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How to compare market efficiency? The Sharpe ratio based on the ARMA-GARCH forecast

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Abstract

This paper derives a new method for comparing the weak-form efficiency of markets. The author derives the formula of the Sharpe ratio from the ARMA-GARCH model and finds that the Sharpe ratio just depends on the coefficients of the AR and MA terms and is not affected by the GARCH process. For empirical purposes, the Sharpe ratio can be formulated with a monotonic increasing function of *R-squared* if the sample size is large enough. One can utilize the Sharpe ratio to compare weak-form efficiency among different markets. The results of stochastic simulation demonstrate the validity of the proposed method. The author also constructs empirical AR-GARCH models and computes the Sharpe ratio for S&P 500 Index and the SSE Composite Index.

Keywords: ARMA, GARCH, Measurement of market efficiency, Sharpe ratio, Stochastic simulation

Mathematics subject classification: 60G10, 62M10

JEL classification: G10, G14, G17, C22

Introduction

Since Fama (1970) published his influential paper, there has been a thorough assessment of market efficiency. Various methods have been proposed over the decades to test the efficient market hypothesis (EMH), but the empirical results vary according to the specific markets chosen, periods of time, and even the selected methods. For instance, like the frictionless state in physics, the efficient market could just be viewed as an ideal state with no real existence in the world. However, it could serve as a useful benchmark for measuring relative efficiency of markets over space and time.

In fact, it is more relevant to measure the relative efficiency of markets from the empirical perspective. Just as Campbell et al. (1997) pointed out, “The notion of relative efficiency—the efficiency of one market measured against another—may be a more useful concept than the all-or-nothing view taken by much of the traditional market-efficiency literature.”



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So far, researchers have tended to measure and investigate market efficiency using various statistical methods. One market efficiency measure is based on autocorrelation of returns. The idea behind this method is that a strong autocorrelation reflects a deviation from random walk pricing and is indicative of violations of the EMH. For example, the Q-statistic of Box and Pierce (1970) is usually used as a measure for market efficiency. Lo and Mackinlay (1988) proposed a test statistic, the ratio of variances, which is also used as a measure for market efficiency. Another type of market efficiency measure reflects the degree to which returns are associated with past market returns; it is similar to the delay measure of Mech (1993) and Hou and Moskowitz (2005). The logic behind it is that a security price has a lower efficiency if it incorporates the information contained in the market index movement slower than another.

The literature mentioned above is based on the EMH, which uses many assumptions to describe the model, such as independence, limited variance, normality, and linear paradigm, among many others. However, some literature that tests the market efficiency relies on an alternative theory known as the fractal market hypothesis (FMH). This theory does not have the aforementioned strict assumptions but instead focuses on the nonlinearities in the financial market and supposes that the behavior of investors is affected by their level of understanding of the information along with the investment period. Based on the FMH, Liu et al. (1999) used a Hurst exponent and a time-varying Hurst exponent to describe the dynamic changes of market efficiency in the Hong Kong REITS market. They determined that the market had not yet reached the state of weak efficiency. Han et al. (2019) compared the efficiency of five primary indexes of different boards of the Chinese stock market using multifractal detrended fluctuation analysis (MF-DFA). This method measures the multifractal degrees. The authors found that the SHSE-SZS300 index that reflects the overall trend of the Shanghai and Shenzhen main board markets was less efficient than the ChiNext index, which had the lowest multifractal lever among five indexes. Moradi et al. (2019) used the L-Co-R algorithm analyze the fractal features of the Tehran Stock Exchange and the London Stock Exchange. In essence, fractal theory is a type of chaos theory to some extent. Ola et al. (2014) applied the local polynomial approximation model, which is a method relating to chaos processes and testing to determine the efficiency of the Tehran Stock Exchange and found that the efficiency of the market could not be confirmed. Moreover, as the efficiency of the market is related to investor behavior, some works have indirectly analyzed the efficiency of the market from this perspective, for example, Wen et al. (2019).

In principle, profitability is at the core of market efficiency. According to Malkiel (1992), "Moreover, efficiency with respect to an information set ... implies that it is impossible to make economic profits by trading on the basis of [that information set]." However, the traditional measures of market efficiency do not reflect profitability. For example, using data from 56 markets, Griffin et al. (2010) found that variance ratios and market delay measures often showed greater deviation from random walk pricing in developed markets than emerging markets. However, they also found that short-term reversal strategies and momentum strategies could earn similar returns in both types of markets. Therefore, they argued that the usual efficiency measures might yield misleading inferences.

It is regarded that a proper measure of market efficiency should directly reflect the expected return and the relevant risk. The Sharpe ratio (SR) is one of the main criteria of portfolio assessment, and it has been widely used to evaluate the performance of mutual funds. Sharpe (1966) proposed the now well-known Sharpe ratio to evaluate a fund's return in excess of the risk-free rate, adjusted by the additional risk the fund assumes. A high Sharpe ratio indicates superior risk-adjusted performance.

The Sharpe ratio can help measure market efficiency. Fortunately, one can derive the formula of the Sharpe ratio from the ARMA-GARCH model. Financial returns are often modeled as autoregressive time series with random disturbances having conditional heteroscedastic variances, especially with GARCH type processes that have been extensively used in the financial and econometric literature as risk models of many financial time series.

The relationship between the SR and the R^2 of the ARMA-GARCH model is investigated in this paper. This is related to the research on return predictability by the R^2 of the regression of stock return on predictors implied by the rational asset pricing model. These research studies compare the observed R^2 of predictive regressions with the maximal bounds permitted by asset pricing models, and through the comparison, they conclude whether the market is effective. Kirby (1998) first proposed the frame of analyzing the restrictions on predictability implied by the rational asset pricing model. Based on similar frameworks, Levich and Potì (2015) constructed an upper bound on the explanatory power of predictive regressions motivated by "no good-deal" restrictions in the currency market and found that the currency predictability exceeded the bound, thereby showing that the market is not effective. Huang and Zhou (2017) provided two theoretical upper bounds on the R^2 of the regression and reported that the predictive R^2 was significantly larger than the upper bounds. Their bounds were tighter than Ross (2005), who is a pioneer of providing the bounds on predictability for all asset pricing models under no arbitrage conditions. Potì (2018) proposed an upper bound on the predictability of asset returns, which was at least as tight as Huang and Zhou (2017).

However, we do not make a comparison between the predicted R^2 and the theoretical boundaries based on asset pricing models under capital market equilibrium because we do not measure the degree of market efficiency precisely but only compare the market efficiency among markets. So, we focus on testing whether the SR can show differences in market efficiency. We find that the SR can be formulated with a monotonic increasing function of R^2 . We also compare the SR with another indicator proposed by Zhang (1999), namely conditional entropy.

This paper proceeds as follows. In "The Sharpe Ratio based on the ARMA-GARCH Forecast" section, the formula of the SR is derived from the ARMA-GARCH model. "Stochastic simulation" section applies the method through stochastic simulation and provides a comparison with the conditional entropy indicator. "Empirical Research" section presents the empirical results of the S&P 500 Index and the SSE Composite Index. "Conclusion" section gives concluding remarks.

The Sharpe ratio based on the ARMA-GARCH forecast

To compare technical analysis with the time series forecast, Fang and Xu (2003) derived the expected value of a one-period excess return from the AR (1) model forecast.

Following on these methods, we derive the expected value of a one-period excess return from the general ARMA(p, q)-GARCH(r, s) model forecast. Furthermore, we deduce its standard deviation so that the formula of the SR can be obtained.

The basic approach

We start with the ARMA (p, q) model.

Let $S_t, t = 0, 1, \dots, T$ denote stock prices observed at discrete moments. Let Y_t denote the continuously compounded return at time t . So:

$$Y_t = \ln\left(\frac{S_t}{S_{t-1}}\right), \quad t = 1, 2, \dots, T. \tag{2.1}$$

Suppose $\{Y_t\}$ satisfies the following ARMA (p, q) process:

$$Y_t = \sum_{i=1}^p a_i Y_{t-i} + \sum_{i=1}^q b_i \varepsilon_{t-i} + \varepsilon_t, \tag{2.2}$$

where $\{\varepsilon_t\}$ is the Gaussian white noise, $WN(0, \sigma^2)$. Neither a_p nor b_q is equal to zero.

Provided that all the roots of

$$1 - \sum_{i=1}^p a_i z^i = 0$$

lie outside the unit circle, it is straightforward to verify that $\{Y_t\}$ is a covariance-stationary process.

According to eq. (2.2), we can express Y_{t+1} as

$$Y_{t+1} = \sum_{i=1}^p a_i Y_{t-i+1} + \sum_{i=1}^q b_i \varepsilon_{t-i+1} + \varepsilon_{t+1}. \tag{2.3}$$

Let I_t denote the information set at time t , which includes the present and past records of both $\{\varepsilon_t\}$ and $\{Y_t\}$. Then the forecast of Y_{t+1} based on I_t is

$$E_t[Y_{t+1}] = E[Y_{t+1}|I_t] = \sum_{i=1}^p a_i Y_{t-i+1} + \sum_{i=1}^q b_i \varepsilon_{t-i+1}. \tag{2.4}$$

Now, eq. (2.3) can be simplified to

$$Y_{t+1} = E_t[Y_{t+1}] + \varepsilon_{t+1}. \tag{2.5}$$

As is known, Y_{t+1} in eq. (2.3) can also be formulated with an MA(∞) process:

$$Y_{t+1} = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i+1}. \tag{2.6}$$

Because $\{\varepsilon_t\}$ is the Gaussian white noise, both Y_{t+1} and $E_t[Y_{t+1}]$ follow the Gaussian distribution with zero mean.

For simplicity, we assume that the risk-free return and the trading cost rate are both zero. Then, it is trivial that the strategy at time t that maximizes the expected value of the one-period excess return can be formulated as follows.

Definition 1: A buy (sell) signal is generated at time t if

$$E_t[Y_{t+1}] > 0 (< 0). \tag{2.7}$$

Accordingly, the expected value of the one-period excess return is

$$\begin{aligned} \mu_g &= \text{Prob}(E_t(Y_{t+1}) > 0) \cdot E\{Y_{t+1}|E_t[Y_{t+1}] > 0\} \\ &\quad - \text{Prob}(E_t(Y_{t+1}) < 0) \cdot E\{Y_{t+1}|E_t[Y_{t+1}] < 0\} \\ &= E\{Y_{t+1}|E_t[Y_{t+1}] > 0\}. \end{aligned} \tag{2.8}$$

If the standard deviation of $E_t[Y_{t+1}]$ is known and denoted by σ_E , it's easy to verify (proof in [Appendix](#)) that

$$\mu_g = \frac{\sigma_E \sqrt{2}}{\sqrt{\pi}}. \tag{2.9}$$

Because ε_{t+1} and $E_t[Y_{t+1}]$ are uncorrelated, we can compute σ_E according to the variance of Y_{t+1} , which is denoted by γ_0 . So,

$$\sigma_E = \sqrt{\gamma_0 - \sigma^2}. \tag{2.10}$$

According to the strategy given by Definition 1, we can verify that the standard deviation of the one-period excess return is (proof in [Appendix](#))

$$\sigma_g = \left[\sigma^2 + \left(1 - \frac{2}{\pi} \right) \sigma_E^2 \right]^{\frac{1}{2}}. \tag{2.11}$$

Therefore, the Sharpe ratio is

$$SR = \frac{\mu_g}{\sigma_g} = \frac{\sigma_E \sqrt{2}}{\sqrt{\pi \sigma^2 + (\pi - 2) \sigma_E^2}} = \left[\frac{\pi}{2} \left(1 + \frac{\sigma^2}{\sigma_E^2} \right) - 1 \right]^{-\frac{1}{2}}, \tag{2.12}$$

or

$$SR = \sqrt{\frac{2(\gamma_0 - \sigma^2)}{2\sigma^2 + (\pi - 2)\gamma_0}}. \tag{2.13}$$

According to formula (2.12) or (2.13), we conclude that the Sharpe ratio lies in a specific scope:

$$SR \in \left[0, \sqrt{\frac{2}{\pi - 2}} \right).$$

In fact, γ_0 is a function of σ^2 . For a few simple processes such as the AR (1), AR (2), MA(q), and ARMA (1,1), we are able to derive the formula of γ_0 and thereby formulate the SR with the parameters of the original model.

For a general ARMA(p, q) process, it is difficult to obtain the formula of γ_0 , but we can adopt the following ways to calculate its value.

First, according to Box et al. (2008), the series $\{\psi_j\}$ in eq. (2.6) can be computed recursively from the following relations:

$$\psi_j = \begin{cases} 1, & j = 0 \\ b_j + \sum_{k=1}^p a_k \psi_{j-k}, & j = 1, 2, \dots \end{cases} \tag{2.14}$$

with $\psi_j = 0$ for $j < 0$, and $b_j = 0$ for $j > q$.

Because the values of $\{\psi_j\}$ decay geometrically toward zero, we need to compute only a few terms.

Secondly, it is easy to calculate the value of γ_0 according to eq. (2.6).

$$\gamma_0 = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2. \tag{2.15}$$

From eq. (2.13), (2.14), and (2.15), we find that the SR just depends on the autoregressive coefficients, a_1, \dots, a_p , and the moving average coefficients, b_1, \dots, b_q .

For empirical purposes, we can estimate the SR by means of formula (2.13), in which σ can be replaced by the standard error of the regression, and γ_0 can be estimated as the sample variance of $\{Y_j\}$. In fact, if the sample size is large enough, formula (2.13) can be reduced to

$$SR \approx \sqrt{\frac{2R^2}{\pi - 2R^2}}. \tag{2.16}$$

Formula (2.16) shows the SR is a monotonic increasing function of the *R-squared*. Therefore, the *R-squared* in time series models is of great significance with respect to market efficiency. The *R-squared* in time series models can be used as a measure of market efficiency.

The relationship between the SR and *R-squared* can also be shown as the curve in Fig. 1.

From Fig. 1, we see that the relationship between the SR and *R-squared* is approximately linear positive dependence when the value of *R-squared* is larger than 0.1.

Cochrane (1999) derived a similar result from the following model:

$$Y_{t+1} = E(Y) + b[X_t - E(X)] + \varepsilon_{t+1}.$$

The author demonstrated that the SR associated with a market timing strategy based on this predictor equation is given by

$$SR = \sqrt{\frac{SR_0^2 + R^2}{1 - R^2}}, \tag{2.16'}$$

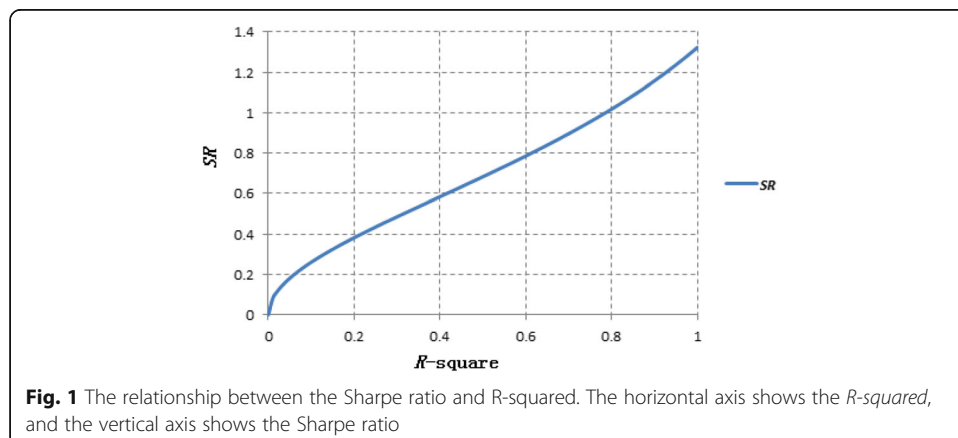


Fig. 1 The relationship between the Sharpe ratio and R-squared. The horizontal axis shows the *R-squared*, and the vertical axis shows the Sharpe ratio

where SR_0 is the SR based on a buy-and-hold strategy, and $R\text{-squared}$ is taken from the predictor equation.

It is obvious that the SR in formula (2.16') is also a monotonic increasing function of $R\text{-squared}$. Nevertheless, if $\{Y_t\}$ satisfies the ARMA(p, q) process, it is improper to use formula (2.16'), which gives an overestimate of the SR.

Several simple examples

The AR (1) process

To start with, we consider the AR (1) process:

$$Y_t = a_1 Y_{t-1} + \varepsilon_t, \tag{2.17}$$

where $0 < |a_1| < 1$.

It is easy to verify

$$\gamma_0 = \frac{\sigma^2}{1 - a_1^2}, \tag{2.18}$$

$$\sigma_E = \frac{|a_1|\sigma}{\sqrt{1 - a_1^2}}. \tag{2.19}$$

First, substituting eq. (2.19) into eq. (2.9) gives

$$\mu_g = \frac{|a_1|\sigma\sqrt{2}}{\sqrt{\pi(1 - a_1^2)}}. \tag{2.20}$$

Then, substituting eq. (2.19) into eq. (2.11) gives

$$\sigma_g = \sigma \left[\frac{\pi - 2a_1^2}{\pi(1 - a_1^2)} \right]^{\frac{1}{2}}. \tag{2.21}$$

Finally, substituting eq. (2.19) into eq. (2.12) gives

$$SR = \frac{\sqrt{2}|a_1|}{\sqrt{\pi - 2a_1^2}}. \tag{2.22}$$

Now it can be seen that the SR just depends on a_1 and is not affected by the variance of ε_t . The smaller the absolute value of a_1 is, the lower the SR is. The SR is equal to zero when a_1 is equal to zero.

The AR (2) process

In order to derive the formula of γ_0 for the AR (2) or ARMA (1,1) processes, we make use of the autocovariance-generating function. According to Hamilton (1994), the autocovariance-generating function for a stationary ARMA(p, q) process can be written as:

$$g_y(z) = \frac{\sigma^2(1 + b_1z + b_2z^2 + \dots)(1 + b_1z^{-1} + b_2z^{-2} + \dots)}{(1 - a_1z - a_2z^2 - \dots)(1 - a_1z^{-1} - a_2z^{-2} - \dots)}. \tag{2.23}$$

If we convert expression (2.23) to a polynomial of z , then the coefficient of z^0 is exactly γ_0 .

Now, we consider the AR (2) process:

$$Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + \varepsilon_t, \tag{2.24}$$

where $0 < |a_2| < 1$, $a_2 \pm a_1 < 1$.

By the way proposed above, we obtain

$$\gamma_0 = \frac{(1 - a_2)\sigma^2}{(1 + a_2)[(1 - a_2)^2 - a_1^2]}. \tag{2.25}$$

Substituting formula (2.25) into formula (2.10) gives

$$\sigma_E = \sigma^* \left\{ \frac{1 - a_2}{(1 + a_2)[(1 - a_2)^2 - a_1^2]} - 1 \right\}^{\frac{1}{2}}. \tag{2.26}$$

Then, substituting formula (2.26) into formula (2.9) gives

$$\mu_g = \frac{\sigma\sqrt{2}}{\sqrt{\pi}} * \sqrt{\frac{1 - a_2}{(1 + a_2)[(1 - a_2)^2 - a_1^2]} - 1}. \tag{2.27}$$

Now, substituting formula (2.26) into formula (2.11) gives

$$\sigma_g = \sigma \left[\frac{\pi(1 - a_2) - 2(a_1^2 + a_2^2) + 2a_2(a_2^2 - a_1^2)}{\pi(1 + a_2)[(1 - a_2)^2 - a_1^2]} \right]^{\frac{1}{2}}. \tag{2.28}$$

In the end, dividing eq. (2.27) by eq. (2.28) gives

$$SR = \frac{\mu_g}{\sigma_g} = \left[\frac{2[(a_1^2 + a_2^2) + a_2(a_1^2 - a_2^2)]}{\pi(1 - a_2) - 2[(a_1^2 + a_2^2) + a_2(a_1^2 - a_2^2)]} \right]^{\frac{1}{2}}. \tag{2.29}$$

Now it can be seen that the SR just depends on a_1 and a_2 , and it is not affected by the variance of ε_t . From formula (2.29), we can see that the AR (2) process is reduced to the AR (1) process when a_2 is equal to zero. Furthermore, both μ_g and SR will be zero if both a_1 and a_2 are equal to zero.

The ARMA (1,1) process

Now, let us consider the ARMA (1,1) process. In a similar way, we obtain

$$\gamma_0 = \frac{(1 + 2a_1b_1 + b_1^2)\sigma^2}{1 - a_1^2}, \tag{2.30}$$

$$\sigma_E = \frac{|a_1 + b_1|\sigma}{\sqrt{1 - a_1^2}}, \tag{2.31}$$

$$\mu_g = \frac{|a_1 + b_1|\sigma\sqrt{2}}{\sqrt{\pi*(1 - a_1^2)}}, \tag{2.32}$$

$$\sigma_g = \sigma \left[1 + \left(1 - \frac{2}{\pi}\right) \frac{(a_1 + b_1)^2}{1 - a_1^2} \right]^{\frac{1}{2}}, \tag{2.33}$$

$$SR = \frac{\sqrt{2}|a_1 + b_1|}{\sqrt{\pi(1 + 2a_1b_1 + b_1^2) - 2(a_1 + b_1)^2}}. \tag{2.34}$$

Now, it can be seen that the SR just depends on a_1 and b_1 , and it is not affected by the variance of ε_t .

It is obvious that the SR will be zero only if

$$a_1 + b_1 = 0.$$

In fact, at this point, the ARMA (1,1) process is equivalent to white noise, so it is impossible to realize any positive return.

In addition, it is easy to verify that formula (2.34) is equivalent to formula (2.22) only if

$$b_1 = -2a_1.$$

Then, the SR is the same for the ARMA (1,1) and AR (1) processes.

The SR for the ARMA (1,1) process is approximately equal to that for the AR (1) process if both a_1 and b_1 are small enough (e.g., less than 0.1) and the coefficient a'_1 in the AR (1) meets the following condition:

$$|a'_1| = |a_1 + b_1|.$$

The MA (2) process

Finally, we consider the MA (2) process and compare the SR with the Q-statistic of Box and Pierce (1970).

For the MA (2) process, it is easy to verify

$$\gamma_0 = (1 + b_1^2 + b_2^2)\sigma^2,$$

$$\gamma_1 = (b_1 + b_1b_2)\sigma^2,$$

$$\gamma_2 = b_2\sigma^2,$$

$$\gamma_j = 0, \text{ for } j > 2.$$

Substituting the expression of γ_0 into formula (2.13) gives

$$SR = \left[\frac{2(b_1^2 + b_2^2)}{\pi + (\pi - 2)(b_1^2 + b_2^2)} \right]^{\frac{1}{2}}. \tag{2.35}$$

Now, it can be seen that the SR just depends on b_1 and b_2 , and it is not affected by the variance of ε_t .

The autocorrelation parameters of $\{Y_t\}$ are as follows:

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{b_1 + b_1b_2}{1 + b_1^2 + b_2^2},$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{b_2}{1 + b_1^2 + b_2^2},$$

$$\rho_j = 0, \text{ for } j > 2.$$

According to Box and Pierce (1970), the Q-statistic at lag k is a test statistic for the null hypothesis that there is no autocorrelation up to order k . It is computed as:

$$Q = T(T + 2) \sum_{j=1}^k \frac{r_j^2}{T - j}, \tag{2.36}$$

where r_j is the j -th sample's autocorrelation and T is the number of observations.

For large T , r_j approximates ρ_j , so expression (2.36) can be reduced to

$$Q \approx T \sum_{j=1}^k \rho_j^2. \tag{2.37}$$

Formula (2.37) shows that the Q -statistic is proportional to the sample size, T . Therefore, it is unreasonable to use the Q -statistic as a measure of market efficiency if the sample size is ignored.

Applying formula (2.37) to the MA (2) process gives

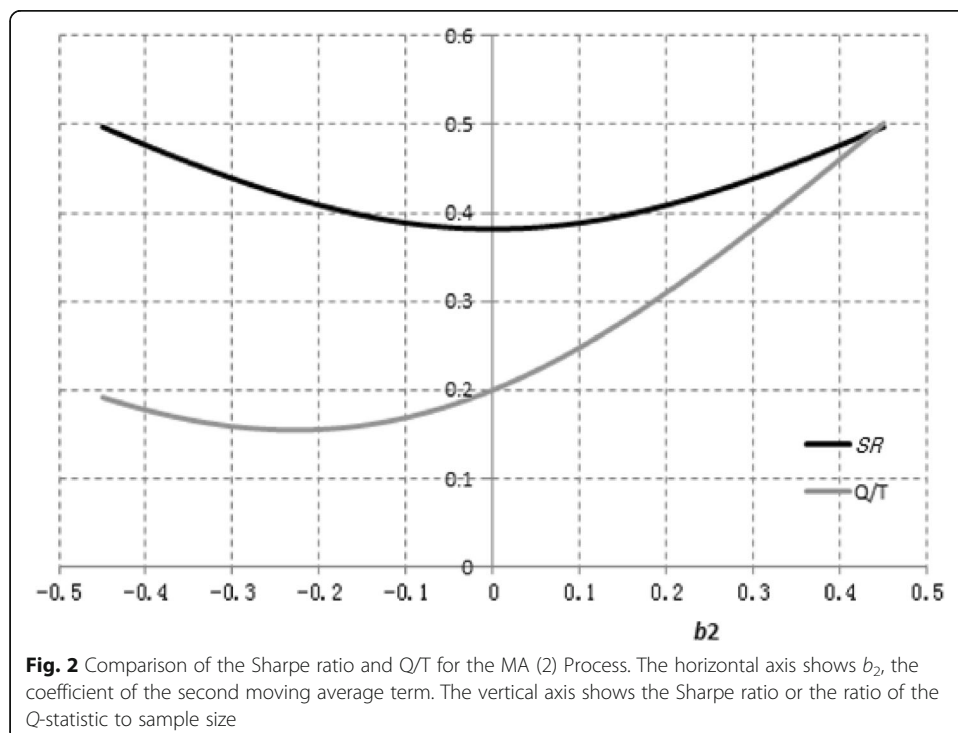
$$Q = \frac{T[b_1^2(1 + b_2)^2 + b_2^2]}{1 + b_1^2 + b_2^2}, \tag{2.38}$$

or

$$Q/T = \frac{b_1^2(1 + b_2)^2 + b_2^2}{1 + b_1^2 + b_2^2}. \tag{2.39}$$

Now, we consider whether the ratio of Q to T can be used to measure market efficiency. For example, given $b_1 = 0.5$, let us change the value of b_2 from -0.45 to 0.45 and observe how the values of SR and Q/T change with it (Fig. 2).

Figure 2 shows that the Sharpe ratio achieves the minimum when b_2 is equal to zero and the SR curve is symmetrical. In contrast, Q/T achieves the minimum when b_2 is



equal to -0.23 , and the curve is asymmetrical. It is obvious that ranking by Q/T is different from ranking by SR . Therefore Q/T is not a proper measure of market efficiency.

The ARMA-GARCH model

Now, we consider the general ARMA-GARCH model.

If the $\{\varepsilon_t\}$ in expression (2.2) follows a GARCH(r,s) process:

$$\varepsilon_t = \sqrt{h_t} \cdot v_t, \tag{2.40}$$

$$h_t = \phi + \sum_{j=1}^r \alpha_j h_{t-j} + \sum_{k=1}^s \beta_k \varepsilon_{t-k}^2, \tag{2.41}$$

where $\{v_t\}$ is the Gaussian white noise, $WN(0, \sigma^2)$, and $\phi > 0$, $\alpha_j > 0$, $j = 1, \dots, r$, $\beta_k > 0$, $k = 1, \dots, s$, $\sum_{j=1}^r \alpha_j + \sum_{k=1}^s \beta_k < 1$.

According to Lu (1999), we can conclude that $\{\varepsilon_t\}$ is a covariance-stationary process:

$$E(\varepsilon_t) = 0,$$

$$Var(\varepsilon_t) = \frac{\phi}{1 - \sum_{j=1}^r \alpha_j - \sum_{k=1}^s \beta_k},$$

$$Cov(\varepsilon_j, \varepsilon_k) = 0, j \neq k.$$

Because v_{t+1} and $E_t[Y_{t+1}]$ are uncorrelated, ε_{t+1} and $E_t[Y_{t+1}]$ are uncorrelated. So, we can compute the standard deviation of $E_t[Y_{t+1}]$ according to formula (2.10) if we

use $\sqrt{\frac{\phi}{1 - \sum_{j=1}^r \alpha_j - \sum_{k=1}^s \beta_k}}$ to replace σ . In fact, all results are the same as the ARMA

model if σ is replaced by $\sqrt{\frac{\phi}{1 - \sum_{j=1}^r \alpha_j - \sum_{k=1}^s \beta_k}}$.

Furthermore, the SR is not affected by the variance of ε_t , so it is not affected by the GARCH process. The SR just depends on the coefficients of the ARMA(p, q) process, namely $a_1, \dots, a_p, b_1, \dots, b_q$.

Stochastic simulation

We explore the proposed method by means of stochastic simulation. We study the AR (1), ARMA (1,1), and AR (1)-GARCH (1,1) processes, which serve as three examples.

Stochastic simulation of the AR (1) process

Simulation about risk

Given a set of parameters, according to eq. (2.17), we simulated the AR (1) process to produce a series consisting of 10,000 observations. Then, we simulated trading in accordance with the strategy described in Definition 1 and generated a series of returns whose standard deviation (denoted by σ_{simul}) can be computed and compared with the value from formula (2.21).

The results for nine sets of parameters are displayed in Table 1.

Table 1 Results on risk for the AR (1) process. The table reports the actual standard deviations of the simulated returns and their theoretical values for nine AR (1) processes with different parameters. The actual standard deviation of the simulated returns is denoted by σ_{simul} , and its theoretical value is denoted by σ_g

Group	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
a_1	0.1	0.4	0.8	0.1	0.4	0.8	0.1	0.4	0.8
σ	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.03
σ_g	0.0100	0.0103	0.0128	0.0200	0.0207	0.0257	0.0301	0.0310	0.0385
σ_{simul}	0.0101	0.0102	0.0128	0.0203	0.0206	0.0255	0.0301	0.0314	0.0381

From Table 1, we can see the values of σ_{simul} are very close to the values of σ_g . The results confirm formula (2.21).

Simulation about return

Now, we specify the parameters as follows:

$$a_1 = 0.1, \sigma = 0.01.$$

Substituting them into formula (2.20) gives

$$\mu_g = 0.0802\%.$$

Similarly, we simulated to produce a series consisting of 1000 observations according to eq. (2.17). Then, we simulated trading according to the strategy described in Definition 1 and generated a series of returns whose average value (denoted by R) can be computed and compared with the value from formula (2.20).

We carried out the simulation 100 times and found the mean of R to be 0.0793%, which approximates 0.0802%.

Figure 3 shows the distribution of R along with its summary statistics.

It can be seen that R approximately follows the Gaussian distribution.

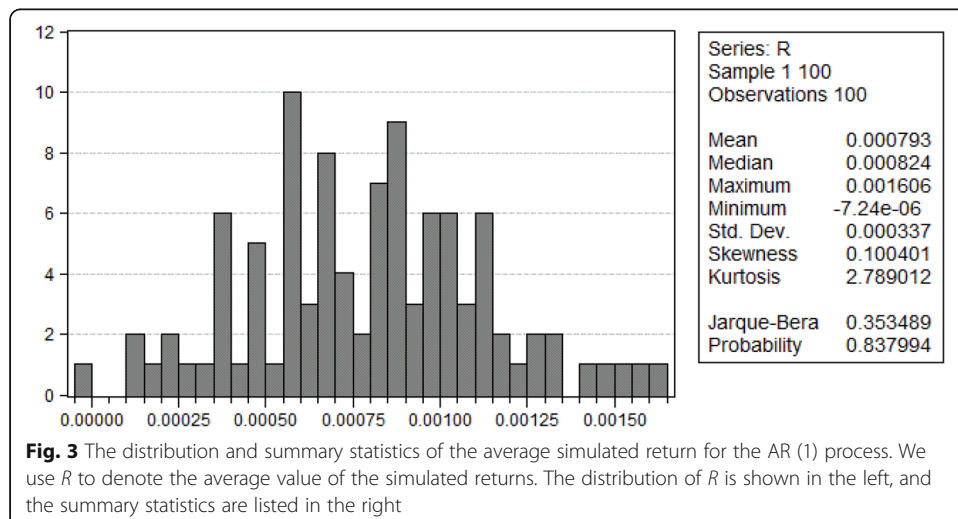


Fig. 3 The distribution and summary statistics of the average simulated return for the AR (1) process. We use R to denote the average value of the simulated returns. The distribution of R is shown in the left, and the summary statistics are listed in the right

Table 2 Results on risk for the ARMA (1,1) process. The table reports the actual standard deviations of the simulated returns and their theoretical values for nine ARMA (1,1) processes with different parameters. The actual standard deviation of the simulated returns is denoted by σ_{simul} , and its theoretical value is denoted by σ_g

Group	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
a_1	0.5	0.5	-0.5	-0.5	0.5	0.5	-0.5	-0.5
b_1	0.3	-0.3	0.3	-0.3	0.3	-0.3	0.3	-0.3
σ	0.01	0.01	0.01	0.01	0.03	0.03	0.03	0.03
σ_g	0.0114	0.0101	0.0101	0.0114	0.0343	0.0303	0.0303	0.0343
σ_{simul}	0.0114	0.0103	0.0101	0.0115	0.0346	0.0301	0.0301	0.0344

Stochastic simulation of the ARMA (1,1) process

Simulation about risk

First, given a set of parameters, we simulated to produce a series consisting of 10,000 observations according to eq. (2.3). Then, we simulated trading according to the strategy described in Definition 1 and generated a series of returns whose standard deviation (denoted by σ_{simul}) can be computed and compared with the value from formula (2.33).

The results for eight sets of parameters are displayed in Table 2.

Just like Tables 1 and 2 shows that the value of σ_{simul} is very close to the value of σ_g . The results confirm formula (2.33).

Simulation about return

Now, we specify the parameters as follows:

$$a_1 = 0.075, b_1 = -0.03, \sigma = 0.015.$$

According to formula (2.32), we obtain

$$\mu_g = 0.0540\%.$$

We simulated to produce a series consisting of 1000 observations according to eq. (2.3). Then, we simulated trading according to the strategy described in Definition 1

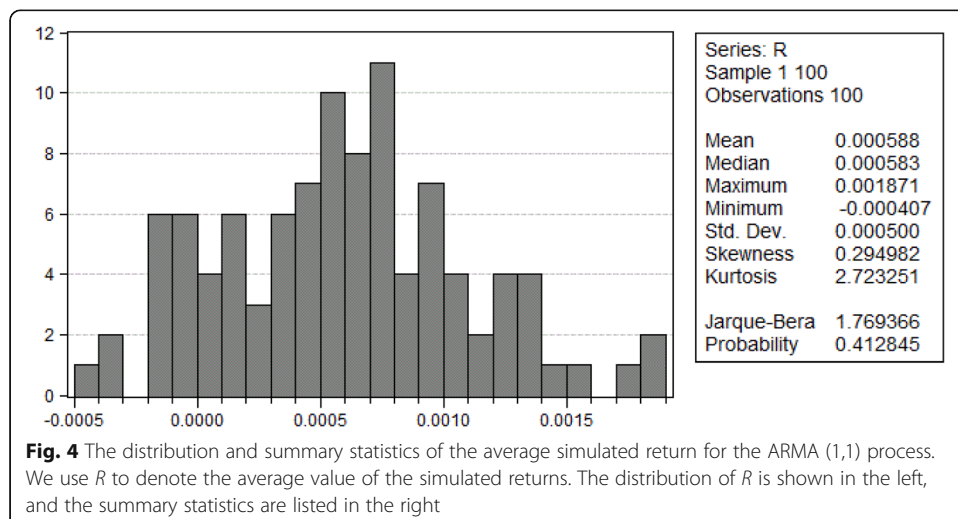


Fig. 4 The distribution and summary statistics of the average simulated return for the ARMA (1,1) process. We use R to denote the average value of the simulated returns. The distribution of R is shown in the left, and the summary statistics are listed in the right

and generate a series of returns whose average value (denoted by R) can be computed and compared with the value from formula (2.32).

We carried out the simulation 100 times and found the mean of R to be 0.0588%, which approximates 0.0540%. In fact, their difference can be accounted for by the low value of the SR, which was only 0.036.

Figure 4 shows the distribution of R along with its summary statistics.

It can be seen that R approximately follows the Gaussian distribution.

Stochastic simulation of the AR (1)-GARCH (1,1) process

Simulation about risk

Given a set of parameters, according to eqs. (2.17), (2.40), and (2.41), we simulated the AR (1)-GARCH (1,1) process to produce a series consisting of 10,000 observations. Then, we simulated trading in accordance with the strategy described in Definition 1 and generated a series of returns whose standard deviation (denoted by σ_{simul}) can be computed and compared with the value from the following formula:

$$\sigma_g = \sqrt{\frac{\phi(\pi - 2a_1^2)}{\pi(1 - a_1^2)(1 - \alpha_1 - \beta_1)}} \tag{3.1}$$

The results for eight sets of parameters are displayed in Table 3.

From Table 3, we can see the values of σ_{simul} are very close to the values of σ_g . The results confirm formula (3.1).

Simulation about return

Now, the expected value of the one-period excess return is

$$\mu_g = \sqrt{\frac{2\phi a_1^2}{\pi(1 - a_1^2)(1 - \alpha_1 - \beta_1)}} \tag{3.2}$$

We specify the parameters as follows:

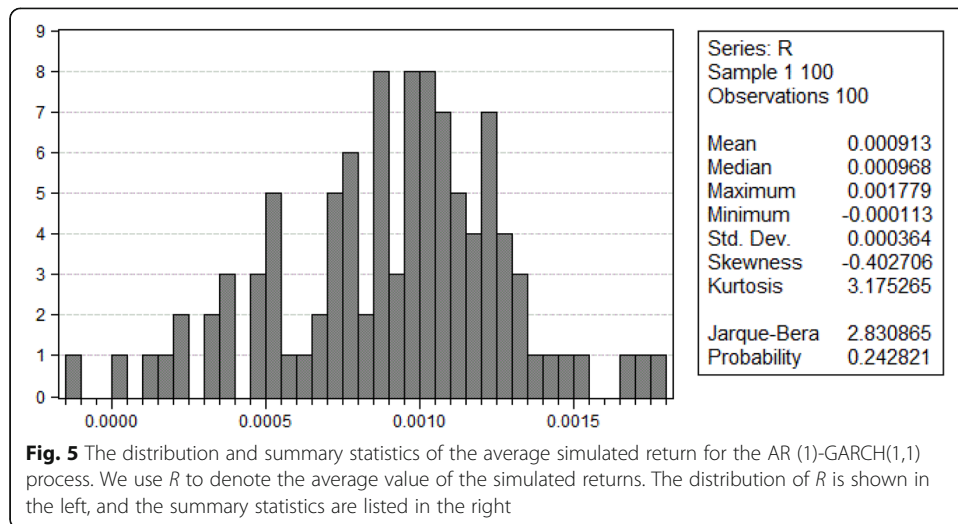
$$a_1 = 0.1, \phi = 0.0001, \alpha_1 = 0.1, \beta_1 = 0.1.$$

Substituting them into formula (3.2) gives

$$\mu_g = 0.0897\%.$$

Table 3 Results on risk for the AR (1)-GARCH (1,1) process. The table reports the actual standard deviations of the simulated returns and their theoretical values for eight AR (1)-GARCH (1,1) processes with different parameters. The actual standard deviation of the simulated returns is denoted by σ_{simul} and its theoretical value is denoted by σ_g

Group	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
a_1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
ϕ	0.0001	0.0001	0.0001	0.0001	0.0004	0.0004	0.0004	0.0004
α_1	0.1	0.1	0.2	0.2	0.1	0.1	0.2	0.2
β_1	0.1	0.2	0.1	0.2	0.1	0.2	0.1	0.2
σ_g	0.0112	0.0120	0.0120	0.0129	0.0224	0.0239	0.0239	0.0259
σ_{simul}	0.0114	0.0120	0.0121	0.0127	0.0224	0.0236	0.0238	0.0262



Similarly, we simulated to produce a series consisting of 1000 observations according to eqs. (2.17), (2.40), and (2.41). Then, we simulated trading according to the strategy described in Definition 1 and generated a series of returns whose average value (denoted by R) can be computed and compared with the value from formula (3.2).

We carried out the simulation 100 times and found the mean of R to be 0.0913%, which approximates 0.0897%.

Figure 5 shows the distribution of R along with its summary statistics.

Comparison with the method of conditional entropy

In physics and information theory, Shannon entropy is used to measure the degree of disorder:

$$S(i) = - \sum_{i=1}^n p(i) \log_2 p(i), \tag{3.3}$$

where $p(i)$ is the probability that the event i appears out of n events.

Based on this concept, Zhang proposed to measure market efficiency with conditional entropy (Wen et al. 2019):

$$H(j) = - \sum_{i=1}^n \left[p(i) \sum_{j=1}^m p(j|i) \log_2 p(j|i) \right], \tag{3.4}$$

where $p(j|i)$ is the conditional probability that the event j would follow, with the event i given. Here, “event” stands for the direction of price movement. For example, $p(+|-,-,+)$ means the conditional probability that price will rise if the price history is “fall,” “fall,” and “rise” in turn for the recent three periods.

Conditional entropy uses the information from a given previous event (i.e., the past price) to help predict future price movement. The higher the conditional entropy is, the better the market efficiency is. For the binary case when i and j are both binary strings, its maximal value is one. Zhang (1999) did not present empirical research on the concept.

Because the MA(q) process is equivalent to an AR(∞) process, it is impossible to compute conditional entropy for the MA(q) or ARMA(p,q) process. Here, we just consider the AR (1) process.

First, given a set of parameters, we computed the expected value of the one-period excess return and the Sharpe ratio according to the corresponding formula.

Then, we simulated to produce a series consisting of 1000 observations according to the corresponding model.

For the AR (1) process, different absolute values of a_1 mean different degrees of market efficiency. The smaller the value of $|a_1|$ is, the better the market efficiency is. If a_1 is equal to zero, then the market is perfectly efficient, meaning the prices conform to a random walk.

Given $\sigma = 0.15$, we simulated five times for each value of a_1 and computed conditional entropy according to formula (3.4).

Table 4 displays the results for nine sets of parameters.

The first column in Table 4 is the result for random walk. As expected, its Sharpe ratio is equal to zero, and its conditional entropy is very close to one.

The SR in the second column is equal to 0.0359, but the conditional entropy is nearly the same as the one in the first column. It shows that the conditional entropy is not a proper measure of market efficiency when the signal-noise ratio of the market is at a low level.

Figure 6 shows the scatter plot corresponding to Table 4.

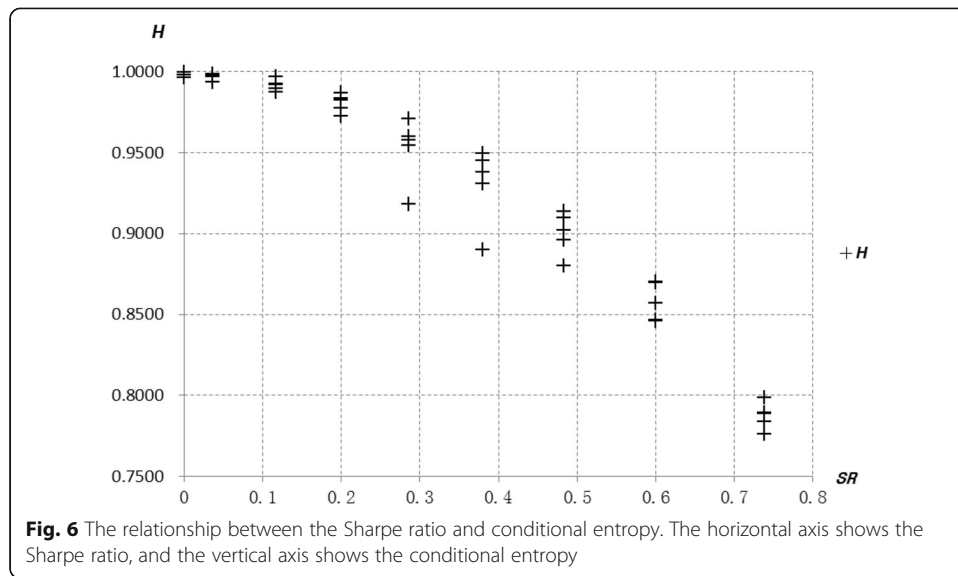
From Fig. 6 and Table 4, we can observe that the conditional entropy declines when the SR rises to a value larger than 0.1 though its range of fluctuation expands correspondingly. In summary, both the Sharpe ratio and conditional entropy can be used to measure market efficiency. However, the former is more accurate than the latter, especially when the degree of market efficiency is very high.

Empirical research

As examples, we measured the market efficiency of the S&P 500 Index and the SSE Composite Index. The two samples span from January 1, 2000 to December 31, 2014. From Yahoo finance, we obtained a total of 782 weekly closing prices of the S&P 500 Index, and a total of 754 weekly closing prices of the SSE Composite Index. Returns were the log differences of the closing prices.

Table 4 Results on conditional entropy. The table reports the Sharpe ratio and conditional entropy for nine AR (1) processes with different parameters. The Sharpe ratio is denoted by SR , and the conditional entropy is denoted by H

Group	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
a_1	0	0.045	0.145	0.245	0.345	0.445	0.545	0.645	0.745
μ_g (%)	0	0.0539	0.1754	0.3024	0.4399	0.5947	0.7780	1.0102	1.3367
SR	0	0.0359	0.1165	0.1993	0.2863	0.3798	0.4829	0.6002	0.7392
H	0.9960	0.9985	0.9968	0.9835	0.9706	0.9453	0.9018	0.8461	0.7835
	0.9995	0.9966	0.9897	0.9869	0.9597	0.8900	0.8958	0.8571	0.7892
	0.9994	0.9978	0.9923	0.9823	0.9546	0.9305	0.8800	0.8704	0.7984
	0.9978	0.9933	0.9919	0.9724	0.9180	0.9493	0.9101	0.8694	0.7760
	0.9979	0.9979	0.9875	0.9773	0.9578	0.9379	0.9138	0.8466	0.7887



We chose the AR(p)-GARCH (1,1) process to apply to the data. Table 5 reports the parameter estimates for the time series model.

Table 5 shows that the ARCH and GARCH effects are always significant at the 1% level in the two models. We can compute the *R-squared* according to the residual error and the data of returns. Then, we can obtain the Sharpe ratio according to formula (2.16). For the S&P 500 Index, the Sharpe ratio is 0.055; for the SSE Composite Index, the Sharpe ratio is 0.086. So, we conclude that the market efficiency of the S&P 500 Index is higher than that of the SSE Composite Index.

Based on the AR-GARCH model, we propose a trading strategy on the S&P 500 Index, which we call the AR-GARCH strategy. We compare the return of the strategy

Table 5 Parameter estimates for the AR(p)-GARCH (1,1) process. The table reports the parameter estimates for the AR(p)-GARCH (1,1) processes. The notations come from expressions (2.2), (2.40), and (2.41). The numbers in parentheses are the *t*-ratios. The *t*-ratios marked with asterisks (double asterisks or three asterisks) indicate that the corresponding coefficients are statistically different from zero at the 10% (5% or 1%) level of significance

	S&P 500 Index	SSE Composite Index
a_1	-0.0782383 (- 1.87)*	0.0511606 (1.38)
a_2	-0.0082874 (- 0.20)	0.0837641 (2.17)**
a_3	-0.0786243 (- 2.16)**	-
Intercept	0.0025749 (3.72)***	0.0006806 (0.65)
α_1	0.7341439 (23.76)***	0.8754173 (28.85)***
β_1	0.2185006 (8.59)***	0.0910423 (4.17)***
φ	0.0000358 (3.92)***	0.0000367 (2.63)***
Log Likelihood	1867.151	1542.614

to the buy-and-hold strategy over the sample period. The merit of the strategy is captured in Definition 1, as also mentioned in "The Sharpe Ratio based on the ARMA-GARCH Forecast" section. It is namely that if the prediction of the AR-GARCH model is positive, we should buy the index at the closing price. Conversely, we should sell the index at the closing price if the prediction is negative. We carried out the strategy on a rolling basis without considering trading friction.

Firstly, for each trading week, we used the previous 50 weekly yields of the S&P 500 Index as a sample window to fit the optimal AR-GARCH model. Secondly, we obtained the predicted yield of the next trading week through the estimated model. Thirdly, if the prediction of the next week was positive, then we went long the index at the latest closing price; if the prediction was negative, we went short the index at the same price. Lastly, for the next trading week, in case we got the same direction of the prediction as the previous week, then we held our position. If not, then we closed our position and made a reverse position at the close price. Based on the strategy, we got the following two curves in the Fig. 7.

The AR-GARCH strategy did not always outperform the buy-and-hold strategy. The excess return of the AR-GARCH strategy most occurred during the financial crisis period of 2008 and in the period of the tech bubble burst of 2001. While in other period, the AR-GARCH strategy could not gain excess return obviously better than the buy-and-hold strategy.. This may mean that during these crisis periods, the market was not as effective as the ordinary period. This may also mean that one can obtain excess returns through the time series strategy only when the market is not efficient. During these periods, the model predicts better because of the available past information.

To show the results' stability, we fit the daily and monthly data to the AR-GARCH process. Table 6 reports the parameter estimates for the four series.

Table 6 shows that the ARCH and GARCH effects are always significant at the 1% level in all four models. Formula (2.16) was used to compute the Sharp ratios for the S&P 500 Index, which were 0.0745 and 0.0515 for the monthly and daily data,

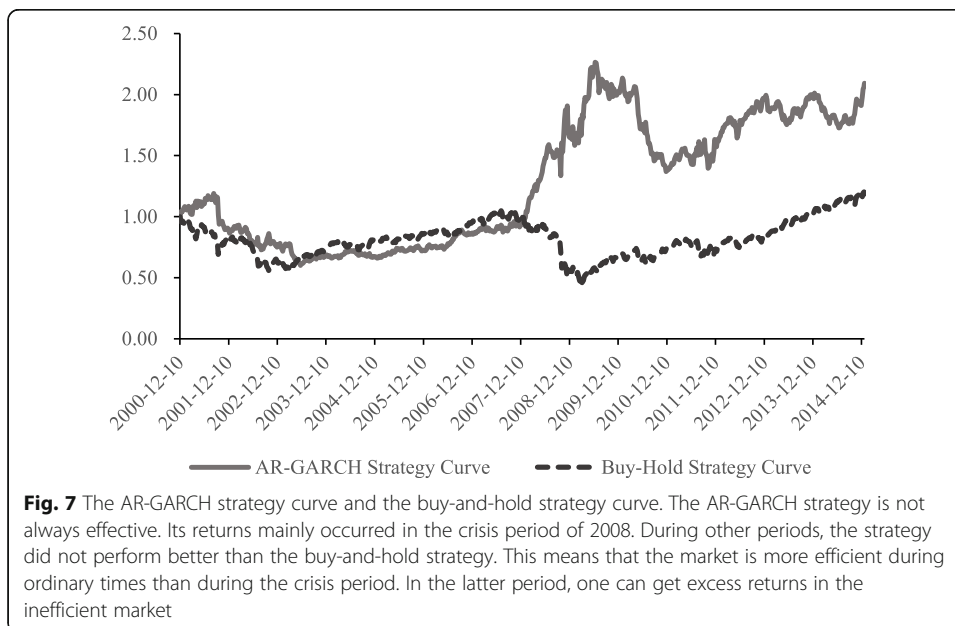


Table 6 Parameter estimates for the AR-GARCH process. The table reports the parameter estimates for the AR(p)-GARCH(1,1) processes for daily and AR(p)-GARCH (1,0) processes for monthly data. The notations come from expressions (2.2), (2.40) and (2.41). The numbers in parentheses are the t -ratios. The t -ratios marked with asterisks (double asterisks or three asterisks) indicate that the corresponding coefficients are statistically different from zero at 10% (5% or 1%) level of significance

	S&P 500 Index -monthly	SSE Composite Index-monthly	S&P 500 Index -daily	SSE Composite Index-daily
a_1	0.151447 (2.54)**	0.080948 (1.11)	-0.058303 (- 3.13)***	0.018532 (1.10)
a_2		0.193746 (2.81)***	-0.032937 (- 1.94)*	- 0.011153 (- 0.67)
a_3		-		0.022778 (1.36)
Intercept	0.002195 (0.60)	0.003046 (0.54)	0.000583 (4.04)***	0.000335 (1.73)
α_1	0.960079 (18.08)***	0.973209 (78.52)***	0.895647 (111.8)***	0.913373 (160.7)***
β_1			0.092683 (12.64)***	0.075154 (14.11)***
φ	0.00007 (0.72)	0.000194 (2.75)***	0.00000164 (7.26)***	0.00000318 (6.99)***
Log Likelihood	306.2535	210.8371	11,967.77	10,311.22

respectively. The Sharpe ratios of the SSE Composite Index were 0.122 and 0.056 for the monthly and daily data, respectively. So, we can also conclude that the market efficiency of the S&P 500 Index is higher than that of the SSE Composite Index.

Conclusion

From the empirical research perspective, it is more meaningful to measure relative efficiency of markets as opposed to testing the EMH. A proper measure of market efficiency can reflect the potential return and the relevant risk. However, the traditional test statistics reflect neither of these two aspects. In this paper, we proposed a new method for comparing the weak-form efficiency of a market. We deduced the formulae of the expected return and risk from the ARMA-GARCH model and thereby acquired the formula of the Sharpe ratio. The study found that the Sharpe ratio is not affected by the variance of ε_t and thus not affected by the GARCH process. The Sharpe ratio just depends on the coefficients of the ARMA (p, q) process, namely $a_1, \dots, a_p, b_1, \dots, b_q$.

In empirical research, the Sharpe ratio can be formulated with a monotonic increasing function of R -squared if the sample size is large enough. This shows that the R -squared in time series models is of great significance for the measure of market efficiency. One can utilize the Sharpe ratio to measure weak-form efficiency and compare different markets. The results of our stochastic simulation demonstrated the validity of the proposed method. It can be applied in both high and low efficiency markets.

Appendix: Proofs

Lemma 1: Provided that $\{Y_t\}$ satisfies the ARMA(p, q) process, and both the risk-free return and the trading cost rate are zero, trading in accordance with the strategy described in Definition 1 will produce a series of excess returns.

If the standard deviation of $E_t[Y_{t+1}]$ is known and denoted by σ_E , then the expected value of one-period excess return is

$$\mu_g = \frac{\sigma_E \sqrt{2}}{\sqrt{\pi}}.$$

Meanwhile, the standard deviation of one-period excess return is

$$\sigma_g = \left[\sigma^2 + \left(1 - \frac{2}{\pi} \right) \sigma_E^2 \right]^{\frac{1}{2}}.$$

Proof of Lemma 1: According to the original assumption, we know

$$E_t[Y_{t+1}] \sim N(0, \sigma_E).$$

According to eq. (2.8), we have

$$\begin{aligned} \mu_g &= E\{Y_{t+1} | E_t(Y_{t+1}) > 0\} \\ &= E\{E_t(Y_{t+1}) | E_t(Y_{t+1}) > 0\} \\ &= \int_0^{+\infty} \frac{2x}{\sqrt{2\pi}\sigma_E} e^{-\frac{x^2}{2\sigma_E^2}} dx \\ &= \frac{\sigma_E \sqrt{2}}{\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{x^2}{2\sigma_E^2}} d\left(\frac{x^2}{2\sigma_E^2}\right) \\ &= \frac{\sigma_E \sqrt{2}}{\sqrt{\pi}} * e^{-\frac{x^2}{2\sigma_E^2}} \Big|_0^{+\infty} \\ &= \frac{\sigma_E \sqrt{2}}{\sqrt{\pi}}. \end{aligned} \tag{A.1}$$

This is just formula (2.9).

Because ε_{t+1} and $E_t[Y_{t+1}]$ are uncorrelated, according to eq. (2.5) and Definition 1, we have

$$\begin{aligned} \sigma_g^2 &= \sigma^2 + Var\{E_t(Y_{t+1}) | E_t(Y_{t+1}) > 0\} \\ &= \sigma^2 + E\{[E_t(Y_{t+1})]^2 | E_t(Y_{t+1}) > 0\} - E\{[E[E_t(Y_{t+1}) | E_t(Y_{t+1}) > 0]]^2\}. \end{aligned} \tag{A.2}$$

Substituting expression (A.1) into eq. (A.2) gives

$$\begin{aligned} \sigma_g^2 &= \sigma^2 + E\{Y_{t+1}^2 | E_t[Y_{t+1}] > 0\} - \frac{2\sigma_E^2}{\pi} \\ &= \sigma^2 - \frac{2\sigma_E^2}{\pi} + \int_0^{+\infty} \frac{2x^2}{\sqrt{2\pi}\sigma_E} e^{-\frac{x^2}{2\sigma_E^2}} dx \\ &= \sigma^2 - \frac{2\sigma_E^2}{\pi} + \int_{-\infty}^{+\infty} \frac{x^2}{\sqrt{2\pi}\sigma_E} e^{-\frac{x^2}{2\sigma_E^2}} dx \\ &= \sigma^2 - \frac{2\sigma_E^2}{\pi} + \sigma_E^2 \\ &= \sigma^2 + \left(1 - \frac{2}{\pi} \right) \sigma_E^2. \end{aligned} \tag{A.3}$$

This is just formula (2.11).

Authors' contributions

Lin Liu contributed the central idea, conceived and designed this research, supervised the study, provided mathematical and model Derivation, and wrote the initial draft of the paper. Qiguang Chen contributed to refining the ideas, performed the data and statistical analysis, carrying out additional analyses and coordination and finalizing this manuscript. Both authors wrote, corrected and agreed to the published version of the manuscript. Both authors have read and agreed to the published version of the manuscript.

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Competing interests

The authors declare that they have no competing interests.

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