}(y) + \hat{\phi}(y)\hat{\phi}(x)\right) \tag{52}$$

With this, the point separated stress-energy tensor operator is defined as

$$\hat{T}_{ab}(x,x') \equiv \frac{1}{2}\mathcal{T}_{ab}\left\{\hat{\phi}(x),\hat{\phi}(x')\right\}. \tag{53}$$

While the classical stress tensor was defined at the coincidence limit $x' \to x$, we cannot attach any physical meaning to the quantum stress tensor at one point until the issue of regularization is dealt with, which will happen in the next section. For now, we will maintain point separation so as to have a mathematically meaningful operator.

The expectation value of the point-separated stress tensor can now be taken. This amounts to replacing the field operators by their expectation value, which is given by the Hadamard (or Schwinger) function

$$G^{(1)}(x,x') = \left\langle\left\{\hat{\phi}(x),\hat{\phi}(x')\right\}\right\rangle, \tag{54}$$

and the point-separated stress tensor is defined as

$$\langle\hat{T}_{ab}(x,x')\rangle = \frac{1}{2}\mathcal{T}_{ab}\, G^{(1)}(x,x'), \tag{55}$$

where, since $\mathcal{T}_{ab}$ is a differential operator, it can be taken "outside" the expectation value. The expectation value of the point-separated quantum stress tensor for a free, massless ($m = 0$) conformally coupled ($\xi = \frac{1}{6}$) scalar field on a four dimension spacetime with scalar curvature R is

$$\begin{aligned}\langle\hat{T}_{ab}(x,x')\rangle = {} & \frac{1}{6}\left(g^{p'}{}_b\, G^{(1)}{}_{;p'a} + g^{p'}{}_a\, G^{(1)}{}_{;p'b}\right) - \frac{1}{12} g^{p'}{}_q\, G^{(1)}{}_{;p'}{}^q\, g_{ab} \\ & -\frac{1}{12}\left(g^{p'}{}_a\, g^{q'}{}_b\, G^{(1)}{}_{;p'q'} + G^{(1)}{}_{;ab}\right) + \frac{1}{12}\left[\left(G^{(1)}{}_{;p'}{}^{p'} + G^{(1)}{}_{;p}{}^p\right) g_{ab}\right] \\ & +\frac{1}{12} G^{(1)}\left(R_{ab} - \frac{1}{2}R\, g_{ab}\right).\end{aligned} \tag{56}$$

5.2.1 Finiteness of the noise kernel

We now turn our attention to the noise kernel introduced in Equation (11), which is the symmetrized product of the (mean subtracted) stress tensor operator:

$$\begin{aligned}8N_{ab,c'd'}(x,y) &= \left\langle\left\{\hat{T}_{ab}(x) - \left\langle\hat{T}_{ab}(x)\right\rangle, \hat{T}_{c'd'}(y) - \left\langle\hat{T}_{c'd'}(y)\right\rangle\right\}\right\rangle \\ &= \left\langle\left\{\hat{T}_{ab}(x), \hat{T}_{c'd'}(y)\right\}\right\rangle - 2\left\langle\hat{T}_{ab}(x)\right\rangle\left\langle\hat{T}_{c'd'}(y)\right\rangle.\end{aligned} \tag{57}$$

Since $\hat{T}_{ab}(x)$ defined at one point can be ill-behaved as it is generally divergent, one can question the soundness of these quantities. But as will be shown later, the noise kernel is finite for $y \neq x$. All field operator products present in the first expectation value that could be divergent, are canceled by similar products in the second term. We will replace each of the stress tensor operators in the above expression for the noise kernel by their point separated versions, effectively separating the two points (x, y) into the four points (x, x', y, y'). This will allow us to express the noise kernel in terms of a pair of differential operators acting on a combination of four and two point functions. Wick's theorem will allow the four point functions to be re-expressed in terms of two point functions. From this we see that all possible divergences for $y \neq x$ will cancel. When the coincidence limit is taken, divergences do occur. The above procedure will allow us to isolate the divergences and to obtain a finite result.

Taking the point-separated quantities as more basic, one should replace each of the stress tensor operators in the above with the corresponding point separated version (53), with $\mathcal{T}_{ab}$ acting at x and x' and $\mathcal{T}_{c'd'}$ acting at y and y'. In this framework the noise kernel is defined as

$$8N_{ab,c'd'}(x, y) = \lim_{x'\to x} \lim_{y'\to y} \mathcal{T}_{ab}\mathcal{T}_{c'd'}\, G(x, x', y, y'), \tag{58}$$

where the four point function is

$$G(x, x', y, y') = \frac{1}{4}\left[\left\langle\left\{\left\{\hat\phi(x), \hat\phi(x')\right\}, \left\{\hat\phi(y), \hat\phi(y')\right\}\right\}\right\rangle - 2\left\langle\left\{\hat\phi(x), \hat\phi(x')\right\}\right\rangle\left\langle\left\{\hat\phi(y), \hat\phi(y')\right\}\right\rangle\right]. \tag{59}$$

We assume that the pairs (x, x') and (y, y') are each within their respective Riemann normal coordinate neighborhoods so as to avoid problems that possible geodesic caustics might be present. When we later turn our attention to computing the limit $y \to x$, after issues of regularization are addressed, we will want to assume that all four points are within the same Riemann normal coordinate neighborhood.

Wick's theorem, for the case of free fields which we are considering, gives the simple product four point function in terms of a sum of products of Wightman functions (we use the shorthand notation $G_{xy} \equiv G_+(x, y) = \langle\hat\phi(x)\,\hat\phi(y)\rangle$):

$$\left\langle\hat\phi(x)\,\hat\phi(y)\,\hat\phi(x')\,\hat\phi(y')\right\rangle = G_{xy'}\,G_{yx'} + G_{xx'}\,G_{yy'} + G_{xy}\,G_{x'y'} \tag{60}$$

Expanding out the anti-commutators in Equation (59) and applying Wick's theorem, the four point function becomes

$$G(x, x', y, y') = G_{xy'}\,G_{x'y} + G_{xy}\,G_{x'y'} + G_{yx'}\,G_{y'x} + G_{yx}\,G_{y'x'}. \tag{61}$$

We can now easily see that the noise kernel defined via this function is indeed well defined for the limit $(x', y') \to (x, y)$:

$$G(x, x, y, y) = 2\left(G_{xy}^2 + G_{yx}^2\right). \tag{62}$$

From this we can see that the noise kernel is also well defined for $y \neq x$; any divergence present in the first expectation value of Equation (59) have been cancelled by those present in the pair of Green functions in the second term, in agreement with the results of Section 3.

5.2.2 Explicit form of the noise kernel

We will let the points separated for a while so we can keep track of which covariant derivative acts on which arguments of which Wightman function. As an example (the complete calculation is quite long), consider the result of the first set of covariant derivative operators in the differential operator (50), from both $\mathcal{T}_{ab}$ and $\mathcal{T}_{c'd'}$, acting on $G(x, x', y, y')$:

$$\frac{1}{4}(1-2\xi)^2\left(g_a{}^{p''}\nabla_{p''}\nabla_b + g_b{}^{p''}\nabla_{p''}\nabla_a\right)\left(g_{c'}{}^{q'''}\nabla_{q'''}\nabla_{d'} + g_{d'}{}^{q'''}\nabla_{q'''}\nabla_{c'}\right)G(x, x', y, y'). \tag{63}$$

(Our notation is that ∇_a acts at x, $\nabla_{c'}$ at y, $\nabla_{b''}$ at x', and $\nabla_{d'''}$ at y'.) Expanding out the differential operator above, we can determine which derivatives act on which Wightman function:

$$\frac{(1-2\,\xi)^2}{4} \times \Big[g_{c'}{}^{p'''} g^{q''}{}_a \left(G_{xy';bp'''}\, G_{x'y;q''d'} + G_{xy;bd'}\, G_{x'y';q''p'''} \right.$$
$$\left. + G_{yx';q''d'}\, G_{y'x;bp'''} + G_{yx;bd'}\, G_{y'x';q''p'''} \right)$$
$$+ g_{d'}{}^{p'''} g^{q''}{}_a \left(G_{xy';bp'''}\, G_{x'y;q''c'} + G_{xy;bc'}\, G_{x'y';q''p'''} \right.$$
$$\left. + G_{yx';q''c'}\, G_{y'x;bp'''} + G_{yx;bc'}\, G_{y'x';q''p'''} \right)$$
$$+ g_{c'}{}^{p'''} g^{q''}{}_b \left(G_{xy';ap'''}\, G_{x'y;q''d'} + G_{xy;ad'}\, G_{x'y';q''p'''} \right.$$
$$\left. + G_{yx';q''d'}\, G_{y'x;ap'''} + G_{yx;ad'}\, G_{y'x';q''p'''} \right)$$
$$+ g_{d'}{}^{p'''} g^{q''}{}_b \left(G_{xy';ap'''}\, G_{x'y;q''c'} + G_{xy;ac'}\, G_{x'y';q''p'''} \right.$$
$$\left. + G_{yx';q''c'}\, G_{y'x;ap'''} + G_{yx;ac'}\, G_{y'x';q''p'''} \right) \Big]. \qquad (64)$$

If we now let $x' \to x$ and $y' \to y$, the contribution to the noise kernel is (including the factor of $\frac{1}{8}$ present in the definition of the noise kernel):

$$\frac{1}{8}\Big[(1-2\xi)^2 \left(G_{xy;ad'} G_{xy;bc'} + G_{xy;ac'} G_{xy;bd'} \right) + (1-2\xi)^2 \left(G_{yx;ad'} G_{yx;bc'} + G_{yx;ac'} G_{yx;bd'} \right) \Big]. \qquad (65)$$

That this term can be written as the sum of a part involving G_{xy} and one involving G_{yx} is a general property of the entire noise kernel. It thus takes the form

$$N_{abc'd'}(x,y) = N_{abc'd'}\left[G_+(x,y)\right] + N_{abc'd'}\left[G_+(y,x)\right]. \qquad (66)$$

We will present the form of the functional $N_{abc'd'}[G]$ shortly. First we note, that for x and y time-like separated, the above split of the noise kernel allows us to express it in terms of the Feynman (time ordered) Green function $G_{\rm F}(x,y)$ and the Dyson (anti-time ordered) Green function $G_{\rm D}(x,y)$:

$$N_{abc'd'}(x,y) = N_{abc'd'}\left[G_{\rm F}(x,y)\right] + N_{abc'd'}\left[G_{\rm D}(x,y)\right]. \qquad (67)$$

This can be connected with the zeta function approach to this problem [242] as follows: Recall when the quantum stress tensor fluctuations determined in the Euclidean section is analytically continued back to Lorentzian signature ($\tau \to it$), the time ordered product results. On the other hand, if the continuation is $\tau \to -it$, the anti-time ordered product results. With this in mind, the noise kernel is seen to be related to the quantum stress tensor fluctuations derived via the effective action as

$$16 N_{abc'd'} = \Delta T^2_{abc'd'}\Big|_{t=-i\tau, t'=-i\tau'} + \Delta T^2_{abc'd'}\Big|_{t=i\tau, t'=i\tau'}. \qquad (68)$$

The complete form of the functional $N_{abc'd'}[G]$ is

$$N_{abc'd'}[G] = \tilde{N}_{abc'd'}[G] + g_{ab}\tilde{N}_{c'd'}[G] + g_{c'd'}\tilde{N}'_{ab}[G] + g_{ab}g_{c'd'}\tilde{N}[G], \qquad (69)$$

with

$$\begin{aligned}
8\tilde{N}_{abc'd'}\,[G] =& \,(1-2\xi)^2\,(G_{;c'b}\,G_{;d'a}+G_{;c'a}\,G_{;d'b})+4\xi^2\,(G_{;c'd'}\,G_{;ab}+G\,G_{;abc'd'})\\
&-2\xi\,(1-2\xi)\,(G_{;b}G_{;c'ad'}+G_{;a}\,G_{;c'bd'}+G_{;d'}\,G_{;abc'}+G_{;c'}\,G_{;abd'})\\
&+2\xi\,(1-2\xi)\,(G_{;a}\,G_{;b}\,R_{c'd'}+G_{;c'}\,G_{;d'}\,R_{ab})\\
&-4\xi^2\,(G_{;ab}\,R_{c'd'}+G_{;c'd'}\,R_{ab})\,G+2\xi^2R_{c'd'}\,R_{ab}G^2, \qquad (70)
\end{aligned}$$

$$\begin{aligned}
8\tilde{N}'_{ab}\,[G] =& \,2(1-2\xi)\left[\left(2\xi-\frac{1}{2}\right)G_{;p'b}\,G_{;}{}^{p'}{}_{a}+\xi\left(G_{;b}\,G_{;p'a}{}^{p'}+G_{;a}\,G_{;p'b}{}^{p'}\right)\right]\\
&-4\xi\left[\left(2\xi-\frac{1}{2}\right)G_{;}{}^{p'}\,G_{;abp'}+\xi\left(G_{;p'}{}^{p'}\,G_{;ab}+G\,G_{;abp'}{}^{p'}\right)\right]\\
&-(m^2+\xi R')\,[(1-2\xi)G_{;a}\,G_{;b}-2G\,\xi\,G_{;ab}]\\
&+2\xi\left[\left(2\xi-\frac{1}{2}\right)G_{;p'}\,G_{;}{}^{p'}+2G\,\xi\,G_{;p'}{}^{p'}\right]R_{ab}-(m^2+\xi R')\,\xi\,R_{ab}\,G^2, \qquad (71)
\end{aligned}$$

$$\begin{aligned}
8\tilde{N}\,[G] =& \,2\left(2\xi-\frac{1}{2}\right)^2 G_{;p'q}\,G_{;}{}^{p'q}+4\xi^2\left(G_{;p'}{}^{p'}\,G_{;q}{}^{q}+G\,G_{;p}{}^{p}{}_{q'}{}^{q'}\right)\\
&+4\xi\left(2\xi-\frac{1}{2}\right)\left(G_{;p}\,G_{;q'}{}^{pq'}+G_{;}{}^{p'}\,G_{;q}{}^{q}{}_{p'}\right)\\
&-\left(2\xi-\frac{1}{2}\right)\left[\left(m^2+\xi R\right)G_{;p'}\,G_{;}{}^{p'}+\left(m^2+\xi R'\right)G_{;p}\,G^{;p}\right]\\
&-2\xi\left[\left(m^2+\xi R\right)G_{;p'}{}^{p'}+\left(m^2+\xi R'\right)G_{;p}{}^{p}\right]G\\
&+\frac{1}{2}\left(m^2+\xi R\right)\left(m^2+\xi R'\right)G^2. \qquad (72)
\end{aligned}$$

5.2.3 Trace of the noise kernel

One of the most interesting and surprising results to come out of the investigations of the quantum stress tensor undertaken in the 1970s was the discovery of the trace anomaly [61, 84]. When the trace of the stress tensor $T = g^{ab}T_{ab}$ is evaluated for a field configuration that satisties the field equation (2), the trace is seen to vanish for massless conformally coupled fields. When this analysis is carried over to the renormalized expectation value of the quantum stress tensor, the trace no longer vanishes. Wald [284] showed that this was due to the failure of the renormalized Hadamard function $G_{\rm ren}(x,x')$ to be symmetric in x and x', implying that it does not necessarily satisfy the field equation (2) in the variable x'. (The definition of $G_{\rm ren}(x,x')$ in the context of point separation will come next.)

With this in mind, we can now determine the noise associated with the trace. Taking the trace at both points x and y of the noise kernel functional (67) yields

$$
\begin{aligned}
N[G] &= g^{ab}\, g^{c'd'}\, N_{abc'd'}[G] \\
&= -3\,G\,\xi \left[\left(m^2 + \frac{1}{2}\xi R\right) G_{;p'}{}^{p'} + \left(m^2 + \frac{1}{2}\xi R'\right) G_{;p}{}^{p} \right] \\
&\quad + \frac{9\xi^2}{2} \left(G_{;p'}{}^{p'}\, G_{;p}{}^{p} + G\, G_{;p}{}^{p}{}_{p'}{}^{p'} \right) + \left(m^2 + \frac{1}{2}\xi R\right) \left(m^2 + \frac{1}{2}\xi R'\right) G^2 \\
&\quad + 3 \left(\frac{1}{6} - \xi\right) \left[3 \left(\frac{1}{6} - \xi\right) G_{;p'p}\, G_{;}{}^{p'p} - 3\xi \left(G_{;p}\, G_{;p'}{}^{pp'} + G_{;p'}\, G_{;p}{}^{pp'} \right) \right. \\
&\qquad \left. + \left(m^2 + \frac{1}{2}\xi R\right) G_{;p'}\, G_{;}{}^{p'} + \left(m^2 + \frac{1}{2}\xi R'\right) G_{;p}\, G^{;p} \right] .
\end{aligned}
\tag{73}
$$

For the massless conformal case, this reduces to

$$
N[G] = \frac{1}{144} \left\{ R R' G^2 - 6G \left(R \Box' + R' \Box \right) G + 18 \left[(\Box G)\,(\Box' G) + \Box' \Box G \right] \right\} ,
\tag{74}
$$

which holds for any function $G(x, y)$. For G being the Green function, it satisfies the field equation (2):

$$
\Box G = (m^2 + \xi R) G.
\tag{75}
$$

We will only assume that the Green function satisfies the field equation in its first variable. Using the fact $\Box' R = 0$ (because the covariant derivatives act at a different point than at which R is supported), it follows that

$$
\Box' \Box G = (m^2 + \xi R) \Box' G.
\tag{76}
$$

With these results, the noise kernel trace becomes

$$
\begin{aligned}
N[G] &= \frac{1}{2} \left[m^2 \,(1 - 3\,\xi) + 3R \left(\frac{1}{6} - \xi\right) \xi \right] \left[G^2 \left(2m^2 + R'\,\xi\right) + (1 - 6\xi)\, G_{;p'}\, G_{;}{}^{p'} - 6G\,\xi\, G_{;p'}{}^{p'} \right] \\
&\quad + \frac{1}{2} \left(\frac{1}{6} - \xi\right) \left[3 \left(2m^2 + R'\,\xi\right) G_{;p}\, G^{;p} - 18\,\xi\, G_{;p}\, G_{;p'}{}^{pp'} + 18 \left(\frac{1}{6} - \xi\right) G_{;p'p}\, G_{;}{}^{p'p} \right] ,
\end{aligned}
\tag{77}
$$

which vanishes for the massless conformal case. We have thus shown, based solely on the definition of the point separated noise kernel, that there is no noise associated with the trace anomaly. This result obtained in [245] is completely general since it is assumed that the Green function is only satisfying the field equations in its first variable; an alternative proof of this result was given in [208]. This condition holds not just for the classical field case, but also for the regularized quantum case, where one does not expect the Green function to satisfy the field equation in both variables. One can see this result from the simple observation used in Section 3: Since the trace anomaly is known to be locally determined and quantum state independent, whereas the noise present in the quantum field is non-local, it is hard to find a noise associated with it. This general result is in agreement with previous findings [43, 167, 58], derived from the Feynman-Vernon influence functional formalism [89, 88] for some particular cases.

6 Metric Fluctuations in Minkowski Spacetime

Although the Minkowski vacuum is an eigenstate of the total four-momentum operator of a field in Minkowski spacetime, it is not an eigenstate of the stress-energy operator. Hence, even for those solutions of semiclassical gravity such as the Minkowski metric, for which the expectation value of the stress-energy operator can always be chosen to be zero, the fluctuations of this operator are non-vanishing. This fact leads to consider the stochastic metric perturbations induced by these fluctuations.

Here we derive the Einstein–Langevin equation for the metric perturbations in a Minkowski background. We solve this equation for the linearized Einstein tensor and compute the associated two-point correlation functions, as well as, the two-point correlation functions for the metric perturbations. Even though, in this case, we expect to have negligibly small values for these correlation functions for points separated by lengths larger than the Planck length, there are several reasons why it is worth carrying out this calculation.

On the one hand, these are the first backreaction solutions of the full Einstein–Langevin equation. There are analogous solutions to a "reduced" version of this equation inspired in a "mini-superspace" model [59, 38], and there is also a previous attempt to obtain a solution to the Einstein–Langevin equation in [58], but there the non-local terms in the Einstein–Langevin equation were neglected.

On the other hand, the results of this calculation, which confirm our expectations that gravitational fluctuations are negligible at length scales larger than the Planck length, but also predict that the fluctuations are strongly suppressed on small scales, can be considered a first test of stochastic semiclassical gravity. In addition, these results reveal an important connection between stochastic gravity and the large N expansion of quantum gravity. We can also extract conclusions on the possible qualitative behavior of the solutions to the Einstein–Langevin equation. Thus, it is interesting to note that the correlation functions at short scales are characterized by correlation lengths of the order of the Planck length; furthermore, such correlation lengths enter in a non-analytic way in the correlation functions.

We advise the reader that his section is rather technical since it deals with an explicit non-trivial backreaction computation in stochastic gravity. We have tried to make it reasonable self-contained and detailed, however a more detailed exposition can be found in [209].

6.1 Perturbations around Minkowski spacetime

The Minkowski metric η_{ab}, in a manifold $\mathcal{M}$ which is topologically $\mathbb{R}^4$, and the usual Minkowski vacuum, denoted as $|0\rangle$, are the class of simplest solutions to the semiclassical Einstein equation (7), the so-called trivial solutions of semiclassical gravity [91]. They constitute the ground state of semiclassical gravity. In fact, we can always choose a renormalization scheme in which the renormalized expectation value $\langle 0|\,\hat{T}_{\mathrm{R}}^{ab}[\eta]|0\rangle = 0$. Thus, Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$ and the vacuum state $|0\rangle$ are a solution to the semiclassical Einstein equation with renormalized cosmological constant $\Lambda = 0$. The fact that the vacuum expectation value of the renormalized stress-energy operator in Minkowski spacetime should vanish was originally proposed by Wald [283], and it may be understood as a renormalization convention [100, 113]. Note that other possible solutions of semiclassical gravity with zero vacuum expectation value of the stress-energy tensor are the exact gravitational plane waves, since they are known to be vacuum solutions of Einstein equations which induce neither particle creation nor vacuum polarization [107, 73, 104].

As we have already mentioned the vacuum $|0\rangle$ is an eigenstate of the total four-momentum operator in Minkowski spacetime, but not an eigenstate of $\hat{T}^R_{ab}[\eta]$. Hence, even in the Minkowski background, there are quantum fluctuations in the stress-energy tensor and, as a result, the noise kernel does not vanish. This fact leads to consider the stochastic corrections to this class of trivial solutions of semiclassical gravity. Since, in this case, the Wightman and Feynman functions (37), their values in the two-point coincidence limit, and the products of derivatives of two of such functions appearing in expressions (38) and (39) are known in dimensional regularization, we can compute the Einstein–Langevin equation using the methods outlined in Sections 3 and 4.

To perform explicit calculations it is convenient to work in a global inertial coordinate system $\{x^\mu\}$ and in the associated basis, in which the components of the flat metric are simply $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, \ldots, 1)$. In Minkowski spacetime, the components of the classical stress-energy tensor (3) reduce to

$$T^{\mu\nu}[\eta, \phi] = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2}\eta^{\mu\nu}\partial^\rho \phi \partial_\rho \phi - \frac{1}{2}\eta^{\mu\nu} m^2 \phi^2 + \xi \left(\eta^{\mu\nu}\Box - \partial^\mu \partial^\nu\right) \phi^2, \tag{78}$$

where $\Box \equiv \partial_\mu \partial^\mu$, and the formal expression for the components of the corresponding "operator" in dimensional regularization, see Equation (4), is

$$\hat{T}_n^{\mu\nu}[\eta] = \frac{1}{2}\left\{\partial^\mu \hat{\phi}_n, \partial^\nu \hat{\phi}_n\right\} + \mathcal{D}^{\mu\nu}\hat{\phi}_n^2, \tag{79}$$

where $\mathcal{D}^{\mu\nu}$ is the differential operator (5), with $g_{\mu\nu} = \eta_{\mu\nu}$, $R_{\mu\nu} = 0$, and $\nabla_\mu = \partial_\mu$. The field $\hat{\phi}_n(x)$ is the field operator in the Heisenberg representation in an n-dimensional Minkowski spacetime, which satisfies the Klein–Gordon equation (2). We use here a stress-energy tensor which differs from the canonical one, which corresponds to $\xi = 0$; both tensors, however, define the same total momentum.

The Wightman and Feynman functions (37) for $g_{\mu\nu} = \eta_{\mu\nu}$ are well known:

$$G_n^+(x, y) = i\Delta_n^+(x - y), \qquad G_{F_n}(x, y) = \Delta_{F_n}(x - y), \tag{80}$$

with

$$\begin{aligned} \Delta_n^+(x) &= -2\pi i \int \frac{d^n k}{(2\pi)^n}\, e^{ikx}\, \delta(k^2 + m^2)\, \theta(k^0), \\ \Delta_{F_n}(x) &= -\int \frac{d^n k}{(2\pi)^n}\, \frac{e^{ikx}}{k^2 + m^2 - i\epsilon} \qquad \text{for } \epsilon \to 0^+, \end{aligned} \tag{81}$$

where $k^2 \equiv \eta_{\mu\nu} k^\mu k^\nu$ and $kx \equiv \eta_{\mu\nu} k^\mu x^\nu$. Note that the derivatives of these functions satisfy $\partial^x_\mu \Delta_n^+(x - y) = \partial_\mu \Delta_n^+(x - y)$ and $\partial^y_\mu \Delta_n^+(x - y) = -\partial_\mu \Delta_n^+(x - y)$, and similarly for the Feynman propagator $\Delta_{F_n}(x - y)$.

To write down the semiclassical Einstein equation (7) in n dimensions for this case, we need to compute the vacuum expectation value of the stress-energy operator components (79). Since, from (80), we have that $\langle 0|\hat{\phi}_n^2(x)|0\rangle = i\Delta_{F_n}(0) = i\Delta_n^+(0)$, which is a constant (independent of x), we have simply

$$\left\langle 0 \left| \hat{T}_n^{\mu\nu}[\eta] \right| 0 \right\rangle = -i \int \frac{d^n k}{(2\pi)^n}\, \frac{k^\mu k^\nu}{k^2 + m^2 - i\epsilon} = \frac{\eta^{\mu\nu}}{2}\left(\frac{m^2}{4\pi}\right)^{n/2} \Gamma\left(-\frac{n}{2}\right), \tag{82}$$

where the integrals in dimensional regularization have been computed in the standard way [209], and where $\Gamma(z)$ is Euler's gamma function. The semiclassical Einstein equation (7) in n dimensions before renormalization reduces now to

$$\frac{\Lambda_{\mathrm{B}}}{8\pi G_{\mathrm{B}}}\, \eta^{\mu\nu} = \mu^{-(n-4)} \left\langle 0 \left| \hat{T}_n^{\mu\nu}[\eta] \right| 0 \right\rangle. \tag{83}$$

This equation, thus, simply sets the value of the bare coupling constant $\Lambda_{\mathrm{B}}/G_{\mathrm{B}}$. Note, from Equation (82), that in order to have $\langle 0|\, \hat{T}_{\mathrm{R}}^{\mu\nu}\, |0\rangle[\eta] = 0$, the renormalized and regularized stress-energy tensor "operator" for a scalar field in Minkowski spacetime, see Equation (6), has to be defined as

$$\hat{T}_{\mathrm{R}}^{\mu\nu}[\eta] = \mu^{-(n-4)}\,\hat{T}_n^{\mu\nu}[\eta] - \frac{\eta^{\mu\nu}}{2}\,\frac{m^4}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{n-4}{2}}\Gamma\left(-\frac{n}{2}\right), \tag{84}$$

which corresponds to a renormalization of the cosmological constant

$$\frac{\Lambda_{\mathrm{B}}}{G_{\mathrm{B}}} = \frac{\Lambda}{G} - \frac{2}{\pi}\,\frac{m^4}{n(n-2)}\,\kappa_n + \mathcal{O}(n-4), \tag{85}$$

where

$$\kappa_n \equiv \frac{1}{n-4}\left(\frac{e^{\gamma}m^2}{4\pi\mu^2}\right)^{\frac{n-4}{2}} = \frac{1}{n-4} + \frac{1}{2}\ln\left(\frac{e^{\gamma}m^2}{4\pi\mu^2}\right) + \mathcal{O}(n-4), \tag{86}$$

with γ being Euler's constant. In the case of a massless scalar field, $m^2 = 0$, one simply has $\Lambda_{\mathrm{B}}/G_{\mathrm{B}} = \Lambda/G$. Introducing this renormalized coupling constant into Equation (83), we can take the limit $n \to 4$. We find that, for $(\mathbb{R}^4, \eta_{ab}, |0\rangle)$ to satisfy the semiclassical Einstein equation, we must take $\Lambda = 0$.

We can now write down the Einstein–Langevin equations for the components $h_{\mu\nu}$ of the stochastic metric perturbation in dimensional regularization. In our case, using $\langle 0|\hat{\phi}_n^2(x)|0\rangle = i\Delta_{F_n}(0)$ and the explicit expression of Equation (34), we obtain

$$\begin{aligned}\frac{1}{8\pi G_{\mathrm{B}}}\left[G^{(1)\mu\nu} + \Lambda_{\mathrm{B}}\left(h^{\mu\nu} - \frac{1}{2}\,\eta^{\mu\nu}h\right)\right](x) - \frac{4}{3}\,\alpha_{\mathrm{B}}D^{(1)\mu\nu}(x) - 2\beta_{\mathrm{B}}B^{(1)\mu\nu}(x)\\ -\xi G^{(1)\mu\nu}(x)\mu^{-(n-4)}i\Delta_{F_n}(0) + \frac{1}{2}\int d^ny\,\mu^{-(n-4)}H_n^{\mu\nu\alpha\beta}(x,y)h_{\alpha\beta}(y) = \xi^{\mu\nu}(x).\end{aligned} \tag{87}$$

The indices in $h_{\mu\nu}$ are raised with the Minkowski metric, and $h \equiv h_\rho^\rho$; here a superindex $^{(1)}$ denotes the components of a tensor linearized around the flat metric. Note that in n dimensions the two-point correlation functions for the field $\xi^{\mu\nu}$ is written as

$$\left\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(y)\right\rangle_{\mathrm{s}} = \mu^{-2(n-4)}N_n^{\mu\nu\alpha\beta}(x,y). \tag{88}$$

Explicit expressions for $D^{(1)\mu\nu}$ and $B^{(1)\mu\nu}$ are given by

$$D^{(1)\mu\nu}(x) = \frac{1}{2}\mathcal{F}_x^{\mu\nu\alpha\beta}\,h_{\alpha\beta}(x), \qquad B^{(1)\mu\nu}(x) = 2\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}h_{\alpha\beta}(x), \tag{89}$$

with the differential operators $\mathcal{F}_x^{\mu\nu} \equiv \eta^{\mu\nu}\Box_x - \partial_x^\mu\partial_x^\nu$ and $\mathcal{F}_x^{\mu\nu\alpha\beta} \equiv 3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}$.

6.2 The kernels in the Minkowski background

Since the two kernels (36) are free of ultraviolet divergences in the limit $n \to 4$, we can deal directly with the $F^{\mu\nu\alpha\beta}(x-y) \equiv \lim_{n\to 4}\mu^{-2(n-4)}\,F_n^{\mu\nu\alpha\beta}$ in Equation (35). The kernels $N^{\mu\nu\alpha\beta}(x,y) = \mathrm{Re}\,F^{\mu\nu\alpha\beta}(x-y)$ and $H_{\mathrm{A}}^{\mu\nu\alpha\beta}(x,y) = \mathrm{Im}\,F^{\mu\nu\alpha\beta}(x-y)$ are actually the components of the "physical" noise and dissipation kernels that will appear in the Einstein–Langevin equations once the renormalization procedure has been carried out. The bi-tensor $F^{\mu\nu\alpha\beta}$ can be expressed in terms of the Wightman function in four spacetime dimensions, according to Equation (38). The different terms in this kernel can be easily computed using the integrals

$$I(p) \equiv \int \frac{d^4k}{(2\pi)^4}\,\delta(k^2+m^2)\,\theta(-k^0)\,\delta[(k-p)^2+m^2]\,\theta(k^0-p^0), \tag{90}$$

and $I^{\mu_1\cdots\mu_r}(p)$, which are defined as the previous one by inserting the momenta $k^{\mu_1}\dots k^{\mu_r}$ with $r=1,\dots,4$ in the integrand. All these integrals can be expressed in terms of $I(p)$; see [209] for the explicit expressions. It is convenient to separate $I(p)$ into its even and odd parts with respect to the variables p^μ as

$$I(p) = I_{\rm S}(p) + I_{\rm A}(p), \tag{91}$$

where $I_{\rm S}(-p) = I_{\rm S}(p)$ and $I_{\rm A}(-p) = -I_{\rm A}(p)$. These two functions are explicitly given by

$$\begin{aligned} I_{\rm S}(p) &= \frac{1}{8(2\pi)^3}\,\theta(-p^2-4m^2)\,\sqrt{1+4\frac{m^2}{p^2}}, \\ I_{\rm A}(p) &= \frac{-1}{8(2\pi)^3}\,{\rm sign}\,p^0\,\theta(-p^2-4m^2)\,\sqrt{1+4\frac{m^2}{p^2}}. \end{aligned} \tag{92}$$

After some manipulations, we find

$$\begin{aligned} F^{\mu\nu\alpha\beta}(x) = {}& \frac{\pi^2}{45}\,\mathcal{F}_x^{\mu\nu\alpha\beta}\int\frac{d^4p}{(2\pi)^4}\,e^{-ipx}\left(1+4\,\frac{m^2}{p^2}\right)^2 I(p) \\ &+\frac{8\pi^2}{9}\,\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\int\frac{d^4p}{(2\pi)^4}\,e^{-ipx}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2 I(p), \end{aligned} \tag{93}$$

where $\Delta\xi \equiv \xi - \frac{1}{6}$. The real and imaginary parts of the last expression, which yield the noise and dissipation kernels, are easily recognized as the terms containing $I_{\rm S}(p)$ and $I_{\rm A}(p)$, respectively. To write them explicitly, it is useful to introduce the new kernels

$$\begin{aligned} N_{\rm A}(x;m^2) &\equiv \frac{1}{480\pi}\int\frac{d^4p}{(2\pi)^4}\,e^{ipx}\,\theta(-p^2-4m^2)\,\sqrt{1+4\frac{m^2}{p^2}}\left(1+4\frac{m^2}{p^2}\right)^2, \\ N_{\rm B}(x;m^2,\Delta\xi) &\equiv \frac{1}{72\pi}\int\frac{d^4p}{(2\pi)^4}\,e^{ipx}\,\theta(-p^2-4m^2)\,\sqrt{1+4\frac{m^2}{p^2}}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2, \\ D_{\rm A}(x;m^2) &\equiv \frac{-i}{480\pi}\int\frac{d^4p}{(2\pi)^4}\,e^{ipx}\,{\rm sign}\,p^0\,\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\left(1+4\frac{m^2}{p^2}\right)^2, \\ D_{\rm B}(x;m^2,\Delta\xi) &\equiv \frac{-i}{72\pi}\int\frac{d^4p}{(2\pi)^4}\,e^{ipx}\,{\rm sign}\,p^0\,\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2, \end{aligned} \tag{94}$$

and we finally get

$$\begin{aligned} N^{\mu\nu\alpha\beta}(x,y) &= \frac{1}{6}\mathcal{F}_x^{\mu\nu\alpha\beta}N_{\rm A}(x-y;m^2) + \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}N_{\rm B}(x-y;m^2,\Delta\xi), \\ H_{\rm A}^{\mu\nu\alpha\beta}(x,y) &= \frac{1}{6}\mathcal{F}_x^{\mu\nu\alpha\beta}D_{\rm A}(x-y;m^2) + \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}D_{\rm B}(x-y;m^2,\Delta\xi). \end{aligned} \tag{95}$$

Notice that the noise and dissipation kernels defined in Equation (94) are actually real because, for the noise kernels, only the $\cos px$ terms of the exponentials e^{ipx} contribute to the integrals, and, for the dissipation kernels, the only contribution of such exponentials comes from the $i\sin px$ terms.

The evaluation of the kernel $H_{{\rm S}_n}^{\mu\nu\alpha\beta}(x,y)$ is a more involved task. Since this kernel contains divergences in the limit $n\to 4$, we use dimensional regularization. Using Equation (39), this kernel can be written in terms of the Feynman propagator (81) as

$$\mu^{-(n-4)}H_{{\rm S}_n}^{\mu\nu\alpha\beta}(x,y) = {\rm Im}\,K^{\mu\nu\alpha\beta}(x-y), \tag{96}$$

where

$$K^{\mu\nu\alpha\beta}(x) \equiv -\mu^{-(n-4)} \Big\{ 2\partial^\mu\partial^{(\alpha}\Delta_{F_n}(x)\,\partial^{\beta)}\partial^\nu\Delta_{F_n}(x) + 2\mathcal{D}^{\mu\nu}\left(\partial^\alpha\Delta_{F_n}(x)\partial^\beta\Delta_{F_n}(x)\right)$$
$$+2\mathcal{D}^{\alpha\beta}\left(\partial^\mu\Delta_{F_n}(x)\,\partial^\nu\Delta_{F_n}(x)\right) + 2\mathcal{D}^{\mu\nu}\mathcal{D}^{\alpha\beta}\left(\Delta^2_{F_n}(x)\right)$$
$$+\Big[\eta^{\mu\nu}\partial^{(\alpha}\Delta_{F_n}(x)\partial^{\beta)} + \eta^{\alpha\beta}\partial^{(\mu}\Delta_{F_n}(x)\partial^{\nu)} + \Delta_{F_n}(0)\left(\eta^{\mu\nu}\mathcal{D}^{\alpha\beta} + \eta^{\alpha\beta}\mathcal{D}^{\mu\nu}\right)$$
$$+\frac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta}\left(\Delta_{F_n}(x)\Box - m^2\Delta_{F_n}(0)\right)\Big]\delta^n(x)\Big\}. \tag{97}$$

Let us define the integrals

$$J_n(p) \equiv \mu^{-(n-4)} \int \frac{d^n k}{(2\pi)^n}\, \frac{1}{(k^2+m^2-i\epsilon)\left[(k-p)^2+m^2-i\epsilon\right]}, \tag{98}$$

and $J_n^{\mu_1\dots\mu_r}(p)$ obtained by inserting the momenta $k^{\mu_1}\dots k^{\mu_r}$ into the previous integral, together with

$$I_{0_n} \equiv \mu^{-(n-4)} \int \frac{d^n k}{(2\pi)^n}\, \frac{1}{(k^2+m^2-i\epsilon)}, \tag{99}$$

and $I_{0_n}^{\mu_1\dots\mu_r}$ which are also obtained by inserting momenta in the integrand. Then, the different terms in Equation (97) can be computed; these integrals are explicitly given in [209]. It is found that $I_{0_n}^{\mu} = 0$, and the remaining integrals can be written in terms of I_{0_n} and $J_n(p)$. It is useful to introduce the projector $P^{\mu\nu}$ orthogonal to p^μ and the tensor $P^{\mu\nu\alpha\beta}$ as

$$p^2 P^{\mu\nu} \equiv \eta^{\mu\nu}p^2 - p^\mu p^\nu, \qquad P^{\mu\nu\alpha\beta} \equiv 3P^{\mu(\alpha}P^{\beta)\nu} - P^{\mu\nu}P^{\alpha\beta}. \tag{100}$$

Then the action of the operator $\mathcal{F}^{\mu\nu}_x$ is simply written as $\mathcal{F}^{\mu\nu}_x \int d^n p\, e^{ipx}\, f(p) = -\int d^n p\, e^{ipx}\, f(p)\, p^2 P^{\mu\nu}$, where $f(p)$ is an arbitrary function of p^μ.

Finally after a rather long but straightforward calculation, and after expanding around $n=4$, we get,

$$K^{\mu\nu\alpha\beta}(x) = \frac{i}{(4\pi)^2}\Bigg\{ \kappa_n \Bigg[\frac{1}{90}\mathcal{F}^{\mu\nu\alpha\beta}_x\delta^n(x) + 4\Delta\xi^2\mathcal{F}^{\mu\nu}_x\mathcal{F}^{\alpha\beta}_x\delta^n(x)$$
$$+\frac{2}{3}\frac{m^2}{(n-2)}\left(\eta^{\mu\nu}\eta^{\alpha\beta}\Box_x - \eta^{\mu(\alpha}\eta^{\beta)\nu}\Box_x + \eta^{\mu(\alpha}\partial^{\beta)}_x\partial^\nu_x\right.$$
$$\left.+\eta^{\nu(\alpha}\partial^{\beta)}_x\partial^\mu_x - \eta^{\mu\nu}\partial^\alpha_x\partial^\beta_x - \eta^{\alpha\beta}\partial^\mu_x\partial^\nu_x\right)\delta^n(x)$$
$$+\frac{4m^4}{n(n-2)}\left(2\eta^{\mu(\alpha}\eta^{\beta)\nu} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)\delta^n(x)\Bigg]$$
$$+\frac{1}{180}\mathcal{F}^{\mu\nu\alpha\beta}_x \int \frac{d^n p}{(2\pi)^n}\, e^{ipx}\left(1+4\frac{m^2}{p^2}\right)^2 \bar{\phi}(p^2)$$
$$+\frac{2}{9}\mathcal{F}^{\mu\nu}_x\mathcal{F}^{\alpha\beta}_x \int \frac{d^n p}{(2\pi)^n}\, e^{ipx}\left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 \bar{\phi}(p^2)$$
$$-\left[\frac{4}{675}\mathcal{F}^{\mu\nu\alpha\beta}_x + \frac{1}{270}\left(60\xi - 11\right)\mathcal{F}^{\mu\nu}_x\mathcal{F}^{\alpha\beta}_x\right]\delta^n(x)$$
$$-m^2\left[\frac{2}{135}\mathcal{F}^{\mu\nu\alpha\beta}_x + \frac{1}{27}\mathcal{F}^{\mu\nu}_x\mathcal{F}^{\alpha\beta}_x\right]\Delta_n(x)\Bigg\} + \mathcal{O}(n-4), \tag{101}$$

where κ_n has been defined in Equation (86), and $\bar{\phi}(p^2)$ and $\Delta_n(x)$ are given by

$$\bar{\phi}(p^2) \equiv \int_0^1 d\alpha \, \ln\left(1 + \frac{p^2}{m^2}\alpha(1-\alpha) - i\epsilon\right) = -i\pi\theta(-p^2 - 4m^2)\sqrt{1 + 4\frac{m^2}{p^2}} + \varphi(p^2), \quad (102)$$

$$\Delta_n(x) \equiv \int \frac{d^n p}{(2\pi)^n}\, e^{ipx}\, \frac{1}{p^2}, \quad (103)$$

where

$$\varphi(p^2) \equiv \int_0^1 d\alpha \, \ln\left|1 + \frac{p^2}{m^2}\,\alpha(1-\alpha)\right|.$$

The imaginary part of Equation (101) gives the kernel components $\mu^{-(n-4)} H_{\mathrm{S}_n}^{\mu\nu\alpha\beta}(x,y)$, according to Equation (96). It can be easily obtained multiplying this expression by $-i$ and retaining only the real part $\varphi(p^2)$ of the function $\bar{\phi}(p^2)$.

6.3 The Einstein–Langevin equation

With the previous results for the kernels we can now write the n-dimensional Einstein–Langevin equation (87), previous to the renormalization. Taking also into account Equations (82) and (83), we may finally write:

$$\begin{aligned}
&\frac{1}{8\pi G_{\mathrm{B}}} G^{(1)\mu\nu}(x) - \frac{4}{3}\,\alpha_{\mathrm{B}} D^{(1)\mu\nu}(x) - 2\beta_{\mathrm{B}} B^{(1)\mu\nu}(x) \\
&+ \frac{\kappa_n}{(4\pi)^2}\left[-4\Delta\xi \frac{m^2}{(n-2)}\, G^{(1)\mu\nu} + \frac{1}{90}\, D^{(1)\mu\nu} \Delta\xi^2 B^{(1)\mu\nu}\right](x) \\
&+ \frac{1}{2880\pi^2}\left\{-\frac{16}{15}\, D^{(1)\mu\nu}(x) + \left(\frac{1}{6} - 10\Delta\xi\right) B^{(1)\mu\nu}(x)\right. \\
&\qquad + \int d^n y \int \frac{d^n p}{(2\pi)^n} e^{ip(x-y)} \varphi(p^2)\left[\left(1 + 4\frac{m^2}{p^2}\right)^2 D^{(1)\mu\nu}(y) + 10\left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 B^{(1)\mu\nu}(y)\right] \\
&\qquad \left. - \frac{m^2}{3} \int d^n y\, \Delta_n(x-y)\, (8D^{(1)\mu\nu} + 5B^{(1)\mu\nu})\,(y)\right\} \\
&+ \frac{1}{2} \int d^n y\, \mu^{-(n-4)} H_{\mathrm{A}_n}^{\mu\nu\alpha\beta}(x,y)\, h_{\alpha\beta}(y) + \mathcal{O}(n-4) = \xi^{\mu\nu}(x). \qquad (104)
\end{aligned}$$

Notice that the terms containing the bare cosmological constant have cancelled. These equations can now be renormalized, that is, we can now write the bare coupling constants as renormalized coupling constants plus some suitably chosen counterterms, and take the limit $n \to 4$. In order to carry out such a procedure, it is convenient to distinguish between massive and massless scalar fields. The details of the calculation can be found in [209].

It is convenient to introduce the two new kernels

$$H_{\rm A}(x;m^2) \equiv \frac{1}{480\pi^2}\int \frac{d^4p}{(2\pi)^4}\, e^{ipx} \times \left\{\left(1+4\frac{m^2}{p^2}\right)^2\left[-i\pi\,{\rm sign}\,p^0\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}+\varphi(p^2)\right]-\frac{8}{3}\frac{m^2}{p^2}\right\}, \tag{105}$$

$$H_{\rm B}(x;m^2,\Delta\xi) \equiv \frac{1}{72\pi^2}\int \frac{d^4p}{(2\pi)^4}\, e^{ipx} \times \left\{\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2\left[-i\pi\,{\rm sign}\,p^0\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}+\varphi(p^2)\right]-\frac{1}{6}\frac{m^2}{p^2}\right\},$$

where $\varphi(p^2)$ is given by the restriction to $n=4$ of expression (102). The renormalized coupling constants $1/G$, α, and β are easily computed as it was done in Equation (85). Substituting their expressions into Equation (104), we can take the limit $n\to 4$, using the fact that, for $n=4$, $D^{(1)\mu\nu}(x)=\frac{3}{2}A^{(1)\mu\nu}(x)$, we obtain the corresponding semiclassical Einstein–Langevin equation.

For the massless case one needs the limit $m\to 0$ of Equation (104). In this case it is convenient to separate κ_n in Equation (86) as $\kappa_n=\tilde{\kappa}_n+\frac{1}{2}\ln(m^2/\mu^2)+\mathcal{O}(n-4)$, where

$$\tilde{\kappa}_n \equiv \frac{1}{n-4}\left(\frac{e^\gamma}{4\pi}\right)^{\frac{n-4}{2}} = \frac{1}{n-4}+\frac{1}{2}\ln\left(\frac{e^\gamma}{4\pi}\right)+\mathcal{O}(n-4), \tag{106}$$

and use that, from Equation (102), we have

$$\lim_{m^2\to 0}\left[\varphi(p^2)+\ln\frac{m^2}{\mu^2}\right] = -2+\ln\left|\frac{p^2}{\mu^2}\right|. \tag{107}$$

The coupling constants are then easily renormalized. We note that in the massless limit, the Newtonian gravitational constant is not renormalized and, in the conformal coupling case, $\Delta\xi=0$, we have that $\beta_{\rm B}=\beta$. Note also that, by making $m=0$ in Equation (94), the noise and dissipation kernels can be written as

$$\begin{aligned} N_{\rm A}(x;m^2=0)&=N(x), \qquad & N_{\rm B}(x;m^2=0,\Delta\xi)&=60\Delta\xi^2 N(x),\\ D_{\rm A}(x;m^2=0)&=D(x), \qquad & D_{\rm B}(x;m^2=0,\Delta\xi)&=60\Delta\xi^2 D(x), \end{aligned} \tag{108}$$

where

$$N(x)\equiv\frac{1}{480\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\theta(-p^2), \qquad D(x)\equiv\frac{-i}{480\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\,{\rm sign}\,p^0\theta(-p^2). \tag{109}$$

It is also convenient to introduce the new kernel

$$\begin{aligned} H(x;\mu^2) &\equiv \frac{1}{480\pi^2}\int\frac{d^4p}{(2\pi)^4}\,e^{ipx}\left[\ln\left|\frac{p^2}{\mu^2}\right|-i\pi\,{\rm sign}\,p^0\,\theta(-p^2)\right]\\ &= \frac{1}{480\pi^2}\lim_{\epsilon\to 0^+}\int\frac{d^4p}{(2\pi)^4}\,e^{ipx}\,\ln\left(\frac{-(p^0+i\epsilon)^2+p^ip_i}{\mu^2}\right). \end{aligned} \tag{110}$$

This kernel is real and can be written as the sum of an even part and an odd part in the variables x^μ, where the odd part is the dissipation kernel $D(x)$. The Fourier transforms (109) and (110) can actually be computed and, thus, in this case we have explicit expressions for the kernels in position space; see, for instance, [179, 56, 137].

Finally, the Einstein–Langevin equation for the physical stochastic perturbations $h_{\mu\nu}$ can be written in both cases, for $m \neq 0$ and for $m = 0$, as

$$\begin{aligned} &\frac{1}{8\pi G} G^{(1)\mu\nu}(x) - 2\left(\bar{\alpha} A^{(1)\mu\nu}(x) + \bar{\beta} B^{(1)\mu\nu}(x)\right) \\ &+ \frac{1}{4}\int d^4y \left[H_{\rm A}(x-y) A^{(1)\mu\nu}(y) + H_{\rm B}(x-y) B^{(1)\mu\nu}(y)\right] = \xi^{\mu\nu}(x), \end{aligned} \qquad (111)$$

where in terms of the renormalized constants α and β the new constants are $\bar{\alpha} = \alpha + (3600\pi^2)^{-1}$ and $\bar{\beta} = \beta - (\frac{1}{12} - 5\Delta\xi)(2880\pi^2)^{-1}$. The kernels $H_{\rm A}(x)$ and $H_{\rm B}(x)$ are given by Equations (105) when $m \neq 0$, and $H_{\rm A}(x) = H(x;\mu^2)$, $H_{\rm B}(x) = 60\Delta\xi^2 H(x;\mu^2)$ when $m = 0$. In the massless case, we can use the arbitrariness of the mass scale μ to eliminate one of the parameters $\bar{\alpha}$ or $\bar{\beta}$. The components of the Gaussian stochastic source $\xi^{\mu\nu}$ have zero mean value, and their two-point correlation functions are given by $\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(y)\rangle_{\rm s} = N^{\mu\nu\alpha\beta}(x,y)$, where the noise kernel is given in Equation (95), which in the massless case reduces to Equation (108).

It is interesting to consider the massless conformally coupled scalar field, i.e., the case $\Delta\xi = 0$, which is of particular interest because of its similarities with the electromagnetic field, and also because of its interest in cosmology: Massive fields become conformally invariant when their masses are negligible compared to the spacetime curvature. We have already mentioned that for a conformally coupled field, the stochastic source tensor must be traceless (up to first order in perturbation theory around semiclassical gravity), in the sense that the stochastic variable $\xi^{\mu}_{\mu} \equiv \eta_{\mu\nu}\xi^{\mu\nu}$ behaves deterministically as a vanishing scalar field. This can be directly checked by noticing, from Equations (95) and (108), that, when $\Delta\xi = 0$, one has $\langle \xi^{\mu}_{\mu}(x)\xi^{\alpha\beta}(y)\rangle_{\rm s} = 0$, since $\mathcal{F}^{\mu}_{\mu} = 3\Box$ and $\mathcal{F}^{\mu\alpha}\mathcal{F}^{\beta}_{\mu} = \Box\mathcal{F}^{\alpha\beta}$. The Einstein–Langevin equations for this particular case (and generalized to a spatially flat Robertson–Walker background) were first obtained in [58], where the coupling constant β was fixed to be zero. See also [169] for a discussion of this result and its connection to the problem of structure formation in the trace anomaly driven inflation [269, 280, 132].

Note that the expectation value of the renormalized stress-energy tensor for a scalar field can be obtained by comparing Equation (111) with the Einstein–Langevin equation (14), its explicit expression is given in [209]. The results agree with the general form found by Horowitz [137, 138] using an axiomatic approach, and coincides with that given in [91]. The particular cases of conformal coupling, $\Delta\xi = 0$, and minimal coupling, $\Delta\xi = -1/6$, are also in agreement with the results for these cases given in [137, 138, 270, 57, 182], modulo local terms proportional to $A^{(1)\mu\nu}$ and $B^{(1)\mu\nu}$ due to different choices of the renormalization scheme. For the case of a massive minimally coupled scalar field, $\Delta\xi = -\frac{1}{6}$, our result is equivalent to that of [276].

6.4 Correlation functions for gravitational perturbations

Here we solve the Einstein–Langevin equations (111) for the components $G^{(1)\mu\nu}$ of the linearized Einstein tensor. Then we use these solutions to compute the corresponding two-point correlation functions, which give a measure of the gravitational fluctuations predicted by the stochastic semiclassical theory of gravity in the present case. Since the linearized Einstein tensor is invariant under gauge transformations of the metric perturbations, these two-point correlation functions are also gauge invariant. Once we have computed the two-point correlation functions for the linearized Einstein tensor, we find the solutions for the metric perturbations and compute the associated two-point correlation functions. The procedure used to solve the Einstein–Langevin equation is similar to the one used by Horowitz [137] (see also [91]) to analyze the stability of Minkowski spacetime in semiclassical gravity.

We first note that the tensors $A^{(1)\mu\nu}$ and $B^{(1)\mu\nu}$ can be written in terms of $G^{(1)\mu\nu}$ as

$$A^{(1)\mu\nu} = \frac{2}{3}\left(\mathcal{F}^{\mu\nu} G^{(1)\alpha}_{\ \ \ \ \alpha} - \mathcal{F}^{\alpha}_{\alpha} G^{(1)\mu\nu}\right), \qquad B^{(1)\mu\nu} = 2\mathcal{F}^{\mu\nu} G^{(1)\alpha}_{\ \ \ \ \alpha}, \tag{112}$$

where we have used that $3\Box = \mathcal{F}^{\alpha}_{\alpha}$. Therefore, the Einstein–Langevin equation (111) can be seen as a linear integro-differential stochastic equation for the components $G^{(1)\mu\nu}$. In order to find solutions to Equation (111), it is convenient to Fourier transform. With the convention $\tilde{f}(p) = \int d^4x e^{-ipx} f(x)$ for a given field $f(x)$, one finds, from Equation (112),

$$\begin{aligned} \tilde{A}^{(1)\mu\nu}(p) &= 2p^2 \tilde{G}^{(1)\mu\nu}(p) - \frac{2}{3} p^2 P^{\mu\nu} \tilde{G}^{(1)\alpha}_{\ \ \ \ \alpha}(p), \\ \tilde{B}^{(1)\mu\nu}(p) &= -2p^2 P^{\mu\nu} \tilde{G}^{(1)\alpha}_{\ \ \ \ \alpha}(p). \end{aligned} \tag{113}$$

The Fourier transform of the Einstein–Langevin Equation (111) now reads

$$F^{\mu\nu}{}_{\alpha\beta}(p)\, \tilde{G}^{(1)\alpha\beta}(p) = 8\pi G\, \tilde{\xi}^{\mu\nu}(p), \tag{114}$$

where

$$F^{\mu\nu}{}_{\alpha\beta}(p) \equiv F_1(p)\, \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} + F_2(p)\, p^2 P^{\mu\nu}\eta_{\alpha\beta}, \tag{115}$$

with

$$\begin{aligned} F_1(p) &\equiv 1 + 16\pi G\, p^2 \left[\frac{1}{4}\tilde{H}_{\mathrm{A}}(p) - 2\bar{\alpha}\right], \\ F_2(p) &\equiv -\frac{16}{3}\pi G \left[\frac{1}{4}\tilde{H}_{\mathrm{A}}(p) + \frac{3}{4}\tilde{H}_{\mathrm{B}}(p) - 2\bar{\alpha} - 6\bar{\beta}\right]. \end{aligned} \tag{116}$$

In the Fourier transformed Einstein–Langevin Equation (114), $\tilde{\xi}^{\mu\nu}(p)$, the Fourier transform of $\xi^{\mu\nu}(x)$, is a Gaussian stochastic source of zero average, and

$$\left\langle \tilde{\xi}^{\mu\nu}(p)\tilde{\xi}^{\alpha\beta}(p')\right\rangle_{\mathrm{s}} = (2\pi)^4\, \delta^4(p+p')\, \tilde{N}^{\mu\nu\alpha\beta}(p), \tag{117}$$

where we have introduced the Fourier transform of the noise kernel. The explicit expression for $\tilde{N}^{\mu\nu\alpha\beta}(p)$ is found from Equations (95) and (94) to be

$$\tilde{N}^{\mu\nu\alpha\beta}(p) = \frac{\theta(-p^2-4m^2)}{720\pi}\sqrt{1+4\frac{m^2}{p^2}}\left[\frac{1}{4}\left(1+4\frac{m^2}{p^2}\right)^2 (p^2)^2 P^{\mu\nu\alpha\beta} + 10\left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 (p^2)^2 P^{\mu\nu}P^{\alpha\beta}\right], \tag{118}$$

which in the massless case reduces to

$$\lim_{m\to 0} \tilde{N}^{\mu\nu\alpha\beta}(p) = \frac{1}{480\pi}\,\theta(-p^2)\left[\frac{1}{6}(p^2)^2\, P^{\mu\nu\alpha\beta} + 60\Delta\xi^2 (p^2)^2 P^{\mu\nu}P^{\alpha\beta}\right]. \tag{119}$$

6.4.1 Correlation functions for the linearized Einstein tensor

In general, we can write $G^{(1)\mu\nu} = \langle G^{(1)\mu\nu}\rangle_{\mathrm{s}} + G^{(1)\mu\nu}_{\mathrm{f}}$, where $G^{(1)\mu\nu}_{\mathrm{f}}$ is a solution to Equations (111) with zero average, or Equation (114) in the Fourier transformed version. The averages $\langle G^{(1)\mu\nu}\rangle_{\mathrm{s}}$ must be a solution of the linearized semiclassical Einstein equations obtained by averaging Equations (111) or (114). Solutions to these equations (specially in the massless case, $m = 0$) have been studied by several authors [137, 138, 141, 247, 248, 273, 274, 128, 262, 182, 91], particularly in connection with the problem of the stability of the ground state of semiclassical gravity. The two-point correlation functions for the linearized Einstein tensor are defined by

$$\begin{aligned}\mathcal{G}^{\mu\nu\alpha\beta}(x,x') &\equiv \left\langle G^{(1)\mu\nu}(x)\, G^{(1)\alpha\beta}(x')\right\rangle_{\rm s} - \langle G^{(1)\mu\nu}(x)\rangle_{\rm s} \left\langle G^{(1)\alpha\beta}(x')\right\rangle_{\rm s} \\ &= \left\langle G_{\rm f}^{(1)\mu\nu}(x)\, G_{\rm f}^{(1)\alpha\beta}(x')\right\rangle_{\rm s}. \end{aligned} \tag{120}$$

Now we shall seek the family of solutions to the Einstein–Langevin equation which can be written as a linear functional of the stochastic source, and whose Fourier transform $\tilde{G}^{(1)\mu\nu}(p)$ depends locally on $\tilde{\xi}^{\alpha\beta}(p)$. Each of such solutions is a Gaussian stochastic field and, thus, it can be completely characterized by the averages $\langle G^{(1)\mu\nu}\rangle_{\rm s}$ and the two-point correlation functions (120). For such a family of solutions, $\tilde{G}_{\rm f}^{(1)\mu\nu}(p)$ is the most general solution to Equation (114) which is linear, homogeneous, and local in $\tilde{\xi}^{\alpha\beta}(p)$. It can be written as

$$\tilde{G}_{\rm f}^{(1)\mu\nu}(p) = 8\pi G\, D^{\mu\nu}{}_{\alpha\beta}(p)\, \tilde{\xi}^{\alpha\beta}(p), \tag{121}$$

where $D^{\mu\nu}{}_{\alpha\beta}(p)$ are the components of a Lorentz invariant tensor field distribution in Minkowski spacetime[3], symmetric under the interchanges $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$, which is the most general solution of

$$F^{\mu\nu}{}_{\rho\sigma}(p)\, D^{\rho\sigma}{}_{\alpha\beta}(p) = \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}. \tag{122}$$

In addition, we must impose the conservation condition, $p_\nu \tilde{G}_{\rm f}^{(1)\mu\nu}(p) = 0$, where this zero must be understood as a stochastic variable which behaves deterministically as a zero vector field. We can write $D^{\mu\nu}{}_{\alpha\beta}(p) = D_{\rm p}^{\mu\nu}{}_{\alpha\beta}(p) + D_{\rm h}^{\mu\nu}{}_{\alpha\beta}(p)$, where $D_{\rm p}^{\mu\nu}{}_{\alpha\beta}(p)$ is a particular solution to Equation (122) and $D_{\rm h}^{\mu\nu}{}_{\alpha\beta}(p)$ is the most general solution to the homogeneous equation. Consequently, see Equation (121), we can write $\tilde{G}_{\rm f}^{(1)\mu\nu}(p) = \tilde{G}_{\rm p}^{(1)\mu\nu}(p) + \tilde{G}_{\rm h}^{(1)\mu\nu}(p)$. To find the particular solution, we try an ansatz of the form

$$D_{\rm p}^{\mu\nu}{}_{\alpha\beta}(p) = d_1(p)\, \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} + d_2(p)\, p^2 P^{\mu\nu}\eta_{\alpha\beta}. \tag{123}$$

Substituting this ansatz into Equations (122), it is easy to see that it solves these equations if we take

$$d_1(p) = \left[\frac{1}{F_1(p)}\right]_{\rm r}, \qquad d_2(p) = -\left[\frac{F_2(p)}{F_1(p)F_3(p)}\right]_{\rm r}, \tag{124}$$

with

$$F_3(p) \equiv F_1(p) + 3p^2 F_2(p) = 1 - 48\pi G\, p^2 \left[\frac{1}{4}\tilde{H}_{\rm B}(p) - 2\bar{\beta}\right], \tag{125}$$

and where the notation $[\ \ ]_{\rm r}$ means that the zeros of the denominators are regulated with appropriate prescriptions in such a way that $d_1(p)$ and $d_2(p)$ are well defined Lorentz invariant scalar distributions. This yields a particular solution to the Einstein–Langevin equations,

$$\tilde{G}_{\rm p}^{(1)\mu\nu}(p) = 8\pi G\, D_{\rm p}^{\mu\nu}{}_{\alpha\beta}(p)\, \tilde{\xi}^{\alpha\beta}(p), \tag{126}$$

which, since the stochastic source is conserved, satisfies the conservation condition. Note that, in the case of a massless scalar field ($m = 0$), the above solution has a functional form analogous to that of the solutions of linearized semiclassical gravity found in the appendix of [91]. Notice also that, for a massless conformally coupled field ($m = 0$ and $\Delta\xi = 0$), the second term on the right-hand side of Equation (123) will not contribute in the correlation functions (120), since in this case the stochastic source is traceless.

[3]By "Lorentz invariant" we mean invariant under the transformations of the orthochronous Lorentz subgroup; see [137] for more details on the definition and properties of these tensor distributions.

A detailed analysis given in [209] concludes that the homogeneous solution $\tilde{G}_{\rm h}^{(1)\mu\nu}(p)$ gives no contribution to the correlation functions (120). Consequently $\mathcal{G}^{\mu\nu\alpha\beta}(x,x') = \langle G_{\rm p}^{(1)\mu\nu}(x) G_{\rm p}^{(1)\alpha\beta}(x')\rangle_{\rm s}$, where $G_{\rm p}^{(1)\mu\nu}(x)$ is the inverse Fourier transform of Equation (126), and the correlation functions (120) are

$$\left\langle \tilde{G}_{\rm p}^{(1)\mu\nu}(p)\, \tilde{G}_{\rm p}^{(1)\alpha\beta}(p')\right\rangle_{\rm s} = 64(2\pi)^6 G^2 \delta^4(p+p') D_{\rm p}^{\mu\nu}{}_{\rho\sigma}(p)\, D_{\rm p}^{\alpha\beta}{}_{\lambda\gamma}(-p)\, \tilde{N}^{\rho\sigma\lambda\gamma}(p). \tag{127}$$

It is easy to see from the above analysis that the prescriptions $[\ \ ]_{\rm r}$ in the factors $D_{\rm p}$ are irrelevant in the last expression and, thus, they can be suppressed. Taking into account that $F_l(-p) = F_l^*(p)$, with $l = 1,2,3$, we get from Equations (123) and (124)

$$\begin{aligned}\left\langle \tilde{G}_{\rm p}^{(1)\mu\nu}(p)\, \tilde{G}_{\rm p}^{(1)\alpha\beta}(p')\right\rangle_{\rm s} = 64(2\pi)^6 G^2 \frac{\delta^4(p+p')}{|F_1(p)|^2}\Bigg[\tilde{N}^{\mu\nu\alpha\beta}(p) - \frac{F_2(p)}{F_3(p)} p^2 P^{\mu\nu} \tilde{N}^{\alpha\beta\rho}{}_{\rho}(p) \\ - \frac{F_2^*(p)}{F_3^*(p)} p^2 P^{\alpha\beta} \tilde{N}^{\mu\nu\rho}{}_{\rho}(p) + \frac{|F_2(p)|^2}{|F_3(p)|^2} p^2 P^{\mu\nu} p^2 P^{\alpha\beta} \tilde{N}^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma}(p)\Bigg]. \end{aligned} \tag{128}$$

This last expression is well defined as a bi-distribution and can be easily evaluated using Equation (118). The final explicit result for the Fourier transformed correlation function for the Einstein tensor is thus

$$\begin{aligned}\left\langle \tilde{G}_{\rm p}^{(1)\mu\nu}(p) \tilde{G}_{\rm p}^{(1)\alpha\beta}(p')\right\rangle_{\rm s} = \frac{2}{45}(2\pi)^5\, G^2\, \frac{\delta^4(p+p')}{|F_1(p)|^2}\, \theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}} \\ \times\Bigg[\frac{1}{4}\left(1+4\frac{m^2}{p^2}\right)^2 (p^2)^2 P^{\mu\nu\alpha\beta} \\ +10\left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 (p^2)^2 P^{\mu\nu}P^{\alpha\beta}\left|1-3p^2\frac{F_2(p)}{F_3(p)}\right|^2\Bigg]. \end{aligned}\tag{129}$$

To obtain the correlation functions in coordinate space, Equation (120), we take the inverse Fourier transform. The final result is

$$\mathcal{G}^{\mu\nu\alpha\beta}(x,x') = \frac{\pi}{45}\, G^2\, \mathcal{F}_x^{\mu\nu\alpha\beta}\, \mathcal{G}_{\rm A}(x-x') + \frac{8\pi}{9}\, G^2\, \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\, \mathcal{G}_{\rm B}(x-x'), \tag{130}$$

with

$$\begin{aligned}\tilde{\mathcal{G}}_{\rm A}(p) &\equiv \theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\left(1+4\frac{m^2}{p^2}\right)^2 \frac{1}{|F_1(p)|^2}, \\ \tilde{\mathcal{G}}_{\rm B}(p) &\equiv \theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2 \frac{1}{|F_1(p)|^2}\left|1-3p^2\frac{F_2(p)}{F_3(p)}\right|^2, \end{aligned}\tag{131}$$

where $F_l(p)$, $l = 1,2,3$, are given in Equations (116) and (125). Notice that, for a massless field ($m = 0$), we have

$$\begin{aligned} F_1(p) &= 1 + 4\pi G p^2 \tilde{H}(p;\bar{\mu}^2), \\ F_2(p) &= -\frac{16}{3}\pi G\left[(1+180\Delta\xi^2)\frac{1}{4}\tilde{H}(p;\bar{\mu}^2) - 6\Upsilon\right], \\ F_3(p) &= 1 - 48\pi G p^2\left[15\Delta\xi^2 \tilde{H}(p;\bar{\mu}^2) - 2\Upsilon\right], \end{aligned}\tag{132}$$

with $\bar{\mu} \equiv \mu\, \exp(1920\pi^2\bar{\alpha})$ and $\Upsilon \equiv \bar{\beta} - 60\Delta\xi^2\bar{\alpha}$, and where $\tilde{H}(p;\mu^2)$ is the Fourier transform of $H(x;\mu^2)$ given in Equation (110).

6.4.2 Correlation functions for the metric perturbations

Starting from the solutions found for the linearized Einstein tensor, which are characterized by the two-point correlation functions (130) (or, in terms of Fourier transforms, Equation (129)), we can now solve the equations for the metric perturbations. Working in the harmonic gauge, $\partial_\nu \bar{h}^{\mu\nu} = 0$ (this zero must be understood in a statistical sense) where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\alpha}_{\alpha}$, the equations for the metric perturbations in terms of the Einstein tensor are

$$\Box \bar{h}^{\mu\nu}(x) = -2G^{(1)\mu\nu}(x), \tag{133}$$

or, in terms of Fourier transforms, $p^2\tilde{\bar{h}}^{\mu\nu}(p) = 2\tilde{G}^{(1)\mu\nu}(p)$. Similarly to the analysis of the equation for the Einstein tensor, we can write $\bar{h}^{\mu\nu} = \langle \bar{h}^{\mu\nu}\rangle_{\rm s} + \bar{h}^{\mu\nu}_{\rm f}$, where $\bar{h}^{\mu\nu}_{\rm f}$ is a solution to these equations with zero average, and the two-point correlation functions are defined by

$$\begin{aligned}\mathcal{H}^{\mu\nu\alpha\beta}(x,x') &\equiv \left\langle \bar{h}^{\mu\nu}(x)\bar{h}^{\alpha\beta}(x')\right\rangle_{\rm s} - \left\langle \bar{h}^{\mu\nu}(x)\right\rangle_{\rm s}\left\langle \bar{h}^{\alpha\beta}(x')\right\rangle_{\rm s} \\ &= \left\langle \bar{h}^{\mu\nu}_{\rm f}(x)\bar{h}^{\alpha\beta}_{\rm f}(x')\right\rangle_{\rm s}.\end{aligned} \tag{134}$$

We can now seek solutions of the Fourier transform of Equation (133) of the form $\tilde{\bar{h}}^{\mu\nu}_{\rm f}(p) = 2D(p)\tilde{G}^{(1)\mu\nu}_{\rm f}(p)$, where $D(p)$ is a Lorentz invariant scalar distribution in Minkowski spacetime, which is the most general solution of $p^2 D(p) = 1$. Note that, since the linearized Einstein tensor is conserved, solutions of this form automatically satisfy the harmonic gauge condition. As in the previous subsection, we can write $D(p) = [1/p^2]_{\rm r} + D_{\rm h}(p)$, where $D_{\rm h}(p)$ is the most general solution to the associated homogeneous equation and, correspondingly, we have $\tilde{\bar{h}}^{\mu\nu}_{\rm f}(p) = \tilde{\bar{h}}^{\mu\nu}_{\rm p}(p) + \tilde{\bar{h}}^{\mu\nu}_{\rm h}(p)$. However, since $D_{\rm h}(p)$ has support on the set of points for which $p^2 = 0$, it is easy to see from Equation (129) (from the factor $\theta(-p^2 - 4m^2)$) that $\langle \tilde{\bar{h}}^{\mu\nu}_{\rm h}(p)\tilde{G}^{(1)\alpha\beta}_{\rm f}(p')\rangle_{\rm s} = 0$ and, thus, the two-point correlation functions (134) can be computed from $\langle \tilde{\bar{h}}^{\mu\nu}_{\rm f}(p)\tilde{\bar{h}}^{\alpha\beta}_{\rm f}(p')\rangle_{\rm s} = \langle \tilde{\bar{h}}^{\mu\nu}_{\rm p}(p)\tilde{\bar{h}}^{\alpha\beta}_{\rm p}(p')\rangle_{\rm s}$. From Equation (129) and due to the factor $\theta(-p^2-4m^2)$, it is also easy to see that the prescription $[\ \ ]_{\rm r}$ is irrelevant in this correlation function, and we obtain

$$\left\langle \tilde{\bar{h}}^{\mu\nu}_{\rm p}(p)\tilde{\bar{h}}^{\alpha\beta}_{\rm p}(p')\right\rangle_{\rm s} = \frac{4}{(p^2)^2}\left\langle \tilde{G}^{(1)\mu\nu}_{\rm p}(p)\,\tilde{G}^{(1)\alpha\beta}_{\rm p}(p')\right\rangle_{\rm s}, \tag{135}$$

where $\langle \tilde{G}^{(1)\mu\nu}_{\rm p}(p)\,\tilde{G}^{(1)\alpha\beta}_{\rm p}(p')\rangle_{\rm s}$ is given by Equation (129). The right-hand side of this equation is a well defined bi-distribution, at least for $m \neq 0$ (the θ function provides the suitable cutoff). In the massless field case, since the noise kernel is obtained as the limit $m \to 0$ of the noise kernel for a massive field, it seems that the natural prescription to avoid the divergences on the lightcone $p^2 = 0$ is a Hadamard finite part (see [256, 302] for its definition). Taking this prescription, we also get a well defined bi-distribution for the massless limit of the last expression.

The final result for the two-point correlation function for the field $\bar{h}^{\mu\nu}$ is

$$\mathcal{H}^{\mu\nu\alpha\beta}(x,x') = \frac{4\pi}{45}\,G^2\,\mathcal{F}^{\mu\nu\alpha\beta}_x\,\mathcal{H}_{\rm A}(x-x') + \frac{32\pi}{9}\,G^2\,\mathcal{F}^{\mu\nu}_x\mathcal{F}^{\alpha\beta}_x\,\mathcal{H}_{\rm B}(x-x'), \tag{136}$$

where $\tilde{\mathcal{H}}_{\rm A}(p) \equiv [1/(p^2)^2]\,\tilde{\mathcal{G}}_{\rm A}(p)$ and $\tilde{\mathcal{H}}_{\rm B}(p) \equiv [1/(p^2)^2]\,\tilde{\mathcal{G}}_{\rm B}(p)$, with $\tilde{\mathcal{G}}_{\rm A}(p)$ and $\tilde{\mathcal{G}}_{\rm B}(p)$ given by Equation (131). The two-point correlation functions for the metric perturbations can be easily obtained using $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{\alpha}_{\alpha}$.

6.4.3 Conformally coupled field

For a conformally coupled field, i.e., when $m = 0$ and $\Delta\xi = 0$, the previous correlation functions are greatly simplified and can be approximated explicitly in terms of analytic functions. The detailed results are given in [209]; here we outline the main features.

When $m = 0$ and $\Delta\xi = 0$, we have $\mathcal{G}_{\rm B}(x) = 0$ and $\tilde{\mathcal{G}}_{\rm A}(p) = \theta(-p^2)\,|F_1(p)|^{-2}$. Thus the two-point correlations functions for the Einstein tensor is

$$\mathcal{G}^{\mu\nu\alpha\beta}(x,x') = \frac{\pi}{45}\, G^2\, \mathcal{F}_x^{\mu\nu\alpha\beta} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-x')}\,\theta(-p^2)}{\left|1 + 4\pi\, G p^2 \tilde{H}(p;\bar{\mu}^2)\right|^2}, \tag{137}$$

where $\tilde{H}(p,\mu^2) = (480\pi^2)^{-1} \ln[-((p^0 + i\epsilon)^2 + p^i p_i)/\mu^2]$ (see Equation (110)).

To estimate this integral, let us consider spacelike separated points $(x - x')^\mu = (0, \mathbf{x} - \mathbf{x}')$, and define $\mathbf{y} = \mathbf{x} - \mathbf{x}'$. We may now formally change the momentum variable p^μ by the dimensionless vector s^μ, $p^\mu = s^\mu/|\mathbf{y}|$. Then the previous integral denominator is $|1 + 16\pi(L_{\rm P}/|\mathbf{y}|)^2 s^2 \tilde{H}(s)|^2$, where we have introduced the Planck length $L_{\rm P} = \sqrt{G}$. It is clear that we can consider two regimes: (a) when $L_{\rm P} \ll |\mathbf{y}|$, and (b) when $|\mathbf{y}| \sim L_{\rm P}$. In case (a) the correlation function, for the 0000 component, say, will be of the order

$$\mathcal{G}^{0000}(\mathbf{y}) \sim \frac{L_{\rm P}^4}{|\mathbf{y}|^8}.$$

In case (b) when the denominator has zeros, a detailed calculation carried out in [209] shows that

$$\mathcal{G}^{0000}(\mathbf{y}) \sim e^{-|\mathbf{y}|/L_{\rm P}} \left(\frac{L_{\rm P}}{|\mathbf{y}|^5} + \cdots + \frac{1}{L_{\rm P}^2 |\mathbf{y}|^2} \right),$$

which indicates an exponential decay at distances around the Planck scale. Thus short scale fluctuations are strongly suppressed.

For the two-point metric correlation the results are similar. In this case we have

$$\mathcal{H}^{\mu\nu\alpha\beta}(x,x') = \frac{4\pi}{45}\, G^2\, \mathcal{F}_x^{\mu\nu\alpha\beta} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-x')}\theta(-p^2)}{(p^2)^2 \left|1 + 4\pi\, G p^2 \tilde{H}(p;\bar{\mu}^2)\right|^2}. \tag{138}$$

The integrand has the same behavior of the correlation function of Equation (137), thus matter fields tends to suppress the short scale metric perturbations. In this case we find, as for the correlation of the Einstein tensor, that for case (a) above we have

$$\mathcal{H}^{0000}(\mathbf{y}) \sim \frac{L_{\rm P}^4}{|\mathbf{y}|^4},$$

and for case (b) we have

$$\mathcal{H}^{0000}(\mathbf{y}) \sim e^{-|\mathbf{y}|/L_{\rm P}} \left(\frac{L_{\rm P}}{|\mathbf{y}|} + \dots \right).$$

It is interesting to write expression (138) in an alternative way. If we use the dimensionless tensor $P^{\mu\nu\alpha\beta}$ introduced in Equation (100), which accounts for the effect of the operator $\mathcal{F}_x^{\mu\nu\alpha\beta}$, we can write

$$\mathcal{H}^{\mu\nu\alpha\beta}(x,x') = \frac{4\pi}{45}\, G^2 \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-x')}\, P^{\mu\nu\alpha\beta}\, \theta(-p^2)}{\left|1 + 4\pi\, G p^2 \tilde{H}(p;\bar{\mu}^2)\right|^2}. \tag{139}$$

This expression allows a direct comparison with the graviton propagator for linearized quantum gravity in the $1/N$ expansion found by Tomboulis [277]. One can see that the imaginary part of the graviton propagator leads, in fact, to Equation (139). In [255] it is shown that the two-point correlation functions for the metric perturbations derived from the Einstein–Langevin equation are equivalent to the symmetrized quantum two-point correlation functions for the metric fluctuations in the large N expansion of quantum gravity interacting with N matter fields.

6.5 Discussion

The main results of this section are the correlation functions (130) and (136). In the case of a conformal field, the correlation functions of the linearized Einstein tensor have been explicitly estimated. From the exponential factors $e^{-|\mathbf{y}|/L_{\mathrm{P}}}$ in these results for scales near the Planck length, we see that the correlation functions of the linearized Einstein tensor have the Planck length as the correlation length. A similar behavior is found for the correlation functions of the metric perturbations. Since these fluctuations are induced by the matter fluctuations, we infer that the effect of the matter fields is to suppress the fluctuations of the metric at very small scales. On the other hand, at scales much larger than the Planck length, the induced metric fluctuations are small compared with the free graviton propagator which goes like $L_{\mathrm{P}}^2/|\mathbf{y}|^2$, since the action for the free graviton goes like $S_{\mathrm{h}} \sim \int d^4x \, L_{\mathrm{P}}^{-2} h \Box h$.

For background solutions of semiclassical gravity with other scales present apart from the Planck scales (for instance, for matter fields in a thermal state), stress-energy fluctuations may be important at larger scales. For such backgrounds, stochastic semiclassical gravity might predict correlation functions with characteristic correlation lengths larger than the Planck scales. It seems quite plausible, nevertheless, that these correlation functions would remain non-analytic in their characteristic correlation lengths. This would imply that these correlation functions could not be obtained from a calculation involving a perturbative expansion in the characteristic correlation lengths. In particular, if these correlation lengths are proportional to the Planck constant $\hbar$, the gravitational correlation functions could not be obtained from an expansion in $\hbar$. Hence, stochastic semiclassical gravity might predict a behavior for gravitational correlation functions different from that of the analogous functions in perturbative quantum gravity [79, 78, 77, 80]. This is not necessarily inconsistent with having neglected action terms of higher order in $\hbar$ when considering semiclassical gravity as an effective theory [91]. It is, in fact, consistent with the closed connection of stochastic gravity with the large N expansion of quantum gravity interacting with N matter fields.

7 Structure Formation

Cosmological structure formation is a key problem in modern cosmology [190, 229] and inflation offers a natural solution to this problem. If an inflationary period is present, the initial seeds for the generation of the primordial inhomogeneities that lead to the large scale structure have their source in the quantum fluctuations of the inflaton field, the field which is generally responsible for driving inflation. Stochastic gravity provides a sound and natural formalism for the derivation of the cosmological perturbations generated during inflation.

In [254] it was shown that the correlation functions that follow from the Einstein–Langevin equation which emerges in the framework of stochastic gravity coincide with that obtained with the usual quantization procedures [218], when both the metric perturbations and the inflaton fluctuations are both linearized. Stochastic gravity, however, can naturally deal with the fluctuations of the inflaton field even beyond the linear approximation.

Here we will illustrate the equivalence with the usual formalism, based on the quantization of the linear cosmological and inflaton perturbations, with one of the simplest chaotic inflationary models in which the background spacetime is a quasi-de Sitter universe [253, 254].

7.1 The model

In this chaotic inflationary model [199] the inflaton field ϕ of mass m is described by the following Lagrangian density:

$$\mathcal{L}(\phi) = \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi + \frac{1}{2} m^2 \phi^2. \tag{140}$$

The conditions for the existence of an inflationary period, which is characterized by an accelerated cosmological expansion, is that the value of the field over a region with the typical size of the Hubble radius is higher than the Planck mass m_{P}. This is because in order to solve the cosmological horizon and flatness problem more than 60 e-folds of expansion are needed; to achieve this the scalar field should begin with a value higher than $3m_{\mathrm{P}}$. The inflaton mass is small: As we will see, the large scale anisotropies measured in the cosmic background radiation [265] restrict the inflaton mass to be of the order of $10^{-6} m_{\mathrm{P}}$. We will not discuss the naturalness of this inflationary model and we will simply assume that if one such region is found (inside a much larger universe) it will inflate to become our observable universe.

We want to study the metric perturbations produced by the stress-energy tensor fluctuations of the inflaton field on the homogeneous background of a flat Friedmann–Robertson–Walker model, described by the cosmological scale factor $a(\eta)$, where η is the conformal time, which is driven by the homogeneous inflaton field $\phi(\eta) = \langle \hat{\phi} \rangle$. Thus we write the inflaton field in the following form:

$$\hat{\phi} = \phi(\eta) + \hat{\varphi}(x), \tag{141}$$

where $\hat{\varphi}(x)$ corresponds to a free massive quantum scalar field with zero expectation value on the homogeneous background metric, $\langle \hat{\varphi} \rangle = 0$. We will restrict ourselves to scalar-type metric perturbations, because these are the ones that couple to the inflaton fluctuations in the linear theory. We note that this is not so if we were to consider inflaton fluctuations beyond the linear approximation; then tensorial and vectorial metric perturbations would also be driven. The perturbed metric $\tilde{g}_{ab} = g_{ab} + h_{ab}$ can be written in the longitudinal gauge as

$$ds^2 = a^2(\eta) \left[-(1 + 2\Phi(x)) d\eta^2 + (1 - 2\Psi(x)) \delta_{ij} dx^i dx^j \right], \tag{142}$$

where the scalar metric perturbations $\Phi(x)$ and $\Psi(x)$ correspond to Bardeen's gauge invariant variables [12].

7.2 The Einstein–Langevin equation for scalar metric perturbations

The Einstein–Langevin equation as described in Section 3 is gauge invariant, and thus we can work in a desired gauge and then extract the gauge invariant quantities. The Einstein–Langevin equation (14) reads now

$$G_{ab}^{(0)} - 8\pi G \left\langle \hat{T}_{ab}^{(0)} \right\rangle + G_{ab}^{(1)}(h) - 8\pi G \left\langle \hat{T}_{ab}^{(1)}(h) \right\rangle = 8\pi G \xi_{ab}, \tag{143}$$

where the two first terms cancel, that is $G_{ab}^{(0)} - 8\pi G \left\langle \hat{T}_{ab}^{(0)} \right\rangle = 0$, as the background metric satisfies the semiclassical Einstein equations. Here the superscripts $^{(0)}$ and $^{(1)}$ refer to functions in the background metric g_{ab} and linear in the metric perturbation h_{ab}, respectively. The stress tensor operator $\hat{T}_{ab}$ for the minimally coupled inflaton field in the perturbed metric is

$$\hat{T}_{ab} = \tilde{\nabla}_a \hat{\phi} \tilde{\nabla}_b \hat{\phi} + \frac{1}{2} \tilde{g}_{ab} \left(\tilde{\nabla}_c \hat{\phi} \tilde{\nabla}^c \hat{\phi} + m^2 \hat{\phi}^2 \right). \tag{144}$$

Using the decomposition of the scalar field into its homogeneous and inhomogeneous part, see Equation (141), and the metric $\tilde{g}_{ab}$ into its homogeneous background g_{ab} and its perturbation h_{ab}, the renormalized expectation value for the stress-energy tensor operator can be written as

$$\left\langle \hat{T}_{ab}^{\mathrm{R}}[\tilde{g}] \right\rangle = \left\langle \hat{T}_{ab}[\tilde{g}] \right\rangle_{\phi\phi} + \left\langle \hat{T}_{ab}[\tilde{g}] \right\rangle_{\phi\varphi} + \left\langle \hat{T}_{ab}^{\mathrm{R}}[\tilde{g}] \right\rangle_{\varphi\varphi}, \tag{145}$$

where the subindices indicate the degree of dependence on the homogeneous field ϕ, and its perturbation φ. The first term in this equation depends only on the homogeneous field and it is given by the classical expression. The second term is proportional to $\langle \hat{\varphi}[\tilde{g}] \rangle$ which is not zero because the field dynamics is considered on the perturbed spacetime, i.e., this term includes the coupling of the field with h_{ab} and may be obtained from the expectation value of the linearized Klein–Gordon equation,

$$\left(\Box_{g+h} - m^2 \right) \hat{\varphi} = 0. \tag{146}$$

The last term in Equation (145) corresponds to the expectation value to the stress tensor for a free scalar field on the spacetime of the perturbed metric.

After using the previous decomposition, the noise kernel $N_{abcd}[g; x, y)$ defined in Equation (11) can be written as

$$\left\langle \left\{ \hat{t}_{ab}[g;x), \hat{t}_{cd}[g;y) \right\} \right\rangle = \left\langle \left\{ \hat{t}_{ab}[g;x), \hat{t}_{cd}[g;y) \right\} \right\rangle_{(\phi\varphi)^2} + \left\langle \left\{ \hat{t}_{ab}[g;x), \hat{t}_{cd}[g;y) \right\} \right\rangle_{(\varphi\varphi)^2}, \tag{147}$$

where we have used the fact that $\langle \hat{\varphi} \rangle = 0 = \langle \hat{\varphi}\hat{\varphi}\hat{\varphi} \rangle$ for Gaussian states on the background geometry. We consider the vacuum state to be the Euclidean vacuum which is preferred in the de Sitter background, and this state is Gaussian. In the above equation the first term is quadratic in $\hat{\varphi}$, whereas the second one is quartic. Both contributions to the noise kernel are separately conserved since both $\phi(\eta)$ and $\hat{\varphi}$ satisfy the Klein–Gordon field equations on the background spacetime. Consequently, the two terms can be considered separately. On the other hand, if one treats $\hat{\varphi}$ as a small perturbation, the second term in Equation (147) is of lower order than the first and may be consistently neglected; this corresponds to neglecting the last term of Equation (145). The stress tensor fluctuations due to a term of that kind were considered in [252].

We can now write down the Einstein–Langevin equations (143) to linear order in the inflaton fluctuations. It is easy to check [254] that the *space-space* components coming from the stress tensor expectation value terms and the stochastic tensor are diagonal, i.e., $\langle \hat{T}_{ij} \rangle = 0 = \xi_{ij}$ for $i \neq j$. This, in turn, implies that the two functions characterizing the scalar metric perturbations are equal,

$\Phi = \Psi$, in agreement with [218]. The equation for Φ can be obtained from the $0i$-component of the Einstein–Langevin equation, which in Fourier space reads

$$2ik_i(\mathcal{H}\Phi_k + \Phi_k') = 8\pi G(\xi_{0i})_k, \tag{148}$$

where k_i is the comoving momentum component associated to the comoving coordinate x^i, and where we have used the definition $\Phi_k(\eta) = \int d^3x \exp(-i\vec{k}\cdot\vec{x})\Phi(\eta,\vec{x})$. Here primes denote derivatives with respect to the conformal time η and $\mathcal{H} = a'/a$. A nonlocal term of dissipative character which comes from the second term in Equation (145) should also appear on the left-hand side of Equation (148), but we have neglected it to simplify the forthcoming expressions. Its inclusion does not change the large scale spectrum in an essential way [254]. Note, however, that the equivalence of the stochastic approach to linear order in $\hat{\varphi}$ and the usual linear cosmological perturbations approach is independent of that approximation [254]. To solve Equation (148), whose left-hand side comes from the linearized Einstein tensor for the perturbed metric [218], we need the retarded propagator for the gravitational potential Φ_k,

$$G_k(\eta,\eta') = -i\frac{4\pi}{k_i m_{\rm P}^2}\left(\theta(\eta-\eta')\frac{a(\eta')}{a(\eta)} + f(\eta,\eta')\right), \tag{149}$$

where f is a homogeneous solution of Equation (148) related to the initial conditions chosen, and $m_{\rm P}^2 = 1/G$. For instance, if we take $f(\eta,\eta') = -\theta(\eta_0-\eta')a(\eta')/a(\eta)$, the solution would correspond to "turning on" the stochastic source at η_0. With the solution of the Einstein–Langevin equation (148) for the scalar metric perturbations we are in the position to compute the two-point correlation functions for these perturbations.

7.3 Correlation functions for scalar metric perturbations

The the two-point correlation function for the scalar metric perturbations induced by the inflaton fluctuations is thus given by

$$\langle\Phi_k(\eta)\Phi_{k'}(\eta')\rangle_{\rm s} = (2\pi)^2\delta(\vec{k}+\vec{k}')\int^{\eta}d\eta_1\int^{\eta'}d\eta_2\, G_k(\eta,\eta_1)\, G_{k'}(\eta',\eta_2)\,\langle(\xi_{0i})_k(\eta_1)(\xi_{0i})_{k'}(\eta_2)\rangle_{\rm s}. \tag{150}$$

Here two-point correlation function for the stochastic source, which is connected to the stress-energy tensor fluctuations through the noise kernel, is given by

$$\begin{aligned}\langle(\xi_{0i})_k(\eta_1)(\xi_{0i})_{-k}(\eta_2)\rangle_{\rm s} &= \frac{1}{2}\langle\{(\hat{t}_{0i})_k(\eta_1),(\hat{t}_{0i})_{-k}(\eta_2)\}\rangle_{\phi\varphi}\\ &= \frac{1}{2}k_ik_i\phi'(\eta_1)\phi'(\eta_2)G_k^{(1)}(\eta_1,\eta_2),\end{aligned} \tag{151}$$

where $G_k^{(1)}(\eta_1,\eta_2) = \langle\{\hat{\varphi}_k(\eta_1),\hat{\varphi}_{-k}(\eta_2)\}\rangle$ is the k-mode Hadamard function for a free minimally coupled scalar field in the appropriate vacuum state on the Friedmann–Robertson–Walker background.

In practice, to make the explicit computation of the Hadamard function, we will assume that the field state is in the Euclidean vacuum and the background spacetime is de Sitter. Furthermore we will compute the Hadamard function for a massless field, and will make a perturbative expansion in terms of the dimensionless parameter $m/m_{\rm P}$. Thus we consider

$$\bar{G}_k^{(1)}(\eta_1,\eta_2) = \langle 0|\{\hat{y}_k(\eta_1),\hat{y}_{-k}(\eta_2)\}|0\rangle = 2\mathcal{R}\left(u_k(\eta_1)u_k^*(\eta_2)\right),$$

with

$$\hat{y}_k(\eta) = a(\eta)\hat{\varphi}_k(\eta) = \hat{a}_k u_k(\eta) + \hat{a}_{-k}^\dagger u_{-k}^*(\eta),$$

and where

$$u_k = (2k)^{-1/2}\, e^{ik\eta}\,(1 - i/\eta)$$

are the positive frequency k-modes for a massless minimally coupled scalar field on a de Sitter background, which define the Euclidean vacuum state, $\hat{a}_k|0\rangle = 0$ [25].

The assumption of a massless field for the computation of the Hadamard function is made because massless modes in de Sitter are much simpler to deal with than massive modes. We can see that this is, however, a reasonable approximation as follows. For a given mode the $m = 0$ approximation is reasonable when its wavelength λ is shorter that the Compton wavelength, $\lambda_c = 1/m$. In our case we have a very small mass m, and the horizon size H^{-1}, where H is the Hubble constant $H = \dot{a}/a$ (here $a(t)$ with t the physical time $dt = ad\eta$), satisfies that $H^{-1} < \lambda_c$. Thus, for modes inside the horizon, $\lambda < \lambda_c$ and $m = 0$ is a reasonable approximation. Outside the horizon massive modes decay in amplitude as $\sim \exp(-m^2 t/H)$, whereas massless modes remain constant, thus when modes leave the horizon the approximation will eventually break down. However, we only need to ensure that the approximation is still valid after 60 e-folds, i.e., $Ht \sim 60$, but this is the case since $60\ m^2 < H^2$ given that $m \sim 10^{-6} m_P$, and $m \ll H$ as in most inflationary models [190, 229].

The background geometry is not exactly that of de Sitter spacetime, for which $a(\eta) = -(H\eta)^{-1}$ with $-\infty < \eta < 0$. One can expand in terms of the "slow-roll" parameters and assume that to first order $\dot{\phi}(t) \simeq m_P^2 (m/m_P)$, where t is the physical time. The correlation function for the metric perturbation (150) can then be easily computed; see [253, 254] for details. The final result, however, is very weakly dependent on the initial conditions, as one may understand from the fact that the accelerated expansion of de quasi-de Sitter spacetime during inflation erases the information about the initial conditions. Thus one may take the initial time to be $\eta_0 = -\infty$, and obtain to lowest order in m/m_P the expression

$$\langle \Phi_k(\eta)\, \Phi_{k'}(\eta')\rangle_s \simeq 8\pi^2 \left(\frac{m}{m_P}\right)^2 k^{-3} (2\pi)^3 \delta(\vec{k} + \vec{k}') \cos k(\eta - \eta'). \tag{152}$$

From this result two main conclusions are derived. First, the prediction of an almost Harrison–Zel'dovich scale-invariant spectrum for large scales, i.e., small values of k. Second, since the correlation function is of order of $(m/m_P)^2$, a severe bound to the mass m is imposed by the gravitational fluctuations derived from the small values of the Cosmic Microwave Background (CMB) anisotropies detected by COBE. This bound is of the order of $(m/m_P) \sim 10^{-6}$ [265, 218].

We should now comment on some differences with those works in [45, 213, 212, 51] which used a self-interacting scalar field or a scalar field interacting nonlinearly with other fields. In those works an important relaxation of the ratio m/m_P was found. The long wavelength modes of the inflaton field were regarded as an open system in an environment made out of the shorter wavelength modes. Then, Langevin type equations were used to compute the correlations of the long wavelength modes driven by the fluctuations of the shorter wavelength modes. In order to get a significant relaxation on the above ratio, however, one had to assume that the correlations of the free long wavelength modes, which correspond to the dispersion of the system initial state, had to be very small. Otherwise they dominate by several orders of magnitude those fluctuations that come from the noise of the environment. This would require a great amount of fine-tuning for the initial quantum state of each mode [254]. We should remark that in the model discussed here there is no environment for the inflaton fluctuations. The inflaton fluctuations, however, are responsible for the noise that induces the metric perturbations.

7.4 Discussion

One important advantage of the Einstein–Langevin approach to the gravitational fluctuations in inflaton over the approach based on the quantization of the linear perturbations of both the metric and the inflaton field [218], is that an exact treatment of the inflaton quantum fluctuations is possible. This leads to corrections to the almost scale invariant spectrum for scalar metric perturbations at large scales, and has implications for the spectrum of the cosmic microwave background anisotropies. However, in the standard inflationary models these corrections are subdominant. Furthermore when the full non linear effect of the quantum field is considered, tensorial metric perturbations are also induced by the inflaton fluctuations. An estimation of this effect, presumably subdominant over the free tensorial fluctuations, has not been performed.

We should remark that although the gravitational fluctuations are here assumed to be classical, the correlation functions obtained correspond to the expectation values of the symmetrized quantum metric perturbations [49, 254]. This means that even in the absence of decoherence the fluctuations predicted by the Einstein–Langevin equation, whose solutions do not describe the actual dynamics of the gravitational field any longer, still give the correct symmetrized quantum two-point functions.

Another important advantage of the stochastic gravity approach is that one may also compute the gravitational fluctuations in inflationary models which are not driven by an inflaton field, such as Starobinsky inflation which is driven by the trace anomaly due to conformally coupled quantum fields. In fact, Einstein's semiclassical equation (7) for a massless quantum field which is conformally coupled to the gravitational field admits an inflationary solution which is almost de Sitter initially and ends up in a matter-dominated-like regime [269, 280]. In these models the standard approach based on the quantization of the gravitational and the matter fields to linear order cannot be used. This is because the calculation of the metric perturbations correspond to having only the last term in the noise kernel in Equation (147), since there is no homogeneous field $\phi(\eta)$ as the expectation value $\langle\hat{\phi}\rangle = 0$, and linearization becomes trivial.

In the trace anomaly induced inflation framework, Hawking et al. [132] were able to compute the two-point quantum correlation function for scalar and tensorial metric perturbations in a spatially closed de Sitter universe, making use of the anti-de Sitter conformal field theory correspondence. They find that short scale metric perturbations are strongly suppressed by the conformal matter fields. This is similar to what we obtained in Section 6 for the induced metric fluctuations in Minkowski spacetime. In the stochastic gravity context, the noise kernel in a spatially closed de Sitter background was derived in [252]. However, in a spatially flat arbitrary Friedmann–Robertson–Walker model the Einstein–Langevin equation describing the metric perturbations was first obtained in [58] (see also [169]). The two-point correlation functions for the metric perturbations can be derived from its solutions, but this is work still in progress.

8 Black Hole Backreaction

As another illustration of the application of stochastic gravity we consider fluctuations and backreaction in black hole spacetimes. The celebrated Hawking effect of particle creation from black holes is constructed from a quantum field theory in a curved spacetime framework. The oft-mentioned 'black hole evaporation' referring to the reduction of the mass of a black hole due to particle creation must entail backreaction considerations. Backreaction of Hawking radiation [118, 13, 297, 298, 299, 135, 136, 8] could alter the evolution of the background spacetime and change the nature of its end state, more drastically so for Planck size black holes. Because of the higher symmetry in cosmological spacetimes, backreaction studies of processes therein have progressed further than the corresponding black hole problems, which to a large degree is still concerned with finding the right approximations for the regularized energy-momentum tensor [177, 233, 210, 6, 7, 5, 134] for even the simplest spacetimes such as the spherically symmetric family including the important Schwarzschild metric (for a summary of the cosmological backreaction problem treated in the stochastic gravity theory, see [169]). The latest important work is that of Hiscock, Larson, and Anderson [134] on backreaction in the interior of a black hole, where one can find a concise summary of earlier work. To name a few of the important landmarks in this endeavor (this is adopted from [134]), Howard and Candelas [145, 144] have computed the stress-energy of a conformally invariant scalar field in the Schwarzschild geometry. Jensen and Ottewill [176] have computed the vacuum stress-energy of a massless vector field in Schwarzschild spacetime. Approximation methods have been developed by Page, Brown, and Ottewill [231, 28, 29] for conformally invariant fields in Schwarzschild spacetime, Frolov and Zel'nikov [99] for conformally invariant fields in a general static spacetime, and Anderson, Hiscock, and Samuel [6, 7] for massless arbitrarily coupled scalar fields in a general static spherically symmetric spacetime. Furthermore the DeWitt–Schwinger approximation has been derived by Frolov and Zel'nikov [97, 98] for massive fields in Kerr spacetime, by Anderson, Hiscock, and Samuel [6, 7] for a general (arbitrary curvature coupling and mass) scalar field in a general static spherically symmetric spacetime and Anderson, Hiscock, and Samuel have applied their method to the Reissner–Nordström geometry [5]. Though arduous and demanding, the effort continues on because of the importance of backreaction effects of Hawking radiation in black holes. They are expected to address some of the most basic issues such as black hole thermodynamics [174, 235, 282, 16, 17, 18, 266, 175, 287, 281, 275, 183, 271, 139, 140, 205, 204] and the black hole end state and information loss puzzles [230].

Here we wish to address the black hole backreaction problem with new insights provided by stochastic semiclassical gravity. (For the latest developments see, e.g., the reviews [151, 154, 168, 169]). It is not our intention to seek better approximations for the regularized energy-momentum tensor, but to point out new ingredients lacking in the existing framework based on semiclassical gravity. In particular one needs to consider both the dissipation and the fluctuations aspects in the backreaction of particle creation or vacuum polarization.

In a short note [164] Hu, Raval, and Sinha discussed the formulation of the problem in this new light, commented on some shortcomings of existing works, and sketched the strategy [264] behind the stochastic gravity theory approach to the black hole fluctuations and backreaction problem. Here we follow their treatment with focus on the class of quasi-static black holes.

From the new perspective provided by statistical field theory and stochastic gravity, it is not difficult to postulate that the backreaction effect is the manifestation of a fluctuation-dissipation relation [85, 86, 220, 35, 34, 288]. This was first conjectured by Candelas and Sciama [60, 258, 259] for a dynamic Kerr black hole emitting Hawking radiation, and by Mottola [217] for a static black hole (in a box) in quasi-equilibrium with its radiation via linear response theory [191, 24, 192, 195, 193]. However, these proposals as originally formulated do not capture the full spirit and content of the self-consistent dynamical backreaction problem. Generally speaking (paraphrasing Mottola), linear response theory is not designed for tackling backreaction problems. More specifically, if one assumes a specified background spacetime (static in this case) and state (thermal) of the matter field(s) as done in [217], one would get a specific self-consistent solution. But in the most general situation which a full backreaction program demands of, the spacetime and the state of matter should be determined by their dynamics under mutual influence on an equal footing, and the solutions checked to be physically sound by some criteria like stability consideration. A recent work of Anderson, Molina-Paris, and Mottola [9, 10] on linear response theory does not make these restrictions. They addressed the issue of the validity of semiclassical gravity (SCG) based on an analysis of the stability of solutions to the semiclassical Einstein equation (SEE). However, on this issue, Hu, Roura, and Verdaguer [165] pointed out the importance of including both the intrinsic and induced fluctuations in the stability analysis, the latter being given by the noise kernel. The fluctuation part represented by the noise kernel is amiss in the fluctuation-dissipation relation proposed by Candelas and Sciama [60, 258, 259] (see below). As will be shown in an explicit example later, the backreaction is an intrinsically dynamic process. The Einstein–Langevin equation in stochastic gravity overcomes both of these deficiencies.

For Candelas and Sciama [60, 258, 259], the classical formula they showed relating the dissipation in area linearly to the squared absolute value of the shear amplitude is suggestive of a fluctuation-dissipation relation. When the gravitational perturbations are quantized (they choose the quantum state to be the Unruh vacuum) they argue that it approximates a flux of radiation from the hole at large radii. Thus the dissipation in area due to the Hawking flux of gravitational radiation is allegedly related to the quantum fluctuations of gravitons. The criticism in [164] is that their's is not a fluctuation-dissipation relation in the truly statistical mechanical sense, because it does not relate dissipation of a certain quantity (in this case, horizon area) to the fluctuations of *the same quantity*. To do so would require one to compute the two point function of the area, which, being a four-point function of the graviton field, is related to a two-point function of the stress tensor. The stress tensor is the true "generalized force" acting on the spacetime via the equations of motion, and the dissipation in the metric must eventually be related to the fluctuations of this generalized force for the relation to qualify as a fluctuation-dissipation relation.

From this reasoning, we see that the stress-energy bi-tensor and its vacuum expectation value known as the noise kernel are the new ingredients in backreaction considerations. But these are exactly the centerpieces in stochastic gravity. Therefore the correct framework to address semiclassical backreaction problems is stochastic gravity theory, where fluctuations and dissipation are the equally essential components. The noise kernel for quantum fields in Minkowski and de Sitter spacetime has been carried out by Martin, Roura, and Verdaguer [207, 209, 254], and for thermal fields in black hole spacetimes and scalar fields in general spacetimes by Campos, Hu, and Phillips [54, 55, 244, 245, 241]. Earlier, for cosmological backreaction problems Hu and Sinha [167] derived a generalized expression relating dissipation (of anisotropy in Bianchi Type I universes) and fluctuations (measured by particle numbers created in neighboring histories). This example shows that one can understand the backreaction of particle creation as a manifestation of a (generalized) fluctuation-dissipation relation.

As an illustration of the application of stochastic gravity theory we outline the steps in a black hole backreaction calculation, focusing on the manageable quasi-static class. We adopt the Hartle–Hawking picture [127] where the black hole is bathed eternally – actually in quasi-thermal equilibrium – in the Hawking radiance it emits. It is described here by a massless scalar quantum field at the Hawking temperature. As is well-known, this quasi-equilibrium condition is possible only if the black hole is enclosed in a box of size suitably larger than the event horizon. We can divide our consideration into the far field case and the near horizon case. Campos and Hu [54, 55] have treated a relativistic thermal plasma in a weak gravitational field. Since the far field limit of a Schwarzschild metric is just the perturbed Minkowski spacetime, one can perform a perturbation expansion off hot flat space using the thermal Green functions [108]. Strictly speaking the location of the box holding the black hole in equilibrium with its thermal radiation is as far as one can go, thus the metric may not reach the perturbed Minkowski form. But one can also put the black hole and its radiation in an anti-de Sitter space [133], which contains such a region. Hot flat space has been studied before for various purposes (see, e.g., [116, 249, 250, 72, 27]). Campos and Hu derived a stochastic CTP effective action and from it an equation of motion, the Einstein–Langevin equation, for the dynamical effect of a scalar quantum field on a background spacetime. To perform calculations leading to the Einstein–Langevin equation, one needs to begin with a self-consistent solution of the semiclassical Einstein equation for the thermal field and the perturbed background spacetime. For a black hole background, a semiclassical gravity solution is provided by York [297, 298, 299]. For a Robertson–Walker background with thermal fields, it is given by Hu [148]. Recently, Sinha, Raval, and Hu [264] outlined a strategy for treating the near horizon case, following the same scheme of Campos and Hu. In both cases two new terms appear which are absent in semiclassical gravity considerations: a nonlocal dissipation and a (generally colored) noise kernel. When one takes the noise average, one recovers York's [297, 298, 299] semiclassical equations for radially perturbed quasi-static black holes. For the near horizon case one cannot obtain the full details yet, because the Green function for a scalar field in the Schwarzschild metric comes only in an approximate form (e.g., Page approximation [231]), which, though reasonably accurate for the stress tensor, fails poorly for the noise kernel [245, 241]. In addition a formula is derived in [264] expressing the CTP effective action in terms of the Bogolyubov coefficients. Since it measures not only the number of particles created, but also the difference of particle creation in alternative histories, this provides a useful avenue to explore the wider set of issues in black hole physics related to noise and fluctuations.

Since backreaction calculations in semiclassical gravity have been under study for a much longer time than in stochastic gravity, we will concentrate on explaining how the new stochastic features arise from the framework of semiclassical gravity, i.e., noise and fluctuations and their consequences. Technically the goal is to obtain an influence action for this model of a black hole coupled to a scalar field and to derive an Einstein–Langevin equation from it. As a by-product, from the fluctuation-dissipation relation, one can derive the vacuum susceptibility function and the isothermal compressibility function for black holes, two quantities of fundamental interest in characterizing the nonequilibrium thermodynamic properties of black holes.

8.1 The model

In this model the black hole spacetime is described by a spherically symmetric static metric with a line element of the following general form written in advanced time Eddington–Finkelstein coordinates,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\psi}\left(1 - \frac{2m}{r}\right)dv^2 + 2e^{2\psi}dv\,dr + r^2 d\Omega^2, \tag{153}$$

where $\psi = \psi(r)$ and $m = m(r)$, $v = t + r + 2M\ln\left(\frac{r}{2M} - 1\right)$, and $d\Omega^2$ is the line element on the two-sphere. Hawking radiation is described by a massless, conformally coupled quantum scalar field ϕ with the classical action

$$S_{\rm m}[\phi, g_{\mu\nu}] = -\frac{1}{2}\int d^n x\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \xi(n)R\phi^2], \tag{154}$$

where $\xi(n) = \frac{(n-2)}{4(n-1)}$ (n is the dimension of spacetime), and R is the curvature scalar of the spacetime it lives in.

Let us consider linear perturbations $h_{\mu\nu}$ off a background Schwarzschild metric $g^{(0)}_{\mu\nu}$,

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{155}$$

with standard line element

$$(ds^2)^0 = \left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr + r^2 d\Omega^2. \tag{156}$$

We look for this class of perturbed metrics in the form given by Equation (153) (thus restricting our consideration only to spherically symmetric perturbations),

$$e^{\psi} \simeq 1 + \epsilon\rho(r), \tag{157}$$

and

$$m \simeq M[1 + \epsilon\mu(r)], \tag{158}$$

where $\epsilon/(\lambda M^2) = \frac{1}{3} a T_{\mathrm{H}}^4$ with $a = \frac{\pi^2}{30}$ and $\lambda = 90(8^4)\pi^2$, and where T_{H} is the Hawking temperature. This particular parametrization of the perturbation is chosen following York's notation [297, 298, 299]. Thus the only non-zero components of $h_{\mu\nu}$ are

$$h_{vv} = -\left[\left(1 - \frac{2M}{r}\right) 2\epsilon\rho(r) + \frac{2M\epsilon\mu(r)}{r}\right], \tag{159}$$

and

$$h_{vr} = \epsilon\rho(r). \tag{160}$$

So this represents a metric with small static and radial perturbations about a Schwarzschild black hole. The initial quantum state of the scalar field is taken to be the Hartle–Hawking vacuum, which is essentially a thermal state at the Hawking temperature and it represents a black hole in (unstable) thermal equilibrium with its own Hawking radiation. In the far field limit, the gravitational field is described by a linear perturbation of Minkowski spacetime. In equilibrium the thermal bath can be characterized by a relativistic fluid with a four-velocity (time-like normalized vector field) u^μ, and temperature in its own rest frame β^{-1}.

To facilitate later comparisons with our program we briefly recall York's work [297, 298, 299]. (See also the work by Hochberg and Kephart [135] for a massless vector field, by Hochberg, Kephart, and York [136] for a massless spinor field, and by Anderson, Hiscock, Whitesell, and York [8] for a quantized massless scalar field with arbitrary coupling to spacetime curvature.) York considered the semiclassical Einstein equation,

$$G_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle T_{\mu\nu}\rangle, \tag{161}$$

with $G_{\mu\nu} \simeq G_{\mu\nu}^{(0)} + \delta G_{\mu\nu}$, where $G_{\mu\nu}^{(0)}$ is the Einstein tensor for the background spacetime. The zeroth order solution gives a background metric in empty space, i.e, the Schwarzschild metric. $\delta G_{\mu\nu}$ is the linear correction to the Einstein tensor in the perturbed metric. The semiclassical Einstein equation in this approximation therefore reduces to

$$\delta G_{\mu\nu}(g^{(0)}, h) = \kappa\langle T_{\mu\nu}\rangle. \tag{162}$$

York solved this equation to first order by using the expectation value of the energy-momentum tensor for a conformally coupled scalar field in the Hartle–Hawking vacuum in the unperturbed (Schwarzschild) spacetime on the right-hand side and using Equations (159) and (160) to calculate $\delta G_{\mu\nu}$ on the left-hand side. Unfortunately, no exact analytical expression is available for the $\langle T_{\mu\nu}\rangle$ in a Schwarzschild metric with the quantum field in the Hartle–Hawking vacuum that goes on the right-hand side. York therefore uses the approximate expression given by Page [231] which is known to give excellent agreement with numerical results. Page's approximate expression for $\langle T_{\mu\nu}\rangle$ was constructed using a thermal Feynman Green's function obtained by a conformal transformation of a WKB approximated Green's function for an optical Schwarzschild metric. York then solves the semiclassical Einstein equation (162) self-consistently to obtain the corrections to the background metric induced by the backreaction encoded in the functions $\mu(r)$ and $\rho(r)$. There was no mention of fluctuations or its effects. As we shall see, in the language of Sec. (4), the semiclassical gravity procedure which York followed working at the equation of motion level is equivalent to looking at the noise-averaged backreaction effects.

8.2 CTP effective action for the black hole

We first derive the CTP effective action for the model described in Sec. (7.1). Using the metric (156) (and neglecting the surface terms that appear in an integration by parts) we have the action for the scalar field written perturbatively as

$$S_{\rm m}[\phi, h_{\mu\nu}] = \frac{1}{2}\int d^n x\,\sqrt{-g^{(0)}}\,\phi\left[\Box^{(0)} + V^{(1)} + V^{(2)} + \dots\right]\phi, \tag{163}$$

where the first and second order perturbative operators $V^{(1)}$ and $V^{(2)}$ are given by

$$\begin{aligned} V^{(1)} &\equiv -\frac{1}{\sqrt{-g^{(0)}}}\left\{\partial_\mu\left(\sqrt{-g^{(0)}}\bar{h}^{\mu\nu}\right)\partial_\nu + \bar{h}^{\mu\nu}\partial_\mu\partial_\nu + \xi(n)R^{(1)}\right\}, \\ V^{(2)} &\equiv -\frac{1}{\sqrt{-g^{(0)}}}\left\{\partial_\mu\left(\sqrt{-g^{(0)}}\hat{h}^{\mu\nu}\right)\partial_\nu + \hat{h}^{\mu\nu}\partial_\mu\partial_\nu - \xi(n)\left(R^{(2)} + \frac{1}{2}hR^{(1)}\right)\right\}. \end{aligned} \tag{164}$$

In the above expressions, $R^{(k)}$ is the k-order term in the perturbation $h_{\mu\nu}(x)$ of the scalar curvature R, and $\bar{h}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ denote a linear and a quadratic combination of the perturbation, respectively:

$$\begin{aligned} \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \frac{1}{2}hg^{(0)}_{\mu\nu}, \\ \hat{h}_{\mu\nu} &\equiv h_\mu{}^\alpha h_{\alpha\nu} - \frac{1}{2}hh_{\mu\nu} + \frac{1}{8}h^2 g^{(0)}_{\mu\nu} - \frac{1}{4}h_{\alpha\beta}h^{\alpha\beta}g^{(0)}_{\mu\nu}. \end{aligned} \tag{165}$$

From quantum field theory in curved spacetime considerations discussed above we take the following action for the gravitational field:

$$\begin{aligned} S_{\rm g}[g_{\mu\nu}] = \frac{1}{(16\pi G)^{\frac{n-2}{2}}}\int d^n x\,\sqrt{-g(x)}R(x) + \frac{\alpha\bar{\mu}^{n-4}}{4(n-4)}\int d^n x\,\sqrt{-g(x)} \\ \times\left\{3R_{\mu\nu\alpha\beta}(x)R^{\mu\nu\alpha\beta}(x) - \left[1 - 360\left(\xi(n) - \frac{1}{6}\right)^2\right]R^2(x)\right\}. \end{aligned} \tag{166}$$

The first term is the classical Einstein–Hilbert action, and the second term is the counterterm in four dimensions used to renormalize the divergent effective action. In this action $\ell_{\rm P}^2 = 16\pi G_{\rm N}$, $\alpha = (2880\pi^2)^{-1}$, and $\bar{\mu}$ is an arbitrary mass scale.

We are interested in computing the CTP effective action (163) for the matter action and when the field ϕ is initially in the Hartle–Hawking vacuum. This is equivalent to saying that the initial state of the field is described by a thermal density matrix at a finite temperature $T = T_{\rm H}$. The CTP effective action at finite temperature $T \equiv 1/\beta$ for this model is given by (for details see [54, 55])

$$S^\beta_{\rm eff}[h^\pm_{\mu\nu}] = S_{\rm g}[h^+_{\mu\nu}] - S_{\rm g}[h^-_{\mu\nu}] - \frac{i}{2}{\rm tr}\left\{\ln\bar{G}^\beta_{ab}\left[h^\pm_{\mu\nu}\right]\right\}, \tag{167}$$

where $\pm$ denote the forward and backward time path of the CTP formalism, and $\bar{G}^\beta_{ab}[h^\pm_{\mu\nu}]$ is the complete 2×2 matrix propagator (a and b take $\pm$ values: G_{++}, G_{+-}, and G_{--} correspond to the Feynman, Wightman, and Schwinger Green's functions respectively) with thermal boundary conditions for the differential operator $\sqrt{-g^{(0)}}(\Box + V^{(1)} + V^{(2)} + \dots)$. The actual form of $\bar{G}^{\beta}_{ab}$ cannot be explicitly given. However, it is easy to obtain a perturbative expansion in terms of $V^{(k)}_{ab}$, the k-order matrix version of the complete differential operator defined by $V^{(k)}_{\pm\pm} \equiv \pm V^{(k)}_{\pm}$ and $V^{(k)}_{\pm\mp} \equiv 0$, and G^{β}_{ab}, the thermal matrix propagator for a massless scalar field in Schwarzschild spacetime. To second order $\bar{G}^{\beta}_{ab}$ reads

$$\bar{G}^{\beta}_{ab} = G^{\beta}_{ab} - G^{\beta}_{ac}\, V^{(1)}_{cd}\, G^{\beta}_{db} - G^{\beta}_{ac}\, V^{(2)}_{cd}\, G^{\beta}_{db} + G^{\beta}_{ac}\, V^{(1)}_{cd}\, G^{\beta}_{de}\, V^{(1)}_{ef}\, G^{\beta}_{fb} + \dots \tag{168}$$

Expanding the logarithm and dropping one term independent of the perturbation $h^{\pm}_{\mu\nu}(x)$, the CTP effective action may be perturbatively written as

$$\begin{aligned} S^{\beta}_{\text{eff}}\left[h^{\pm}_{\mu\nu}\right] = {} & S_{\text{g}}\left[h^{+}_{\mu\nu}\right] - S_{\text{g}}\left[h^{-}_{\mu\nu}\right] \\ & + \frac{i}{2}\text{tr}\left[V^{(1)}_{+}\, G^{\beta}_{++} - V^{(1)}_{-}\, G^{\beta}_{--} + V^{(2)}_{+}\, G^{\beta}_{++} - V^{(2)}_{-}\, G^{\beta}_{--}\right] \\ & - \frac{i}{4}\text{tr}\left[V^{(1)}_{+}\, G^{\beta}_{++}\, V^{(1)}_{+}\, G^{\beta}_{++} + V^{(1)}_{-}\, G^{\beta}_{--}\, V^{(1)}_{-}\, G^{\beta}_{--} - 2V^{(1)}_{+}\, G^{\beta}_{+-}\, V^{(1)}_{-}\, G^{\beta}_{-+}\right]. \end{aligned} \tag{169}$$

In computing the traces, some terms containing divergences are canceled using counterterms introduced in the classical gravitational action after dimensional regularization.

8.3 Near flat case

At this point we divide our considerations into two cases. In the far field limit $h_{\mu\nu}$ represent perturbations about flat space, i.e., $g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$. The exact "unperturbed" thermal propagators for scalar fields are known, i.e., the Euclidean propagator with periodicity β. Using the Fourier transformed form (those quantities are denoted with a tilde) of the thermal propagators $\tilde{G}^{\beta}_{ab}(k)$, the trace terms of the form $\text{tr}[V^{(1)}_{a} G^{\beta}_{mn} V^{(1)}_{b} G^{\beta}_{rs}]$ can be written as [54, 55]

$$\begin{aligned} & \text{tr}\left[V^{(1)}_{a}\, G^{\beta}_{mn}\, V^{(1)}_{b}\, G^{\beta}_{rs}\right] = \\ & \quad \int d^n x\, d^n x'\, h^{a}_{\mu\nu}(x)\, h^{b}_{\alpha\beta}(x') \int \frac{d^n k}{(2\pi)^n} \frac{d^n q}{(2\pi)^n}\, e^{ik(x-x')}\, \tilde{G}^{\beta}_{mn}(k+q)\, \tilde{G}^{\beta}_{rs}(q)\, \text{T}^{\mu\nu,\alpha\beta}(q,k), \end{aligned} \tag{170}$$

where the tensor $\text{T}^{\mu\nu,\alpha\beta}(q,k)$ is defined in [54, 55] after an expansion in terms of a basis of 14 tensors [249, 250]. In particular, the last trace of Equation (169) may be split in two different kernels $\text{N}^{\mu\nu,\alpha\beta}(x-x')$ and $\text{D}^{\mu\nu,\alpha\beta}(x-x')$,

$$\frac{i}{2}\text{tr}\left[V^{(1)}_{+}\, G^{\beta}_{+-}\, V^{(1)}_{-}\, G^{\beta}_{-+}\right] = -\int d^4x\, d^4x'\, h^{+}_{\mu\nu}(x)\, h^{-}_{\alpha\beta}(x')\left[\text{D}^{\mu\nu,\alpha\beta}(x-x') + i\text{N}^{\mu\nu,\alpha\beta}(x-x')\right]. \tag{171}$$

One can express the Fourier transforms of these kernels as

$$\begin{aligned} \tilde{\text{N}}^{\mu\nu,\alpha\beta}(k) = {} & \pi^2 \int \frac{d^4q}{(2\pi)^4}\, \{\theta(k^o+q^o)\,\theta(-q^o) + \theta(-k^o-q^o)\,\theta(q^o) + n_\beta(|q^o|) + n_\beta(|k^o+q^o|) \\ & + 2\, n_\beta(|q^o|)\, n_\beta(|k^o+q^o|)\}\ \delta(q^2)\,\delta\left[(k+q)^2\right]\ \text{T}^{\mu\nu,\alpha\beta}(q,k), \end{aligned} \tag{172}$$

$$\begin{aligned} \tilde{\text{D}}^{\mu\nu,\alpha\beta}(k) = {} & -i\pi^2 \int \frac{d^4q}{(2\pi)^4}\, \{\theta(k^o+q^o)\,\theta(-q^o) - \theta(-k^o-q^o)\,\theta(q^o) + \text{sig}\,(k^o+q^o)\, n_\beta(|q^o|) \\ & - \text{sig}\,(q^o)\, n_\beta(|k^o+q^o|)\}\, \delta(q^2)\delta\left[(k+q)^2\right]\ \text{T}^{\mu\nu,\alpha\beta}(q,k), \end{aligned} \tag{173}$$

respectively.

Using the property $\mathrm{T}^{\mu\nu,\alpha\beta}(q,k) = \mathrm{T}^{\mu\nu,\alpha\beta}(-q,-k)$, it is easy to see that the kernel $\mathrm{N}^{\mu\nu,\alpha\beta}(x-x')$ is symmetric and $\mathrm{D}^{\mu\nu,\alpha\beta}(x-x')$ is antisymmetric in its arguments; that is, $\mathrm{N}^{\mu\nu,\alpha\beta}(x) = \mathrm{N}^{\mu\nu,\alpha\beta}(-x)$ and $\mathrm{D}^{\mu\nu,\alpha\beta}(x) = -\mathrm{D}^{\mu\nu,\alpha\beta}(-x)$.

The physical meanings of these kernels can be extracted if we write the renormalized CTP effective action at finite temperature (169) in an influence functional form [32, 111, 161, 162]. N, the imaginary part of the CTP effective action can be identified with the noise kernel and D, the antisymmetric piece of the real part with the dissipation kernel. Campos and Hu [54, 55] have shown that these kernels identified as such indeed satisfy a thermal fluctuation-dissipation relation.

If we denote the difference and the sum of the perturbations $h^{\pm}_{\mu\nu}$ defined along each branch $C_{\pm}$ of the complex time path of integration C by $[h_{\mu\nu}] \equiv h^{+}_{\mu\nu} - h^{-}_{\mu\nu}$ and $\{h_{\mu\nu}\} \equiv h^{+}_{\mu\nu} + h^{-}_{\mu\nu}$, respectively, the influence functional form of the thermal CTP effective action may be written to second order in $h_{\mu\nu}$ as

$$\begin{aligned} S^{\beta}_{\mathrm{eff}}\left[h^{\pm}_{\mu\nu}\right] \simeq & \frac{1}{2(16\pi G_{\mathrm{N}})} \int d^4x\, d^4x'\, [h_{\mu\nu}](x)\, \mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x-x')\{h_{\alpha\beta}\}(x') \\ & +\frac{1}{2}\int d^4x\, [h_{\mu\nu}](x)\, T^{\mu\nu}_{(\beta)} \\ & +\frac{1}{2}\int d^4x\, d^4x'\, [h_{\mu\nu}](x)\, \mathrm{H}^{\mu\nu,\alpha\beta}(x-x')\{h_{\alpha\beta}\}(x') \\ & -\frac{1}{2}\int d^4x\, d^4x'\, [h_{\mu\nu}](x)\, \mathrm{D}^{\mu\nu,\alpha\beta}(x-x')\{h_{\alpha\beta}\}(x') \\ & +\frac{i}{2}\int d^4x\, d^4x'\, [h_{\mu\nu}](x)\, \mathrm{N}^{\mu\nu,\alpha\beta}(x-x')[h_{\alpha\beta}](x'). \end{aligned} \tag{174}$$

The first line is the Einstein–Hilbert action to second order in the perturbation $h^{\pm}_{\mu\nu}(x)$. $\mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x)$ is a symmetric kernel (i.e., $\mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x) = \mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(-x)$). In the near flat case its Fourier transform is given by

$$\tilde{\mathrm{L}}^{\mu\nu,\alpha\beta}_{(o)}(k) = \frac{1}{4}\left[-k^2\mathrm{T}^{\mu\nu,\alpha\beta}_{1}(q,k) + 2k^2\mathrm{T}^{\mu\nu,\alpha\beta}_{4}(q,k) + \mathrm{T}^{\mu\nu,\alpha\beta}_{8}(q,k) - 2\mathrm{T}^{\mu\nu,\alpha\beta}_{13}(q,k)\right]. \tag{175}$$

The 14 elements of the tensor basis $\mathrm{T}^{\mu\nu,\alpha\beta}_{i}(q,k)$, $i = 1,\dots,14$, are defined in [249, 250]. The second is a local term linear in $h^{\pm}_{\mu\nu}(x)$. Only far away from the hole it takes the form of the stress tensor of massless scalar particles at temperature β^{-1}, which has the form of a perfect fluid stress-energy tensor,

$$T^{\mu\nu}_{(\beta)} = \frac{\pi^2}{30\beta^4}\left[u^{\mu}u^{\nu} + \frac{1}{3}(\eta^{\mu\nu} + u^{\mu}u^{\nu})\right], \tag{176}$$

where u^{μ} is the four-velocity of the plasma and the factor $\frac{\pi^2}{30\beta^4}$ is the familiar thermal energy density for massless scalar particles at temperature β^{-1}. In the far field limit, taking into account the four-velocity u^{μ} of the fluid, a manifestly Lorentz-covariant approach to thermal field theory may be used [292]. However, in order to simplify the involved tensorial structure, we work in the co-moving coordinate system of the fluid where $u^{\mu} = (1,0,0,0)$. In the third line, the Fourier transform of the symmetric kernel $\mathrm{H}^{\mu\nu,\alpha\beta}(x)$ can be expressed as

$$\begin{aligned}\tilde{\mathrm{H}}^{\mu\nu,\alpha\beta}(k) = & -\frac{\alpha k^4}{4}\left\{\frac{1}{2}\ln\frac{|k^2|}{\mu^2}\mathrm{Q}^{\mu\nu,\alpha\beta}(k)+\frac{1}{3}\bar{\mathrm{Q}}^{\mu\nu,\alpha\beta}(k)\right\}\\ & +\frac{\pi^2}{180\beta^4}\left\{-\mathrm{T}_1^{\mu\nu,\alpha\beta}(u,k)-2\mathrm{T}_2^{\mu\nu,\alpha\beta}(u,k)+\mathrm{T}_4^{\mu\nu,\alpha\beta}(u,k)+2\mathrm{T}_5^{\mu\nu,\alpha\beta}(u,k)\right\}\\ & +\frac{\xi}{96\beta^2}\left\{k^2\mathrm{T}_1^{\mu\nu,\alpha\beta}(u,k)-2k^2\mathrm{T}_4^{\mu\nu,\alpha\beta}(u,k)-\mathrm{T}_8^{\mu\nu,\alpha\beta}(u,k)+2\mathrm{T}_{13}^{\mu\nu,\alpha\beta}(u,k)\right\}\\ & +\pi\int\frac{d^4q}{(2\pi)^4}\left\{\delta(q^2)n_\beta(|q^o|)\mathcal{P}\left[\frac{1}{(k+q)^2}\right]+\delta[(k+q)^2]n_\beta(|k^o+q^o|)\mathcal{P}\left[\frac{1}{q^2}\right]\right\}\mathrm{T}^{\mu\nu,\alpha\beta}(q,k),\end{aligned} \tag{177}$$

where μ is a simple redefinition of the renormalization parameter $\bar{\mu}$ given by $\mu \equiv \bar{\mu}\exp\left(\frac{23}{15}+\frac{1}{2}\ln 4\pi - \frac{1}{2}\gamma\right)$, and the tensors $\mathrm{Q}^{\mu\nu,\alpha\beta}(k)$ and $\bar{\mathrm{Q}}^{\mu\nu,\alpha\beta}(k)$ are defined by

$$\begin{aligned}\mathrm{Q}^{\mu\nu,\alpha\beta}(k) = & \frac{3}{2}\left\{\mathrm{T}_1^{\mu\nu,\alpha\beta}(q,k)-\frac{1}{k^2}\mathrm{T}_8^{\mu\nu,\alpha\beta}(q,k)+\frac{2}{k^4}\mathrm{T}_{12}^{\mu\nu,\alpha\beta}(q,k)\right\}\\ & -\left[1-360\left(\xi-\frac{1}{6}\right)^2\right]\left\{\mathrm{T}_4^{\mu\nu,\alpha\beta}(q,k)+\frac{1}{k^4}\mathrm{T}_{12}^{\mu\nu,\alpha\beta}(q,k)-\frac{1}{k^2}\mathrm{T}_{13}^{\mu\nu,\alpha\beta}(q,k)\right\},\end{aligned} \tag{178}$$

$$\begin{aligned}\bar{\mathrm{Q}}^{\mu\nu,\alpha\beta}(k) = & \left[1+576\left(\xi-\frac{1}{6}\right)^2-60(\xi-\frac{1}{6})(1-36\xi')\right]\\ & \times\left\{\mathrm{T}_4^{\mu\nu,\alpha\beta}(q,k)+\frac{1}{k^4}\mathrm{T}_{12}^{\mu\nu,\alpha\beta}(q,k)-\frac{1}{k^2}\mathrm{T}_{13}^{\mu\nu,\alpha\beta}(q,k)\right\},\end{aligned} \tag{179}$$

respectively.

In the above and subsequent equations, we denote the coupling parameter in four dimensions $\xi(4)$ by ξ, and consequently ξ' means $d\xi(n)/dn$ evaluated at $n = 4$. $\tilde{\mathrm{H}}^{\mu\nu,\alpha\beta}(k)$ is the complete contribution of a free massless quantum scalar field to the thermal graviton polarization tensor [249, 250, 72, 27], and it is responsible for the instabilities found in flat spacetime at finite temperature [116, 249, 250, 72, 27]. Note that the addition of the contribution of other kinds of matter fields to the effective action, even graviton contributions, does not change the tensor structure of these kernels, and only the overall factors are different to leading order [249, 250]. Equation (177) reflects the fact that the kernel $\tilde{\mathrm{H}}^{\mu\nu,\alpha\beta}(k)$ has thermal as well as non-thermal contributions. Note that it reduces to the first term in the zero temperature limit ($\beta \to \infty$),

$$\tilde{\mathrm{H}}^{\mu\nu,\alpha\beta}(k) \simeq -\frac{\alpha k^4}{4}\left\{\frac{1}{2}\ln\frac{|k^2|}{\mu^2}\mathrm{Q}^{\mu\nu,\alpha\beta}(k)+\frac{1}{3}\bar{\mathrm{Q}}^{\mu\nu,\alpha\beta}(k)\right\}, \tag{180}$$

and at high temperatures the leading term (β^{-4}) may be written as

$$\tilde{\mathrm{H}}^{\mu\nu,\alpha\beta}(k) \simeq \frac{\pi^2}{30\beta^4}\sum_{i=1}^{14}\mathrm{H}_i(r)\mathrm{T}_i^{\mu\nu,\alpha\beta}(u,K), \tag{181}$$

where we have introduced the dimensionless external momentum $K^\mu \equiv k^\mu/|\vec{k}| \equiv (r,\hat{k})$. The $\mathrm{H}_i(r)$ coefficients were first given in [249, 250] and generalized to the next-to-leading order β^{-2} in [72, 27]. (They are given with the MTW sign convention in [54, 55].)

Finally, as defined above, $\mathrm{N}^{\mu\nu,\alpha\beta}(x)$ is the noise kernel representing the random fluctuations of the thermal radiance and $\mathrm{D}^{\mu\nu,\alpha\beta}(x)$ is the dissipation kernel, describing the dissipation of energy of the gravitational field.

8.4 Near horizon case

In this case, since the perturbation is taken around the Schwarzschild spacetime, exact expressions for the corresponding unperturbed propagators $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ are not known. Therefore apart from the approximation of computing the CTP effective action to certain order in perturbation theory, an appropriate approximation scheme for the unperturbed Green's functions is also required. This feature manifested itself in York's calculation of backreaction as well, where, in writing the $\langle T_{\mu\nu}\rangle$ on the right-hand side of the semiclassical Einstein equation in the unperturbed Schwarzschild metric, he had to use an approximate expression for $\langle T_{\mu\nu}\rangle$ in the Schwarzschild metric given by Page [231]. The additional complication here is that while to obtain $\langle T_{\mu\nu}\rangle$ as in York's calculation the knowledge of only the thermal Feynman Green's function is required; however, to calculate the CTP effective action one needs the knowledge of the full matrix propagator, which involves the Feynman, Schwinger, and Wightman functions.

It is indeed possible to construct the full thermal matrix propagator $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ based on Page's approximate Feynman Green's function by using identities relating the Feynman Green's function with the other Green's functions with different boundary conditions. One can then proceed to explicitly compute a CTP effective action and hence the influence functional based on this approximation. However, we desist from delving into such a calculation for the following reason. Our main interest in performing such a calculation is to identify and analyze the noise term which is the new ingredient in the backreaction. We have mentioned that the noise term gives a stochastic contribution $\xi^{\mu\nu}$ to the Einstein–Langevin equation (14). We had also stated that this term is related to the variance of fluctuations in $T_{\mu\nu}$, i.e, schematically, to $\langle T^2_{\mu\nu}\rangle$. However, a calculation of $\langle T^2_{\mu\nu}\rangle$ in the Hartle-Hawking state in a Schwarzschild background using the Page approximation was performed by Phillips and Hu [244, 245, 241], and it was shown that though the approximation is excellent as far as $\langle T_{\mu\nu}\rangle$ is concerned, it gives unacceptably large errors for $\langle T^2_{\mu\nu}\rangle$ at the horizon. In fact, similar errors will be propagated in the non-local dissipation term as well, because both terms originate from the same source, that is, they come from the last trace term in Equation (169) which contains terms quadratic in the Green's function. However, the Influence Functional or CTP formalism itself does not depend on the nature of the approximation, so we will attempt to exhibit the general structure of the calculation without resorting to a specific form for the Green's function and conjecture on what is to be expected. A more accurate computation can be performed using this formal structure once a better approximation becomes available.

The general structure of the CTP effective action arising from the calculation of the traces in equation (169) remains the same. But to write down explicit expressions for the non-local kernels one requires the input of the explicit form of $G^{\beta}_{ab}[h^{\pm}_{\mu\nu}]$ in the Schwarzschild metric, which is not available in closed form. We can make some general observations about the terms in there. The first line containing L does not have an explicit Fourier representation as given in the far field case, neither will $T^{\mu\nu}_{(\beta)}$ in the second line representing the zeroth order contribution to $\langle T_{\mu\nu}\rangle$ have a perfect fluid form. The third and fourth terms containing the remaining quadratic component of the real part of the effective action will not have any simple or even complicated analytic form. The symmetry properties of the kernels $H^{\mu\nu,\alpha\beta}(x,x')$ and $D^{\mu\nu,\alpha\beta}(x,x')$ remain intact, i.e., they are even and odd in x,x', respectively. The last term in the CTP effective action gives the imaginary part of the effective action and the kernel $N(x,x')$ is symmetric.

Continuing our general observations from this CTP effective action, using the connection between this thermal CTP effective action to the influence functional [272, 43] via an equation in the schematic form (17), we see that the nonlocal imaginary term containing the kernel $N^{\mu\nu,\alpha\beta}(x,x')$ is responsible for the generation of the stochastic noise term in the Einstein–Langevin equation, and the real non-local term containing kernel $D^{\mu\nu,\alpha\beta}(x,x')$ is responsible for the non-local dissipation term. To derive the Einstein–Langevin equation we first construct the stochastic effective action (27). We then derive the equation of motion, as shown earlier in Equation (29), by taking its functional derivative with respect to $[h_{\mu\nu}]$ and equating it to zero. With the identification of noise and dissipation kernels, one can write down a linear, non-local relation of the form

$$N(t-t') = \int d(s-s')\, K(t-t', s-s')\, \gamma(s-s'), \tag{182}$$

where $D(t,t') = -\partial_{t'}\gamma(t,t')$. This is the general functional form of a fluctuation-dissipation relation, and $K(t,s)$ is called the fluctuation-dissipation kernel [32, 111, 161, 162]. In the present context this relation depicts the backreaction of thermal Hawking radiance for a black hole in quasi-equilibrium.

8.5 The Einstein–Langevin equation

In this section we show how a semiclassical Einstein–Langevin equation can be derived from the previous thermal CTP effective action. This equation depicts the stochastic evolution of the perturbations of the black hole under the influence of the fluctuations of the thermal scalar field.

The influence functional $\mathcal{F}_{\mathrm{IF}} \equiv \exp(iS_{\mathrm{IF}})$ previously introduced in Equation (16) can be written in terms of the the CTP effective action $S^{\beta}_{\mathrm{eff}}[h^{\pm}_{\mu\nu}]$ derived in Equation (174) using Equation (17). The Einstein–Langevin equation follows from taking the functional derivative of the stochastic effective action (27) with respect to $[h_{\mu\nu}](x)$ and imposing $[h_{\mu\nu}](x) = 0$. This leads to

$$\begin{aligned}\frac{1}{16\pi G_{\mathrm{N}}} \int d^4x'\, \mathrm{L}^{\mu\nu,\alpha\beta}_{(o)}(x-x')\, h_{\alpha\beta}(x') + \frac{1}{2} T^{\mu\nu}_{(\beta)} \\ + \int d^4x' \left(\mathrm{H}^{\mu\nu,\alpha\beta}(x-x') - \mathrm{D}^{\mu\nu,\alpha\beta}(x-x') \right) h_{\alpha\beta}(x') + \xi^{\mu\nu}(x) = 0,\end{aligned} \tag{183}$$

where

$$\left\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(x') \right\rangle_j = \mathrm{N}^{\mu\nu,\alpha\beta}(x-x'). \tag{184}$$

In the far field limit this equation should reduce to that obtained by Campos and Hu [54, 55]: For gravitational perturbations $h^{\mu\nu}$ defined in Equation (165) under the harmonic gauge $\bar{h}^{\mu\nu}{}_{,\nu} = 0$, their Einstein–Langevin equation is given by

$$\Box \bar{h}^{\mu\nu}(x) + \frac{1}{16\pi G_{\mathrm{N}}^2} \left\{ T^{\mu\nu}_{(\beta)} + 2P_{\rho\sigma,\alpha\beta} \int d^4x' \left(\mathrm{H}^{\mu\nu,\alpha\beta}(x-x') - \mathrm{D}^{\mu\nu,\alpha\beta}(x-x') \right) \bar{h}^{\rho\sigma}(x') + 2\xi^{\mu\nu}(x) \right\} = 0, \tag{185}$$

where the tensor $P_{\rho\sigma,\alpha\beta}$ is given by

$$P_{\rho\sigma,\alpha\beta} = \frac{1}{2} \left(\eta_{\rho\alpha}\, \eta_{\sigma\beta} + \eta_{\rho\beta}\, \eta_{\sigma\alpha} - \eta_{\rho\sigma}\, \eta_{\alpha\beta} \right). \tag{186}$$

The expression for $P_{\rho\sigma,\alpha\beta}$ in the near horizon limit of course cannot be expressed in such a simple form. Note that this differential stochastic equation includes a non-local term responsible for the dissipation of the gravitational field and a noise source term which accounts for the fluctuations of the quantum field. Note also that this equation in combination with the correlation for the stochastic variable (184) determines the two-point correlation for the stochastic metric fluctuations $\langle \bar{h}_{\mu\nu}(x)\bar{h}_{\alpha\beta}(x')\rangle_{\xi}$ self-consistently.

As we have seen before and here, the Einstein–Langevin equation is a dynamical equation governing the dissipative evolution of the gravitational field under the influence of the fluctuations of the quantum field, which, in the case of black holes, takes the form of thermal radiance. From its form we can see that even for the quasi-static case under study the backreaction of Hawking radiation on the black hole spacetime has an innate dynamical nature.

For the far field case, making use of the explicit forms available for the noise and dissipation kernels, Campos and Hu [54, 55] formally proved the existence of a fluctuation-dissipation relation at all temperatures between the quantum fluctuations of the thermal radiance and the dissipation of the gravitational field. They also showed the formal equivalence of this method with linear response theory for lowest order perturbations of a near-equilibrium system, and how the response functions such as the contribution of the quantum scalar field to the thermal graviton polarization tensor can be derived. An important quantity not usually obtained in linear response theory, but of equal importance, manifest in the CTP stochastic approach is the noise term arising from the quantum and statistical fluctuations in the thermal field. The example given in this section shows that the backreaction is intrinsically a dynamic process described (at this level of sophistication) by the Einstein–Langevin equation.

8.6 Discussions

We make a few remarks here and draw some connection with related work on black hole fluctuations.

8.6.1 Black hole backreaction

As remarked earlier, except for the near-flat case, an analytic form of the Green function is not available. Even the Page approximation [231], which gives unexpectedly good results for the stress-energy tensor, has been shown to fail in the fluctuations of the energy density [245, 241]. Thus, using such an approximation for the noise kernel will give unreliable results for the Einstein–Langevin equation. If we confine ourselves to Page's approximation and derive the equation of motion without the stochastic term, we expect to recover York's semiclassical Einstein equation if one retains only the zeroth order contribution, i.e, the first two terms in the expression for the CTP effective action in Equation (174). Thus, this offers a new route to arrive at York's semiclassical Einstein equations. Not only is it a derivation of York's result from a different point of view, but it also shows how his result arises as an appropriate limit of a more complete framework, i.e, it arises when one averages over the noise. Another point worth noting is that our treatment will also yield a non-local dissipation term arising from the fourth term in Equation (174) in the CTP effective action which is absent in York's treatment. This difference is primarily due to the difference in the way backreaction is treated, at the level of iterative approximations on the equation of motion as in York, versus the treatment at the effective action level as pursued here. In York's treatment, the Einstein tensor is computed to first order in perturbation theory, while $\langle T_{\mu\nu} \rangle$ on the right-hand side of the semiclassical Einstein equation is replaced by the zeroth order term. In the effective action treatment the full effective action is computed to second order in perturbation, and hence includes the higher order non-local terms.

The other important conceptual point that comes to light from this approach is that related to the fluctuation-dissipation relation. In the quantum Brownian motion analog (see, e.g., [32, 111, 161, 162] and references therein), the dissipation of the energy of the Brownian particle as it approaches equilibrium and the fluctuations at equilibrium are connected by the fluctuation-dissipation relation. Here the backreaction of quantum fields on black holes also consists of two forms – dissipation and fluctuation or noise – corresponding to the real and imaginary parts of the influence functional as embodied in the dissipation and noise kernels. A fluctuation-dissipation relation has been shown to exist for the near flat case by Campos and Hu [54, 55] and we anticipate that it should also exist between the noise and dissipation kernels for the general case, as it is a categorical relation [32, 111, 161, 162, 151]. Martin and Verdaguer have also proved the existence of a fluctuation-dissipation relation when the semiclassical background is a stationary spacetime and the quantum field is in thermal equilibrium. Their result was then extended to a conformal field in a conformally stationary background [207]. The existence of a fluctuation-dissipation relation for the black hole case has been discussed by some authors previously [60, 258, 259, 217]. In [164], Hu, Raval, and Sinha have described how this approach and its results differ from those of previous authors. The fluctuation-dissipation relation reveals an interesting connection between black holes interacting with quantum fields and non-equilibrium statistical mechanics. Even in its restricted quasi-static form, this relation will allow us to study *nonequilibrium* thermodynamic properties of the black hole under the influence of stochastic fluctuations of the energy-momentum tensor dictated by the noise terms.

There are limitations of a technical nature in the specific example invoked here. For one we have to confine ourselves to small perturbations about a background metric. For another, as mentioned above, there is no reliable approximation to the Schwarzschild thermal Green's function to explicitly compute the noise and dissipation kernels. This limits our ability to present explicit analytical expressions for these kernels. One can try to improve on Page's approximation by retaining terms to higher order. A less ambitious first step could be to confine attention to the horizon and using approximations that are restricted to near the horizon and work out the Influence Functional in this regime.

Yet another technical limitation of the specific example is the following. Although we have allowed for backreaction effects to modify the initial state in the sense that the temperature of the Hartle-Hawking state gets affected by the backreaction, we have essentially confined our analysis to a Hartle-Hawking thermal state of the field. This analysis does not directly extend to a more general class of states, for example to the case where the initial state of the field is in the Unruh vacuum. Thus, we will not be able to comment on issues of the stability of an *isolated* radiating black hole under the influence of stochastic fluctuations.

8.6.2 Metric fluctuations in black holes

In addition to the work described above by Campos, Hu, Raval, and Sinha [54, 55, 164, 264] and earlier work quoted therein, we mention also some recent work on black hole metric fluctuations and their effect on Hawking radiation. For example, Casher et al. [64] and Sorkin [267, 268] have concentrated on the issue of fluctuations of the horizon induced by a fluctuating metric. Casher et al. [64] consider the fluctuations of the horizon induced by the "atmosphere" of high angular momentum particles near the horizon, while Sorkin [267, 268] calculates fluctuations of the shape of the horizon induced by the quantum field fluctuations under a Newtonian approximation. Both group of authors come to the conclusion that horizon fluctuations become large at scales much larger than the Planck scale (note that Ford and Svaiter [94] later presented results contrary to this claim). However, though these works do deal with backreaction, the fluctuations considered do not arise as an explicit stochastic noise term as in our treatment. It may be worthwhile exploring the horizon fluctuations induced by the stochastic metric in our model and comparing the conclusions with the above authors. Barrabes et al. [14, 15] have considered the propagation of null rays and massless fields in a black hole fluctuating geometry, and have shown that the stochastic nature of the metric leads to a modified dispersion relation and helps to confront the trans-Planckian frequency problem. However, in this case the stochastic noise is put in by hand and does not naturally arise from coarse graining as in the quantum open systems approach. It also does not take backreaction into account. It will be interesting to explore how a stochastic black hole metric, arising as a solution to the Einstein–Langevin equation, hence fully incorporating backreaction, would affect the trans-Planckian problem.

Ford and his collaborators [94, 95, 294] have also explored the issue of metric fluctuations in detail and in particular have studied the fluctuations of the black hole horizon induced by metric fluctuations. However, the fluctuations they have considered are in the context of a fixed background and do not relate to the backreaction.

Another work originating from the same vein of stochastic gravity but not complying with the backreaction spirit is that of Hu and Shiokawa [166], who study effects associated with electromagnetic wave propagation in a Robertson-Walker universe and the Schwarzschild spacetime with a small amount of given metric stochasticity. They find that time-independent randomness can decrease the total luminosity of Hawking radiation due to multiple scattering of waves outside the black hole and gives rise to event horizon fluctuations and fluctuations in the Hawking temperature. The stochasticity in a background metric in their work is assumed rather than derived (from quantum field fluctuations, as in this work), and so is not in the same spirit of backreaction. But it is interesting to compare their results with that of backreaction, so one can begin to get a sense of the different sources of stochasticity and their weights (see, e.g., [154] for a list of possible sources of stochasticity).

In a subsequent paper Shiokawa [261] showed that the scalar and spinor waves in a stochastic spacetime behave similarly to the electrons in a disordered system. Viewing this as a quantum transport problem, he expressed the conductance and its fluctuations in terms of a nonlinear sigma model in the closed time path formalism and showed that the conductance fluctuations are universal, independent of the volume of the stochastic region and the amount of stochasticity. This result can have significant importance in characterizing the mesoscopic behavior of spacetimes resting between the semiclassical and the quantum regimes.

9 Concluding Remarks

In the first part of this review on the fundamentals of theory we have given two routes to the establishment of stochastic gravity and derived a general (finite) expression for the noise kernel. In the second part we gave three applications, the correlation functions of gravitons in a perturbed Minkowski metric, structure formation in stochastic gravity theory, and the outline of a program for the study of black hole fluctuations and backreaction. A central issue which stochastic gravity can perhaps best address is the validity of semiclassical gravity as measured by the fluctuations of stress-energy compared to the mean. We will include a review of this topic in a future update.

There is ongoing research related to the topics discussed in this review. On the theory side, Roura and Verdaguer [255] have recently shown how stochastic gravity can be related to the large N limit of quantum metric fluctuations. Given N free matter fields interacting with the gravitational field, Hartle and Horowitz [128], and Tomboulis [277] have shown that semiclassical gravity can be obtained as the leading order large N limit (while keeping N times the gravitational coupling constant fixed). It is of interest to find out where in this setting can one place the fluctuations of the quantum fields and the metric fluctuations they induce; specifically, whether the inclusion of these sources will lead to an Einstein–Langevin equation [43, 157, 167, 58, 202], as it was derived historically in other ways, as described in the first part of this review. This is useful because it may provide another pathway or angle in connecting semiclassical to quantum gravity (a related idea is the kinetic theory approach to quantum gravity described in [155]).

Theoretically, stochastic gravity is at the frontline of the 'bottom-up' approach to quantum gravity [146, 154, 155]. Structurally, as can be seen from the issues discussed and the applications given, stochastic gravity has a very rich constituency because it is based on quantum field theory and nonequilibrium statistical mechanics in a curved spacetime context. The open systems concepts and the closed-time-path/influence functional methods constitute an extended framework suitable for treating the backreaction and fluctuations problems of dynamical spacetimes interacting with quantum fields. We have seen it applied to cosmological backreaction problems. It can also be applied to treat the backreaction of Hawking radiation in a fully dynamical black hole collapse situation. One can then address related issues such as the black hole end state and information loss puzzles (see, e.g., [230, 152] and references therein). The main reason why this program has not progressed as swiftly as desired is due more to technical rather than programatic difficulties (such as finding reasonable analytic approximations for the Green function or numerical evaluation of mode-sums near the black hole horizon). Finally, the multiplex structure of this theory could be used to explore new lines of inquiry and launch new programs of research, such as *nonequilibrium* black hole thermodynamics and statistical mechanics.

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