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Further results on edge even graceful labeling of the join of two graphs

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Abstract
In this paper, we investigated the edge even graceful labeling property of the join of two graphs. A function f is called *an edge even graceful labeling* of a graph $G=(V(G),E(G))$ with $p=|V(G)|$ vertices and $q=|E(G)|$ edges if $f:E(G)\rightarrow\{2,4,...,2q\}$ is bijective and the induced function $f^*:V(G)\rightarrow\{0,2,4,\cdots,2q-2\}$, defined as $f^*(x)=(\sum_{xy\in E(G)}f(xy))\bmod(2k)$, where $k=\max(p,q)$, is an injective function. Sufficient conditions for the complete bipartite graph $K_{m,n}=mK_1+nK_1$ to have an edge even graceful labeling are established. Also, we introduced an edge even graceful labeling of the join of the graph K_1 with the star graph $K_{1,n}$, the wheel graph W_n and the sunflower graph sf_n for all $n\in\mathbb{N}$. Finally, we proved that the join of the graph $\overline{K}_2$ with the star graph $K_{1,n}$, the wheel graph W_n and the cyclic graph C_n are edge even graceful graphs.

Keywords: Complete bipartite graph, Wheel graph, Sunflower graph, Edge even graceful labeling, Join of two graphs

Mathematics Subject Classification: 05 C 78, 05 C 76, 05 C 90, 05 C 99

Introduction

A labeling of a graph is a mapping that carries graph elements (edges or vertices, or both) to positive integers subject to certain constraints. Recently, graph labeling has much attention from different researches in graph theory because it has rigorous applications in many disciplines, e.g., coding theory, X-ray, radar, communication networks, circuit design, astronomy, communication network addressing, and graph decomposition problems. For more interesting applications of graph labeling, see [1–3].

In graph theory, one can generate many new graphs from given ones by using graph operation. For a graph G, let $q=|E(G)|$ be the cardinality of $E(G)$ and $p=|V(G)|$ be that of $V(G)$. Let G and H be two graphs with no vertex in common. *The join of G and H*, denoted by $G+H$, defined to be the graph with vertex set and edge set given as follows: $V(G+H)=V(G)\cup V(H), E(G+H)=E(G)\cup E(H)\cup\{x_1x_2 : x_1\in V(G),\ x_2\in V(H)\}$. If G and H are (p_1,q_1) and (p_2,q_2) graphs, respectively, then the number of vertices and edges in the join graph are p_1+p_2 and $q_1+q_2+p_1p_2$ [1].

Elsonbaty and Daoud [4] introduced a new type of labeling of a graph G with p vertices and q edges called *an edge even graceful labeling* if there is a bijection f from the edges of the graph to the set $\{2,4,\cdots,2q\}$ such that, when each vertex is assigned the sum of all edges incident to it mod $2k$ where $k=\max(p,q)$, the resulting vertex labels are distinct.

The graph that admits edge even graceful labeling is called *an edge even graceful graph.* They introduced some path and cycle-related graphs which are edge even graceful, then Zeen El Deen [5] studied more graphs having an edge even graceful labeling.

Furthermore, Elsonbaty and Daoud [6] investigate edge even graceful labeling of cylinder grid graphs also, Daoud [7] studied the edge even graceful labeling of Polar grid graphs after that, Zeen El Deen and Omar N. [8] extended the edge even graceful labeling into r- edge even graceful labeling. For a summary of graph labeling, we refer to the dynamic survey by Gallian [9].

It should be noted that the join graph is not necessarily an edge even graceful graph. For example, *the wheel graph* $W_3 = K_1 + C_3$ is not an edge even graceful graph. In [4], they proved that *the fan graph* $F_n = K_1 + P_n;\ n \geq 2$ and $W_n = K_1 + C_n;\ n > 3$ are edge even graceful graphs. Now, we will study the edge even graceful labeling of the join of the graph K_1 with *the star graph* $K_{1,n}$, *the wheel graph* W_n , and *the sunflower graph* sf_n. Also, we will study the edge even graceful labeling of *the complete bipartite graph* $K_{m,n} = mK_1 + nK_1$. Since the *double fan graph* $F_{2,n} = \overline{K}_2 + P_n;\ n \geq 2$ is an edge even[5], so we will study the edge even graceful labeling of the join of the graph $\overline{K}_2$ with the graphs $K_{1,n},\ C_n$ and W_n.

Edge even graceful labeling of the graph $K_{n,n} = nk_1 + nK_1$

Theorem 1 *The complete bipartite graph* $K_{n,n} = nK_1 + nK_1$ *has an edge even graceful labeling when* $n > 1$ *is an odd number.*

Proof Let us use the standard notation $p = |V(K_{n,n})| = 2n$ and $q = |E(K_{n,n})| = n^2$. The vertices of $K_{n,n}$ were divided into two disjoint sets $\{v_1, v_2, \cdots, v_n\}$ $\{u_1, u_2, \cdots, u_n\}$ such that every pair of graph vertices in the two sets are adjacent. There are three cases:

Case (1) If $n \equiv 3 \pmod 4$, $n > 3$, we define the function $f : E(K_{n,n}) \longrightarrow \{2, 4, \cdots, 2n^2\}$ as follows:

If i is an odd number, $1 \leq i \leq n$

$$f(u_i v_j) = \begin{cases} (i-1)n + j + 1 & \text{if } j = 1, 3, \cdots, n; \\ 2n^2 - [(i-1)n + j] & \text{if } j = 2, 4, \cdots, n-1, \end{cases}$$

and if i is an even number, $2 \leq i \leq n-1$

$$f(u_i v_j) = \begin{cases} (i-1)n + j + 2 & \text{if } j = 1, 3, \cdots, n-2; \\ 2n^2 - [(i-1)n + j + 1] & \text{if } j = 2, 4, \cdots, n-1; \\ 2n^2 - [(i-1)n + 1] & \text{if } j = n. \end{cases}$$

The following matrix $X = (a_{ij})$ shows the methods of labeling, where a_{ij} represents the label of the edge $u_i v_j$.

$$X = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ u_{n-2} \\ u_{n-1} \\ u_n \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-2} & v_{n-1} & v_n \end{array} \\ \left(\begin{array}{cccccccc} 2 & 2n^2-2 & 4 & 2n^2-4 & \ldots & n-1 & 2n^2-n+1 & n+1 \\ n+3 & 2n^2-n-3 & n+5 & 2n^2-n-5 & \ldots & 2n & 2n^2-2n & 2n^2-n-1 \\ 2n+2 & 2n^2-2n-2 & 2n+4 & 2n^2-2n+4 & \ldots & 3n-1 & 2n^2-3n+1 & 3n+1 \\ 3n+3 & 2n^2-3n-3 & 3n+5 & 2n^2-3n-5 & \ldots & 4n & 2n^2-4n & 2n^2-3n-1 \\ \vdots & \vdots & \ldots & \ldots & \vdots & \vdots & \ldots & \ldots \\ \vdots & \vdots & \ldots & \ldots & \vdots & \ddots & \ldots & \ldots \\ n^2-3n+2 & n^2+3n-2 & n^2-3n+4 & n^2+3n-4 & \ldots & n^2-2n-1 & n^2+2n+1 & n^2-2n+1 \\ n^2-2n+3 & n^2+2n-3 & n^2-2n+5 & n^2+2n-5 & \ldots & n^2-n & n^2+n & n^2+2n-1 \\ n^2-n+2 & n^2+n-2 & n^2-n+4 & n^2+n-4 & \ldots & n^2-1 & 2n^2 & n^2+1 \end{array} \right) \end{array}$$

In this case, the equality of the labeling of the two vertices u_n and v_n forces us to change the labels of the two edges u_nv_{n-1} and u_nv_n, that is, $f(u_nv_{n-1}) = 2n^2$ and $f(u_nv_n) = n^2 + 1$. Thus, the induced vertex labels are

(i) The labels of the vertices u_i, $i = 1, 2, \cdots, n$ is the sum of rows in the matrix, i.e.,
$f^*(u_i) = [\ \sum_{j=1}^{n-1} f(u_iv_j)\ ] \bmod (2n^2) = f(u_iv_n)$, so the labels of the vertices $u_1, u_2, u_3, u_4, \cdots, u_{n-2}, u_{n-1}$ are
$n+1,\ 2n^2 - n - 1,\ 3n+1,\ 2n^2 - 3n - 1, \cdots, n^2 - 2n + 1, n^2 + 2n - 1$, respectively, and $f^*(u_n) = 0$.

(ii) The labels of the vertices v_i is the sum of columns in the matrix and since $n \equiv 3 \pmod 4 \Rightarrow n = 3 + 4k \Rightarrow 2q = 2n^2 = 32k^2 + 48k + 18$, then, we have

$$f^*(v_i) = \begin{cases} \frac{2n^2 + (2i+3)n - 1}{2} & \text{if } i = 1, 3, \cdots, n-2\,; \\ \frac{2n^2 - (2i+1)n + 1}{2} & \text{if } i = 2, 4, \cdots, n-3. \end{cases}$$

Hence, the labels of the vertices $v_1, v_2, v_3, v_4, \cdots, v_{n-3}, v_{n-2}$ are $\frac{2n^2+5n-1}{2}$, $\frac{2n^2-5n+1}{2}$, $\frac{2n^2+9n-1}{2}$, $\frac{2n^2-9n+1}{2}, \cdots,$ $\frac{5n+1}{2}$, $\frac{4n^2-n-1}{2}$, respectively.

Also $f^*(v_n) = [\ \sum_{i=1}^{n} f(u_i\, v_n)\ ] \bmod (2n^2) = f(u_n\, v_n) = n^2 + 1$ and $f^*(v_{n-1}) = [\ \frac{-n^3+n^2+n-1}{2}\ ] \bmod (2n^2) = [\ \frac{-2n^2+n-1}{2}\ ] \bmod (2n^2) = \frac{n^2+n-1}{2}$.

Case (2) If $n = 3$, the graph $K_{3,3}$ is an edge even graceful graph, see the following Fig. 1.

Case (3) If $n \equiv 1 \pmod 4$. The labels of the edges incident to the vertices $\{u_i\ ,\ i = 1, 2, \cdots, n-1\ \}$ are similar to the first case, but there are some changes in the label of the edges in the last row in the matrix. The labels of the edges u_nv_1, u_nv_2, u_nv_3, u_nv_4, $\cdots$, $u_nv_{\frac{n-3}{2}}$, $u_nv_{\frac{n-1}{2}}$ are given by $(n-1)n + 2$, $2n^2 - [\,(n-1)n + 2]$, $(n-1)n + 4$, $2n^2 - [\,(n-1)n + 4]$, $\cdots$, $n(n-1) + (\frac{n-1}{2})$, $2n^2 - [\,n(n-1) + (\frac{n-1}{2})]$ and the edge $u_nv_{\frac{n+1}{2}}$ label by $2n^2$. Finally, we swap the direction of labeling to start labels from the end of the row, so the edges u_nv_n, u_nv_{n-1}, u_nv_{n-2}, u_nv_{n-3}, $\cdots$, $u_nv_{\frac{n+5}{2}}$, $u_nv_{\frac{n+3}{2}}$ will label by $(n-1)n +$

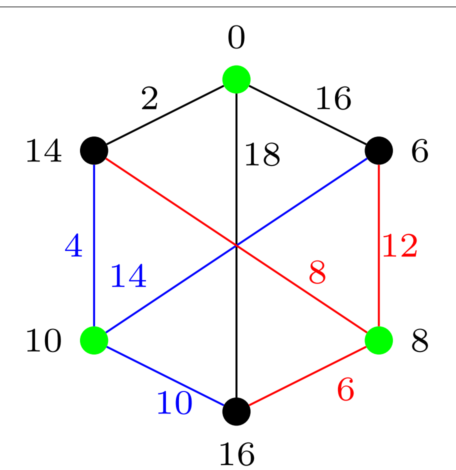


Fig. 1 An edge even graceful labeling of the graph $K_{3,3}$

$\frac{n+3}{2}$, $2n^2-[(n-1)n+\frac{n+3}{2}]$, $(n-1)n+\frac{n+7}{2}$, $2n^2-[(n-1)n+\frac{n+7}{2}]$, $\cdots$, $(n+1)(n-1)$, $2n^2-[(n+1)(n-1)]$ as shown in the following matrix $X=(a_{ij})$.

$$X=\begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{n-1} \\ u_n \end{array}\begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & v_3 & \dots & v_{\frac{n+1}{2}} & \dots & v_{n-2} & v_{n-1} & v_n \end{array} \\ \begin{pmatrix} 2 & 2n^2-2 & 4 & \dots & \frac{n+3}{2} & \dots & n-1 & 2n^2-n+1 & n+1 \\ n+3 & 2n^2-(n+3) & n+5 & \dots & \frac{3n+5}{2} & \dots & 2n & 2n^2-(2n) & 2n^2-n-1 \\ \vdots & \vdots & \dots & \ddots & \vdots & \ddots & \vdots & \dots & \dots \\ \vdots & \vdots & \dots & \ddots & \vdots & \ddots & \vdots & \dots & \dots \\ n^2-2n+3 & n^2+2n-3 & n^2-2n+5 & \dots & \frac{(2n-1)n+5}{2} & \dots & (n-1)n & n^2+n & n^2+2n-1 \\ n^2-n+2 & n^2+n-2 & n^2-n+4 & \dots & 2n^2 & \dots & n^2-(\frac{n-7}{2}) & n^2+\frac{n-3}{2} & n^2-(\frac{n-3}{2}) \end{pmatrix} \end{array}$$

The algorithm for the matrix $X=(a_{ij})$ of the graph $K_{n,n}$ when $n\equiv 1(\text{mod}4)$ is shown below.

n is odd number.
$X\rightarrow$ square matrix $[n\times n]$.
for i 1 $\longrightarrow$ n,
for j 1 $\longrightarrow$ n,
if i $\longrightarrow$ odd & $i\leq n-2$.
if j $\longrightarrow$ odd,
$X(i,j)=(i-1)n+j+1$.
else if j $\longrightarrow$ even ,
$X(i,j)=2n^2-[(i-1)n+j]$.
if $i==n$,
if j $\longrightarrow$ odd & $j\leq\frac{n-3}{2}$,
$X(i,j)=(n-1)n+j+1$.
else if j $\longrightarrow$ even & $j\leq\frac{n-1}{2}$,
$X(i,j)=2n^2-[(i-1)n+j]$.
else if $j=\frac{n+1}{2}$,
$X(i,j)=2n^2$.
else if j $\longrightarrow$ even & $\frac{n+3}{2}\leq j\leq n-1$,
$X(i,j)=n^2-(\frac{n+1}{2})+j$.
else if j $\longrightarrow$ odd & & $\frac{n+5}{2}\leq j\leq n$,
$X(i,j)=n^2+(\frac{n+3}{2})-j$.

if i $\longrightarrow$ even.
if j $\longrightarrow$ odd & $j\leq n-2$,
$X(i,j)=(i-1)n+j+2$.
else if j $\longrightarrow$ even ,
$X(i,j)=2n^2-[(i-1)n+j+1]$.
else if $j==n$,
$X(i,j)=2n^2-[(i-1)n+1]$.

Thus, the induced vertex labels are

(i) The labels of the vertices $u_i, i = 1, 2, \cdots, n$ is the sum of rows in the matrix, so the labels of these vertices are
$n+1,\ 2n^2-n-1,\ 3n+1,\ 2n^2-3n-1, \cdots, n^2-2n+1, n^2+2n-1, 0,$
respectively.

(ii) The labels of the vertices v_i is the sum of columns in the matrix and since $n \equiv 1 \pmod 4 \Rightarrow n = 1+4k \Rightarrow 2q = 2n^2 = 32k^2+16k+2,$ then, we have

$$f^*(v_i) = \begin{cases} \frac{(2i+3)n-1}{2} & \text{if } i = 1, 3, \cdots, \frac{n-3}{2}; \\ \frac{4n^2-(2i+1)n+1}{2} & \text{if } i = 2, 4, \cdots, \frac{n-1}{2}. \end{cases}$$

Hence, the labels of the vertices $v_1, v_2, v_3, v_4, \cdots, v_{\frac{n-3}{2}}, v_{\frac{n-1}{2}}$ are
$\frac{5n-1}{2}, \frac{4n^2-5m+1}{2}, \frac{9n-1}{2}, \frac{4n^2-9m+1}{2}, \cdots, \frac{n^2+1}{2}, \frac{3n^2-1}{2},$, respectively.

Also, $f^*(v_{\frac{n+1}{2}}) = [\ \frac{n^3-2n^2+5n-4}{2}\ ] \bmod (2n^2)$

$= [\ \frac{-n^2+5n-4}{2}\ ] \bmod (2n^2) = \frac{3n^2+5n-4}{2},$

$f^*(v_n) = [\ \sum_{i=1}^{n} f(u_i\, v_n)\ ] \bmod (2n^2) = f(u_n\, v_n) = \frac{2n^2-n+3}{2},$

$f^*(v_{n-1}) = [\ \frac{-n^3+3n^2+2n-4}{2}\ ] \bmod (2n^2) = n^2+n-2,$

$f^*(v_{n-2}) = [\ \frac{n^3+n^2-2n+8}{2}\ ] \bmod (2n^2) = n^2-n+4,$

$f^*(v_{n-3}) = [\ \frac{-n^3-n^2+6n-12}{2}\ ] \bmod (2n^2) = n^2+3n-6,$

and

$f^*(v_{n-4}) = [\ \frac{n^3+n^2-6n+16}{4}\ ] \bmod (2n^2) = n^2-3n+8.$

In the general case, we have

$$f^*(v_{n-i}) = \begin{cases} n^2+in+2i & \text{if } i = 1, 3, \cdots, \frac{n-3}{2}; \\ n^2-(i-1)n+2i & \text{if } i = 2, 4, \cdots, \frac{n-4}{2}. \end{cases}$$

Here, we notice that $f^*(v_{n-i}) + f^*(v_{n-(i+1)}) = 2,\ i = 1, 2, 3, \cdots, \frac{n-3}{2}$

Clearly, all the label of the vertices are even and distinct. Hence, the graph $K_{n,n} = nK_1 + nK_1$ has an edge even graceful labeling. □

Illustration: we present an edge even graceful labeling of the graph $K_{13,13}$ in the following matrix

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
u_1	2	336	4	334	6	332	8	330	10	328	12	326	14
u_2	16	322	18	320	20	318	22	316	24	314	26	312	324
u_3	28	310	30	308	32	306	34	304	36	302	38	300	40
u_4	42	296	44	294	46	292	48	290	50	288	52	286	298
u_5	54	284	56	282	58	280	60	278	62	276	64	244	66
u_6	68	270	70	268	72	266	74	264	76	262	78	260	272
u_7	80	258	82	256	84	254	86	252	88	250	90	248	92
u_8	94	244	96	242	98	240	100	238	102	236	104	234	246
u_9	106	232	108	230	110	228	112	226	114	224	116	222	118
u_{10}	120	218	122	216	124	214	126	212	128	210	130	208	220
u_{11}	132	206	134	204	136	202	138	200	140	198	142	196	144
u_{12}	146	192	148	190	150	188	152	186	154	184	156	182	194
u_{13}	158	180	160	178	162	176	338	170	168	172	166	174	164

.

Theorem 2 *The graph $K_{m,n} = mK_1 + nK_1$ has an edge even graceful labeling when m and n are distinct odd numbers, $m \neq 2n-3$ and $km \neq ln$, $k = 1,3,\cdots,n-1$, $l = 1,3,\cdots,m-1$.*

Proof In the graph $K_{m,n} = mK_1 + nK_1$, we have $p = |V(K_{m,n})| = m+n$ and $q = |E(K_{m,n})| = m\,n$. Without loss of generality, assume that $m > n$ and the vertices of $K_{m,n}$ divided into two disjoint sets $\{v_1, v_2, \cdots, v_m\}$ and $\{u_1, u_2, \cdots, u_n\}$. Put v_i as the columns of the matrix and u_i as the rows. We define the labeling function $f : E(K_{m,n}) \longrightarrow \{2, 4, \cdots, 2mn\}$ as follows: first, label the edges incident to the vertex u_1, i.e., $u_1v_1, u_1v_2, u_1v_3, u_1v_4, \cdots, u_1v_{n-2}, u_1v_{n-1}, u_1v_n$ by $2, 2mn-2, 4, 2mn-4, \cdots, m-1, 2mn-(n-1), n+1,$ respectively, then reverse the direction of labeling to label the edges incident to the vertex u_2 as $u_2v_m, u_2v_{m-1}, u_2v_{m-2}, u_2v_{m-3}, \cdots, u_2v_3, u_2v_2, u_2v_1$ by $2mn-(n+1), m+3, 2mn-(n+3), n+5, \cdots, 2m+4, 2mn-(2m+2), 2m+2$ and so on.

The following matrix shows the methods of labeling

$$
\begin{array}{c}
 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ u_{n-1} \\ u_n
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
v_1 & v_2 & v_3 & v_4 & \cdots & v_{m-2} & v_{m-1} & v_m
\end{array} \\
\begin{pmatrix}
2 & 2mn-2 & 4 & 2mn-4 & \ldots & m-1 & 2mn-m+1 & m+1 \\
2mn-2m & 2m & 2mn-2m+2 & 2mn-2 & \ldots & 2mn-m-3 & m+3 & 2mn-m-1 \\
2m+2 & 2mn-2m-2 & 2m+4 & 2mn-2n-4 & \ldots & 3m-1 & 2mn-3n+1 & 3n+1 \\
4m & 2mn-4m+2 & 4m-2 & 2mn-4m+4 & \ldots & 2mn-3m-3 & 3m+3 & 2mn-3n-1 \\
\vdots & \vdots & \ldots & \vdots & \ddots & \vdots & \ldots & \ldots \\
\vdots & \vdots & \ldots & \vdots & \ddots & \vdots & \ldots & \ldots \\
mn+m & nm-m & mn+m+2 & nm-m-2 & \ldots & mn+2m-3 & nm-2m+3 & mn+2m-1 \\
nm-m+2 & mn+m-2 & nm-m+4 & mn+m-4 & \ldots & mn-1 & 2mn & mn+1
\end{pmatrix}
\end{array}.
$$

In this case, the label of the vertex u_n will repeat with the labels of the vertex v_m. To avoid this problem we replace the labels of the two edges u_nv_{m-1} and u_nv_m, that is $f(u_nv_{m-1}) = 2mn$ and $f(u_nv_m) = nm+1$. Thus, the induced vertex labels are

(i) The labels of the vertices u_i is the sum of rows in the matrix, so the labels of these vertices are
$m+1,\ 2mn-m-1,\ 3m+1,\ 2mn-3m-1, \cdots, (n-2)m+1, mn+2n-1,\ 0,$
respectively.

(ii) The labels of the vertices $v_i, i = 1, 2, \cdots, m$ is the sum of columns in the matrix, we have

$$
f^*(v_i) = \begin{cases} in+1 & \text{if } i = 1,3,,5\cdots,m-2\,; \\ 2mn-(i-1)n-1 & \text{if } i = 2,4,\cdots,m-3. \end{cases}
$$

Hence, the labels of the vertices $v_1, v_2, v_3, v_4, \cdots, v_{m-3}, v_{m-2}$ are $n+1,\ 2mn-(n+1),\ 3n+1,\ 2mn-(3m+1), \cdots, 2mn-[\,(m-4)n+1],\ (m-2)n+1,$ respectively, and $f^*(v_{m-1}) = 2n-2, f^*(v_m) = nm+1$.

□

Illustration: If we take $m = 15$, then we can label the graphs $K_{15,7}$, $K_{15,11}$, and $K_{15,13}$, while we can not find labels of $K_{15,3}$, $K_{15,5}$, and $K_{15,9}$. We present an edge even graceful labeling of the graph $K_{15,7} = 15K_1 + 7\,K_1$ in the following matrix

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
u_1	2	208	4	206	6	204	8	202	10	200	12	198	14	196	16
u_2	180	30	182	28	184	26	186	24	188	22	190	20	192	18	194
u_3	32	178	34	176	36	174	38	172	40	170	42	168	44	166	46
u_4	150	60	152	58	154	56	156	54	158	52	160	50	162	45	164
u_5	62	148	64	146	66	144	68	142	70	140	72	138	74	136	76
u_6	120	90	122	88	124	86	126	84	128	82	130	80	132	78	134
u_7	92	118	94	116	96	114	98	112	100	110	102	108	104	210	106

Edge even graceful labeling of the join graph $k_1 + k_{1,n}$.

Theorem 3 *The graph* $K_1 + K_{1,n}$ *has an edge even graceful labeling.*

Proof Let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertices of the graph $K_{1,n}$ with central vertex v_0 and $\{x\}$ be the vertex of K_1 so the edges of $K_1+K_{1,n}$ are $\{x\,v_0,\ x\,v_i,\ v_0\,v_i\ , i = 1, 2, \cdots, n\}$. Here, $p = |V(K_1+K_{1,n})| = n+2$ and $q = |E(K_1+K_{1,n})| = 2n+1$. There are two cases:

Case (1) : when n is even. We define the labeling function $f : E(K_1 + K_{1,n}) \longrightarrow \{2, 4, \cdots, 4n + 2\}$ as follows:

$$f(x\,v_0) = 2n,$$

$$f\,(x\,v_i) = \begin{cases} 2i & \text{if } i = 1, 2, \cdots, \frac{n}{2}\,; \\ 2n + 2i & \text{if } i = \frac{n}{2} + 1, \cdots, n\,. \end{cases}$$

and

$$f\,(v_0\,v_i) = \begin{cases} 2n + 2i & \text{if } i = 1, 2, \cdots, \frac{n}{2}\,; \\ 4n + 2 & \text{if } i = \frac{n}{2} + 1\,; \\ 2i - 2 & \text{if } i = \frac{n}{2} + 2, \cdots, n\,. \end{cases}$$

Therefore, the induced vertex labels are

$$f^*(x) = [\,f(x\,v_0) + \sum_{i=1}^{n} f(xv_i)\,] \bmod (4n+2) = f(x\,v_0) \bmod (4n+2) = 2n,$$

$$f^*(v_0) = [\,f(x\,v_0) + \sum_{i=1}^{n} f(v_0\,v_i)\,] \bmod (4n+2)$$

$$= [f(x\,v_0) + f(a_1)] \bmod (4n+2) = 0,$$

$$f^*(v_i) = [f(xv_i) + f(v_0\,v_i)] \bmod (4n+2)$$

$$= \begin{cases} (2n + 4i) \bmod (4n+2) & \text{if } 1 \le i \le \frac{n}{2}; \\ (2n + 4i - 2) \bmod (4n+2) & \text{if } \frac{n}{2} + 2 \le i < n. \end{cases}$$

Hence, the labels of the vertices $v_1, v_2, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}$ are $2n+4, 2n+8, \cdots, 4n-4, 4n$, respectively, and the labels of the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \cdots, v_{n-1}, v_n$ are $3n+2, 4, 8 \cdots, 2n-8, 2n-4$, respectively. Finally,

$$f^*(v_{\frac{n}{2}+1}) = [f(x\,v_{\frac{n}{2}+1}) + f(v_0\,v_{\frac{n}{2}+1})] \bmod (4n+2) = 3n+2.$$

Case (2) : when n is odd. We define the labeling function $f : E(K_1 + K_{1,n}) \longrightarrow \{2, 4, \cdots, 4n + 2\}$ as follows:

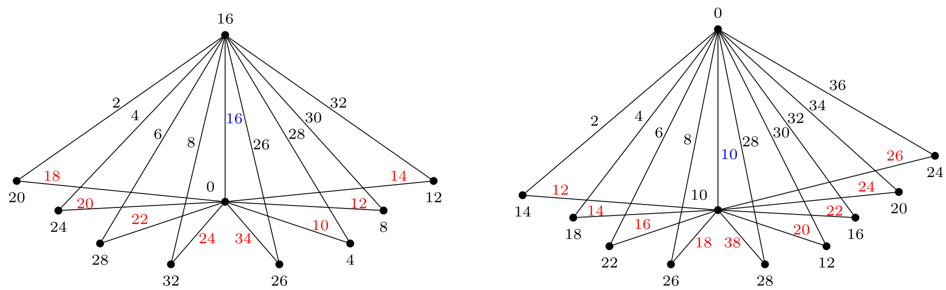


Fig. 2 An edge even graceful labeling of $K_1 + K_{1,8}$, and $K_1 + K_{1,9}$

$$f(x\,v_0) = n+1,$$

$$f\,(x\,v_i) = \begin{cases} 2i & \text{if } \ i = 1, 2, \cdots, \frac{n-1}{2}; \\ 2n+2i & \text{if } \ i = \frac{n+1}{2}, \cdots, n. \end{cases}$$

and

$$f\,(v_0\,v_i) = \begin{cases} n+1+2i & \text{if } \ i = 1, 2, \cdots, \frac{n-1}{2}\,; \\ 4n+2 & \text{if } \ i = \frac{n+1}{2}\,; \\ n-1+2i & \text{if } \ i = \frac{n+3}{2}, \cdots, n\,. \end{cases}$$

Considering the vertex labels we find

$f^*(v_0) = [f(x\,v_0) + \sum_{i=1}^{n} f(v_0 v_i)] \mod (4n+2) = n+1, \ \ f^*(x) = 0$ and

$$f^*(v_i) = \begin{cases} (n+1+4i) \mod (4n+2) & \text{if } \ 1 \le i \le \frac{n-1}{2}; \\ (3n-1+4i) \mod (4n+2) & \text{if } \ \frac{n+3}{2} \le i \le n\,. \end{cases}$$

Then, the labels of the vertices $v_1, v_2, \cdots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}$ are $n+5, n+9, \cdots, 3n-1, 3n+1,$ respectively, and the labels of the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \cdots, v_{n-1}, v_n$ are $n+3, \ n+7 \cdots, 3n-7, \ 3n-3$, respectively.
Finally, $f^*(v_{\frac{n+1}{2}}) = [f(e_{\frac{n+1}{2}}) + f(a_{\frac{n+}{2}})] \mod (4n+2) = 3n+1$.

Overall, all vertex labels are distinct even numbers, also $f^*(x)$ and $f^*(v_0)$ are different from all the labels of the vertices v_i. Hence, the graph $K_1 + K_{1,n}$ is edge even graceful for all n. □

Illustration: In Fig. 2, we present an edge even graceful labeling of the graphs $K_1 + K_{1,8}$ and $K_1 + K_{1,9}$

Edge even graceful labeling of the join graph $K_1 + w_n$

Theorem 4 *The join graph* $K_1 + W_n$ *has an edge even graceful labeling for all* n.

Proof Let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertices of W_n with central vertex v_0 and $\{x\}$ be the vertex of K_1 so the edges of $K_1 + W_n$ will be $\{x\,v_0, \ x\,v_i, \ v_0\,v_i\,, v_i\,v_{i+1} \ , i = 1, 2, \cdots, n\}$. So, $p = |V(K_1 + W_n)| = n+2$ and $q = |E(K_1 + W_n)| = 3n+1$. There are five cases:

Case (1) : For $n \equiv 0 \pmod 6$, *or* $n \equiv 4 \pmod 6$, we define the labeling function $f : E(K_1 + W_n) \longrightarrow \{2, 4, \cdots, 6n+2\}$ as follows:

$$f(v_0\,x) = 6n+2\,,$$

$$f(v_1\,v_n) = 3n, \quad f(v_i\,v_{i+1}) = n+2i \qquad \text{for } \ i = 1, 2, \cdots n-1,$$

$$f\,(x\,v_i) = \begin{cases} 3n+2 & \text{if } \ i = 1\,; \\ 5n-2i+4 & \text{if } \ 2 \le i \le n\,. \end{cases},$$

and

$$f\ (v_0\ v_i) = \begin{cases} 2i & \text{if } \ 1 \le i \le \frac{n}{2}\ ; \\ 4n + 2i & \text{if } \ \frac{n}{2} < i \le \ n\ . \end{cases}$$

Then, the induced vertex labels are

$f^*(v_0) = [\ \sum_{i=1}^{n} f(v_0\ v_i) + f(v_0\ x)\ ]\ \bmod (6n+2) =\ 0$ and

$f^*(x) = [\sum_{i=1}^{n} f(x\ v_i) + f(xv_0)]\bmod (6n+2)$

$= [\sum_{i=1}^{n}(3n+2i)\ + (6n+2)]\ \bmod (6n+2) = (4n^2+n)\bmod(6n+2).$

If $n \equiv 0 \pmod 6\ \Rightarrow n = 6k\ \Rightarrow 2q = 6n+2 = 36k+2,$ then,

$$\begin{aligned} f^*(x) &= [\ 4\ (6k)^2 + (6k)\ ]\ \bmod (36k+2\ ) \\ &= [\ 4k(36k+2) - (2k)\ ]\ \bmod (36k+2\ ) \\ &\equiv\ (-2k) \bmod (36k+2\ ) \equiv (34k+2) \bmod (36k+2\ ) \\ &=\ \left(\frac{17n+6}{3}\right). \end{aligned}$$

Similarly, if $n \equiv 4 \pmod 6\ \Rightarrow n = 6k+4\ \Rightarrow 2q = 6n+2 = 36k+26,$ then,

$f^*(x) = [\ 4\ (6k+4)^2 + (6k+4)\ ]\ \bmod (36k+26\ )\ =\ \left(\frac{11n+4}{3}\right).$

Also, $f^*(v_1) = [f(v_1\ v_2)+f(v_n\ v_1)+f(x\ v_1)+f(v_0\ v_1)]\ \bmod (6n+2)\ =\ n+4$

and $f^*(v_i) = [f(v_iv_{i+1}) + f(v_{i-1}v_i) + f(xv_i) + f(v_0v_i)]\ \bmod (6n+2)$

$$= \begin{cases} (n+4i) \bmod (6n+2) & \text{if } \ 2 \le i \le \frac{n}{2}; \\ (5n+4i) \bmod (6n+2) & \text{if } \ \frac{n}{2}+1 \le i \le n. \end{cases}$$

Hence, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}}$ will be $n+4, n+8, n+12, \cdots, 3n,$ respectively, and the labels of the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \cdots, v_{n-1}, v_n$ will be $n+2, n+6, \cdots, 3n-6, 3n-2$, respectively, which are all even and distinct numbers.

Case (2) : For $n \equiv 2 \pmod 6,$ we define the labeling function f as follows:

$f(x\ v_0\ ) = 6n+2\ ,$

$$f\ (x\ v_i) = \begin{cases} 2i & \text{if } \ 1 \le i \le \frac{n}{2}\ ; \\ 4n + 2i & \text{if } \ \frac{n}{2} < i \le \ n\ , \end{cases}$$

$$f\ (v_i\ v_{i+1}) = \begin{cases} 3n + 2i & \text{if } \ 1 \le i \le n-1; \\ n + 2 & \text{if } \ i = \ n, \end{cases}$$

and

$$f\ (v_0\ v_i) = \begin{cases} 5n & \text{if } \ i = 1\ ; \\ 3n + 4 - 2i & \text{if } \ 2 \le i \le \ n\ . \end{cases}$$

In view of the above labeling patten and since $n \equiv 2 \pmod 6\ \Rightarrow n = 6k+2\ \Rightarrow 2q = 6n+2 = 36k+14$ then the induced vertex labels are

$f^*(v_0) =\ (2n^2+5n-2) \bmod (6n+2) =$

$[\ 2k(36k+14) + (94k+68)\ ]\ \bmod (36k+14\ )$

$\equiv (14k+2) \bmod (36k+14\ ) = (\frac{7n-8}{3}).$

By the same way, in the first case, we have

$$f^*(v_i) = \begin{cases} (3n+4i) \bmod (6n+2) & \text{if } \ 2 \le i \le \frac{n}{2}; \\ (n+4i-2) \bmod (6n+2) & \text{if } \ \frac{n}{2}+1 \le i < n. \end{cases}$$

Finally, $f^*(x) =\ 0$, $f^*(v_1) =\ 3n+4$ and $f^*(v_n) =\ n.$

Therefore, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}}$ are $3n+4, 3n+8, 3n+12, \cdots, 5n,$ respectively, and the labels of the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \cdots, v_{n-1}, v_n$ are $3n+2, 3n+6, \cdots,\ 5n-6,\ n,$ respectively.

Case (3) : For $n \equiv 3 \pmod 6$, we define the labeling
$f : E(K_1 + W_n) \to \{2, 4, \cdots, 6n + 2\}$ as follows:
$f(x\,v_0) = 6n$, $f(x\,v_i) = 3n + 3 - 2i$ for $i = 1, 2, \cdots n$,

$$f(v_i\,v_{i+1}) = \begin{cases} 3n + 1 + 2i & \text{if } i = 1, 2, \cdots, n - 1; \\ 6n + 2 & \text{if } i = n, \end{cases}$$

and

$$f(v_0\,v_i) = \begin{cases} 2i & \text{if } 1 \le i \le \frac{n+1}{2}; \\ 4n + 2i - 2 & \text{if } \frac{n+3}{2} \le i \le n. \end{cases}$$

In this labeling, the induced vertex labels are
$f^*(v_0) = [\sum_{i=1}^{n} f(v_0\,v_i) + f(x\,v_0)] \bmod (6n + 2) = 0$,
$f^*(v_1) = [f(v_1\,v_2) + f(v_n\,v_1) + f(v_0\,v_1) + f(x\,v_1)] \bmod (6n + 2) = 4$,
$f^*(x) = [\sum_{i=1}^{n} f(x\,v_i) + f(v_0\,x)] \bmod (6n+2) = (2n^2+2n-2) \bmod (6n+2)$.
Since $n \equiv 3 \pmod 6 \Rightarrow n = 6k + 3 \Rightarrow 2q = 6n + 2 = 36k + 20$.
Then, $f^*(x) = \left(\frac{4n-6}{3}\right)$.
And $f^*(v_i) = [f(v_iv_{i+1}) + f(v_{i-1}v_i) + f(v_0v_i) + f(xv_i)] \bmod (6n + 2)$

$$= \begin{cases} (3n + 4i + 1) \bmod (6n + 2) & \text{if } 2 \le i \le \frac{n+1}{2}; \\ (n + 4i - 3) \bmod (6n + 2) & \text{if } \frac{n+3}{2} \le i \le n - 1. \end{cases}$$

Finally,
$f^*(v_n) = [f(v_{n-1}v_n) + f(v_nv_1) + f(v_0v_n) + f(xv_n)] \bmod (6n + 2) = 6n - 2$.
Hence, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}$ are
$4, 3n+9, 3n+13, \cdots, 5n-1, 5n+3$, respectively, and the labels of the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, \cdots, v_{n-1}, v_n$ are $3n + 3, 3n + 7, 3n + 11, \cdots, 5n - 7, 6n - 2$, respectively. There is no repetition in the vertex label, also $f^*(x)$ and $f^*(v_0)$ are different from all the labels of the vertices v_i.

Case (4) : For $n \equiv 5 \pmod 6$, $n > 5$, we define the labeling function f as follows:
$f(x\,v_0) = 2$, $f(x\,v_i) = (3n + 3) - 2i$ *for* $i = 1, 2, \cdots n$,

$$f(v_i\,v_{i+1}) = \begin{cases} 3n + 1 + 2i & \text{if } i = 1, 2, \cdots, n - 1; \\ 6n + 2 & \text{if } i = n, \end{cases}$$

and

$$f(v_0\,v_i) = \begin{cases} 2i + 2 & \text{if } 1 \le i \le \frac{n-1}{2}; \\ 4n + 2i & \text{if } \frac{n+1}{2} \le i \le n. \end{cases}$$

In view of the above labeling pattern, the induced vertex labels are
Since $n \equiv 5 \pmod 6 \Rightarrow n = 6k + 5 \Rightarrow 2q = 6n + 2 = 36k + 30$, then
$f^*(x) = (2n^2 + 2n + 2) \bmod (6n + 2) = (\frac{16n+10}{3}) \bmod (6n + 2)$.

$$\text{Also, } f^*(v_i) = \begin{cases} (3n + 4i + 3) \bmod (6n + 2) & \text{if } 2 \le i \le \frac{n-1}{2}; \\ (n + 4i - 1) \bmod (6n + 2) & \text{if } \frac{n+1}{2} \le i < n. \end{cases}$$

Finally, $f^*(v_0) = 0$, $f^*(v_1) = 6$ and $f^*(v_n) = 6n$.
Hence, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-3}{2}}, v_{\frac{n-1}{2}}$ are
$6, 3n + 11, 3n + 15, \cdots, 5n - 3, 5n + 1$, respectively, and the labels of the vertices $v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}, \cdots, v_{n-1}, v_n$ are $3n + 1, 3n + 5, \cdots, 5n - 5, 6n$, respectively. Clearly, $f^*(x)$ and $f^*(v_0)$ are different from all the labels of the vertices v_i.
When $n = 5$, the graph $E(K_1 + W_5)$ is an edge even graceful graph but it does not follow this rule, see Fig. 4.

Case (5) : For $n \equiv 1 \pmod 6$, we defined the labeling function f as follows:
$f(v_0\,x) = 6n$,
$f(v_1\,v_n) = 4n$, $f(v_i\,v_{i+1}) = 2n + 2i$ for $i = 1, 2, \cdots n - 1$,

$$f\left(x\,v_i\right)=\begin{cases}2i & \text{if } \ 1\le i\le \ \frac{n+1}{2};\\ 4n-2+2i & \text{if } \ \frac{n+3}{2}\le i\le n,\end{cases}$$

and

$$f\left(v_0\,v_i\right)=\begin{cases}5n+1-2i & \text{if } \ 1\le i\le \frac{n-1}{2};\\ 6n+2 & \text{if } \ i=\frac{n+1}{2};\\ 3n+3-2i & \text{if } \ \frac{n+3}{2}\le i\le n.\end{cases}$$

In this case, the induced vertex labels are

$$f^*(x)=\ 0\,,\quad f^*(v_0)=\ 6n\,,\quad f^*(v_1)\ =\ 5n+1\,,$$

and

$$f^*(v_i)=\begin{cases}(3n+4i-3)\ \mathrm{mod}\ (6n+2) & \text{if } \ 2\le i\le \frac{n-1}{2}\\ (5n+4i-3)\ \mathrm{mod}\ (6n+2)\equiv\ (4i-n-5) & \text{if } \ \frac{n+3}{2}\le i\le n\end{cases}$$

Finally,

$$f^*(v_{\frac{n+1}{2}})=\left[f\left(v_{\frac{n+1}{2}}\,v_{\frac{n+3}{2}}\right)+f\left(v_{\frac{n-1}{2}}\,v_{\frac{n+1}{2}}\right)+f\left(v_0v_{\frac{n+1}{2}}\right)+f\left(xv_{\frac{n+1}{2}}\right)\right]\mathrm{mod}(6n+2)=\ n-1.$$

It is clear that for all $i\in\{1,2,3,...,n\}$, the labels of the vertices v_i are all distinct, even and different from $f^*(x)$ and $f^*(v_0)$ which complete the proof. □

Illustration: In Fig. 3, we present an edge even graceful labeling of the graphs $K_1+\ W_8$ and $K_1+\ W_{10}$.

Illustration: In Fig. 4, we present an edge even graceful labeling of the graphs $K_1+\ W_9$, $K_1+\ W_{11}$, $K_1+\ W_5$ and $K_1+\ W_7$.

Edge even graceful labeling of the join graph K_1+sf_n

The sunflower graph, sf_n, is defined as a graph obtained by starting with an n-cycle C_n with a consecutive vertices $v_1,v_2,\cdots,v_n$ and creating new vertices $u_1,u_2,\cdots,u_n$, with u_i connected to v_i and v_{i+1}. The graph sf_n has number of vertices $p=2n$ and a number of edges $q=3n$.

Theorem 5 *The join graph K_1+sf_n has an edge even graceful labeling for all n.*

Proof Let $\{\ x\ \}$ be the vertex of K_1 and the edges of the graph $K_1+\ sf_n$ will be $\{\ x\,v_i,\ x\,u_i,\ v_i\,v_{i+1},\ v_i\,u_i,\ v_{i+1}\,u_i,\quad i=1,2,\cdots,n\ \}$. Let us use the standard notation $p=|V(\,K_1+\ sf_n)|=2n+1$ and $q\ =|E(K_1+\ sf_n)|=5n$.

We define the labeling function $f:E(K_1+\ sf_n)\longrightarrow\{2,4,\cdots,10n\}$ as follows:

for $\ i=1,\ 2,\ \cdots,\ n,\quad f(x\,v_i\,)=2i,\quad f(x\,u_i)=10\,n-2i,$

for $\ i=1,\ 2,\ \cdots,\ n,\quad f(v_i\,u_i)=4n+2i,\quad f(u_i\,v_{i+1})=2n+2i,$

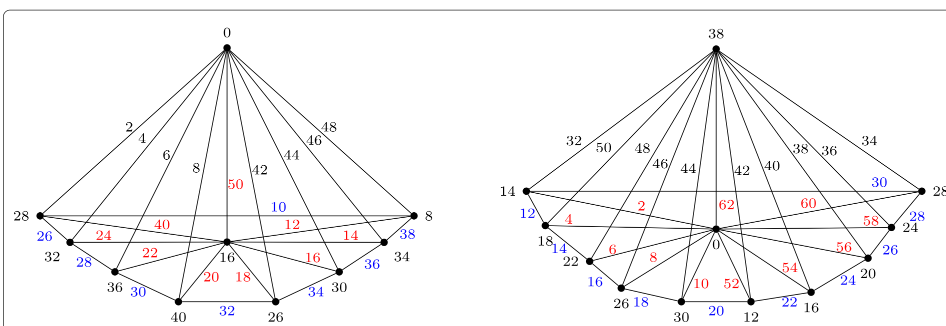


Fig. 3 An edge even graceful labeling of the graphs $K_1+\ W_8$ and $K_1+\ W_{10}$

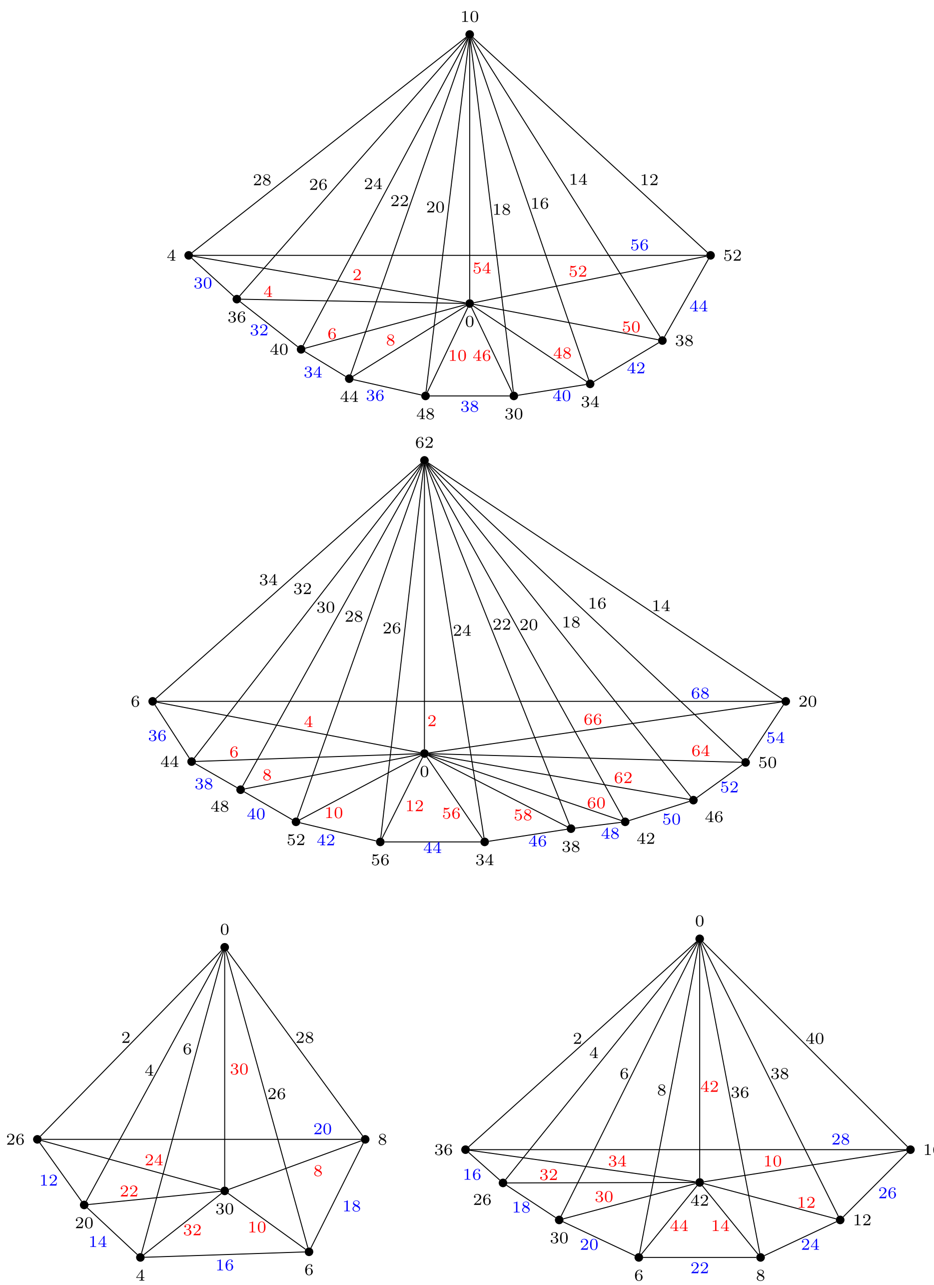


Fig. 4 An edge even graceful labeling of the graphs $K_1 + W_9$, $K_1 + W_{11}$, $K_1 + W_5$ and $K_1 + W_7$

and

$$f\,(v_1\,v_n) = 6n + 2, \qquad f\,(v_i\,v_{i+1}) = \begin{cases} 10\,n & \text{if } \ i = 1\,; \\ 8n + 2 - 2i & \text{if } \ i = 2, 3, \cdots, \ n - 1. \end{cases}$$

Considering the given vertex labels, the induced vertex labels are

$$f^*(x) = [\, \textstyle\sum_{i=1}^{n} f(x\,v_i) + \sum_{i=1}^{n} f(x\,u_i)] \ \text{ mod } (10\,n) \ = 0,$$

$$f^*(v_1) = 4n + 6\,, \qquad f^*(v_2) = 4n + 8,$$

$$f^*(v_i) = 2n + 2i + 4 \qquad \text{for } \ i = 3, \ 4, \ \cdots, \ n,$$

and

$$f^*(u_i) = 6n + 2i \qquad \text{for } \ i = 1, 2, \ \cdots, \ n.$$

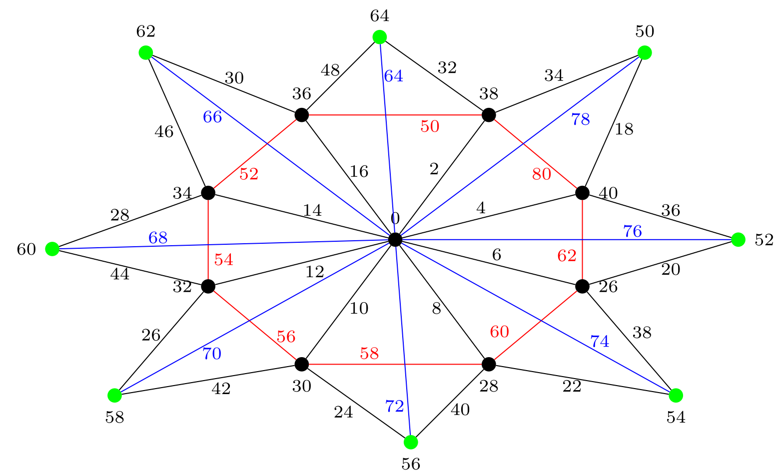


Fig. 5 An edge even graceful labeling of the graph $K_1 + sf_8$

Thus, the labels of the vertices $v_3, v_4, \cdots, v_n$ are $2n+10, 2n+12, \cdots, 4n+4$, respectively, and the labels of the vertices $u_1, u_2, \cdots, u_n$ are $6n+2, 6n+4, \cdots, 8n$, respectively. Clearly, all the labels are even and distinct. Thus, the graph $K_1 + sf_n$ is an edge even graceful labeling. □

Illustration: In Fig. 5, we present an edge even graceful labeling of the graph $K_1 + sf_8$.

Edge even graceful labeling of the join graph $\overline{K}_2 + K_{1,n}$

Theorem 6 *The graph $\overline{K}_2 + K_{1,n}$ has an edge even graceful labeling for all n.*

Proof Let $\{v_0, v_1, v_2, \cdots, v_n\}$ be the vertices of the graph $K_{1,n}$ with central vertex v_0 and $\{x, y\}$ be the vertices of $\overline{K}_2$ so the edges of $\overline{K}_2 + K_{1,n}$ are $\{x\,v_0,\ x\,v_i,\ v_0\,v_i,\ y\,v_0, y\,v_i,\ i = 1, 2, \cdots, n\}$. Let us use the standard notation $p = |V(\overline{K}_2 + K_{1,n})| = n+3$ and $q = |E(\overline{K}_1 + K_{1,n})| = 3n+2$. There are two cases:

Case (1) : when n is even. We define the labeling function

$f : E(\overline{K}_2 + K_{1,n}) \longrightarrow \{2, 4, \cdots, 6n+4\}$ as follows:

$$f(x\,v_i) = \begin{cases} 6n+4 & \text{if } i = 0; \\ 2i & \text{if } 1 \le i \le \frac{n}{2}; \\ 4n+2+2i & \text{if } \frac{n}{2}+1 \le i \le n, \end{cases}$$

$$f(y\,v_i) = \begin{cases} 3n+4 & \text{if } i = 0; \\ n+2i & \text{if } 1 \le i \le \frac{n}{2}; \\ 3n+2+2i & \text{if } \frac{n}{2}+1 \le i \le n, \end{cases}$$

and

$$f\,(v_0\,v_i) = \begin{cases} 2n+2i & \text{if } 1 \le i \le \frac{n}{2}+1; \\ 2n+2+2i & \text{if } \frac{n}{2}+2 \le i \le n. \end{cases}$$

Considering the given vertex labels, the induced vertex labels are

$$f^*(x) = [\, f(x\,v_0) + \textstyle\sum_{i=1}^{n} f(x\,v_i)\,] \bmod (6n+4) = f(x\,v_0) \bmod (6n+4) = 0,$$
$$f^*(y) = [f(yv_0) + \textstyle\sum_{i=1}^{n} f(yv_i)] \bmod (6n+4) = f(yv_0) \bmod (6n+4) = 3n+4,$$
$$f^*(v_0) = [\textstyle\sum_{i=1}^{n} f(v_0\,v_i) + f(x\,v_0) + f(y\,v_0)] \bmod (6n+4) = 3n+2,$$
$$f^*(v_i) = [f(v_0\,v_i) + f(x\,v_i) + f(y\,v_i)] \bmod (6n+4)$$

$$= \begin{cases} (3n+6i) \bmod (6n+4) & \text{if } \ 2 \le i \le \frac{n}{2}; \\ (3n+2+6i) \bmod (6n+4) & \text{if } \ \frac{n}{2}+2 \le i \le n. \end{cases}$$

and

$f^*(v_{\frac{n}{2}+1}) = \left[f\left(v_0 v_{\frac{n}{2}+1}\right) + f\left(x\, v_{\frac{n}{2}+1}\right) + f\left(y\, v_{\frac{n}{2}+1}\right)\right] \bmod (6n+4) \ = \ 2.$

Thus, the labels of the vertices $v_1,\ v_2,\ \cdots,\ v_{\frac{n}{2}-1},\ v_{\frac{n}{2}}$ are $3n+6,\ 3n+12,\ \cdots,\ 6n-6,\ 6n,$ respectively, and the labels of the vertices $v_{\frac{n}{2}+1},\ v_{\frac{n}{2}+2},\ v_{\frac{n}{2}+3},\ \cdots,\ v_{n-1},\ v_n$ are $2,\ 10,\ 16,\ \cdots,\ 3n-8,\ 3n-2$, respectively. Clearly, $f^*(x)$, $f^*(y)$ and $f^*(v_0)$ are different from all the labels of the vertices v_i.

Case (2) : when n is odd. We introduce two different labeling

Method 1: We define the labeling function f as follows:

$$f\,(v_0\,v_i) = \begin{cases} 2n+2+2i & \text{if } \ 1 \le i \le \frac{n-1}{2}; \\ 5n+1-2i & \text{if } \ \frac{n+1}{2} \le i \le \ n-1; \,, \\ 6n+2 & \text{if } \ i = \ n\,, \end{cases}$$

$$f(x\,v_i\,) = \begin{cases} 4 & \text{if } \ i = \ 0\,; \\ 4+2i & \text{if } \ 1 \le i \le \frac{n-1}{2}\,; \\ 4n+2i & \text{if } \ \frac{n+1}{2} \le i \le \ n, \end{cases}$$

and

$$f(y\,v_i\,) = \begin{cases} 2 & \text{if } \ i = \ 0\,; \\ 5n+1-2i & \text{if } \ 1 \le i \le \frac{n-1}{2}; \\ 3n+3-2i & \text{if } \ \frac{n+1}{2} \le i \le \ n-1\,; \\ 6n+4 & \text{if } \ i = \ n. \end{cases}$$

Hence, the induced vertex labels are

$$f^*(x) = \left[f(x\,v_0) \ + \sum_{i=1}^{n} f(x\,v_i)\right] \bmod (6n+4) = f(x\,v_0\,) \bmod (6n+4) = \ 0,$$

$$f^*(y) = \left[f(y\,v_0) \ + \sum_{i=1}^{n} f(y\,v_i\,)\ \right] \bmod (6n+4) = \ f(y\,v_0\,) \bmod (6n+4) = 2,$$

$$f^*(v_0) = \left[\sum_{i=1}^{n} f(v_0 v_i) + f(xv_0) + f(yv_0)\right] \bmod (6n+4) = \ 4,$$

$$f^*(v_i) = \begin{cases} (n+3+2i) \bmod (6n+4) & \text{if } \ 1 \le i \le \frac{n-1}{2}; \\ (6n-2i) \bmod (6n+4) & \text{if } \ \frac{n+1}{2} \le i \le n-1\,. \end{cases}$$

and $f^*(v_n) = [f(v_0\,v_n) + f(x\,v_n) + f(y\,v_n)] \bmod (6n+4) \ = \ 6n-2.$

Then the labels of the vertices $v_1,\ v_2,\ \cdots,\ v_{\frac{n-3}{2}},\ v_{\frac{n-1}{2}}$ are $n+5,\ n+7,\ \cdots,\ 2n,\ 2n+2,$ respectively, and the labels of the vertices $v_{\frac{n+1}{2}},\ v_{\frac{n+3}{2}},\ \cdots, v_{n-2},\ v_{n-1},\ v_n$ are $5n-1,\ 5n-3,\ \cdots,\ 4n+4,\ 4n+2,\ 6n-2$, respectively. Clearly $f^*(x)$, $f^*(y)$ and $f^*(v_0)$ are different from all the labels of the vertices v_i . Thus the graph $\overline{K}_2 + K_{1,n}$ has an edge even graceful labeling for all n.

Method 2: We can find another labeling when n is an odd number, by redefining the labeling function as follows:

$$f(x\,v_i\,) = \begin{cases} 5n+3 & \text{if } \ i = \ 0\,; \\ 2i & \text{if } \ 1 \le i \le \frac{n+1}{2}\,; \\ 4n+2i+2 & \text{if } \ \frac{n+3}{2} \le i \le \ n, \end{cases}$$

$$f(y\,v_i\,) = \begin{cases} 6n+4 & \text{if } \ i = \ 0\,; \\ n+1+2i & \text{if } \ 1 \le i \le \frac{n+1}{2}\,; \\ 3n+1+2i & \text{if } \ \frac{n+3}{2} \le i \le \ n\,, \end{cases}$$

and

$$f\,(v_0\,v_i)=\begin{cases}2n+2+2i & \text{if} \quad 1\le i\le \frac{n+1}{2};\\ 5n+5-2i & \text{if} \quad \frac{n+3}{2}\le i\le\ n.\end{cases}$$

Then, by the same way, we can calculate $f^*(x), f^*(y)$ and $f^*(v_0)$ and prove that they are different from all the labels of the vertices v_i. □

Illustration: In Fig. 6, we present an edge even graceful labeling of $\overline{K}_2 +\ K_{1,10}$ and $\overline{K}_2 +\ K_{1,11}$.

Edge even graceful labeling of the join graph $\overline{K}_2 + w_n$

Theorem 7 *The graph* $\overline{K}_2 +\ W_n$ *has an edge even graceful labeling for all* n.

Proof Let $\{v_0,\ v_1, v_2,\ \cdots,\ v_n\}$ be the vertices of the wheel graph W_n with central vertex v_0 and $\{x,\ y\}$ be the vertices of the graph $\overline{K}_2$, so $E(\overline{K}_2 +\ W_n) = \{x\,v_0,\ x\,v_i,\ y\,v_0, y\,v_i, v_0\,v_i\,, v_i\,v_{i+1},\ \ i=1,2,\cdots,\ n\,\}$. In this graph, $p=|V(\,\overline{K}_2 +\ w_n)| = n+3$ and $q\ =|E(\,\overline{K}_2 +\ W_n)| = 4n+2$. There are two cases:

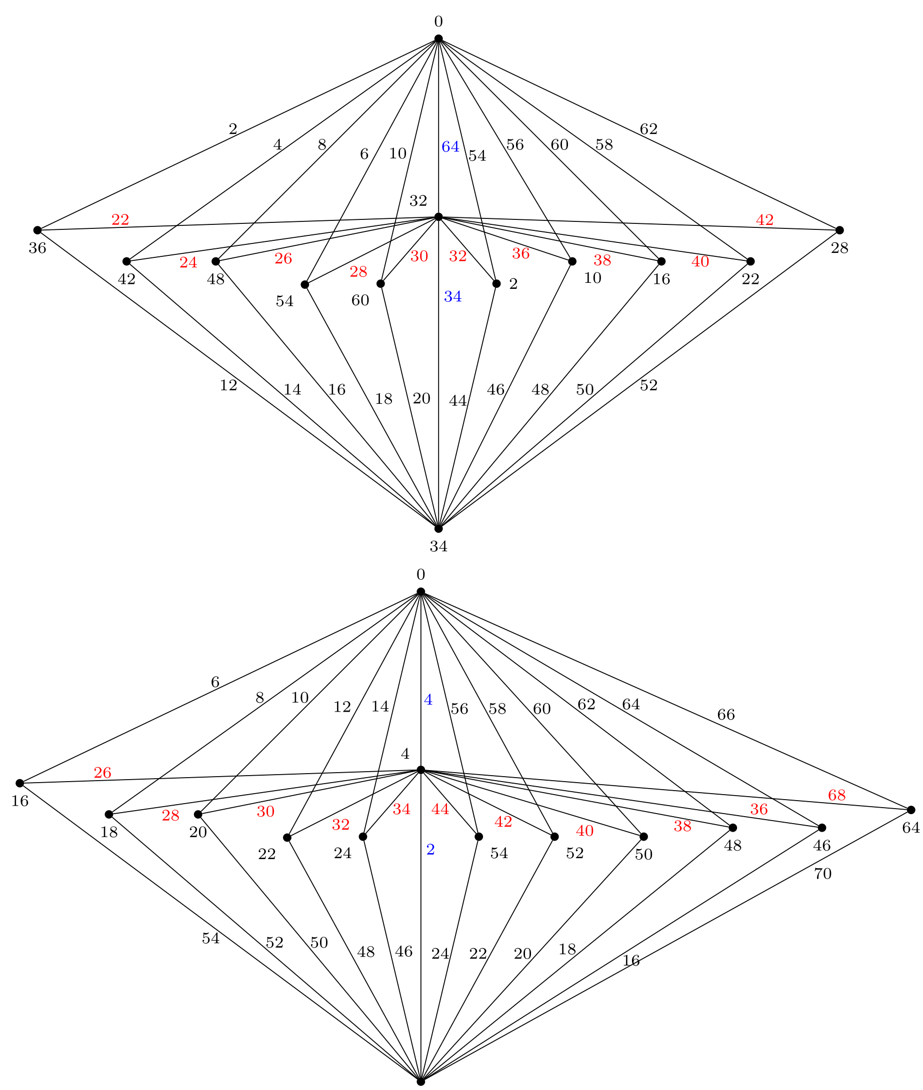


Fig. 6 An edge even graceful labeling of the graphs $\overline{K}_2 +\ K_{1,10}$ and $\overline{K}_2 +\ K_{1,11}$

Case (1) : when n is even, we define the labeling $f : E(\overline{K}_2 + W_n) \longrightarrow \{2, 4, \cdots, 8n+4\}$ as follows:

$f(v_1\, v_n) = 8n+4, \quad f(v_i\, v_{i+1}) = 3n+2i+2 \qquad for \;\; i = 1, 2, \cdots n-1,$

$$f\,(v_0\, v_i) = \begin{cases} 2i+2 & \text{if } 1 \le i \le \frac{n}{2}; \\ 6n+2i & \text{if } \frac{n}{2} < i \le n, \end{cases}$$

$$f(x\, v_i\,) = \begin{cases} 2 & \text{if } i = 0; \\ n+2+2i & \text{if } 1 \le i \le \frac{n}{2}; \\ 5n+2i & \text{if } \frac{n}{2} < i \le n, \end{cases},$$

and

$$f(y\, v_i\,) = \begin{cases} 8n & \text{if } i = 0; \\ 2n+2+2i & \text{if } 1 \le i \le \frac{n}{2}; \\ 4n+2i & \text{if } \frac{n}{2} < i \le n. \end{cases}$$

In view of the above labeling pattern we have

$f^*(v_1) = 6n+16\,,\; f^*(v_n) = 2n-12,$

$f^*(x) = \left[f(x\, v_0) + \sum_{i=1}^{n} f(x\, v_i\,) \right] \bmod (8n{+}4) = f(x\, v_0\,) \bmod (8n{+}4) = 2,$

$f^*(y) = \left[f(yv_0) + \sum_{i=1}^{n} f(yv_i) \right] \bmod (8n{+}4) = f(yv_0) \bmod (8n{+}4) = 8n{+}2,$

$f^*(v_0) = \left[\sum_{i=1}^{n} f(v_0\, v_i) + f(x\, v_0) + f(y\, v_0) \right] \bmod (8n+4) = 0,$

and $f^*(v_i) = [f(v_i\, v_{i+1}) + f(v_{i-1}\, v_i) + f(v_0\, v_i) + f(x\, v_i) + f(y\, v_i)] \bmod (8n{+}4).$

Therefore, $$f^*(v_i) = \begin{cases} (n+4+10i) \bmod (8n+4) & \text{if } 2 \le i \le \frac{n}{2}; \\ (5n-6+10i) \bmod (8n+4) & \text{if } \frac{n}{2}+1 \le i < n. \end{cases}$$

Hence, the labels of the vertices $v_2, v_3, v_4, \cdots, v_{\frac{n}{2}}$ are $n+24, n+34, n+44, \cdots, 6n+4$, respectively, and the labels of the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \cdots, v_{n-1}, v_n$ are $2n, 2n+10, \cdots, 7n-20$, respectively, which are even and distinct numbers. Clearly, $f^*(x)$ and $f^*(v_0)$ are even and different from all the labels of the vertices v_i.

Case (2) : when n is odd, we define the labeling function f as follows:

$f(v_1\, v_n) = 5n+1, \quad f(v_i\, v_{i+1}) = 5n-2i+1 \qquad \text{for } \; i = 1, 2, \cdots n-1,$

$$f\,(v_0\, v_i) = \begin{cases} n+2i+1 & \text{if } 1 \le i \le \frac{n+1}{2}; \\ 5n-1+2i & \text{if } \frac{n+3}{2} \le i \le n, \end{cases}$$

$$f(x\, v_i\,) = \begin{cases} n+1 & \text{if } i = 0; \\ 2i & \text{if } 1 \le i \le \frac{n-1}{2}; \\ 6n+2+2i & \text{if } \frac{n+1}{2} \le i \le n, \end{cases}$$

and

$$f(y\, v_i\,) = \begin{cases} 7n+1 & \text{if } i = 0; \\ 2n+2+2i & \text{if } 1 \le i \le \frac{n-1}{2}; \\ 8n+4 & \text{if } i = \frac{n+1}{2}; \\ 4n+2i & \text{if } \frac{n+3}{2} \le i \le n. \end{cases}$$

In view of the above labeling pattern and by the same way in **Case (1)**, we have $f^*(x) = 0,\; f^*(y) = 7n+1,\; f^*(v_0) = n+1,\; f^*(v_{\frac{n+1}{2}}) = n-1$ and $f^*(v_n) = 5n-7$.

Finally, $$f^*(v_i) = \begin{cases} (5n+3+2i) \bmod (8n+4) & \text{if } 1 \le i \le \frac{n-1}{2}; \\ (n-7+2i) \bmod (8n+4) & \text{if } \frac{n+3}{2} \le i \le n-1\,. \end{cases}$$

Hence, the labels of the vertices $v_1,\; v_2,\; v_3,\; \cdots,\; v_{\frac{n-3}{2}},\; v_{\frac{n-1}{2}}$ will be $5n+5,\; 5n+7, 5n+9\; \cdots,\; 6n,\; 6n+2,$ respectively. Also, the labels of the vertices $v_{\frac{n+3}{2}},\; v_{\frac{n+5}{2}},\; \cdots,\; v_{n-2},\; v_{n-1}$ will be $2n-4,\; 2n-2,\; \cdots,\; 3n-11,\; 3n-9,$ respectively. It is clear that $f^*(x)$, $f^*(y)$ and $f^*(v_0)$ are even and different from all the labels of the vertices v_i.

Obviously, the vertex labels are all even and distinct. Also $f^*(x)$ and $f^*(y)$ are even and different from all the labels of the vertices v_i. Thus, the graph $\overline{K}_2 + W_n$ is an edge even graceful graph for all n. □

Illustration: In Fig. 7, we present an edge even graceful labeling of $\overline{K}_2 + W_{10}$ and $\overline{K}_2 + W_{11}$.

Edge even graceful labeling of the double cone $\overline{K}_2 + C_n$

Theorem 8 *The double cone $\overline{K}_2 + C_n$ has an edge even graceful labeling for all n.*

Proof Let $\{x, y\}$ be the vertices of $\overline{K}_2$ and $\{v_1, v_2, \cdots, v_n\}$ be the vertices of the graph C_n so the edges are $\{x\,v_i, y\,v_i, v_i\,v_{i+1}, i = 1, 2, \cdots, n\}$. Let us use the standard notation $p = |V(\overline{K}_2 + C_n)| = n + 2$ and $q = |E(\overline{K}_2 + C_n)| = 3n$. There are three cases:

Case (1): When $n \equiv 1 \pmod 6$ *or* $n \equiv 3 \pmod 6$. We define the labeling $f : E(\overline{K}_2 + C_n) \longrightarrow \{2, 4, \cdots, 6n\}$ as follows:

$$f(x\,v_i) = 2i \qquad \text{for} \quad i = 1, 2, \cdots n,$$
$$f(y\,v_i) = 6n - 2i \qquad \text{for} \quad i = 1, 2, \cdots n,$$

and

$$f(v_1\,v_n) = 6n, \quad f(v_i\,v_{i+1}) = 2n + 2i \qquad \text{for} \quad i = 1, 2, \cdots n-1,$$

The induced vertex labels are

$$f^*(x) = \left[\sum_{i=1}^{n} f(xv_i)\right] \bmod (6n) = \left[\sum_{i=1}^{n} (2i)\right] \bmod (6n) = (n^2 + n) \bmod (6n),$$

If $n \equiv 1 \pmod 6 \Rightarrow 2q = 6n = 36k + 6$, then

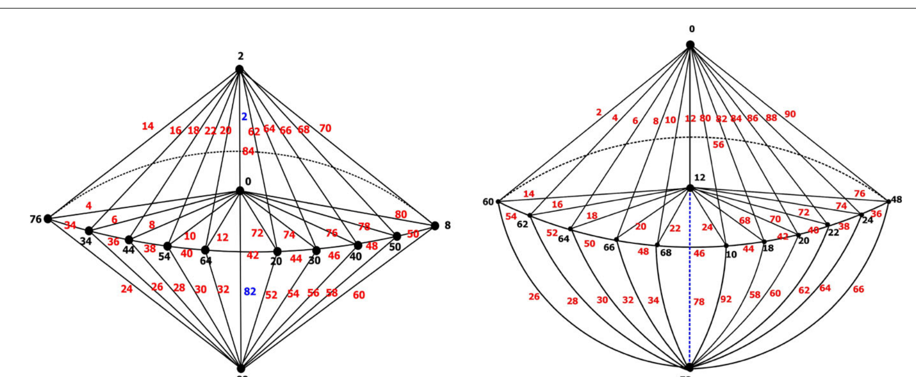
$f^*(x) \equiv [\,12k + 2] \bmod (36k + 6) = 2n.$

If $n \equiv 3 \pmod 6 \Rightarrow 2q = 6n = 36k + 18$, then

$f^*(x) = \equiv [\,24k + 12] \bmod (36k + 12) = 4n.$

Similarly, $f^*(y) = [\,\sum_{i=1}^{n} f(yv_i)\,] \bmod (6n) = \left[\,\sum_{i=1}^{n} (6n - 2i)\right] \bmod (6n)$.

$$\therefore \quad f^*(y) = (5n^2 - n) \bmod (6n) = \begin{cases} 4n & \text{if } n \equiv 1 \pmod 6; \\ 2n & \text{if } n \equiv 3 \pmod 6. \end{cases}$$



(a) $\overline{K}_2 + W_{10}$ (b) $\overline{K}_2 + W_{11}$

Fig. 7 An edge even graceful labeling of the graphs $\overline{K}_2 + W_{10}$ and $\overline{K}_2 + W_{11}$

$$f^*(v_1) = [f(v_1v_2) + f(v_nv_1) + f(x\,v_1) + f(y\,v_1)] \mod (6n) = 2n+2,$$

$$\begin{aligned} f^*(v_i) &= [f(v_i\,v_{i+1}) + f(v_{i-1}v_i) + f(x\,v_i) + f(y\,v_i)] \mod (6n) \\ &= [\,f(v_i\,v_{i+1}) + f(v_{i-1}v_i)] \mod (6n) \\ &= (4n+4i-2) \mod \quad (6n)\,, \quad 2 \le i \le n-1, \end{aligned}$$

and

$f^*(v_n) = [f(v_n\,v_1) + f(v_{n-1}\,v_n) + f(x\,v_n) + f(y\,v_n)] \mod (6n) = 4n-2.$

Hence, the labels of the vertices $v_1,\ v_2,\ v_3,\ \cdots,\ v_{\frac{n-1}{2}}$ will be $2n+2,\ 4n+6,\ 4n+10,\ \cdots,\ 6n-4$, respectively, and the labels of the vertices $v_{\frac{n+1}{2}},\ v_{\frac{n+3}{2}},\ \cdots,\ v_{n-1},\ v_n$ will be $0, 4, \cdots, 4n-6, 4n-2$, respectively. Clearly, $f^*(x)$ and $f^*(y)$ are different from all the labels of the vertices v_i.

Case (2) : When $n \equiv 5 \pmod 6$, we define the labeling f as follows:

$f(y\,v_i) = 2n+2i$ for $i = 1, 2, \cdots n$,

$f(v_1\,v_n) = 4n+2,\quad f(v_i\,v_{i+1}) = 6n-2i+2$ *for* $i = 1, 2, \cdots n-1$,

and

$$f\,(x\,v_i) = \begin{cases} 2 & \text{if } i = 1 \\ 2n-2i+4 & \text{if } i = 2, 3, \cdots n \end{cases}$$

Since $n \equiv 5 \pmod 6 \Rightarrow n = 6k+5 \Rightarrow 2q = 6n = 36k+30$.

Then the induced vertex labels are

$$\begin{aligned} f^*(y) &= \left[\ \textstyle\sum_{i=1}^{n}(2n+2i)\ \right] \mod (6n) = (3n^2+n) \mod (6n) \\ &\equiv \left[\ 24(\tfrac{n-5}{6}) + 20\ \right] \mod (6n\,) = 4n, \end{aligned}$$

$f^*(x) = (n^2+n) \mod (6n) \equiv 0,\quad f^*(v_1) = 6$ and

$f^*(v_i) = (\,4n-4i+10) \mod (6n)$ *for* $2 \le i \le n$.

Hence the labels of the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}, \cdots, v_{n-1}, v_n$ are $6,\ 4n+2,\ 4n-2,\ \cdots,\ 2n+14,\ 2n+8,\ 2n+4,\ \cdots,\ 14,\ 10,$ respectively. Clearly, $f^*(x)$ and $f^*(y)$ are even and different from all the labels of the vertices v_i. Thus, the graph $\overline{K}_2 + C_n$ is an edge even graceful graph.

Case (3) : When n is even, $n \ge 4$, we define the labeling function f as follows:

$f(v_1\,v_n) = 4n,\quad f(v_i\,v_{i+1}) = 2n+2i$ for $i = 1, 2, \cdots n-1$,

$$f\,(x\,v_i) = \begin{cases} 2i & \text{if } i = 1, 2, \cdots, \frac{n}{2}; \\ 4n+2(i-1) & \text{if } \frac{n}{2}+1 \le i \le n, \end{cases}$$

and

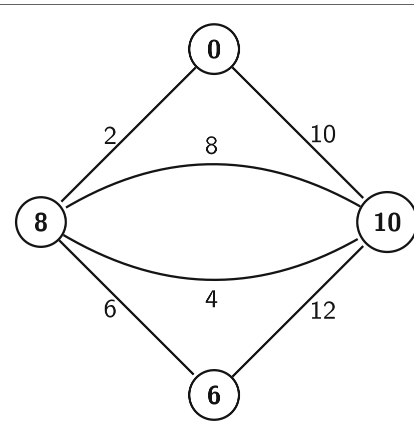


Fig. 8 An edge even graceful labeling of the double cone $\overline{K}_2 + C_2$

$$f\,(y\,v_i)=\begin{cases}2n-2(i-1) & \text{if } \ 1\le i\le \frac{n}{2}\,;\\ 6n & \text{if } \ i\ =\frac{n}{2}+1;\\ 6n-2(i-1) & \text{if } \ \frac{n}{2}+2\le i\le\ n.\end{cases}$$

Thus, the induced vertex labels are

$$f^*(x)=\left[\ \sum_{i=1}^{n} f(x\,v_i)\ \right] \bmod (6n)=0,$$

$$f^*(y)=\left[\ \sum_{i=1}^{n} f(y\,v_i)\ \right] \bmod (6n)=$$

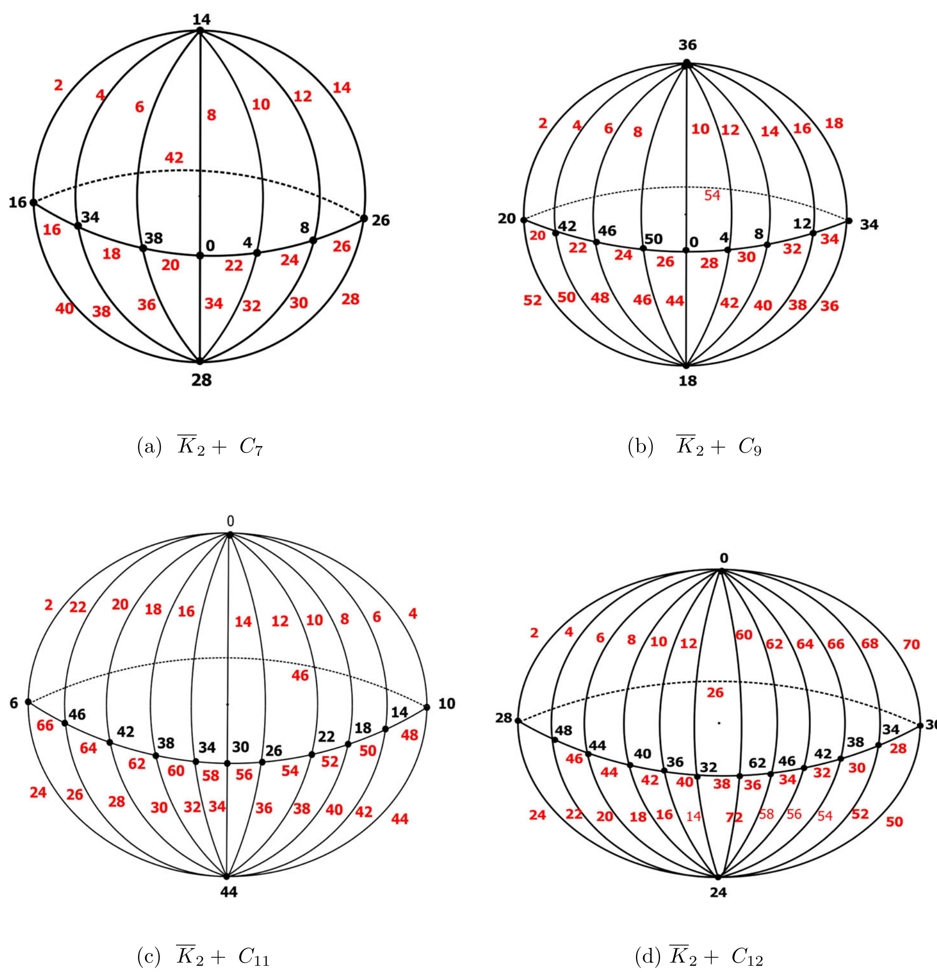
$$\left[f(y\,v_1)+f(y\,v_{\frac{n}{2}})\right] \bmod (6n)=\ 2n,$$

$$f^*(v_1)=\left[f(x\,v_1)+f(y\,v_1)+f(v_1\,v_2)+f(v_n\,v_1)\right] \bmod (6n)=2n+4,$$

and

$$f^*(v_i)=\begin{cases}4n-4i+8 & \text{if } \ 2\le i\le\frac{n}{2}\,;\\ 5n+2 & \text{if } \ i\ =\frac{n}{2}+1\,;\\ 6n-4i+6 & \text{if } \ \frac{n}{2}+2\le i\le\ n\,.\end{cases}$$

Hence, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}$ are $2n+4,\ 4n,\ 4n-4, \cdots, 2n+12, 2n+8,$ respectively, and the labels of the vertices $v_{\frac{n}{2}+1},\ v_{\frac{n}{2}+2},\ v_{\frac{n}{2}+3},\ \cdots,\ v_{n-1},\ v_n$ are $5n+2,\ 4n-2,\ 4n-6,\ \cdots, 2n+10, 2n+6,$ respectively. Obviously the vertex labels are all even and distinct. Also, $f^*(x)$ and $f^*(y)$ are even and different from all the labels of the vertices v_i . Thus, the double cone $\overline{K}_2+\ C_n$ is an edge even graceful labeling when n is even.



(a) $\overline{K}_2+\ C_7$ (b) $\overline{K}_2+\ C_9$

(c) $\overline{K}_2+\ C_{11}$ (d) $\overline{K}_2+\ C_{12}$

Fig. 9 An edge even graceful labeling of the double cones $\overline{K}_2+\ C_7$, $\overline{K}_2+\ C_9$, $\overline{K}_2+\ C_{11}$, and $\overline{K}_2+\ C_{12}$

- If $n = 2$, the double cone $\overline{K}_2 + C_2$ has an edge even graceful labeling, see the following Fig. 8. □

Illustration: In Fig. 9, we present an edge even graceful labeling of $\overline{K}_2 + C_7, \overline{K}_2 + C_9, \overline{K}_2 + C_{11}$ and $\overline{K}_2 + C_{12}$

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