Ramadan *et al. J Egypt Math Soc* (2021) 29:9
https://doi.org/10.1186/s42787-021-00117-8

Journal of the Egyptian Mathematical Society

ORIGINAL RESEARCH

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On L-fuzzy closure operators and L-fuzzy pre-proximities

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Abstract

The aim of this paper is to investigate the relations among the L-fuzzy pre-proximities, L-fuzzy closure operators and L-fuzzy co-topologies in complete residuated lattices. We show that there is a Galois correspondence between the category of separated L-fuzzy closure spaces and that of separated L-fuzzy pre-proximity spaces and we give their examples.

Keywords: Complete residuated lattice, L-fuzzy pre-proximity, L-fuzzy closure operators, L-fuzzy co-topologies, Galois correspondence

Mathematics Subject Classification: 54F05, 54D10, 54B05, 54B10

Introduction

Closure operators are very useful tool in several areas of mathematical structures with direct applications, both mathematical (e.g, topology, logic) and extra-mathematical (e.g, data mining, knowledge representation). In fuzzy set theory [1, 2], several particular kinds such as general theory of closure operators which operate with fuzzy sets (so-called fuzzy closure operators) are studied [3–6].

Ward et al. [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [8] investigated information systems, decision rules and developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattices. Höhle [9] introduced L-fuzzy topologies with algebraic structure L (cqm, quantales, MV-algebra). It has developed in many directions [10–12]. Recently, Bělohlávek [13, 14] outlined a general theory of fuzzy closure operators by using the structure of the residuated lattice in place of the usual structure of truth value on [0, 1]. Fang and Yue [15] studied the relationship between L-fuzzy closure systems and L-fuzzy topological spaces from a category viewpoint for a complete residuated lattice L (see also [16]). Ramadan [17] studied the relationship between L-fuzzy interior systems and L-fuzzy topological spaces over complete residuated lattices.

Proximity is an important concept in topology, and it can be considered either as axiomatizations of geometric notions, close to but quite independent of topology, or as convenient tools for an investigation of topological spaces. Hence, proximity has close relations with topology, uniformity and metric. With the development of topology, the theory of proximity makes a massive progress. In the framework of L-topology, many authors generalized the crisp proximity to L-fuzzy setting. Katsaras [18, 19] introduced the concepts of fuzzy topogenous order and fuzzy topogenous structures in completely distributive lattice which are a unified approach to the three spaces: Chang's fuzzy topologies [20], Katsaras's fuzzy proximities [21] and Hutton's fuzzy uniformities [22] (see also [23]) . Subsequently, Liu [24], Artico and Moresco [25] extended it into L-fuzzy set theory in view points of Lowen's fuzzy topology [26]. As an extension of Katsaras's definition, El-Dardery [27] introduced L-fuzzy topogenous order in view points of Sostak's fuzzy topology [28], smooth fuzzy topology [29] and Kim's L-fuzzy proximities [30] on strictly two-sided, commutative quantales. L-fuzzy topogenous structures and L-fuzzy proximities [23, 31–34] have been developed in a slightly different sense.

In this paper, we introduce the notions of L-fuzzy pre-proximities and L-fuzzy closure operators in complete residuated lattices. Moreover, we investigate the relations among the L-fuzzy pre-proximities, L-fuzzy closure operators and L-fuzzy co-topologies. We show that there is a Galois correspondence between the category of separated L-fuzzy closure spaces and that of separated L-fuzzy pre-proximity spaces. In Example 19, as an information system as an extension of Pawlak's rough set [35, 36], L-fuzzy pre-proximities, L-fuzzy co-topologies and L-fuzzy closure operators are introduced. By using these concepts, we can apply them to information systems and decision makings [37].

Preliminaries

Definition 1 ([8–11, 38]) An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

- (C1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
- (C2) $(L, \odot, \top)$ is a commutative monoid;
- (C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume that $(L, \leq, \odot, ^*)$ is a complete residuated lattice with an order reversing involution * which is defined by

$$x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \rightarrow \perp.$$

For $\alpha \in L$ *and* $f \in L^X$, we denote $(\alpha \rightarrow f), (\alpha \odot f), \alpha_X \in L^X$ as $(\alpha \rightarrow f)(x) = \alpha \rightarrow f(x)$, $(\alpha \odot f)(x) = \alpha \odot f(x), \alpha_X(x) = \alpha$, *respectively*

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \perp, & \text{otherwise} \end{cases}, \quad \top^*_x(y) = \begin{cases} \perp, & \text{if } y = x, \\ \top, & \text{otherwise} . \end{cases}$$

Some basic properties of the binary operation $\odot$ and residuated operation $\rightarrow$ are collected in the following lemma, and they can be found in many works, for instance [8–11, 38].

Lemma 2 *For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.*

(1) $\top \to x = x, \bot \odot x = \bot$,
(2) *If* $y \leq z$, *then* $x \odot y \leq x \odot z, x \oplus y \leq x \oplus z, x \to y \leq x \to z$ *and* $z \to x \leq y \to x$,
(3) $x \leq y$ *iff* $x \to y = \top$.
(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*, (\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
(5) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$,
(6) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$,
(7) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$,
(8) $(\bigwedge_i x_i) \oplus y = \bigwedge_i (x_i \oplus y)$,
(9) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,
(10) $x \odot y = (x \to y^*)^*, x \oplus y = x^* \to y$ *and* $x \to y = y^* \to x^*$,
(11) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$,
(12) $x \to y \leq (x \odot z) \to (y \odot z)$ *and* $(x \to y) \odot (y \to z) \leq x \to z$,
(13) $(x \to y) \odot (z \to w) \leq (x \oplus z) \to (y \oplus w)$,
(14) $x \odot (x \to y) \leq y$ *and* $y \leq x \to (x \odot y)$,
(15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$,
(16) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i), \bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$,
(17) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$,
(18) $z \to x \leq (x \to y) \to (z \to y)$ *and* $y \to z \leq (x \to y) \to (x \to z)$.

Definition 3 [14, 16, 39] A map $\mathcal{C} : L^X \to L^X$ is called an *L-fuzzy closure operator* on X if $\mathcal{C}$ satisfies the following conditions:

(C1) $\mathcal{C}(\bot_X) = \bot_X$,
(C2) $\mathcal{C}(f) \geq f$ for all $f \in L^X$,
(C3) If $f \leq g$, then $\mathcal{C}(f) \leq \mathcal{C}(g))$ for all $f, g \in L^X$,
(C4) $\mathcal{C}(f \oplus g) \leq \mathcal{C}(f) \oplus \mathcal{C}(g)$.

The pair $(X, \mathcal{C})$ is called *L-fuzzy closure space*. An L-fuzzy closure space is called

(T) *topological* if $\mathcal{C}(\mathcal{C}(f)) = \mathcal{C}(f)\ \ \forall f \in L^X$,
(U) *stratified* if $\mathcal{C}(\alpha \to f) \leq \alpha \to \mathcal{C}(f)$ for all $f \in L^X$ and $\alpha \in L$,
(V) *co-stratified* if $\mathcal{C}(\alpha \odot f) \leq \alpha \odot \mathcal{C}(f)$ for all $f \in L^X$ and $\alpha \in L$,
(W) *strong* if it is both stratified and co-stratified, i.e, $\mathcal{C}(\alpha \odot f) = \alpha \odot \mathcal{C}(f)$ for all $f \in L^X$ and $\alpha \in L$,
(X) *separated* if $\mathcal{C}(\top_x^*) = \top_x^*$ for all $x \in X$,
(Y) *generalized* if $\mathcal{C}(f)(x) \geq \bigvee_{x\in X} f(x)$,
(Z) *Alexandrov* if $\mathcal{C}(\bigvee_{i\in\Gamma} f_i) = \bigvee_{i\in\Gamma} \mathcal{C}(f_i)$.

Definition 4 Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L-fuzzy closure spaces and $\varphi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ be a mapping. Then, $D_C(\varphi)$ defined by

$$D_C(\varphi) = \bigwedge_{f\in L^Y} \bigwedge_{x\in X} \Big(\mathcal{C}_X(\varphi^{\leftarrow}(f))(x) \to \varphi^{\leftarrow}(\mathcal{C}_Y(f))(x)\Big)$$

is called the degree of LF-closure for φ. If $D_C(\varphi) = \top$, then $\mathcal{C}_X(\varphi^{\leftarrow}(f)) \leq \phi^{\leftarrow}(\mathcal{C}_Y(f))$ for each $f \in L^Y$, which is exactly the definition of LF-closure mappings between L-fuzzy closure spaces.

Remark 5

An L-fuzzy closure space $(X, \mathcal{C})$ is stratified if and only if $\mathcal{C}(\alpha \odot f) \geq \alpha \odot \mathcal{C}(f)$.

Definition 6 [16, 17, 39] A mapping $\mathcal{F} : L^X \to L$ is called *L-fuzzy co-topology* on X if it satisfies the following conditions:

(T1) $\mathcal{F}(\perp_X) = \mathcal{F}(\top_X) = \top$,
(T2) $\mathcal{F}(f \oplus g) \geq \mathcal{F}(f) \odot \mathcal{F}(g)$ *forall* $f, g \in L^X$,
(T3) $\mathcal{F}(\bigwedge_i f_i) \geq \bigwedge_i \mathcal{F}(f_i)$ *forall* $\{f_i\}_{i\in\Gamma} \subseteq L^X$.

The pair $(X, \mathcal{F})$ is called *L-fuzzy co-topological space*. An L-fuzzy co-topological space is said to be

(A) *stratified* if $\mathcal{F}(\alpha \odot f) \geq \mathcal{F}(f)$,
(B) *co-stratified* if $\mathcal{F}(\alpha \to f) \geq \mathcal{F}(f)$,
(C) *strong* if it is both stratified and co-stratified,
(D) *separated* if $\mathcal{F}(\top_x) = \top$ for all $x \in X$,
(E) *Alexandrov* if $\mathcal{F}(\bigvee_i f_i) \geq \bigwedge_i \mathcal{F}(f_i)$ *forall* $\{f_i\}_{i\in\Gamma} \subseteq L^X$.

Definition 7 Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be L-fuzzy co-topological spaces and $\varphi : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ be a mapping. Then, $D_{\mathcal{F}}(\phi)$ defined by

$$D_{\mathcal{F}}(\varphi) = \bigwedge_{f\in L^Y} \left(\mathcal{F}_Y(f) \to \mathcal{F}_X(\varphi^{\leftarrow}(f))\right)$$

is called the degree of LF-continuous for φ. If $D_{\mathcal{F}}(\varphi) = \top$, then $\mathcal{F}_Y(f) \leq \mathcal{F}_Y(\varphi^{\leftarrow}(f))$ for each $f \in L^Y$, which is exactly the definition of LF-continuous mappings between L-fuzzy co-topological spaces.

Definition 8 [8, 36] Let X be a set. A map $R : X \times X \to L$ is called an *L-partial order* if it satisfies the following conditions

(E1) reflexive if $R(x, x) = \top$ for all $x \in X$,
(E2) transitive if $R(x, y) \odot R(y, z) \leq R(x, z)$ for all $x, y, z \in X$,
(E3) antisymmetric if $R(x, y) = R(y, x) = \top$, then $x = y$.

The relationships between *L*-fuzzy pre-proximities and topological structures

Definition 9 A mapping $\delta : L^X \times L^X \to L$ is called an *L-fuzzy pre-proximity* on X if it satisfies the following axioms.

(P1) $\delta(\top_X, \bot_X) = \delta(\bot_X, \top_X) = \bot$,
(P2) $\delta(f,g) \geq \bigvee_{x\in X}(f \odot g)(x)$,
(P3) If $f_1 \leq f_2, h_1 \leq h_2$, then $\delta(f_1,h_1) \leq \delta(f_2,h_2)$. The pair (X,δ) is called *L-fuzzy pre-proximity space.* An L-fuzzy pre-proximity is called an $(L, \odot, \oplus)$-fuzzy pre-proximity if
(P4) For every $f_1, f_2, h_1, h_2 \in L^X$ we have

$$\delta(f_1 \odot f_2, h_1 \oplus h_2) \leq \delta(f_1,h_1) \oplus \delta(f_2,h_2),$$
$$\delta(f_1 \oplus f_2, h_1 \odot h_2) \leq \delta(f_1,h_1) \oplus \delta(f_2,h_2).$$

An L-fuzzy pre-proximity is called an *L-fuzzy quasi-proximity* on X if it satisfies (P4) and
(Q) $\delta(f,g) \geq \bigwedge_h\{\delta(f,h) \oplus \delta(h^*,g)\}$. An L-fuzzy quasi-proximity is called an *L-fuzzy proximity* on X if
(P) $\delta^s = \delta$ where $\delta^s(f,g) = \delta(g,f)$. An L-fuzzy pre-proximity is called
(St) stratified if $\delta(\alpha \odot f, \alpha \to g) \leq \delta(f,g)$ and $\delta(\alpha \to f, \alpha \odot g) \leq \delta(f,g)$,
(SE) separated if $\delta(\top_x, \top_x^*) = \delta(\top_x^*, \top_x) = \bot$ for each $x \in X$,
(AL) Alexandrov if $\delta(\bigvee_{i\in\Gamma} f_i, g) \leq \bigvee_{i\in\Gamma}\delta(f_i,g)$, $\delta(f, \bigvee_{i\in\Gamma} g_i) \leq \bigvee_{i\in\Gamma}\delta(f,g_i)$,
(GL) generalized if $\delta(f,g) \leq \bigvee_{x\in X} f(x) \odot \bigvee_{x\in X} g(x)$.

Definition 10 Let (X,δ_X) and (Y,δ_Y) be L-fuzzy pre-proximity spaces and $\varphi : (X,\delta_X) \to (Y,\delta_Y)$ be a mapping. Then, $D_\delta(\varphi)$ defined by

$$D_\delta(\varphi) = \bigwedge_{f,g\in L^Y} \Big(\delta_X(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(g)) \to \delta_Y(f,g)\Big)$$

is called the degree of *LF*-proximity for φ. If $D_\delta(\varphi) = \top$, then $\delta_X(\varphi^{\leftarrow}(f), \varphi^{\leftarrow}(g)) \leq \delta_Y(f,g)$ for each $f, g \in L^Y$, which is exactly the definition of *LF*-proximity mappings between L-fuzzy pre-proximity spaces.

Lemma 11 *Let (X,δ) be an L-fuzzy pre-proximity space. Then,*

$$\delta(\alpha \odot f, g) \geq \alpha \odot \delta(f,g) \text{ iff } \delta(\alpha \to f, g) \leq \alpha \to \delta(f,g).$$

Proof
(1) Let $\delta(\alpha \odot f, g) \geq \alpha \odot \delta(f,g)$. Then, $\alpha \odot \delta(\alpha \to f, g) \leq \delta(\alpha \odot (\alpha \to f), g) \leq \delta(f,g)$. Thus, $\delta(\alpha \to f, g) \leq \alpha \to \delta(f,g)$.

Let $\delta(\alpha \to f, g) \leq \alpha \to \delta(f,g)$. Then, $\delta(f,g) \leq \delta(\alpha \to \alpha \odot f, g) \leq \alpha \to \delta(\alpha \odot f, g)$. Thus, $\alpha \odot \delta(f,g) \leq \delta(\alpha \odot f, g)$.

From the following theorem, we obtain the L-fuzzy closure operator induced by an L-fuzzy pre-proximity.

Theorem 12 *Let δ be an L-fuzzy pre-proximity on X. Define $\mathcal{C}_\delta : L^X \to L^X$ as follows:*

$$\mathcal{C}_\delta(f)(x) = \bigwedge_{g \in L^X} \{g(x) \to \delta(g, g^*) \mid f \le g^*\}.$$

Then,

1. $(X, \mathcal{C}_\delta)$ *is an L-fuzzy closure space,*
2. *If δ is stratified, then $\mathcal{C}_\delta$ is stratified,*
3. *If δ is separated, then $\mathcal{C}_\delta$ is separated.*

Proof
(1)(C1) Since $\delta(\top_X, \bot_X) = \bot$,

$$\begin{aligned} \mathcal{C}_\delta(\bot_X)(x) &= \bigwedge\nolimits_{g \in L^X} \{g(x) \to \delta(g, g^*) \mid \bot_X \le g^*\} \\ &\le (\top_X(x) \to \delta(\top_X, \bot_X)) = \bot_X(x). \end{aligned}$$

(C2) Since $g \le f^*$, then $g \to \delta(g, g^*) \ge f^* \to \bot = f$.

(C3) If $f \le h$, then

$$\begin{aligned} \mathcal{C}_\delta(h)(x) &= \bigwedge\nolimits_{g \in L^X} \{(g(x) \to \delta(g, g^*)) \mid h \le g^*\} \\ &\ge \bigwedge\nolimits_{g \in L^X} \{(g(x) \to \delta(g, g^*)) \mid f \le g^*\} = \mathcal{C}_\delta(f)(x). \end{aligned}$$

(C4) Since

$$\begin{aligned} &((a \to b) \oplus (c \to d))^* = (a \to b)^* \odot (c \to d)^* \\ &= (a \odot b^*) \odot (c \odot d^*) = (a \odot c) \odot (b^* \odot d^*), \end{aligned}$$

then we have $(a \to b) \oplus (c \to d) = (a \odot c) \to (b \oplus d)$. From Lemma 2, we obtain

$$\begin{aligned} \mathcal{C}_\delta(f)(x) \oplus \mathcal{C}_\delta(h)(x) &= \bigwedge\nolimits_{g \in L^X} \{(g(x) \to \delta(g, g^*)) \mid f \le g^*\} \\ &\oplus \bigwedge\nolimits_{k \in L^X} \{(k(x) \to \delta(k, k^*)) \mid h \le k^*\} \\ &= \bigwedge\nolimits_{g,k \in L^X} \{(g(x) \odot k(x)) \to (\delta(g, g^*) \oplus \delta(k, k^*)) \mid f \le g^*, h \le k^*\} \\ &\ge \bigwedge\nolimits_{g,k \in L^X} \{(g \odot k)(x)) \to \delta(g \odot k, g^* \oplus k^*)) \mid f \oplus h \le g^* \oplus k^*\} \\ &\ge \mathcal{C}_\delta(f \oplus h)(x). \end{aligned}$$

Hence, $\mathcal{C}_\delta$ is an L-fuzzy closure operator on X.

(2)

$$\begin{aligned} \alpha \to \mathcal{C}_\delta(f) &= \alpha \to \bigwedge\nolimits_{g \in L^X} \{(g(x) \to \delta(g, g^*)) \mid f \le g^*\} \\ &= \bigwedge\nolimits_{g \in L^X} \{((\alpha \odot g(x)) \to \delta(g, g^*)) \mid f \le g^*\} \\ &\ge \bigwedge\nolimits_{g \in L^X} \{(\alpha \odot g(x)) \to \delta((\alpha \odot g, \alpha \to g^*)) \mid \alpha \to f \le \alpha \to g^*\} \\ &\ge \mathcal{C}_\delta(\alpha \to f). \end{aligned}$$

(3) By (C2) and

$$\mathcal{C}_\delta(\top_x^*)(x) = \bigwedge_{g\in L^X}\{g(x) \to \delta(g, g^*) \mid \top_x^* \le g^*\} \le \top_x(x) \to \delta(\top_x, \top_x^*) = \top_x^*,$$

we have $\mathcal{C}_\delta(\top_x^*) = \top_x^*$.

Example 13
Let X be a set and $R \in L^{X\times X}$ be an L-fuzzy pre-order. Define $\delta : L^X \times L^X \to L$ as

$$\delta(f, g) = \bigvee_{x,y\in X} R(x, y) \odot f(x) \odot g(y).$$

(P1) and (P3) are easily proved.

(P2) For all $f, g \in L^X$,

$$\begin{aligned}\delta(f, g) &= \bigvee_{x,y\in X} R(x, y) \odot f(x) \odot g(y)\\ &\ge \bigvee_{x\in X} R(x, x) \odot f(x) \odot g(x) = \bigvee_{x\in X} f(x) \odot g(x).\end{aligned}$$

(P4) For all $f_1, f_2, h_1, h_2 \in L^X$, by Lemma 2 (17),

$$\begin{aligned}\delta(f_1, h_1) \oplus \delta(f_2, h_2) &= (\bigvee_{x,y\in X} R(x, y) \odot f_1(x) \odot h_1(y))\oplus\\ &(\bigvee_{z,w\in X} R(z, w) \odot f_2(z) \odot h_2(w))\\ &\ge \bigvee_{x,y,z,w\in X}(R(x, y) \odot R(z, w) \odot f_1(x) \odot f_2(z))\odot\\ &(h_1(y) \oplus h_2(w))\\ &\ge \bigvee_{x,y,w\in X}(R(x, y) \odot R(y, w) \odot f_1(x) \odot f_2(x))\odot\\ &(h_1(w) \oplus h_2(w))\\ &= \bigvee_{x,w\in X}(\bigvee_{y\in X}(R(x, y) \odot R(y, w))\\ &\odot (f_1(x) \odot f_2(x)) \odot (h_1(w) \oplus h_2(w))\\ &= \bigvee_{x,w\in X}(R(x, w) \odot f_1(x) \odot f_2(x)) \odot (h_1(w) \oplus h_2(w))\\ &= \delta(f_1 \odot f_2, h_1 \oplus h_2).\end{aligned}$$

Hence, δ is an L-fuzzy pre-proximity on X. Since

$$\begin{aligned}\delta(\alpha \odot f, \alpha \to g) &= \bigvee_{x,y\in X}(R(x, y) \odot (\alpha \odot f)(x) \odot (\alpha \to g)(y))\\ &\le \bigvee_{x,y\in X}(R(x, y) \odot f(x) \odot g(y)) = \delta(f, g),\end{aligned}$$

δ is stratified. Moreover, δ is Alexandrov and generalized. By Theorem 12, we obtain a stratified L-fuzzy closure operator $\mathcal{C}_\delta : L^X \to L^X$ as

$$\mathcal{C}_\delta(f)(x) = \bigwedge_{f\le g^*}(g(x) \to \bigvee_{x,y\in X}(R(x, y) \odot g(x) \odot g^*(y))).$$

(1) Let $R = \top_{X\times X}$ be given. Then, $\delta_1(f, g) = \bigvee_{x,y\in X} f(x) \odot g(y)$.

Hence, δ_1 is an L-fuzzy pre-proximity on X. Moreover, δ_1 is stratified, Alexandrov and generalized. Since $\delta_1(\top_x, \top_x^*) = \top$, δ_1 is not separated.

By Theorem 12, we obtain a stratified L-fuzzy closure operator $\mathcal{C}_{\delta_1} : L^X \to L^X$ as

$$\mathcal{C}_{\delta_1}(f) = \bigwedge_{f\le g^*}(g(x) \to (\bigvee_{x,y\in X} g(x) \odot g^*(y))).$$

(2) Let $R = \triangle_{X\times X}$ be given, where

$$\triangle_{X\times X}(x,y)=\begin{cases}\top, & \text{if } y=x,\\ \bot, & \text{otherwise}.\end{cases}$$

Then, $\delta_2(f,g)=\bigvee_{x\in X} f(x)\odot g(x)$. Hence, δ_2 is an L-fuzzy pre-proximity on X. Moreover,

(Q) For all $f,g\in L^X$,

$$\begin{aligned}&\bigwedge_{h\in L^X}(\delta_2(f,h)\oplus\delta_2(h^*,g))\\ &=\bigwedge_{h\in L^X}(\bigvee_{x\in X}(f(x)\odot h(x))\oplus\bigvee_{x\in X}(h^*(x)\odot g(x)))\quad(\text{Put } h=g)\\ &\le\bigvee_{x\in X}(f(x)\odot g(x))\oplus\bigvee_{x\in X}(g^*(x)\odot g(x))\\ &=\bigvee_{x\in X}(f(x)\odot g(x))\oplus\bot=\delta_2(f,g).\end{aligned}$$

Hence, δ_2 is an L-fuzzy proximity on X. Since $\delta_2(\top_x,\top_x^*)=\bot$, δ_2 is separated. Hence, δ_2 is separated, stratified, Alexandrov and generalized. By Theorem 12, we obtain a strong, separated, generalized and Alexandrov L-fuzzy closure operator $\mathcal{C}_{\delta_2}:L^X\to L^X$ as follows:

$$\mathcal{C}_{\delta_2}(f)=\bigwedge_{f\le g^*}(g(x)\to(\bigvee_{x\in X} g(x)\odot g^*(x)))=\bigwedge_{f\le g^*}(g(x)\to\bot)=f.$$

From the following theorem, we obtain the L-fuzzy pre-proximity induced by an L-fuzzy closure operator.

Theorem 14 *Let $(X,\mathcal{C})$ be an L-fuzzy closure space. Define a mapping $\delta_{\mathcal{C}}:L^X\times L^X\to L$ by*

$$\delta_{\mathcal{C}}(f,g)=\bigvee_{x\in X} f(x)\odot\mathcal{C}(g)(x)\quad\forall f,g\in L^X.$$

Then, we have the following properties.

(1) *$\delta_{\mathcal{C}}$ is an L-fuzzy pre-proximity,*
(2) *If $\mathcal{C}$ is stratified, then so is $\delta_{\mathcal{C}}$* and $\delta_{\mathcal{C}}(f,\alpha\odot g)\ge\alpha\odot\delta_{\mathcal{C}}(f,g)$,
(3) *$\delta_{\mathcal{C}}(f,g)\le\bigvee_{h\in L^X}(\delta_{\mathcal{C}}(f,h)\odot\delta_{\mathcal{C}}(h^*,g))$,the equality holds if $\mathcal{C}$ is topological,*
(4) *If $\mathcal{C}$ is topological, then $\delta_{\mathcal{C}}$ is an L-fuzzy quasi-proximity on X,*
(5) *$\mathcal{C}\le\mathcal{C}_{\delta_{\mathcal{C}}}$, the equality holds if $\mathcal{C}$ is topological,*
(6) *If $\mathcal{C}$ is separated, then $\delta_{\mathcal{C}}$ is separated,*
(7) *$\delta_{\mathcal{C}_\delta}\le\delta$,*
(8) *If $\mathcal{C}$ is generalized (resp. Alexandrov), then $\delta_{\mathcal{C}}$ is generalized (resp. Alexandrov).*

Proof
(1) (P1) Since $\mathcal{C}(\bot_X)=\bot_X$ and $\mathcal{C}(\top_X)=\top_X$, we have

$$\begin{aligned}\delta_{\mathcal{C}}(\top_X,\bot_X)&=\bigvee_{x\in X}(\top_X(x)\odot\mathcal{C}(\bot_X)(x))=\bot,\\ \delta_{\mathcal{C}}(\bot_X,\top_X)&=\bigvee_{x\in X}(\bot_X(x)\odot\mathcal{C}(\top_X)(x))=\bot.\end{aligned}$$

(P2) Since $\mathcal{C}(f)\ge f$, we have

$$\delta_{\mathcal{C}}(f,g) = \bigvee_{x\in X} f(x) \odot \mathcal{C}(g)(x) \geq \bigvee_{x\in X} f(x) \odot g(x).$$

(P3) If $f \leq f_1$ and $g \leq g_1$, then $\mathcal{C}(g) \leq \mathcal{C}(g_1)$. Thus,

$$\delta_{\mathcal{C}}(f,g) = \bigvee_{x\in X} f(x) \odot \mathcal{C}(g)(x) \leq \bigvee_{x\in X} f_1(x) \odot \mathcal{C}(g_1)(x) = \delta_{\mathcal{C}}(f_1,g_1).$$

(P4)

$$\begin{aligned}\delta_{\mathcal{C}}(f_1,g_1) \oplus \delta_{\mathcal{C}}(f_2,g_2) &= \bigvee_{x\in X}(f_1(x) \odot \mathcal{C}(g_1)(x)) \oplus (\bigvee_{x\in X} f_2(x) \odot \mathcal{C}(g_2)(x))\\ &\geq \bigvee_{x\in X}(f_1(x) \odot \mathcal{C}(g_1)(x)) \oplus (f_2(x) \odot \mathcal{C}(g_2)(x))\\ &\text{(by Lemma 2(13))}\\ &\geq \bigvee_{x\in X}(f_1(x) \odot f_2(x)) \odot (\mathcal{C}(g_1)(x) \oplus \mathcal{C}(g_2)(x))\\ &\geq \bigvee_{x\in X}(f_1(x) \odot f_2(x)) \odot \mathcal{C}(g_1 \oplus g_2)(x) = \delta_{\mathcal{C}}(f_1 \oplus f_2, g_1 \oplus g_2).\end{aligned}$$

Hence, $\delta_{\mathcal{C}}$ is an L-fuzzy pre-proximity on X.

(2) If $\mathcal{C}$ is a stratified, we have

$$\begin{aligned}\delta_{\mathcal{C}}(\alpha \odot f, \alpha \to g) &= \bigvee_{x\in X}(\alpha \odot f)(x) \odot \mathcal{C}(\alpha \to g)(x)\\ &\leq \bigvee_{x\in X} \alpha \odot f(x) \odot (\alpha \to \mathcal{C}(g)(x))\\ &\leq \bigvee_{x\in X} f(x) \odot \mathcal{C}(g)(x) = \delta_{\mathcal{C}}(f,g),\end{aligned}$$

$$\begin{aligned}\delta_{\mathcal{C}}(f, \alpha \odot g) &= \bigvee_{x\in X} f(x) \odot \mathcal{C}(\alpha \odot g)(x)\\ &\geq \bigvee_{x\in X} f(x) \odot \alpha \odot \mathcal{C}(g)(x)\\ &= \alpha \odot (\bigvee_{x\in X} f(x) \odot \mathcal{C}(g)(x)) = \alpha \odot \delta_{\mathcal{C}}(f,g).\end{aligned}$$

(3)

$$\begin{aligned}&\delta^*_{\mathcal{C}}(f,h) \odot \delta^*_{\mathcal{C}}(h^*,g)\\ &= \Big(\bigvee_{x\in X} f(x) \odot \mathcal{C}(h)(x)\Big)^* \odot \Big(\bigvee_{x\in X} h^*(x) \odot \mathcal{C}(g)(x)\Big)^*\\ &= \bigwedge_{x\in X}(f(x) \to \mathcal{C}^*(h)(x)) \odot \bigwedge_{x\in X}(h^*(x) \to \mathcal{C}^*(g)(x))\\ &\quad \text{(Since } \mathcal{C}^*(h) \leq h^*)\\ &\leq \bigwedge_{x\in X}(f(x) \to h^*(x)) \odot \bigwedge_{x\in X}(h^*(x) \to \mathcal{C}^*(g)(x))\\ &\leq \bigwedge_{x\in X}(f(x) \to \mathcal{C}^*(g)(x)) = \delta^*_{\mathcal{C}}(f,g).\end{aligned}$$

Hence, $\delta_{\mathcal{C}}(f,g) \leq \bigwedge_{h\in L^X}(\delta_{\mathcal{C}}(f,h) \oplus \delta_{\mathcal{C}}(h^*,g))$.

If $\mathcal{C}$ is topological, then

$$\begin{aligned}&\bigvee_{h\in L^X}(\delta^*_{\mathcal{C}}(f,h) \odot \delta^*_{\mathcal{C}}(h^*,g))\\ &= \bigvee_{h\in L^X}(\bigwedge_{x\in X}(f(x) \to \mathcal{C}^*(h)(x))) \odot (\bigwedge_{x\in X}(h^*(x) \to \mathcal{C}^*(g)(x)))\\ &\quad \text{(put } \mathcal{C}(g) = h)\\ &\geq \bigwedge_{x\in X}(f(x) \to \mathcal{C}^*(\mathcal{C}(g))(x)) \odot (\bigwedge_{x\in X}(\mathcal{C}^*(g)(x) \to \mathcal{C}(g^*)(x))\\ &= \bigwedge_{x\in X}(f(x) \to \mathcal{C}^*(g)(x)) = \delta^*_{\mathcal{C}}(f,g).\end{aligned}$$

(4) By (3), it is trivial.

(5) From Lemma 2, we have,

$$\begin{aligned} \mathcal{C}_{\delta_{\mathcal{C}}}(f)(x) &= \bigwedge_{g\in L^X}\{\bigwedge_{x\in X}(g(x)\to \delta_{\mathcal{C}}(g,g^*))\mid f\le g^*\}\\ &= \bigwedge_{g\in L^X}\{\bigwedge_{x\in X}(g(x)\to (\bigvee_{x\in X} g(x)\odot \mathcal{C}(g^*)(x)))\mid f\le g^*\}\\ &= \{\big(\bigvee_{g\in L^X} g(x)\odot \bigwedge_{x\in X}(\mathcal{C}(g^*)(x)\to g^*(x))\big)^*\mid f\le g^*\}\\ &\ge \{\big(\bigvee_{g\in L^X}(\bigwedge_{x\in X}(\mathcal{C}(f)(x)\to g^*(x))\odot g(x))\big)^*\mid \mathcal{C}(f)\le \mathcal{C}(g^*)\}\\ &= \Big(\bigvee_{g\in L^X}(\bigwedge_{x\in X}(g(x)\to \mathcal{C}^*(f)(x))\odot g(x))\Big)^*\ge \mathcal{C}(f)(x). \end{aligned}$$

If $\mathcal{C}$ is topological, then

$$\begin{aligned} \mathcal{C}_{\delta_{\mathcal{C}}}(f)(x) &= \bigwedge_{g\in L^X}\{g(x)\to \delta_{\mathcal{C}}(g,g^*)\mid f\le g^*\}\\ &= \{\Big(\bigvee_{g\in L^X} g(x)\odot \bigwedge_{x\in X}(\mathcal{C}(g^*)(x)\to g^*(x))\Big)^*\mid f\le g^*\}\\ &\quad (\text{Put } g^*=\mathcal{C}(f))\\ &\le \Big(\mathcal{C}^*(f)(x)\odot \bigwedge_{x\in X}(\mathcal{C}(\mathcal{C}(f)(x))\to \mathcal{C}(f)(x))\Big)^*=\mathcal{C}(f)(x). \end{aligned}$$

(6) $\delta^*_{\mathcal{C}_\delta}(\top_x,\top_x^*)=\bigwedge_{x\in X}(\mathcal{C}_\delta(\top_x^*)(x)\to \top_x^*(x))=\top.$

(7)

$$\begin{aligned} \delta_{\mathcal{C}_\delta}(f,g) &= \bigvee_{x\in X} f(x)\odot \mathcal{C}_\delta(g)(x)\\ &= \bigvee_{x\in X} f(x)\odot \Big(\bigvee_{h\le g^*}\delta^*(h,h^*)\odot h(x)\Big)^*\\ &\le \bigvee_{x\in X} f(x)\odot \Big(\bigvee_{h\le g^*}(\bigwedge_{x\in X}(h(x)\to h(x))\odot h(x))\Big)^*\\ &\le \bigvee_{x\in X} f(x)\odot g(x)\le \delta(f,g). \end{aligned}$$

(8) It is easily proved from definitions.

Corollary 15 *Let $(X,\mathcal{C})$ be an L-fuzzy closure space. Define a mapping $\delta^s_{\mathcal{C}}: L^X\times L^X\to L$ by*

$$\delta^s_{\mathcal{C}}(f,g)=\bigvee_{x\in X} g(x)\odot \mathcal{C}(f)(x)\quad \forall f,g\in L^X.$$

Then, we have the following properties.

1. *$\delta^s_{\mathcal{C}}$ is an L-fuzzy pre-proximity,*
2. *If $\mathcal{C}$ is stratified, then δ^s is a stratified,*
3. *$\delta^s_{\mathcal{C}}(f,g)\le \bigvee_{h\in L^X}(\delta^s_{\mathcal{C}}(f,h)\odot \delta^s_{\mathcal{C}}(h^*,g))$, the equality holds if $\mathcal{C}$ is topological,*
4. *If $\mathcal{C}$ is topological, then $\delta^s_{\mathcal{C}}$ is a L-fuzzy quasi-proximity on X,*
5. *$\mathcal{C}\le \mathcal{C}_{\delta^s_{\mathcal{C}}}$, the equality holds if $\mathcal{C}$ is topological,*
6. *If $\mathcal{C}$ is separated, then $\delta^s_{\mathcal{C}}$ is separated,*
7. *$\delta^s_{\mathcal{C}_\delta}\le \delta^s$,*
8. *If $\mathcal{C}$ is generalized (resp. Alexandrov), then $\delta^s_{\mathcal{C}}$ is generalized (resp. Alexandrov).*

The relationships between *L*-fuzzy pre-proximities and *L*-fuzzy co-topologies

Theorem 16 *Let δ be an Alexandrov L-fuzzy pre-proximity on X. Define a mapping $\mathcal{F}_\delta: L^X\to L$ by $\mathcal{F}_\delta(f)=\delta^*(f^*,f)$. Then,*

(1) $\mathcal{F}_\delta$ *is an L-fuzzy co-topology on X,*
(2) *If* δ *is stratified, then* $\mathcal{F}_\delta$ *is strong,*
(3) *If* δ *is separated, then* $\mathcal{F}_\delta$ *is separated.*

Proof
(1) (T1) $\mathcal{F}_\delta(\bot_X) = \delta^*(\bot_X^*, \bot_X) = \top, \ \mathcal{F}_\delta(\top_X) = \delta^*(\top_X^*, \top_X) = \top.$

(T2) $\mathcal{F}_\delta(f \oplus g) = \delta^*(f^* \odot g^*, f \oplus g) \geq \delta^*(f^*, f) \odot \delta^*(g^*, g) = \mathcal{F}_\delta(f) \odot \mathcal{F}_\delta(g).$

(T3) $\mathcal{F}_\delta(\bigwedge_{i\in\Gamma} f_i) = \delta^*(\bigvee_{i\in\Gamma} f_i^*, \bigwedge_{i\in\Gamma} f_i) \geq \bigwedge_{i\in\Gamma} \delta^*(f_i^*, f_i) = \bigwedge_{i\in\Gamma} \mathcal{F}_\delta(f_i).$

(2) $\mathcal{F}_\delta(\alpha \odot f) = \delta^*(\alpha \to f^*, \alpha \odot f) \geq \delta^*(f^*, f) = \mathcal{F}_\delta(f),$

$$\mathcal{F}_\delta(\alpha \to f) = \delta^*(\alpha \odot f^*, \alpha \to f) \geq \delta^*(f^*, f) = \mathcal{F}_\delta(f).$$

(3) It is easy.

Theorem 17 *Let* $(X, \mathcal{C})$ *be an L-fuzzy closure space. Define the mapping* $\mathcal{F}_{\mathcal{C}_\delta} : L^X \to L$ *by*

$$\mathcal{F}_{\mathcal{C}_\delta}(f) = \bigwedge_{x\in X} (\mathcal{C}_\delta(f)(x) \to f(x)).$$

Then,

(1) $\mathcal{F}_{\mathcal{C}_\delta}$ *is an L-fuzzy co-topology on X with* $\mathcal{F}_{\mathcal{C}_\delta} \geq \mathcal{F}_\delta$,
(2) *If* $\mathcal{C}$ *is Alexandrov (resp. strong, separated), then* $\mathcal{F}_{\delta_\mathcal{C}}$ *is Alexandrov (resp. strong, separated).*

Proof
(1) (T1) $\mathcal{F}_{\mathcal{C}_\delta}(\top_X) = \bigwedge_{x\in X}(\mathcal{C}_\delta(\top_X)(x) \to \top_X(x)) = \top,$

$$\mathcal{F}_{\mathcal{C}_\delta}(\bot_X) = \bigwedge_{x\in X}(\mathcal{C}_\delta(\bot_X)(x) \to \bot_X(x)) = \top.$$

(T2)

$$\begin{aligned}\mathcal{F}_{\mathcal{C}_\delta}(f \oplus g) &= \textstyle\bigwedge_{x\in X}(\mathcal{C}_\delta(f \oplus g)(x) \to (f \oplus g)(x)) \\ &\geq \textstyle\bigwedge_{x\in X}((\mathcal{C}_\delta(f)(x) \oplus \mathcal{C}_\delta(g)(x)) \to (f(x) \oplus g(x))) \quad \text{(by Lemma 2(13))} \\ &\geq \textstyle\bigwedge_{x\in X}(\mathcal{C}_\delta(f)(x) \to f(x)) \odot \bigwedge_{x\in X}(\mathcal{C}_\delta(g)(x) \to g(x)) \\ &= \mathcal{F}_{\mathcal{C}_\delta}(f) \odot \mathcal{F}_{\mathcal{C}_\delta}(g).\end{aligned}$$

(T3) By Lemma 2(16), we have

$$\begin{aligned}\mathcal{F}_{\mathcal{C}_\delta}(\textstyle\bigwedge_{i\in\Gamma} f_i) &= \textstyle\bigwedge_{x\in X}(\mathcal{C}_\delta(\bigwedge_{i\in\Gamma} f_i)(x) \to (\bigwedge_{i\in\Gamma} f_i)(x)) \\ &\geq \textstyle\bigwedge_{x\in X}(\bigwedge_{i\in\Gamma} \mathcal{C}_\delta(f_i)(x) \to \bigwedge_{i\in\Gamma} f_i(x)) \\ &\geq \textstyle\bigwedge_{i\in\Gamma}(\bigwedge_{x\in X}(\mathcal{C}_\delta(f_i)(x) \to f_i(x))) = \bigwedge_{i\in\Gamma} \mathcal{F}_{\mathcal{C}_\delta}(f_i).\end{aligned}$$

Hence, $\mathcal{F}_{\mathcal{C}_\delta}$ is an L-fuzzy co-topology on X.

Moreover,

$$\begin{aligned}\mathcal{F}_{\mathcal{C}_\delta}(f) &= \bigwedge_{x\in X}(\mathcal{C}_\delta(f)(x) \to f(x)) \\ &= \big(\bigvee_{x\in X}\mathcal{C}_\delta(f)(x) \odot f^*(x)\big)^* \\ &\geq \big(\bigvee_{x\in X} f(x) \odot f^*(x)\big)^* \geq \delta^*(f^*, f) = \mathcal{F}_\delta(f).\end{aligned}$$

(2)

$$\begin{aligned}\mathcal{F}_{\mathcal{C}_\delta}(\bigvee_{i\in\Gamma} f_i) &= \bigwedge_{x\in X}(\mathcal{C}_\delta(\bigvee_{i\in\Gamma} f_i)(x) \to \bigvee_{i\in\Gamma} f_i(x)) \\ &= \bigwedge_{x\in X}(\bigvee_{i\in\Gamma}\mathcal{C}_\delta(f_i)(x) \to \bigvee_{i\in\Gamma} f_i(x))) \\ &\geq \bigwedge_{i\in\Gamma}(\bigwedge_{x\in X}(\mathcal{C}_\delta(f_i)(x) \to f_i(x))) = \bigwedge_{i\in\Gamma}\mathcal{F}_{\mathcal{C}_\delta}(f_i).\end{aligned}$$

Hence, $\mathcal{F}_{\mathcal{C}_\delta}$ is an Alexandrov L-fuzzy co-topology on X. By Lemma 2(14)(18), we have

$$\begin{aligned}\mathcal{F}_{\mathcal{C}_\delta}(\alpha \odot f) &= \bigwedge_{x\in X}(\mathcal{C}_\delta(\alpha \odot f)(x) \to (\alpha \odot f(x))) \\ &\geq \bigwedge_{x\in X}((\alpha \odot \mathcal{C}_\delta(f)(x)) \to (\alpha \odot f(x))) \\ &\geq \bigwedge_{x\in X}(f(x) \to \mathcal{C}_\delta(f)(x)) = \mathcal{F}_{\mathcal{C}_\delta}(f),\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\mathcal{C}_\delta}(\alpha \to f) &= \bigwedge_{x\in X}(\mathcal{C}_\delta(\alpha \to f)(x) \to (\alpha \to f(x))) \\ &\geq \bigwedge_{x\in X}((\alpha \to \mathcal{C}_\delta(f)(x)) \to (\alpha \to f(x))) \\ &\geq \bigwedge_{x\in X}(\mathcal{C}_\delta(f)(x) \to f(x)) = \mathcal{F}_{\mathcal{C}_\delta}(f).\end{aligned}$$

Other cases are easily proved.

Theorem 18 *Let (X, δ) be an L-fuzzy pre-proximity space. Then, the mapping $\mathcal{F}_\delta^{(1)} : L^X \to L$ defined by $\mathcal{F}_\delta^{(1)}(f) = \bigwedge_{x\in X}(\delta(f, \top_x) \to f(x))$ is an L-fuzzy co-topology on X. Moreover, if δ is Alexandrov and $\delta(\alpha \odot f, g) \geq \alpha \odot \delta(f, g)$, then $\mathcal{F}_\delta^{(1)}(f^*) \geq \mathcal{F}_\delta(f)$.*

Proof

(1) (T1) $\mathcal{F}_\delta^{(1)}(\bot_X) = \bigwedge_{x\in X}(\delta(\bot_X, \top_x) \to \bot_X(x)) = \top$,

$$\mathcal{F}_\delta^{(1)}(\top_x) = \bigwedge_{x\in X}(\delta(\top_X, \top_x) \to \top_X(x)) = \top.$$

(T2)

$$\begin{aligned}\mathcal{F}_\delta^{(1)}(f \oplus g) &= \bigwedge_{x\in X}(\delta(f \oplus g, \top_x) \to (f \oplus g)(x)) \\ &\geq \bigwedge_{x\in X}((\delta(f, \top_x) \oplus \delta(g, \top_x)) \to (f(x) \oplus g(x))) \\ &\geq \bigwedge_{x\in X}(\delta(f, \top_x) \to f(x)) \odot \bigwedge_{x\in X}(\delta(g, \top_x) \to g(x)) \\ &\geq \mathcal{F}_\delta^{(1)}(f) \odot \mathcal{F}_\delta^{(1)}(g).\end{aligned}$$

(T3)

$$\begin{aligned}\mathcal{F}_\delta^{(1)}(\bigwedge_{i\in\Gamma} f_i) &= \bigwedge_{x\in X}(\delta(\bigwedge_{i\in\Gamma} f_i, \top_x) \to \bigwedge_{i\in\Gamma} f_i(x)) \\ &= \bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(\delta(\bigwedge_{i\in\Gamma} f_i, \top_x) \to f_i(x)) \\ &\geq \bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(\delta(f_i, \top_x) \to f_i(x)) = \bigwedge_{i\in\Gamma}\mathcal{F}_\delta^{(1)}(f_i).\end{aligned}$$

Moreover, if δ is Alexandrov, then

$$\begin{aligned}\mathcal{F}^{(1)}_\delta(\bigvee_{i\in\Gamma} f_i) &= \bigwedge_{x\in X}(\delta(\bigvee_{i\in\Gamma} f_i, \top_x) \to \bigvee_{i\in\Gamma} f_i(x)) \\ &= \bigwedge_{x\in X}(\bigvee_{i\in\Gamma}\delta(f_i, \top_x) \to \bigvee_{i\in\Gamma} f_i(x)) \\ &\geq \bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(\delta(f_i, \top_x) \to f_i(x)) = \bigwedge_{i\in\Gamma}\mathcal{F}^{(1)}_\delta(f_i).\end{aligned}$$

Hence, $\mathcal{F}^{(1)}_\delta$ is Alexandrov L-fuzzy co-topology on X.

If $\delta(\alpha\odot f, g) \geq \alpha\odot\delta(f,g)$, then

$$\begin{aligned}\mathcal{F}_\delta(f) = \delta^*(f^*, f) &= \delta^*(f^*, \bigvee_{x\in X} f(x)\odot\top_x) \\ &\leq \bigwedge_{x\in X}(f(x) \to \delta^*(f^*, \top_x)) \\ &= \bigwedge_{x\in X}(\delta(f^*, \top_x) \to f^*(x)) = \mathcal{F}^{(1)}_\delta(f^*).\end{aligned}$$

Example 19

Let $X = \{h_i \mid i = \{1, ..., 3\}\}$ *with* h_i = *house and* $Y = \{e, b, w, c, i\}$ *with* e = *expensive,* b = *beautiful,* w=*wooden,* c = *creative,* i = *in the green surroundings. Let* $([0,1], \odot, \to, ^*, 0, 1)$ *be a complete residuated lattice as*

$$x\odot y = \max\{0, x+y-1\},\ x\to y = \min\{1-x+y, 1\},\ x^* = 1-x.$$

Let $I \in [0,1]^{X\times Y}$ be a fuzzy information as follows:

I	e	b	w	c	i
h_1	0.7	0.6	0.5	0.9	0.2
h_2	0.6	0.8	0.4	0.3	0.5
h_3	0.4	0.9	0.8	0.6	0.6

Define [0, 1]-fuzzy pre-orders $R^Y_X,\ R^{\{b,w\}}_X \in [0,1]^{X\times X}$ by

$$\begin{aligned}R^Y_X(h_i, h_j) &= \bigwedge_{y\in Y}(I(h_i, y) \to I(h_j, y)), \\ R^{\{b,w\}}_X(h_i, h_j) &= \bigwedge_{y\in\{b,w\}}(R(h_i, y) \to R(h_j, y)),\end{aligned}$$

$$R^Y_X = \begin{pmatrix} 1 & 0.4 & 0.7 \\ 0.7 & 1 & 0.8 \\ 0.6 & 0.6 & 1\end{pmatrix}, R^{\{b,w\}}_X = \begin{pmatrix} 1 & 0.9 & 1 \\ 0.8 & 1 & 1 \\ 0.7 & 0.6 & 1\end{pmatrix}.$$

(1) For each $R \in \{R^Y_X, R^{\{b,w\}}_X\}$, by Example 13, we obtain a stratified, Alexandrov and generalized [0, 1]-fuzzy pre-proximity $\delta_R : [0,1]^X \times [0,1]^X \to [0,1]$ as

$$\delta_R(f, g) = \bigvee_{h_i, h_j\in X} R^Y_X(h_i, h_j)\odot f(h_i)\odot g(h_j).$$

By Theorem 12, we obtain a stratified [0, 1]-fuzzy closure operator $C_{\delta_R} : [0,1]^X \to [0,1]^X$ as

$$\begin{aligned}C_{\delta_R}(f)(h_i) &= \bigwedge_{g\in L^X}((S(f, g^*)\odot g(h_i)) \to \delta_R(g, g^*)) \\ &= \bigwedge_{g\in L^X}\Big((S(f, g^*)\odot g(h_i)) \\ &\to \big(\bigvee_{h_j, h_k\in X} R^Y_X(h_j, h_k)\odot g(h_j)\odot g^*(h_k)\big)\Big).\end{aligned}$$

By Theorem 16, we obtain a strong [0, 1]-fuzzy co-topology $\mathcal{F}_{\delta_R} : [0,1]^X \to [0,1]$ as

$$\begin{aligned}\mathcal{F}_{\delta_R}(f)=\delta_R^*(f^*,f) &= (\bigvee_{h_i,h_j\in X} R_X^Y(h_i,h_j)\odot f^*(h_i)\odot f(h_j))^* \\ &= \bigwedge_{h_i,h_j\in X}(R_X^Y(h_i,h_j)\odot f(h_j)\to f(h_i)).\end{aligned}$$

Since

$$\delta_R(f,\top_{h_j})=\bigvee_{h_i,h_j\in X} R_X^Y(h_i,h_j)\odot f(h_i)\odot\top_{h_j}(h_j)=\bigvee_{h_i\in X} R(h_i,h_j)\odot f(h_i),$$

by Theorem 18, we obtain [0, 1]-fuzzy co-topology $\mathcal{F}^{(1)}_{\delta_R}:[0,1]^X\to[0,1]$ as

$$\begin{aligned}\mathcal{F}^{(1)}_{\delta_R}(f) &= \bigwedge_{h_j\in X}(\delta_R(f,\top_{h_j})\to f(h_j)) \\ &= \bigwedge_{h_j\in X}\left((\bigvee_{h_i\in X} R(h_i,h_j)\odot f(h_i))\to f(h_j)\right) \\ &= \bigwedge_{h_i,h_j\in X}((R(h_i,h_j)\odot f(h_i))\to f(h_j)).\end{aligned}$$

(2) For each $R\in\{R_X^Y,R_X^{\{b,w\}}\}$, we obtain a strong, generalized, topological and Alexandrov [0, 1]-fuzzy closure operator $C_R:[0,1]^X\to[0,1]^X$ as

$$C_R(f)(h_j)=\bigvee_{h_i\in X} R(h_i,h_j)\odot f(h_i).$$

By Theorem 14, we obtain a generalized, topological and Alexandrov [0, 1]-fuzzy quasi-proximity δ_{C_R} as

$$\begin{aligned}\delta_{C_R}(f,g) &= \bigvee_{h_i\in X} f(h_i)\odot C_R(g)(h_i) \\ &= \bigvee_{h_i\in X} f(h_i)\odot(\bigvee_{h_j\in X} R(h_j,h_i)\odot g(h_j)) \\ &= \bigvee_{h_i,h_j\in X} R(h_j,h_i)\odot f(h_i)\odot g(h_j).\end{aligned}$$

By Theorem 16, we obtain [0, 1]-fuzzy co-topologies $\mathcal{F}_{\delta_{C_R}}$ and $\mathcal{F}^{(1)}_{\delta_{C_R}}$ as follows:

$$\begin{aligned}\mathcal{F}_{\delta_{C_R}}(f)=\delta_{C_R}^*(f^*,f) &= (\bigvee_{h_i,h_j\in X} R(h_i,h_j)\odot f^*(h_i)\odot f(h_j))^* \\ &= \bigwedge_{h_i,h_j\in X}(R_X^Y(h_i,h_j)\odot f(h_j)\to f(h_i)).\end{aligned}$$

Also we have

$$\begin{aligned}\mathcal{F}^{(1)}_{\delta_{C_R}}(f) &= \bigwedge_{h_j\in X}(\delta_R(f,\top_{h_j})\to f(h_j)) \\ &= \bigwedge_{h_j\in X}((\bigvee_{h_i\in X} R(h_i,h_j)\odot f(h_i))\to f(h_j)) \\ &= \bigwedge_{h_i,h_j\in X}((R(h_i,h_j)\odot f(h_i))\to f(h_j)).\end{aligned}$$

Galois correspondences

Theorem 20 *Let $\varphi:X\to Y$ be a mapping. Then*

(1) $D_\delta(\varphi)\le D_{C_\delta}(\varphi)$,
(2) $D_\delta(\varphi)=D_{\mathcal{F}_\delta}(\varphi)$,
(3) $D_\delta(\varphi)\le D_{\mathcal{F}_\delta{}^{(1)}}(\varphi)$.

Proof
(1) By Lemma 2(18), we have

$$\begin{aligned}
D_{\mathcal{C}_\delta}(\varphi) &= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big(\mathcal{C}_{\delta_X}(\varphi^{\leftarrow}(f))(x)\to\varphi^{\leftarrow}(\mathcal{C}_{\delta_Y}(f))(x)\Big)\\
&= \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big(\bigwedge_{h\in L^X}\{h(x)\to\delta_X(h,h^*)\ \mid\ \varphi^{\leftarrow}(f)\le h^*\}\\
&\to \bigwedge_{g\in L^Y}\{g(y)\to\delta_Y(g,g^*)\ \mid\ f\le g^*\}\Big)\\
&\ge \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big(\bigwedge_{h\in L^X}\{h(x)\to\delta_X(h,h^*)\ \mid\ \varphi^{\leftarrow}(f)\le h^*\}\\
&\to \bigwedge_{g\in L^Y}\{g(\varphi(x))\to\delta_Y(g,g^*)\ \mid\ \varphi^{\leftarrow}(f)\le\varphi^{\leftarrow}(g^*)\}\Big)\\
&\ge \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big((\varphi^{\leftarrow}(g)(x)\to\delta_X(\varphi^{\leftarrow}(g),\varphi^{\leftarrow}(g^*)))\to(\varphi^{\leftarrow}(g)(x)\to\delta_Y(g,g^*))\Big)\\
&\ge \bigwedge_{f,g\in L^Y}\Big(\delta_X(\varphi^{\leftarrow}(g),\varphi^{\leftarrow}(g^*))\to\delta_Y(g,g^*)\Big)=D_\delta(\varphi).
\end{aligned}$$

(2)

$$\begin{aligned}
D_{\mathcal{F}_\delta}(\varphi) &= \bigwedge_{f\in L^Y}\Big(\mathcal{F}_{\delta_Y}(f)\to\mathcal{F}_{\delta_X}(\varphi^{\leftarrow}(f))\Big)\\
&= \bigwedge_{f\in L^Y}\Big(\delta^*_Y(f^*,f)\to\delta^*_X(\varphi^{\leftarrow}(f^*),\varphi^{\leftarrow}(f))\Big)\\
&= \bigwedge_{f\in L^Y}\Big(\delta_X(\varphi^{\leftarrow}(f^*),\varphi^{\leftarrow}(f))\to\delta_Y(f^*,f))\Big)=D_\delta(\varphi).
\end{aligned}$$

(3)

$$\begin{aligned}
D_{\mathcal{F}^{(1)}_\delta}(\varphi) &= \bigwedge_{f\in L^Y}\Big(\mathcal{F}^{(1)}_{\delta_Y}(f)\to\mathcal{F}^{(1)}_{\delta_X}(\varphi^{\leftarrow}(f))\Big)\\
&= \bigwedge_{f\in L^Y}\Big(\bigwedge_{y\in Y}(\delta_Y(f,\top_y)\to f(y))\\
&\to \bigwedge_{x\in X}(\delta_X(\varphi^{\leftarrow}(f),\varphi^{\leftarrow}(\top_{\varphi(x)})\to\varphi^{\leftarrow}(f)(x))\Big)\\
&\ge \bigwedge_{f\in L^Y}\Big(\bigwedge_{x\in X}(\delta_Y(f,\top_{\varphi(x)})\to f(\varphi(x)))\\
&\to \bigwedge_{x\in X}(\delta_X(\varphi^{\leftarrow}(f),\varphi^{\leftarrow}(\top_{\varphi(x)})\to f(\varphi(x))\Big)\\
&\ge \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\Big(\delta_X(\varphi^{\leftarrow}(f),\varphi^{\leftarrow}(\top_{\varphi(x)})\to\delta_Y(f,\top_{\varphi(x)})\Big)\\
&= \bigwedge_{f\in L^Y}\Big(\delta_X(\varphi^{\leftarrow}(f),\varphi^{\leftarrow}(\top_{\varphi(x)})\to\delta_Y(f,\top_{\varphi(x)})\Big)=D_\delta(\varphi).
\end{aligned}$$

Theorem 21 *Let $\varphi : X \to Y$ be a mapping. Then,*

(1) $D_{\mathcal{C}}(\varphi)\le D_{\delta_{\mathcal{C}}}(\varphi)$,
(2) $D_{\mathcal{C}_\delta}(\varphi)\le D_{\mathcal{F}_\delta}(\varphi)$.

Proof
(1) From Lemma 2(18), we have

$$\begin{aligned}
D_{\delta_{\mathcal{C}}}(\varphi) &= \bigwedge_{f,g\in L^Y}\Big(\delta_{\mathcal{C}_X}(\varphi^{\leftarrow}(f),\phi^{\leftarrow}(g))\to\delta_{\mathcal{C}_Y}(f,g)\Big)\\
&= \bigwedge_{f,g\in L^Y}\Big(\bigvee_{x\in X}\varphi^{\leftarrow}(f)(x)\odot\mathcal{C}_X(\varphi^{\leftarrow}(g))(x)\to\bigvee_{y\in Y}f(y)\odot\mathcal{C}_Y(g)(y)\Big)\\
&\ge \bigwedge_{f,g\in L^Y}\Big(\bigvee_{x\in X}f(\varphi(x))\odot\mathcal{C}_X(\varphi^{\leftarrow}(g))(x)\to\bigvee_{x\in X}f(\varphi(x))\odot\mathcal{C}_Y(g)(\varphi(x))\Big)\\
.\quad &\ge \bigwedge_{g\in L^Y}\bigwedge_{x\in X}\Big(\mathcal{C}_X(\varphi^{\leftarrow}(g))(x)\to\varphi^{\leftarrow}(\mathcal{C}_Y(g))(x)\Big)=D_{\mathcal{C}}(\varphi).
\end{aligned}$$

(2) From Lemma 2(18), we have

$$\begin{aligned} D_{\mathcal{F}_{\mathcal{C}_\delta}}(\varphi) &= \bigwedge_{f\in L^Y}\left(\mathcal{F}_{\mathcal{C}_{\delta_Y}}(f) \to \mathcal{F}_{\mathcal{C}_{\delta_X}}(\varphi^{\leftarrow}(f))\right) \\ &= \bigwedge_{f\in L^Y}\Big(\bigwedge_{y\in Y}(\mathcal{C}_{\delta_Y}(f)(y) \to f(y)) \\ &\to \bigwedge_{x\in X}(\mathcal{C}_{\delta_X}(\varphi^{\leftarrow}(f))(x) \to \varphi^{\leftarrow}(f)(x))\Big) \\ &\geq \bigwedge_{f\in L^Y}\Big(\bigwedge_{x\in X}(\mathcal{C}_{\delta_Y}(f)(\varphi(x)) \to f(\varphi(x))) \\ &\to \bigwedge_{x\in X}(\mathcal{C}_{\delta_X}(\varphi^{\leftarrow}(f))(x) \to \varphi^{\leftarrow}(f)(x))\Big) \\ &= \bigwedge_{f\in L^Y}\Big(\bigwedge_{x\in X}(\varphi^{\leftarrow}(\mathcal{C}_{\delta_Y}(f))(x) \to \varphi^{\leftarrow}(f)(x)) \\ &\to \bigwedge_{x\in X}(\mathcal{C}_{\delta_X}(\varphi^{\leftarrow}(f))(x) \to \varphi^{\leftarrow}(f)(x))\Big) \\ &\geq \bigwedge_{f\in L^Y}\bigwedge_{x\in X}\left(\mathcal{C}_{\delta_X}(\varphi^{\leftarrow}(f))(x) \to \varphi^{\leftarrow}(\mathcal{C}_{\delta_Y}(f))(x)\right) = D_{\mathcal{C}_\delta}(\varphi). \end{aligned}$$

Definition 22 [40] Suppose that $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ are concrete functors. The pair (F, G) is called a *Galois correspondence* between $\mathcal{C}$ and $\mathcal{D}$ if for each $Y \in \mathcal{C}$, $id_Y : F \circ G(Y) \to Y$ is a $\mathcal{C}$-morphism, and for each $X \in \mathcal{D}$, $id_X : X \to G \circ F(X)$ is a $\mathcal{D}$ -morphism.

If (F, G) is a Galois correspondence, then it is easy to check that F is a left adjoint of G, or equivalently that G is a right adjoint of F.

The category of separated L-fuzzy pre-proximity spaces with LF-proximity mappings as morphisms is denoted by **SPROX**.

The category of separated LF-fuzzy closure spaces with LF-closure mappings as morphisms is denoted by **SFC**.

From Theorems 12 and 20, we obtain a concrete functor $\Theta : \mathbf{SPROX} \to \mathbf{SFC}$ defined as

$$\Theta(X, \delta) = (X, \mathcal{C}_\delta), \Theta(\phi) = \phi.$$

From Theorems 14 and 21, we obtain a concrete functor $\Gamma : \mathbf{SFC} \to \mathbf{SPROX}$ defined as

$$\Gamma(X, \mathcal{C}) = (X, \delta_{\mathcal{C}}), \Gamma(\phi) = \phi.$$

Theorem 23 $\Gamma : \mathbf{SFC} \to \mathbf{SPROX}$ *is a left adjoint of* $\Theta : \mathbf{SPROX} \to \mathbf{SFC}$, *i.e*, (Θ, Γ) *is a Galois correspondence.*

Proof

By Theorem 14(5), if $\mathcal{C}_X$ *is a separated L-fuzzy closure operator on a set X, then* $\Theta(\Gamma(\mathcal{C}_X)) = \mathcal{C}_{\delta_{\mathcal{C}_X}} \geq \mathcal{C}_X$. *Hence, the identity map* $id_X : (X, \mathcal{C}_X) \to (X, \mathcal{C}_{\delta_{\mathcal{C}_X}}) = (X, \Theta(\Gamma(\mathcal{C}_X)))$ *is an LF-closure map. Moreover, if* δ_Y *is a separated L-fuzzy pre-proximity on a set Y, by Theorem 14(7), we have* $\Gamma(\Theta(\delta_Y)) = \delta_{\mathcal{C}_{\delta_Y}} \leq \delta_Y$. *Hence, the identity map* $id_Y : (Y, \Gamma(\Theta(\delta_Y))) \to (Y, \delta_Y)$ *is an LF-proximity map. Therefore,* (Θ, Γ) *is a Galois correspondence.*

Conclusion.

In this paper, L-fuzzy pre-proximities and L-fuzzy closure operators in complete residuated lattice are investigated. From a given L-fuzzy pre-proximity δ, we can obtain an L-fuzzy closure operator $\mathcal{C}_\delta$ (see Theorem 12). Conversely, for given L-fuzzy closure space $\mathcal{C}$, we obtain L-fuzzy pre-proximity $\delta_\mathcal{C}$ (see Theorem 14) and L-fuzzy co-topologies $\mathcal{F}_\delta$ and $\mathcal{F}_{\mathcal{C}_\delta}$ (Theorems 16, 17, 18). It is also shown that there is a Galois correspondence between the category of (separated) L-fuzzy closure spaces and that of (separated) L-fuzzy pre-proximity spaces (Theorem 21). We give Example 19 as a viewpoint of the topological structure for fuzzy information and fuzzy rough sets in a complete residuated lattice.

In the future, the concepts of L-fuzzy pre-proximity spaces, information systems and decision rules with a view point of applications to multi-attribute decision-making will be investigated in residuated lattices.

Acknowledgements
Not applicable.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Received: 21 November 2019 Accepted: 15 February 2021
Published online: 24 March 2021

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