Allah and Nawar *Journal of the Egyptian Mathematical Society* (2020) 28:38
https://doi.org/10.1186/s42787-020-00087-3

Journal of the Egyptian Mathematical Society

ORIGINAL RESEARCH Open Access

ψ*-closed sets in fuzzy topological spaces

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Abstract

In this paper, we introduce a new class of fuzzy sets, namely, fuzzy ψ*-closed sets for fuzzy topological spaces, and some of their properties have been proved. Further, we introduce fuzzy ψ*-continuous, fuzzy ψ*-irresolute functions, and fuzzy ψ*-closed (open) functions, as applications of these fuzzy sets, fuzzy $T_{1/5}$-spaces, fuzzy $T^{\psi *}_{1/5}$-spaces, and fuzzy ${}^{\psi^*}T_{1/5}$-spaces.

Keywords: Fuzzy ψ*-closed sets, Fuzzy $T_{1/5}$-spaces, Fuzzy ψ*-continuous, Fuzzy ψ*-closed functions

Mathematical subject classification: 54 C 10, 54 A 40, 54 A 05

Introduction

Zadeh [1] introduced the fundamental concept of fuzzy sets and fuzzy set operations in 1965. Fuzzy topology was introduced by Chang [2] in 1965. Subsequently, many researchers have worked on various basic concepts from general topology using fuzzy sets and developed the theory of fuzzy topological spaces [3–7]. Muthukumaraswamy and Devi [8] introduced fuzzy generalized α–closed and fuzzy α–generalized closed (briefly fgα-closed and fαg-closed) sets in fuzzy topological space in 2004. Abd Allah and Nawar [9] introduced and studied ψ*-closed sets in topological space in 2014. In this paper, we introduced another new notion of fuzzy generalized closed set called fuzzy ψ*-closed sets, which is properly placed in between the class of fuzzy α-closed sets and the class of fuzzy generalized α-closed sets. The structure of the rest of this paper is as follows. The "Preliminaries" section introduces the necessary definitions of fuzzy α-closed sets and fuzzy generalized α-closed sets. In the "Fuzzy ψ*-closed sets in fts" section, we introduce the definition of fuzzy ψ*-closed sets in fuzzy topological spaces and proved some of their properties. In the "Fuzzy ψ*-continuous and fuzzy ψ*-irresolute functions in fts" section, we identify the concept of fuzzy ψ*-continuous and fuzzy ψ*-irresolute functions and fuzzy ψ*-closed (open) functions and introducing some of their properties. Further, new classes of spaces, namely, fuzzy $T_{1/5}$-spaces, fuzzy $T^{\psi *}_{1/5}$-spaces, and fuzzy ${}^{\psi^*}T_{1/5}$-spaces, are introduced in the "Applications of Fψ*-closed sets" section.

Preliminaries

Throughout this paper, (G, τ) and (H, σ) (or simply, G and H) always mean fuzzy topological spaces. The members of τ are called fuzzy open sets, and their complements are fuzzy closed sets. And $\phi : (G, \tau) \to (H, \sigma)$ (or simply, ϕ: $G \to H$) denotes a mapping ϕ from fts G to fts H.

For a fuzzy set D of (G, τ), fuzzy closure and fuzzy interior of D denoted by cl(D) and int(D), respectively and are defined by $\mathrm{cl}(D) = \wedge\{E : E$ is fuzzy closed set of $G, E \geq D, 1 - E \in \tau\}$ and $\mathrm{int}(D) = \vee\{S : S$ is fuzzy open set of $G, S \leq D, S \in \tau\}$ [10].

Definition 2.1 A fuzzy set D of a fts G is called fuzzy α-open (briefly, Fα-open) if $D \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}(D)))$ and a fuzzy α-closed (briefly, Fα-closed) if $D \geq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(D)))$ [4]; the intersection of all fuzzy α-closed sets of (G, τ) containing D is called fuzzy α-closure of a fuzzy subset D of G and is denoted by αcl(D).

Definition 2.2 A fuzzy set D of a fts G is called fuzzy generalized α-closed (briefly, Fgα-closed) [8] if $\alpha cl(D) \leq U$ whenever $D \leq U$ and U is fuzzy α-open in (G, τ). The complement of Fgα-closed set is called Fgα-open set.

Definition 2.3 Let (G, τ) and (H, σ) be two fuzzy topological spaces. A function ϕ: $(G, \tau) \to (H, \sigma)$ is called as follows:

(i) Fα-continuous [10] if $\phi^{-1}(V)$ is Fα-closed in G, for each $V \in FC\,(H)$;
(ii) Fgα-continuous [8] if $\phi^{-1}(V)$ is Fgα-closed in G, for each $V \in FC\,(H)$;
(iii) F-irresolute [11] if $\phi^{-1}(V)$ is F-closed in G, for each $V \in FC\,(H)$.

Definition 2.4 A function $\phi : (G, \tau) \to (H, \sigma)$ is said to be fuzzy-open (fuzzy-closed) [2] if the image of every fuzzy open (fuzzy-closed) set in G is fuzzy-open (fuzzy-closed) set in H.

Fuzzy *ψ**-closed sets in fts

In this section, we introduce fuzzy ψ^*-closed sets in fuzzy topological space and discuss some of its characterizations and relationships with other notions.

Definition 3.1 A fuzzy set D in (G, τ) is called fuzzy ψ^*-closed (Fψ^*-closed) if αcl(D) $\leq U$ whenever $D \leq U$ and U is Fgα-open in (G, τ). The complement of Fψ^*-closed set is called Fψ^*-open set.

The class of fuzzy ψ^*-closed sets of fts (G, τ) is denoted by $F\psi^*C(G)$.

Proposition 3.1 Every fuzzy α-closed set is fuzzy ψ^*-closed.

Proof Let D be a Fα-closed set in (G, τ), and since every Fα-closed set is Fgα-closed. Then, αcl(D) $\leq U$ whenever $D \leq U$ and U is Fα-open in (G, τ), and since every Fα-open set is Fgα-open. So, αcl(D) $\leq U$ whenever $D \leq U$ and U is Fgα-open in (G, τ). Thus, D is Fψ^*-closed.

The converse of Proposition 3.1 needs not be true as seen from the following example.

Example 3.1 Let $G = \{a, b, c\}$ with fuzzy topology $\tau = \{0, 1, \{a_{0.5}, b_{0.2}, c_{0.7}\}, \{a_{0.7}, b_{0.8}, c_{0.3}\}, \{a_{0.5}, b_{0.2}, c_{0.3}\}, \{a_{0.7}, b_{0.8}, c_{0.7}\}\}$. The fuzzy subset $D = \{a_{0.4}, b_{0.8}, c_{0.7}\}$ is Fψ^*-closed set in (G, τ) but not Fα-closed set since $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(D))) = \{a_{0.5}, b_{0.8}, c_{0.7}\}$.

Proposition 3.2 Every fuzzy ψ^*-closed set is fuzzy gα-closed set.

Proof Follows from the fact that every Fα-open set is Fgα-open.

The converse of Proposition 3,2 needs not be true as seen from the following example.

Example 3.2 In Example 3.1, the fuzzy subset $D = \{a_{0.5}, b_{0.3}, c_{0.7}\}$ is Fgα-closed set in (G, τ) but not Fψ*-closed set.

Proposition 3.3 If D and E are Fψ*-closed sets in (G, τ), then $D \cup E$ is also Fψ*-closed set in (G, τ).

Proof If $D \vee E \leq U$ and U are Fgα-open, then $D \leq$ and $E \leq U$. Since D and E are Fψ*-closed, $\alpha cl(D) \leq U$ and $\alpha cl(E) \leq U$, and hence $\alpha cl(D \vee E) = \alpha d(D) \vee \alpha d(E) \leq U$. Thus, $D \vee E$ is Fψ*-closed set in (G, τ).

Proposition 3.4 If D is Fgα-open set and fuzzy ψ*-closed set in (G, τ), then D is fuzzy α-closed set in (G, τ).

Proof Since $D \leq D$ and D is Fgα-open set and Fψ*-closed, then $\alpha\mathrm{cl}(D) \leq D$. Since $D \leq \alpha\mathrm{cl}(D)$, then $D = \alpha\mathrm{cl}(D)$, and thus D is Fα-closed set in (G, τ).

Proposition 3.5 Every fuzzy ψ*-open set is fuzzy gα-open.

Proof Let $D \in F\psi^*O(G)$. Then, $1 - D \in F\psi^*C(G)$ and hence Fgα-closed set in (G, τ) by Proposition 3.2. This implies that D is Fgα-open set in (G, τ). Hence, every Fψ*-open set in G is Fgα-open set in G.

Proposition 3.6 If D is Fψ*-closed set in (G, τ) and $D \leq E \leq \alpha cl(A)$, then E is Fψ*-closed set of (G, τ).

Proof Let U be a Fgα-open subset of (G, τ) such that $E \leq U$. Then, $D \leq U$ and since $D \in F\psi^*C(G)$, then $\alpha cl(D) \leq U$. Now, $\alpha cl(E) \leq \alpha cl(D) \leq U$. Then, $E \in F\psi^*C(G)$.

Corollary 3.1 If D is Fψ*-open set in (G, τ) and $\alpha int(D) \leq E \leq D$, then E is Fψ*-open set.

Proof Let $D \in F\psi^*O(G)$, and $\alpha int(D) \leq E \leq D$. Then, $1 - D \in F\psi^*C(G)$, and $1 - D \leq 1 - E \leq \alpha cl(1 - D)$. By Proposition 3.6, $1 - B \in F\psi^*C(G)$. Hence, $E \in F\psi^*O(G)$.

Definition 3.2 For any fuzzy set D in a fts G, we have the fuzzy ψ*-interior of D (briefly ψ*-int(D)) is the union of all fuzzy ψ*-open sets of G contained in D. That is, $\psi^* - int(D) = \vee\{E : E \leq D,\ E \text{ is } F\psi^* - \text{open in } G\}$.

Definition 3.3 Let (G, τ) be a fuzzy topological space. Then, for a fuzzy subset D of G, the fuzzy ψ*-closure of D (briefly ψ*-cl(D)) is the intersection of all fuzzy ψ*-closed sets of G containing D. That is, $\psi^* - cl(D) = \wedge\{E : E \geq D,\ E \text{ is fuzzy } \psi^* - \text{closed in } G\}$.

Proposition 3.7 For any fuzzy sets D and B in a fts G, we have as follows:

(i) $\psi^* - int(D) \leq D$.

(ii) D is $F\psi^*$–open$\Leftrightarrow \psi^* - int(D) = D$.

(iii) $\psi^* - int(\psi^* - int(D)) = \psi^* - int(D)$.

(iv)If $D \leq B$, then $\psi^* - int(D) \leq \psi^* - int(B)$.

Proof (i) Follows from Definition 3.3.

(ii) Let $D \in F\psi^*O(G)$. Then, $D \leq \psi^* - int(D)$. By using (i), we get $D = \psi^* - int(D)$. Conversely, assume that $D = \psi^* - int(D)$. By using Definition 3.3, $D \in F\psi^*O(G)$.

(iii) By using (ii), we get $\psi^* - int(\psi^* - int(D)) = \psi^* - int(D)$.

(iv) Since $D \leq E$ by using (i), $\psi^* - int(D) \leq D \leq E$. That is, $\psi^* - int(D) \leq E$. By (iii), $\psi^* - int(\psi^* - int(D)) \leq \psi^* - int(E)$. Thus, $\psi^* - int(D) \leq \psi^* - int(E)$.

Proposition 3.8 For any fuzzy sets D and E in a fts G, we have as follows:

(i)$\psi^*-\ int(D\wedge E)=\psi^*-\ \mathrm{int}(\mathrm{D})\wedge\psi^*-\ int(E)$.

(ii) $\psi^*-\ int(D\vee E)\geq\psi^*-\ int(D)\vee\psi^*-\ int(E)$.

Proof (i) Since $D\wedge E\leq D$ and $D\wedge E\leq E$, by using Proposition 3.7 (iv), we get $\psi^*-\ int\,(D\wedge E)\leq\psi^*-\ int\,(D)$ and $\psi^*-\ int\,(D\wedge E)\leq\psi^*-\ int\,(E)$. Thus,

$$\psi^*-\ \mathrm{int}(D\wedge E)\leq\psi^*-\ int(D)\wedge\psi^*-\ \mathrm{int}(E). \tag{1}$$

By using Proposition 3.7 (i), we have $\psi^*-\ int\,(D)\leq D$ and $\psi^*-\ int\,(E)\leq E$. This implies that $\psi^*-\ int\,(D)\wedge\psi^*-\ int\,(E)\leq D\wedge E$. Now applying Proposition 3.7 (iv), we get $\psi^*-\ int\,(\psi^*-\ int\,(D)\wedge\psi^*-\ int\,(E))\leq\psi^*-\ int\,(D\wedge E)$. By (1), $\psi^*-\ int\,(\psi^*-\ int\,(D)\wedge\psi^*-\ int\,(\psi^*-\ int\,(E))\leq\psi^*-\ int\,(D\wedge E)$. By using Proposition 3.7 (iii),

$$\psi^*-\ int(D)\wedge\psi^*-\ \mathrm{int}(E)\leq\psi^*-\ \mathrm{int}(D\wedge E). \tag{2}$$

Forms (1) and (2), $\psi^*-\ int\,(D\wedge E)=\psi^*-\ int\,(D)\wedge\psi^*-\ int\,(E)$.

(ii) Since $D\leq D\vee E$ and $E\leq D\vee E$, by using Proposition 3.7 (iv), we have $\psi^*-\ int\,(D)\leq\psi^*-\ int\,(D\vee E)$ and $\psi^*-\ int\,(E)\leq\psi^*-\ int\,(D\vee E)$. Thus, $\psi^*-\ int\,(D)\vee\psi^*-\ int\,(E)\leq\psi^*-\ int\,(D\vee E)$.

The equality in Proposition 3.8 (ii) need not be hold as seen from the following example.

Example 3.3 In Example 3.1, consider $D=\{a_{0.4},b_{0.8},c_{0.7}\}$, and $E=\{a_{0.6},b_{0.8},c_{0.5}\}$. Then, $\psi^*-\ int\,(D)=0$, and $\psi^*-\ int\,(E)=\{a_{0.6},b_{0.2},c_{0.3}\}$. That implies $\psi^*-\ int\,(D)\vee\psi^*-\ int\,(E)=\{a_{0.6},b_{0.2},c_{0.3}\}$. Now, $D\vee E=\{a_{0.6},b_{0.8},c_{0.7}\}$; it follows that $\psi^*-\ int\,(D\vee E)=\{a_{0.6},b_{0.2},c_{0.7}\}$. Then, $\psi^*-\ int\,(D\vee E)\neq\psi^*-\ int\,(D)\vee\psi^*-\ int\,(E)$.

Proposition 3.9 For any fuzzy set D in a fts G, we have as follows:

(i) $(\psi^*\text{-}\ \mathrm{int}(D))^{\mathrm{c}}=\psi^*\text{-}\mathrm{cl}(D^{\mathrm{c}})$

(ii) $(\psi^*\text{-}\mathrm{cl}(D))^{\mathrm{c}}=\psi^*\text{-}\ \mathrm{int}(D^{\mathrm{c}})$

Proof (i) By using Definition 3.3, $\psi^*-\ int\,(D)=\vee\{\ E:E\leq D,\ \ E\in F\psi^*O(G)\}$. Taking complement on both sides, we get as follows:

$$\begin{aligned}(\psi^*\ \mathrm{int}(D))^{\mathrm{c}}&=(\sup\{\ E:B\leq\ D,\ \ E\text{ is }\mathrm{F}\psi^*\text{-open in }G\ \})^{\mathrm{C}}\\&=\ \inf\{E^c:E^c\geq\ D^c,\ \ E^c\text{ is F }\psi^*\text{-closed in }G\ \}.\end{aligned}$$

Replacing E^{c} by C, we get

$(\psi^*\mathrm{int}(D))^{\mathrm{c}}=\wedge\{C:C\geq\ D^{\mathrm{c}},\ \ C\ \text{ is }\ \mathrm{F}\psi^*\text{ - closed in }\ G\ \}$. By Definition 3.4, $(\psi^*\text{ - }\mathrm{int}(D))^{\mathrm{c}}=\psi^*\text{ - }\mathrm{cl}(D^{\mathrm{c}})$.

(ii) By using (i), $(\psi^*\text{ - }\mathrm{int}(D^{\mathrm{c}}))^{\mathrm{c}}=\psi^*\text{ - }\mathrm{cl}(D^{\mathrm{c}})^{\mathrm{c}}=\psi^*\text{ - }\mathrm{cl}(D)$. Taking complement on both sides, we get $\psi^*\text{ - }\mathrm{int}(D^c)=(\psi^*\text{ - }\mathrm{cl}(D))^{\mathrm{c}}$.

Proposition 3.10 Let D be a fuzzy set in a fts G. Then, $D\in F\psi^*C(G)$ if and only if D^{c} is Fψ^*-open.

Proposition 3.11 For any fuzzy sets D and E in a fts G, we have as follows:

(i) $D\leq\psi^*-cl(D)$

(ii) D is $F\psi^*-\text{closed}\Leftrightarrow\psi^*-cl(D)=D$

(iii) $\psi^*-cl(\psi^*-cl(D))=\psi^*-cl(D)$

(iv)If $D \leq E$, then $\psi^{*}-cl(D) \leq \psi^{*}-cl(E)$

Proof (i) Follows from Definition 3.4.

(ii) Let $D \in F\psi^{*}C(G)$. By using Proposition 3.10, $D^{c} \in F\psi^{*}O(G)$. By using Proposition 3.9 (ii), ψ^{*} - $\mathrm{int}(D^{c}) = D^{c} \Leftrightarrow (\psi^{*}$ - $\mathrm{cl}(D))^{c} = D^{c} \Leftrightarrow \psi^{*}$ - $\mathrm{cl}(D) = D$.

(iii) By using (ii), we get $\psi^{*}-cl(\psi^{*}-cl(D)) = \psi^{*}-cl(D)$.

(iv) If $D \wedge E \leq D$ and $D \wedge E \leq E$ By using Proposition 3.7 (iv), ψ^{*} - $\mathrm{int}(E^{c}) \leq \psi^{*}$ - $\mathrm{int}(D^{c})$. Taking complement on both sides, we get $(\psi^{*}$ - $\mathrm{int}(E^{c}))^{c} \geq (\psi^{*}$ - $\mathrm{int}(D^{c}))^{c}$. By using Proposition 3.9 (ii), $\psi^{*}-cl(E) \geq \psi^{*}-cl(D)$.

Proposition 3.12 Let D be a fuzzy set in a fts G. Then, $int(D) \leq \alpha-\ int\,(D) \leq \psi^{*}-\ int\,(D) \leq D \leq \psi^{*}-cl(D) \leq \alpha-cl(D\ ) \leq cl(D)$.

Proof It follows from the definition of corresponding operators.

Proposition 3.13 For any fuzzy sets D and E in a fts G, we have as follows:

(i) $\psi^{*}-cl(D \vee E) = \psi^{*}-d(D) \vee \psi^{*}-cl(E)$

(ii) $\psi^{*}-cl(D \wedge E) \leq \psi^{*}-d(D) \wedge \psi^{*}-cl(E)$

Proof (i) Since ψ^{*} - $\mathrm{cl}(D \vee E) = \psi^{*}$ - $\mathrm{cl}((D \vee E)^{c})^{c}$, by using Proposition 3.9 (i), we have ψ^{*} - $\mathrm{cl}(D \vee E) = (\psi^{*}$ - $\mathrm{int}(D \vee E)^{c})^{c} = (\psi^{*}$ - $\mathrm{int}(D^{c} \wedge E^{c}))^{c}$. By using Proposition 3.8 (i), we have ψ^{*} - $\mathrm{cl}(D \vee E) = (\psi^{*}$ - $\mathrm{int}(D^{c}) \wedge \psi^{*}$ - $\mathrm{int}(E^{c}))^{c} = (\psi^{*}$ - $\mathrm{int}(D^{c}))^{c} \vee (\psi_{*}$ - $\mathrm{int}(E^{c}))^{c}$.

By using Proposition 3.9 (i), we have ψ^{*} - $\mathrm{cl}(D \vee E) = \psi^{*}$ - $\mathrm{cl}(D^{c})^{c} \vee \psi^{*}$ - $\mathrm{cl}(E^{c})^{c} = \psi^{*}$ - $\mathrm{cl}(D) \vee \psi^{*}$ - $\mathrm{cl}(E)$.

(ii) Since $D \wedge E \leq D$ and $D \wedge E \leq E$, by using Proposition 3.11 (iv), we have $\psi^{*}-cl(D \wedge E) \leq \psi^{*}-cl(D)$ and $\psi^{*}-cl(D \wedge E) \leq \psi^{*}-cl(E)$. This implies that $\psi^{*}-cl(D \wedge E) \leq \psi^{*}-cl(D) \wedge \psi^{*}-cl(E)$.

Proposition 3.14 For any fuzzy sets D and E in a fts G, we have as follows:

(i) $\psi^{*}-cl(D) \geq D \vee \psi^{*}-cl(\psi^{*}-\ int(D))$

(ii) $\psi^{*}-\ int(D) \leq D \vee \psi^{*}-\ int(\psi^{*}-cl(D))$

(iii) $int(\psi^{*}-cl(D)) \leq\ int(cl(D))$

(iv) $int(\psi^{*}-cl(D)) \geq\ int(\psi^{*}-cl(\psi^{*}-\ int(D)))$

Proof (i) By Proposition 3.11 (i), $D \leq \psi^{*}-cl(A)$. Again, using Proposition 3.7 (i), $\psi^{*}-\ int\,(D) \leq D$. Then, $\psi^{*}-cl(\psi^{*}-\ int\,(D)) \leq \psi^{*}-cl(D)$.

Then, we have $D \vee \psi^{*}-cl(\psi^{*}-\ int\,(D)) \leq \psi^{*}-cl(D)$.

(ii) By Proposition 3.7 (i), $\psi^{*}-\ int\,(D) \leq D$. Again, using Proposition 3.11(i), $D \leq \psi^{*}-cl(D)$. Then, $\psi^{*}-\ int\,(D) \leq \psi^{*}-\ int\,(\psi^{*}-cl(D))$. Then, we have $\psi^{*}-\ int\,(D) \leq D \vee \psi^{*}-int\,(\psi^{*}-cl(D))$.

(iii) By Proposition 3.12, $\psi^{*}-cl(D) \leq cl(D)$. We get $int(\psi^{*}-cl(D)) \leq\ int\,(cl(D))$.

(iv) By (i), $\psi^{*}-cl(D) \geq D \vee \psi^{*}-cl(\psi^{*}-\ int\,(D))$. Then, we have $int(\psi^{*}-cl(D)) \geq\ int\,(D \vee \psi^{*}-cl(\psi^{*}-\ int\,(D)))$. Since $int(D \vee E) \geq\ int\,(D) \vee\ int\,(E)$, $int\,(\psi^{*}-cl(D)) \geq\ int\,(D) \vee int\,(\psi^{*}-cl(\psi^{*}-\ int\,(D))) \geq\ int\,(\psi^{*}-cl(\psi^{*}-\ int\,(D)))$.

Fuzzy ψ*-continuous and fuzzy ψ*-irresolute functions in FTS

As application of fuzzy ψ^{*}-closed set, we identify some types of fuzzy functions and introducing some of their properties.

Definition 4.1 A function $\phi : (G, \tau) \to (H, \sigma)$ is said to be fuzzy ψ*-continuous (Fψ*-continuous) if $\phi^{-1}(V)$ is Fψ*-closed in *G*, for each fuzzy closed set *V* in *H*.

Proposition 4.1 Every Fα-continuous function is Fψ*-continuous.

Proof Let $V \in FC\ (H)$. Since ϕ is Fα-continuous, then $\phi^{-1}(V)$ is Fα-closed in *G*. Since every Fα-closed set is Fψ*-closed set, then $\phi^{-1}(V) \in F\psi^*C(G)$. Thus, ϕ is Fψ*-continuous.

The converse of Proposition 4.1 need not be true as seen from the following example.

Example 4.1 Suppose that G = {a, b, c} with fuzzy topology τ = {0, 1, {$a_{0.5}, b_{0.2}, c_{0.7}$}, {$a_{0.7}, b_{0.8}, c_{0.3}$}, {$a_{0.5}, b_{0.2}, c_{0.3}$}, {$a_{0.7}, b_{0.8}, c_{0.7}$} and H = {x, y, z} with fuzzy topology σ = {0, 1, {$x_{0.8}, y_{0.2}, z_{0.3}$}}. Let $\phi : (G, \tau) \to (H, \sigma)$ be defined by $\phi\ (a) = x$, $\phi\ (b) = y$, and $\phi\ (c) = z$. ϕ is Fψ*-continuous function, but it is not a Fα-continuous function, since $V = \{x_{0.2}, y_{0.8}, z_{0.7}\} \in FC(H)$ but $\phi^{-1}(V) \notin F\alpha C(G)$.

Proposition 4.2 Every Fψ*-continuous function is Fgα-continuous.

Proof Let $V \in FC\ (H)$. Since ϕ is Fψ*-continuous, then $\phi^{-1}(V) \in F\psi^*C(G)$. By Proposition 3.1, every Fψ*-closed set is Fgα-closed set; then, $\phi^{-1}(V)$ is Fgα-closed. Thus, ϕ is Fgα-continuous.

The converse of Proposition 4.2 need not be true as seen from the following example.

Example 4.2 Suppose that G = {a, b, c} with fuzzy topology τ = {0, 1, {$a_{0.5}, b_{0.2}, c_{0.7}$}, {$a_{0.7}, b_{0.8}, c_{0.3}$}, {$a_{0.5}, b_{0.2}, c_{0.3}$}, {$a_{0.7}, b_{0.8}, c_{0.7}$} and H = {x, y, z} with fuzzy topology σ = {0, 1, {$x_{0.5}, y_{0.6}, z_{0.3}$}}. Let $\phi : (G, \tau) \to (H, \sigma)$ be defined by $\phi\ (a) = x$, $\phi\ (b) = y$, and $\phi\ (c) = z$. ϕ is Fgα-continuous function, but it is not a Fψ*-continuous function, since $V = \{x_{0.5}, y_{0.4}, z_{0.7}\} \in FC(Y)$ but $\phi^{-1}(V) \notin F\ \psi^*\ C(X)$.

Definition 4.2 A function $\phi : (G, \tau) \to (H, \sigma)$ is said to be Fψ*-irresolute (Fψ*-irresolute) if $\phi^{-1}(V) \in F\psi^*C(G)$, for each F$\psi$*-closed set *V* in *H*.

Proposition 4.3 Every Fψ*-irresolute function is Fψ*-continuous.

Proof It follows from the definitions.

The converse of Proposition 4.3 need not be true as seen from the following example.

Example 4.3 In the Example 4.1, Let $\phi : (G, \tau) \to (H, \sigma)$ be defined by $\phi\ (a) = x$, $\phi\ (b) = y$, and $\phi\ (c) = z$. ϕ is Fψ*-continuous function, but it is not a Fψ*-irresolute function, since $V = \{x_{0.2}, y_{0.7}, z_{0.4}\} \in F\psi^*\ C(H)$ but $\phi^{-1}(V) \notin F\ \psi^*\ C(G)$.

Proposition 4.4 Let $\phi : G \to H$ and $\gamma : H \to W$ be any two functions. Then, as follows:

(i) γ o ϕ is *Fψ*-continuous* if g is fuzzy continuous, and ϕ is *Fψ*-continuous*.

(ii) γ o ϕ is *Fψ*-irresolute* if both ϕ and *g* are *Fψ*-irresolute*.

(iii) γ o ϕ is *Fψ*-continuous* if *g* is *Fψ*-continuous*, and ϕ is *Fψ*-irresolute*.

Proof Let $V \in$ FC(*W*). Since γ is fuzzy continuous, then $\gamma^{-1}(V) \in FC(H)$. Since ϕ is *Fψ*-continuous*, then we have $\phi^{-1}(\gamma^{-1}(V)) \in F\psi^*\ C(G)$. Consequently, γ o ϕ is *Fψ*-continuous*.

(ii) - (iii) By similarity.

Applications of *Fψ*-closed sets*

As applications of Fψ*-closed sets, three fuzzy spaces, namely, fuzzy $T_{1/5}$-spaces, fuzzy $T_{1/5}^{\psi^*}$-spaces, and fuzzy ${}^{\psi^*}T_{1/5}$-spaces are introduced.

We introduce the following definitions.

Definition 5.1 A fuzzy topological space (G, τ) is called as follows:

(i) Fuzzy $T_{1/5}$-space if every Fgα-closed set in G is a Fα-closed set in G.

(ii) Fuzzy $T^{\psi*}_{1/5}$-space if every Fψ*-closed set in G is a Fα-closed set in G.

(iii) Fuzzy ${}^{\psi^*}T_{1/5}$-space if every Fgα-closed set in G is a Fψ*-closed set in G.

Proposition 5.1 If $\phi : G \to H$ is Fψ*-continuous and G is fuzzy $T^{\psi*}_{1/5}$-space; then ϕ is Fα-continuous.

Proof Let $V \in \mathrm{F}C\ (H)$; since f is Fψ*-continuous, then $\phi^{-1}(V) \in F\psi{*}C(G)$. Since G is F $T^{\psi*}_{1/5}$-space, then $\phi^{-1}(V)$ is Fα-closed set in G. Thus, ϕ is Fα-continuous.

Proposition 5.2 If $\phi : G \to H$ is Fψ*-irresolute and G is fuzzy $T^{\psi*}_{1/5}$-space, then ϕ is Fα-continuous.

Proof **By** Theorem 5.1.

Proposition 5.3 If $\phi : G \to H$ is Fgα-continuous and G is fuzzy ${}^{\psi^*}T_{1/5}$-space, then ϕ is Fψ*-continuous.

Proof Let $V \in FC\ (H)$; since ϕ is Fgα-continuous, then $\phi^{-1}(V)$ is Fgα-closed set in G. Since G is F${}^{\psi^*}T_{1/5}$-space, then $\phi^{-1}(V) \in F\psi{*}C(G)$. Thus, ϕ is Fψ*-continuous.

Proposition 5.4 Let $\phi : G \to H$ be onto Fψ*-irresolute and Fα-closed. If G is fuzzy $T^{\psi*}_{1/5}$-space, then H is also a fuzzy $T^{\psi*}_{1/5}$-space.

Proof Let $V \in F\psi{*}C(H)$; since f is Fψ*-irresolute, then $\phi^{-1}(V) \in F\psi{*}C(G)$. Since G is F $T^{\psi*}_{1/5}$-space, then $\phi^{-1}(V)$ is Fα-closed set in G. Since ϕ is Fα-closed and onto, then we have V is Fα-closed. Therefore, H is also a F$T^{\psi*}_{1/5}$-space.

Proposition 5.5 Let G, H, and W be ftss, and $\phi : G \to H$, $\gamma : H \to W$ and $\gamma \circ \phi : G \to W$ be functions, then if ϕ is Fα-irresolute function and γ is Fψ*-continuous function, such that H is fuzzy $T^{\psi*}_{1/5}$-space. Then, $\gamma \circ \phi$ is Fα-continuous function.

Proof Let $U \in FC\ (W)$; since γ is Fψ*-continuous, then $\gamma^{-1}(U)$ is $\in F\psi{*}C(H)$. Since H is fuzzy $T^{\psi*}_{1/5}$-space, then $\gamma^{-1}(U)$ is Fα-closed set in H. But ϕ is Fα-irresolute function, then $\phi^{-1}(\gamma^{-1}(U))$ is Fα-closed set in H. But $\phi^{-1}(\gamma^{-1}(U)) = (\gamma \circ \phi)^{-1}(U)$. Therefore, $\gamma \circ \phi$ is Fα-continuous function.

Definition 5.2 A map $\phi : (G, \tau) \to (H, \sigma)$ is said to be Fψ*-open (Fψ*-closed) if the image of every open (closed) fuzzy set in G is Fψ*-open (closed) set in H.

Proposition 5.6 Every fuzzy-open map is fuzzy ψ*-open map.

Proof The proof follows from the Definition 5.2.

The converse of Proposition 5.6 need not be true as seen from the following example.

Example 5.1 Suppose that $G = \{a, b, c\}$ with fuzzy topology $\tau = \{0, 1, \{a_{0.8}, b_{0.2}, c_{0.3}\}\}$, and $H = \{x, y, z\}$ with fuzzy topology $\sigma = \{0,\ 1,\ \{x_{0.5}, y_{0.2}, z_{0.7}\},\ \{x_{0.7}, y_{0.8}, z_{0.3}\},\ \{x_{0.5}, y_{0.2}, z_{0.3}\},\ \{x_{0.7}, y_{0.8}, z_{0.7}\}$. Let $\phi : (G, \tau) \to (H, \sigma)$ be defined by $\phi\ (a) = x$, $\phi\ (b) = y$, and $\phi\ (c) = z$. ϕ is Fψ*-open map, but it is not a F-open map, since $G = \{a_{0.8}, b_{0.2}, c_{0.3}\} \in FO(G)$ but $\phi(G) \notin FO(H)$.

Proposition 5.7 Every fuzzy-closed map is Fψ*-closed map.

Proof The proof follows from the Definition 5.2.

The converse of Proposition 5.7 need not be true as seen from the following example.

Example 5.2 In the Example 5.1, let $\phi : (G, \tau) \to (H, \sigma)$ be defined by $\phi\ (a) = x$, $\phi\ (b) = y$, and $\phi\ (c) = z$. ϕ is Fψ*-closed map, but it is not an F-closed map, since $V = \{a_{0.2}, b_{0.8}, c_{0.7}\} \in FC(G)$ but $\phi(V) \notin F\ C(H)$.

Proposition 5.8 If $\phi : G \to H$ is F-closed map and $\gamma : H \to W$ is Fψ*-closed map, then $\gamma\ o\ \phi : G \to W$ is Fψ*-closed map.

Conclusion

In this paper, we have defined a new class of fuzzy sets, namely, fuzzy ψ*-closed sets for fuzzy topological spaces, which is properly placed in between the class of fuzzy α-closed sets and the class of fuzzy generalized α-closed sets. We have also investigated some properties of these fuzzy sets. Fuzzy ψ*-continuous, fuzzy ψ*-irresolute functions, and fuzzy ψ*-closed (open) functions have been introduced. We have proved that every Fψ*-continuous function is Fgα-continuous, but the converse need not be true, and the composition of two $F\psi$*-irresolute functions is $F\psi$*-irresolute. Fuzzy $T_{1/5}$-spaces, fuzzy $T^{\psi *}_{1/5}$-spaces, and fuzzy ${}^{\psi^*}T_{1/5}$-spaces have been established as applications of fuzzy ψ*-closed set. In the future, we will generalize this class of fuzzy sets in fuzzy bitopological spaces, and some applied examples should be given.

Acknowledgements
The authors are grateful to the referees for their valuable suggestions.

Authors' contributions
All authors contributed to the draft of the manuscript; all authors read and approved the final manuscript.

Funding
No funding was received.

Availability of data and materials
All data generated or analyzed during this study are included in this published article.

Competing interests
The authors declare that they have no competing interests.

Received: 15 January 2020 Accepted: 28 April 2020
Published online: 20 July 2020

References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control. **8**, 338–353 (1965)
2. Chang, C.L.: Fuzzy topological spaces. J. Math. Anal. Appl. **24**(1), 182–190 (1968)
3. Azad, K.K.: On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity. J. Math. Anal. Appl. **82**(1), 14–32 (1981)
4. Shahna, A.S.B.: On fuzzy strong semicontinuity and fuzzy precontinuity. Fuzzy Sets Syst. **44**(2), 303–308 (1991)
5. Thakur, S.S., Singh, S.: On fuzzy semi-preopen sets and fuzzy semi-precontinuity. Fuzzy Sets Syst. **98**(3), 383–391 (1998)
6. Balasubramanian, G., Sundaram, P.: On some generalizations of fuzzy continuous functions. Fuzzy Sets Syst. **86**(1), 93–100 (1997)
7. El-Shafei, M.E., Zakari, A.: Semi-generalized continuous mappings in fuzzy topological space. J. Egyptian Math. Soc. **15**(1), 57–67 (2007)
8. K. Muthukumaraswamy and R. Devi "On fuzzy α-generalized closed sets" Acta Ciencia Indica. Mathematics, 30(2004).
9. Abd Allah, M.A., Nawar, A.S.: "ψ*-closed sets in topological spaces" *Wulfenia Journal* 21(9), 391–401(2014).
10. Signal, M.K., Rajvansi, N.: Fuzzy alpha-sets and alpha-continuous maps. Fuzzy Sets Syst. **48**(3), 383–390 (1992)
11. T. H. Yalvac "Semi-interior and semiclosure of a fuzzy set", J. Math. Anal. Appl.. 132(2), 356–364(1988).

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