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Higgs bundles and exceptional isogenies

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Abstract

We explore relations between Higgs bundles that result from isogenies between low-dimensional Lie groups, with special attention to the spectral data for the Higgs bundles. We focus on isogenies onto SO(4, ℂ) and SO(6, ℂ) and their split real forms. Using fiber products of spectral curves, we obtain directly the desingularizations of the (necessarily singular) spectral curves associated with orthogonal Higgs bundles. In the case of SO(6, ℂ), our construction can be interpreted as a new description of Recillas' trigonal construction.

1 Background

Exceptional isomorphisms between low-rank Lie algebras, and the corresponding isogenies between Lie groups, have been a source of fascination since first catalogued by Cartan a century ago ([6, pp. 352–355], see also [12, p. 519]). In this paper, we explore the implications for Higgs bundles of the isogenies between Lie groups of rank 2 and 3:

$$\mathcal{I}_2 : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C}); \tag{1}$$

$$\mathcal{I}_3 : \mathrm{SL}(4,\mathbb{C}) \to \mathrm{SO}(6,\mathbb{C}); \tag{2}$$

as well as their restrictions to the split real forms of these complex semi-simple groups.

Higgs bundles over a compact Riemann surface Σ of genus $g \geq 2$ were introduced in [13] and in a more general setting in [28]. For any Lie group G, a G-Higgs bundle on Σ is a pair (P, Φ) where P is a holomorphic principal bundle and Φ (the Higgs field) is a holomorphic section of an associated bundle twisted by K, the canonical bundle of the surface Σ. If G is a complex group, then P is a principal G-bundle, but if G is a real form of a complex group, then the structure group of P is the complexification of a maximal compact subgroup of G. In this paper, we consider only matrix groups, in particular $G = \mathrm{SL}(n,\mathbb{C})$ or $\mathrm{SO}(2n,\mathbb{C})$, and real forms of these groups. In these cases (described in Sect. 2), the Higgs bundles can be seen as holomorphic vector bundles with extra structure, where the precise nature of the extra structure is determined by the group G, and the Higgs fields are appropriately constrained sections of the endomorphism bundle twisted by K.

The defining data for G-Higgs bundle serve to construct G-local systems on Σ. Indeed, this is the crux of non-abelian Hodge theory (NAHT), whereby the moduli space of semistable G-Higgs bundles on Σ is identified with the moduli space of reductive representations of $\pi_1(\Sigma)$ into G. The implications of any group homomorphism $h : G_1 \to G_2$ are clear for surface group representations, since composition with h induces a map from representations into G_1 to representations into G_2. It follows from NAHT that there must

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be a corresponding induced map between the Higgs bundle moduli spaces, but the transcendental nature of the NAHT correspondence means that the clarity of the induced map on representations does not transfer so easily to the induced map on Higgs bundles. Our goal is to understand this map in the cases where the group homomorphism is given by the above isogenies.

In [14], Hitchin showed how the defining data of Higgs bundles can be re-encoded into a so-called spectral data set consisting of a ramified covering S of Σ and a bundle on S. For the groups of interest in this paper, the spectral bundles are line bundles and hence lie in Jacobians of the spectral curve. In fact, they must lie in Prym varieties determined by the S and Σ. These abelian varieties form the generic fibers in a fibration of the Higgs bundle moduli space over a half-dimensional linear space.

In terms of vector bundles, the effect of some low-rank isogenies has been studied in [1]. In this paper, we expand those result in the case of (2) through the use of Hodge star-type operators. For both (1) and (2), we give a novel description of the spectral data correspondences, shedding new light on the maps between Higgs bundles and their moduli spaces. Our key tool for understanding the induced maps on the spectral data for the Higgs bundles is a fiber product construction, at the heart of which is the following diagram:

$$\begin{array}{ccccc} L_1 & & \mathcal{L} = p_1^* L_1 \otimes p_2^* L_2 & & L_2 \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \xleftarrow{p_1} & S_1 \times_\Sigma S_2 & \xrightarrow{p_2} & S_2 \\ & {\scriptstyle \pi_1} \searrow & \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \pi_2} & \\ & & \Sigma & & \end{array} \tag{3}$$

In our applications, the curves S_i and line bundles L_i come from spectral data for $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SL}(4,\mathbb{C})$-Higgs bundles, while the fiber product $S_1 \times_\Sigma S_2$ and the line bundle $\mathcal{L}$ yield the spectral data sets for the $\mathrm{SO}(4,\mathbb{C})$ or $\mathrm{SO}(6,\mathbb{C})$-Higgs bundles. This construction clearly has wider applicability than the cases inspired by the isogenies (1) and (2). One notable feature of the construction is that it generically yields smooth curves, even though curves defined by spectral data for $\mathrm{SO}(2n,\mathbb{C})$-Higgs bundles are necessarily singular. Indeed, the curves provided by our construction are the normalizations of the singular $\mathrm{SO}(2n,\mathbb{C})$-spectral curves.

As seen in [1], the isogeny $\mathcal{I}_2$ induces a map on $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles (E_i,Φ_i) given by

$$\mathcal{I}_2((E_1,\Phi_1),(E_2,\Phi_2)) = [E_1 \otimes E_2, \Phi_1 \otimes 1 + 1 \otimes \Phi_2]. \tag{4}$$

where $E_1 \otimes E_2$ has orthogonal structure determined by the symplectic structures on E_i. Hence, in the case of the rank 2 isogeny $\mathcal{I}_2$, on the generic fibers of the Hitchin fibration for the moduli space $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})}$, our main result is the following (see Propositions 16, 17, 35):

Theorem 1 *Let S_i be the spectral curve of the $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles (E_i,Φ_i), and $L_i \in Prym(S_i,\Sigma)$ the corresponding spectral line bundle. Then, the spectral data for the the image $\mathcal{I}_2((E_1,\Phi_1),(E_2,\Phi_2))$ are given by $(\hat{S}_4,\mathcal{L})$ where*

- *$\hat{S}_4 := S_1 \times_\Sigma S_2$ is a smooth ramified fourfold cover, and*
- *$\mathcal{L} := p_1^*(L_1) \otimes p_2^*(L_2)$ where $p_i : S_1 \times_\Sigma S_2 \to S_i$ are the projection maps.*

The map $\mathcal{I}_2$ is a 2^{2g} fold-coverings onto its images. Restricted to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$*-Higgs bundles, the image lies only in components of $\mathcal{M}_{SO_0(2,2)}$ in which the Higgs bundles satisfy a topological constraint.*

In the case of the rank 3 isogeny, as seen in [1], the map $\mathcal{I}_3$ induces a morphism on $\mathrm{SL}(4,\mathbb{C})$-Higgs bundles (E,Φ) given by

$$\mathcal{I}_3(E,\Phi) = \left[\Lambda^2 E, \Phi \otimes 1 + 1 \otimes \Phi\right]. \tag{5}$$

where $\Lambda^2 E$ has orthogonal structure determined by the isomorphism $\Lambda^2 E^* \otimes \det(E) \simeq \Lambda^2 E$.

This case differs from the previous one in two notable ways. The first is seen in the restriction of (5) to $\mathrm{SL}(4,\mathbb{R})$-Higgs bundles, where the map reflects the fact that $\mathrm{SO}(4)$ is not a simple Lie group. This leads to a decomposition of the bundle $\Lambda^2 E$ that is analogous to the decomposition of 2-forms on a Riemannian 4-manifold into self-dual and anti-self-dual forms.

The other new wrinkle is in the induced map on spectral data where the fiber product in diagram (3) is now taken on two copies of the same covering, say S. The resulting curve can never be smooth since it has a diagonally embedded copy of S, which intersects other components at the ramification points for the covering $S \to \Sigma$. It turns out though that this component plays no part in the map induced by $\mathcal{I}_3$. We thus modify our construction by removing the diagonally embedded component in the fiber product of an $\mathrm{SL}(4,\mathbb{C})$-spectral data set with itself (see Sect. 6, Proposition 19, 27, 41 for details):

Theorem 2 *Restricted to Higgs bundles for the split real form* $\mathrm{SL}(4,\mathbb{R}) \subset \mathrm{SL}(4,\mathbb{C})$*, the induced map yields*

$$\mathcal{I}_3(E,\Phi) = \left[\Lambda^2_+ E \oplus \Lambda^2_- E, \begin{pmatrix} 0 & \alpha \\ -\alpha^{\mathrm{T}} & 0 \end{pmatrix}\right],$$

where $\Lambda^2_\pm E$ are the ± 1-eigenbundles for an involution $* : \Lambda^2 E \to \Lambda^2 E$ determined by an orthogonal structure on E, with canonically determined orthogonal structures, and with Φ and α related as in Eq. (32). Over the smooth loci in the Hitchin base, given the spectral data (S, L) corresponding to (E,Φ), the spectral data corresponding to the $\mathrm{SO}(6,\mathbb{C})$*-Higgs bundle $\mathcal{I}_3(E,\Phi)$ are given by $(\hat{S}_6, \mathcal{I}_3(L))$, where*

- *$\hat{S}_6$ is the symmetrization of the non-diagonal component in the fiber product $S \times_\Sigma S$;*
- *$\mathcal{I}_3(L)$ is a canonical twist of the line bundle generated by local sections of $\mathcal{L} = p_1^*(L) \otimes p_2^*(L)$ that are anti-invariant with respect to the symmetry of $S \times_\Sigma S$.*

The map $\mathcal{I}_3$ is a 2^{2g} fold-coverings onto its images. Restricted to points representing $\mathrm{SL}(4,\mathbb{R})$*-Higgs bundles, the image lies only in components of $\mathcal{M}_{SO_0(3,3)}$ in which the Higgs bundles satisfy a topological constraint.*

While the two cases covered by (1) and (2) are by no means the only interesting ones, they have some unique features and serve to illustrate phenomena that we expect to apply in greater generality. The group $\mathrm{SL}(2,\mathbb{R})$ has a distinguished place in any discussion of surface group representations because of its relation to hyperbolic structures and Teichmuller space, while the group $\mathrm{SO}_0(2,2)$ is the isometry group of the anti-de Sitter space AdS^3. Moreover, both groups are split real forms of complex semisimple groups (viz. $\mathrm{SL}(2,\mathbb{C})$

and SO(4, $\mathbb{C}$), respectively) and also groups of Hermitian type. The only other groups that lie in both classes are the symplectic groups Sp($2n$, $\mathbb{R}$). For both classes of real Lie group, the representation variety or, equivalently, the moduli space of Higgs bundles has a distinguished set of components. In the case of the split real forms, the distinguished components are called "higher Teichmuller components" because they generalize the copies of Teichmuller space which occur in the case of SL(2, $\mathbb{R}$). For the groups of Hermitian type, the G-Higgs bundles (or the surface group representations) carry a discrete invariant known as a Toledo invariant. This invariant satisfies a so-called Milnor–Wood bound, and the distinguished components are those in which the invariant has maximal value.

In the two cases considered in this article, the induced maps on the spectral curves yield a pairing between two different ramified coverings of a common base curve, together with isogenies between the associated Prym varieties. The spectral curves in each pair are, moreover, determined by different representations of the same group. This kind of situation, with pairings between coverings of a given curve and isogenies between the associated Prym varieties, has been considered in the context of integrable systems and also by Donagi (see, e.g., [8,9]). As noted in [8, Example 3], the correspondence we see for the pair SL(4, $\mathbb{C}$) and SO(6, $\mathbb{C}$), namely between a fourfold covering and a sixfold covering with a fixed-point-free involution, is essentially the correspondence described by Recillas in his trigonal construction [22] and generalized by Donagi in [9]. The novelty in our version of the correspondence—and resulting relation between Prym varieties—lies in the use of fiber products to get explicit descriptions of the maps.

The first part of this paper covers background material on Higgs bundles (Sect. 2), spectral curves (Sect. 3) and the isogenies (Sect. 4). In Sect. 5, we describe the maps induced by the isogeny $\mathcal{I}_2$ on both the complex group SL(2, $\mathbb{C}$) $\times$ SL(2, $\mathbb{C}$) and its split real form, and in Sect. 6, we do the same for $\mathcal{I}_3$. We include a discussion of the relation between our construction on spectral data and the trigonal construction of Recillas and show how the map we obtain between Prym varieties can be interpreted in terms of a correspondence between curves. We conclude, in Sect. 7, with a discussion of maps between moduli spaces.

2 Higgs bundles and the Hitchin fibration

Let Σ be a compact Riemann surface of genus $g \geq 2$, and $\pi : K := T^*\Sigma \to \Sigma$ its canonical bundle. For G_c a complex reductive Lie group with Lie algebra $\mathfrak{g}_c$, from [14] one has the following definition:

Definition 3 A G_c-Higgs bundle on Σ is given by a pair (P, Φ) for P a principal G_c-bundle on Σ, and Φ a holomorphic section of Ad$P \otimes K$, for Ad$P = P \times_{Ad} \mathfrak{g}_c$ the adjoint bundle associated with P.

For $GL(n, \mathbb{C})$ one recovers classical Higgs bundles as introduced in [13]. For matrix groups, the definition can be reformulated in terms of vector bundles rather than principal bundles. In particular,

- An SL(n, $\mathbb{C}$)-Higgs bundle is a pair (E, Φ) where E is a rank n holomorphic bundle on Σ with fixed trivial determinant, and Φ a traceless holomorphic section of End$(E) \otimes K$;
- An SO(n, $\mathbb{C}$)-Higgs bundle is a pair (E, Φ) where E is a rank n holomorphic bundle on Σ with an orthogonal structure Q and a compatible trivialization of its

determinant bundle,[1] and Φ is a holomorphic section of $\mathrm{End}(E) \otimes K$ satisfying $Q(u, \Phi v) = -Q(\Phi u, w)$.

The construction of G-Higgs bundles for a real form G of G_c goes back to [13,15] (see also [5,10], for further details). The definition requires a choice of a maximal compact subgroup of $H \subset G$, and the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, for $\mathfrak{h}$ the Lie algebra of H, and $\mathfrak{m}$ its orthogonal complement. Note that the Lie algebras satisfy the symmetric space relations $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Hence, there is an induced isotropy representation given by $\mathrm{Ad}|_{H^{\mathbb{C}}} : H^{\mathbb{C}} \to GL(\mathfrak{m}^{\mathbb{C}})$.

Definition 4 Given G a real form of a complex Lie group G_c, a principal *G-Higgs bundle* is a pair (P, Φ) where P is a holomorphic principal $H^{\mathbb{C}}$-bundle on Σ, and Φ is a holomorphic section of $P \times_{Ad} \mathfrak{m}^{\mathbb{C}} \otimes K$.

Alternatively, as done in [15], we may regard real Higgs bundles as classical Higgs bundles (E, Φ), with extra conditions reflecting the structure of the real group and its isotropy representation. In this paper, we shall mainly consider $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SO}(n, n)$-Higgs bundles, for which we recall their main properties in Sects. 3.2 and 3.4 (for further references, see [1,15,25]).

Following [13], a classical Higgs bundle (E, Φ) is said to be (semi)-stable if all subbundles $F \subset E$ such that $\Phi(F) \subset F \otimes K$ satisfy $\deg(F)/\mathrm{rk}(F) < (\leq) \deg(E)/\mathrm{rk}(E)$. Moreover, the pair is said to be poly-stable if it can be written as a direct sum of stable Higgs bundles (E_i, Φ_i) for which $\deg(E_i)/\mathrm{rk}(E_i) = \deg(E)/\mathrm{rk}(E)$. The notion of stability can be extended to G_c-Higgs bundles as in [14], as well as to G-Higgs bundles (e.g., see [4]), and used to construct the corresponding moduli spaces. As explained below in Remark 7, stability considerations will not play a role in this paper. We shall denote by $\mathcal{M}_{G_c}$ and $\mathcal{M}_G$ the moduli space of G_c-Higgs bundles and $\mathcal{M}_G$-Higgs bundles, respectively.

2.1 The Hitchin fibration and spectral curves

Given a homogenous basis $r_1, \ldots, r_k$ of invariant polynomials for the Lie algebra of a complex Lie group G_c, let $d_1, \ldots, d_k$ be their degrees. Then, the Hitchin fibration (introduced in [14]) is defined as

$$\begin{aligned} h : \mathcal{M}_{G_c} &\to \mathcal{A}_{G_c} := \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}) \\ (E, \Phi) &\mapsto (r_1(\Phi), \ldots, r_k(\Phi)). \end{aligned}$$

A point in the Hitchin base $\mathcal{A}_{G_c}$ determines a section of the line bundle $\pi^* K^{d_k}$ defined by $\eta^{d_k} + \cdots + r_{k-1}(\Phi)\eta^{d_1} + r_k(\Phi)$, where η is the tautological section of $\pi^* K$. The zero locus of this section is the so-called spectral curve S associated with the Higgs bundle. The curve S lies in the total space of K, and the projection $\pi : S \to \Sigma$ is a d_k-fold cover of Σ. By considering vector bundles on it (or its desingularization), one can give a geometric description of the fibers of the Hitchin fibration [14]. In the case of classical Higgs bundles, the generic fibers are given by the Jacobian varieties of the spectral curves S, on which Φ has a single-valued eigenvalue η. Indeed, given a line bundle L on S one may recover the corresponding classical Higgs bundle (E, Φ) by taking $E := \pi_* L$ and Φ the direct image of

[1] A trivialization $\delta : \det(E) \simeq \mathcal{O}_\Sigma$ is compatible with Q if δ^2 agrees with the trivialization of $(\Lambda^n E)^2$ given by the discriminant $Q : \Lambda^n E \to \Lambda^n E^*$ (see Remark 2.6 in [21]).

$\eta : L \to L \otimes \pi^* K$. In Sect. 3, we describe more fully the spectral data for the specific cases of interest in this paper.

By considering the moduli space $\mathcal{M}_G$ in $\mathcal{M}_{G_c}$, one may identify the G-Higgs bundles as points in the Hitchin fibration satisfying additional constraints (see [14] for classical complex Lie groups, and [27] and references therein for real Higgs bundles). In particular, in the case of the split real form of G_c, the line bundles are the torsion two points in the Jacobian (see [25, Theorem 4.12]). It should be noted that $\mathcal{M}_G$ does not always embed in $\mathcal{M}_{G_c}$, and an example of when one does not have an embedding can be seen in [24].

Remark 5 Spectral curves are defined for all Higgs bundles represented by points in the moduli spaces $\mathcal{M}_G$, but they are not necessarily smooth. It follows, however, from Bertini's theorem that on the generic fiber of the Hitchin fibration the spectral curves are smooth for $G = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(2n+1, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$. Throughout the paper, we shall consider Higgs bundles over the smooth loci of the Hitchin fibration, i.e., points defining smooth spectral curves for the above groups, since we will further restrict our attention to those curves for which the most generic type of ramification behavior occurs.

Remark 6 In the case of $\mathrm{SO}(2n, \mathbb{C})$-Higgs bundles, the associated spectral curves are always singular, and one needs to work with a canonical normalization, as explained in Sect. 3.3. It should be noted that through the fiber product process described in this paper, one recovers the normalization of the singular curve in a natural way.

Remark 7 Note that if the spectral curve for a Higgs bundle is smooth, then the characteristic polynomial for the Higgs field is irreducible. The Higgs bundle thus has no invariant subbundles and is therefore automatically stable. Hence, while stability is needed to define the moduli spaces, it will play no role in our discussions, and we shall omit any further comment on it.

3 Spectral data for complex and real Higgs bundles

We shall recall here how to study the fibers of the Hitchin fibration through spectral data, which we shall do by reviewing the methods introduced in [14,15] for complex Higgs bundles, and in [25,27] for real Higgs bundles. Since our main focus is on $\mathrm{SO}(2n, \mathbb{C})$ and $\mathrm{SL}(2n, \mathbb{C})$-Higgs bundles, we shall restrict attention to those groups and their split real forms.

3.1 Spectral data for SL(n, $\mathbb{C}$)-Higgs bundles

The spectral curve $\pi : S \to \Sigma$ associated with an $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle (E, Φ) has equation

$$\det(I\eta - \Phi) = \eta^n + a_2\eta^{n-2} + \cdots + a_{n-1}\eta + a_n = 0. \tag{6}$$

Here $a_i \in H^0(\Sigma, K^i)$, and η is the tautological section of $\pi^* K$. The $(n-1)$-tuple $(a_2, \ldots, a_n)$ defines a point in the base of the Hitchin fibration (see [14]). Over a dense open set in the base of the Hitchin fibration, the spectral curve S is smooth and the fiber can be identified (biholomorphically) with the Prym variety $\mathrm{Prym}(S, \Sigma)$, the subset of $\mathrm{Jac}(S)$ of line bundles whose direct image sheaf has trivial determinant. The relation between $L \in \mathrm{Prym}(S, \Sigma)$ and E (once a choice of $K^{1/2}$ is made) is then

$$E := \pi_*(L \otimes \pi^*(K^{(n-1)/2})). \tag{7}$$

Moreover, $\mathrm{Prym}(S, \Sigma) = \{L \in \mathrm{Jac}(S) \mid Nm(L) = \mathcal{O}_\Sigma\}$, where Nm denotes the norm map defined by the projection $\pi : S \to \Sigma$, the trivialization of $\det(E)$ determined by the trivialization of $Nm(L)$. Note that in the case of twofold covers, a line bundle is in the Prym variety if and only if its dual is isomorphic to the pull back by the involution switching the sheets of the cover.

3.2 Spectral data for SL(n, $\mathbb{R}$)-Higgs bundles

From Definition 4, Higgs bundles with structure group $\mathrm{SL}(n, \mathbb{R})$ are given by classical Higgs bundles (E, Φ) together with an oriented orthogonal structure on E, with respect to which the Higgs field is traceless and symmetric. Moreover, by [25, Theorem 4.12] one has that the intersection of the moduli space $\mathcal{M}_{\mathrm{SL}(n,\mathbb{R})}$ with the smooth fibers of the $\mathrm{SL}(n, \mathbb{C})$ Hitchin fibration is given by line bundles $L \in \mathrm{Prym}(S, \Sigma)$ such that $L^2 \cong \mathcal{O}$. Following [3] and [14], a torsion 2 line bundle L induces an $\mathrm{SL}(n, \mathbb{R})$-Higgs bundle (E, Φ) with $E = \pi_*(L \otimes \pi^* K^{(n-1)/2})$ and Φ the push down of the tautological section η. Moreover, the orthogonal structure on E comes from an $O(1)$ structure on L.

3.3 Spectral data for SO(2n, $\mathbb{C}$)-Higgs bundles

From the characteristic polynomial of an $\mathrm{SO}(2n, \mathbb{C})$-Higgs field Φ, one obtains a $2n$-fold cover $\pi : S \to \Sigma$ whose equation is given by

$$\det(\eta I - \Phi) = \eta^{2n} + b_1 \eta^{2n-2} + \cdots + b_{n-1}\eta^2 + p_n^2 = 0, \tag{8}$$

where η is as in (6), the sections $b_i \in H^0(\Sigma, K^{2i})$, and $p_n \in H^0(\Sigma, K^n)$ is the Pfaffian of Φ. This curve has a natural involution $\sigma : \eta \mapsto -\eta$, and is singular at $\eta = p_n = 0$, i.e., at the fixed points of σ. The normalization of S, which we denote by $\hat{\pi} : \hat{S} \to \Sigma$, is what we shall refer to as the spectral curve. As in the previous cases, $\hat{S}$ is generically smooth, and the involution σ extends to an involution $\hat{\sigma}$ on $\hat{S}$ which does not have fixed points.

The generic fibers of the Hitchin fibration can be identified with an abelian variety defined by a connected component of $\mathrm{Prym}(\hat{S}, \hat{S}/\sigma)$, i.e., the kernel of the norm map $Nm : Pic(\hat{S}) \to Pic(\hat{S}/\sigma)$. Given a line bundle $L \in \mathrm{Prym}(\hat{S}, \hat{S}/\sigma)$ and a choice of $K^{1/2}$, the vector bundle E is recovered as

$$E := \pi_*(L \otimes (K_{\hat{S}} \otimes \pi^*(K^*))^{1/2}). \tag{9}$$

In this case, the orthogonal structure on E comes from the isomorphism $\sigma^* L \simeq L^{-1}$ (by virtue of which $Nm(L) = 0$). The spectral data associated with an $\mathrm{SO}(2n, \mathbb{C})$-Higgs bundle is defined on the desingularization $\hat{S}$ of S. From Eq. (8), each pair of points $(b_1, \ldots, b_{n-1}, p_n)$ and $(b_1, \ldots, b_{n-1}, -p_n)$ defines the same curve S and desingularization $\hat{S}$. The Prym variety $\mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$ has two connected components. Moreover, from [18, Lemma 1] all line bundles L on $\hat{S}$ which satisfy $Nm(L) \simeq \mathcal{O}$ are of the form $L = N \otimes \hat{\sigma}^*(N^*)$, for some line bundle N on $\hat{S}$ of degree 0 or 1. The two connected components of $\mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$ correspond to the two possibilities for the parity of the degree $\deg(N)$. This can also be seen as a reflection of the fact that $\mathcal{M}_{\mathrm{SO}(2n,\mathbb{C})}$ has two components corresponding to the possible values for the second Stiefel–Whitney class of an $\mathrm{SO}(2n, \mathbb{C})$-bundle. We say that the spectral data are given by $(\hat{S}, L)$ for $\hat{S}$ the normalized curve and $L \in \mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$.

Remark 8 For any m-fold ramified cover $\pi : S \to \Sigma$, the ramification divisor R in S relates the canonical bundles K_S and K of S and Σ, respectively, by $[R] = K_S \otimes \pi^* K^*$, where $[R]$ denotes the line bundle defined by the divisor R. If S is a spectral curve, i.e., contained in the total space of K, then $K_S = \pi^*K^m$, and $[R] = \pi^*K^{(m-1)}$. The relation between L and E in both (7) and (9) can thus be given[2] as

$$E = \pi_*(L \otimes [R]^{1/2}).$$

3.4 Spectral data for $SO_0(n, n)$-Higgs bundles

From Definition 4, one has that $SO(n, n)$-Higgs bundles can be viewed as $SO(2n, \mathbb{C})$-Higgs bundles of the form $(W_1 \oplus W_2, \Phi)$ where W_i are vector bundles of rank n with orthogonal structure, say q_i, and

$$\Phi = \begin{pmatrix} 0 & \alpha \\ -\alpha^{\mathrm{T}} & 0 \end{pmatrix}, \tag{10}$$

for α^{T} the orthogonal transpose of α, i.e., $\alpha^T = q_2^{-1} \cdot \alpha^* \cdot q_1$ where α^* denotes the dual map. By further requiring that $\det(W_i) \simeq \mathcal{O}$, one obtains an $SO_0(n, n)$ pair. In this case, from [25, Theorem 4.12] and along the lines of Sect. 3.2, one has that the intersection of the moduli space $\mathcal{M}_{SO_0(n,n)}$ with the smooth fibers of the $SO(n, \mathbb{C})$ Hitchin fibration is given by $L \in \mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$ which satisfy $L^2 \cong \mathcal{O}$. As for classical Higgs bundles, given a torsion two line bundle $L \in \mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$ one obtains an $SO_0(n, n)$-Higgs bundle (E, Φ) by taking the direct image of the line bundle $L \otimes (K_{\hat{S}} \otimes \pi^*K^*)^{1/2}$ and the push forward of η.

Given the spectral data $(\hat{S}, L)$ of an $SO_0(n, n)$-Higgs bundle, since L is of order two and in the Prym variety of a twofold cover $p : \hat{S} \to \hat{S}/\sigma$, it is invariant under the involution σ on $\hat{S}$. Hence, its local sections decompose into invariant and anti-invariant local sections, and thus, the direct image decomposes as $p_*L = \hat{L}_1 \oplus \hat{L}_2$, where the summands are generated by local invariant and anti-invariant sections (see [26]). Moreover, considering the n-fold cover $\hat{\pi} : \hat{S}/\hat{\sigma} \to \Sigma$, the orthogonal bundles W_i are recovered by taking $\hat{\pi}_*(L_i \otimes p_*(K_{\hat{S}} \otimes \hat{\pi}^*K^*)^{1/2})$ for $i = 1, 2$.

4 Homomorphisms of groups and induced maps

As noted in the Introduction, given a fixed surface Σ and a homomorphism between two Lie groups $\Psi : G \to G'$, there is clearly an induced map

$$\Psi : \mathrm{Rep}(\Sigma, G) \to \mathrm{Rep}(\Sigma, G'),$$

where $\mathrm{Rep}(\Sigma, G)$ denotes the space of representations modulo conjugation. The correspondence between surface group representations and Higgs bundles thus implies a similar induced map between G-Higgs bundles and G'-Higgs bundles.

From Definition 3, one sees that for a homomorphism between complex groups there is in fact an induced map from G-Higgs bundles to G'-Higgs bundles given by

$$\Psi_* : (P_G, \Phi) \mapsto (P_{G'} = P_G \times_\Psi G', \Phi' = \mathrm{d}\Psi(\Phi)), \tag{11}$$

where

$$\mathrm{d}\Psi : ad(P_G) \to ad(P_{G'}) \tag{12}$$

is the map defined by the derivative at the identity of the map Ψ, i.e., by the map on Lie algebras. Moreover, if the homomorphism Ψ restricts to a map between real forms of G

[2] Note that the line bundle corresponding to our L is denoted in [14] by U, so that the line bundle denoted in [14] by L corresponds to $L[R]^{-1/2}$ in our notation.

and G' which respects the Cartan decompositions of the Lie algebras, then the map Ψ induces a map from G_r-Higgs bundles to G'_r-Higgs bundles. From Definition 4, following the notation of Sect. 2, the map is given by

$$\Psi_* : (P_{H_{\mathbb{C}}}, \Phi) \mapsto \left(P_{H'_{\mathbb{C}}} = P_{H_{\mathbb{C}}} \times_\Psi H'_{\mathbb{C}}, \Phi' = \mathrm{d}\Psi(\Phi)\right). \tag{13}$$

The maps in (11) and (13) ought to acquire descriptions purely in terms of spectral data, and this shall be investigated in forthcoming sections. The relation between the spectral curves can be deduced from its relation to the eigenvalues of the Higgs field, once it is viewed as a classical Higgs bundle to which one imposes conditions reflecting the nature of the group. Thus, if the spectral curve S for $(P_{H_{\mathbb{C}}}, \Phi)$ is defined by

$$0 = \det(\eta I - \Phi) = \prod_{i=1}^{n}(\eta - \eta_i), \tag{14}$$

where $\{\eta_1, \dots, \eta_n\}$ are the eigenvalues of Φ, then the spectral curve S' for $\Psi_*(P_{H_{\mathbb{C}}}, \Phi)$ is defined by an equation of the same form except with the eigenvalues replaced by the eigenvalues of $\mathrm{d}\Psi(\Phi)$, leading to a map $S \mapsto S'$.

Alternatively, one may seek a more intrinsic understanding of the map purely in terms of the information encoded in the geometry of the covering $S \to \Sigma$ and the restrictions on the spectral line bundle L. This is our goal for the special cases of the isogenies in (1) and (2). The isogenies can be described in several ways, including from coincidences of Dynkin diagrams or in terms of representations, e.g., in terms of Schur functors as in [20, Chapter 9–10]. For our purposes, the representation theoretic point of view is convenient, as described in the next two sections.

4.1 The isogeny between SL(2, $\mathbb{C}$) × SL(2, $\mathbb{C}$) and SO(4, $\mathbb{C}$)

The group $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ by $(g,h)(v \otimes w) := (gv) \otimes (hw)$. The isogeny onto $\mathrm{SO}(4,\mathbb{C})$ can be seen as coming from the fact that $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Sp}(2,\mathbb{C})$. If ω is the symplectic form on $\mathbb{C}^2$ preserved by matrices with unit determinant, then $Q_4 := \omega \otimes \omega$, defines a symmetric, non-degenerate bilinear form on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Hence, one has a map

$$\begin{aligned} \mathcal{I}_2 &: \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C}), \\ &(A_1, A_2) \mapsto A_1 \otimes A_2, \end{aligned} \tag{15}$$

where $\mathrm{SO}(4,\mathbb{C})$ is the group of (orientation preserving) linear maps $\mathbb{C}^4 \to \mathbb{C}^4$ preserving the form Q_4. The derivative at the identity yields an isomorphism of Lie algebras given by

$$\begin{aligned} \mathrm{d}\mathcal{I}_2 &: \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(4,\mathbb{C}), \\ &(\dot{A}_1, \dot{A}_2) \mapsto \dot{A}_1 \otimes I + I \otimes \dot{A}_2. \end{aligned} \tag{16}$$

Remark 9 If $\dot{A}_i$ has eigenvalues $\{\lambda_1^i, \lambda_2^i\}$, for $i = 1, 2$, then the image $\mathrm{d}\mathcal{I}_2(\dot{A}_1, \dot{A}_2)$ has eigenvalues $\{\lambda_a^1 + \lambda_b^2 \,|1 \le a, b \le 2\}$. In particular, if $Tr(\dot{A}_i) = 0$, then $\lambda_2^i = -\lambda_1^i$ and the eigenvalues for $\mathrm{d}\mathcal{I}_2(\dot{A}_1, \dot{A}_2)$ are $\{\pm\lambda_1^1 \pm \lambda_1^2\}$.

Restricted to $\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{R}^4$, the quadratic form Q_4 has signature $(2, 2)$, and thus, the above map (15) between complex Lie groups restricts to

$$\mathcal{I}_2 : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_0(2,2), \tag{17}$$

where the subscript in $\mathrm{SO}_0(2,2)$ denotes the connected component of the identity (see [1, Section 5.2] for more details). The map on Lie algebras similarly restricts. Indeed, given

$\dot{a}_i \in \mathfrak{sl}(2,\mathbb{R})$ symmetric and trace-free for $i = 1, 2$, and fixing a basis for $\mathbb{R}^4$ such that the orthogonal structure has the form $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, then the map in (16) is given by

$$\mathrm{d}\mathcal{I}_2(\dot{a}_1, \dot{a}_2) = \begin{bmatrix} 0 & \alpha \\ \alpha^t & 0 \end{bmatrix} \in \mathfrak{so}(2,2), \tag{18}$$

where α^t denotes the transpose.[3] The precise form of α depends on the orientation chosen for $\mathbb{R}^4$, i.e., on the identification $\Lambda^4\mathbb{R}^4 \simeq \mathbb{R}$. A concrete description of the implications of this identification, as well as of α, shall be given in the following section (see Remark 13). Finally, it should be noted that while $\mathrm{Pf}(\mathrm{d}\mathcal{I}_2(\dot{a}_1, \dot{a}_2))^2 = \det(\alpha)^2$, the sign of $\mathrm{Pf}(\mathrm{d}\mathcal{I}_2(\dot{a}_1, \dot{a}_2))$ depends on the choice of orientation.

4.2 The isogeny between SL(4, $\mathbb{C}$) and SO(6, $\mathbb{C}$)

The group SL(4, $\mathbb{C}$) of volume-preserving linear maps $T : \mathbb{C}^4 \to \mathbb{C}^4$ has a 6-dimensional representation on the exterior power $\Lambda^2\mathbb{C}^4$ given by $g(v \wedge w) := (gv) \wedge (gw)$. Taking $n = 4$ and $k = 2$ in the isomorphism

$$\Lambda^k(\mathbb{C}^*) \otimes \Lambda^n(\mathbb{C}) \longrightarrow \Lambda^{n-k}(\mathbb{C}), \tag{19}$$

and fixing an identification $\Lambda^4\mathbb{C}^4 \simeq \mathbb{C}$ (i.e., fixing a volume form), one gets a bilinear form Q_6 on $\Lambda^2\mathbb{C}^2$, which is symmetric and non-degenerate. Since an element of SL(4, $\mathbb{C}$) preserves the volume form, it preserves a bilinear form, and thus, one has a map

$$\begin{aligned} \mathcal{I}_3 : \mathrm{SL}(4,\mathbb{C}) &\to \mathrm{SO}(6,\mathbb{C}) \\ A &\mapsto \Lambda^2 A, \end{aligned} \tag{20}$$

where SO(6, $\mathbb{C}$) is the group of (orientation preserving) linear maps $\mathbb{C}^6 \to \mathbb{C}^6$ preserving the non-degenerate symmetric form Q_6. Moreover, the map gives a double cover SL(4, $\mathbb{C}$) of SO(6, C), thus realizing SL(4, $\mathbb{C}$) as Spin(6, $\mathbb{C}$). The derivative at the identity gives the Lie algebra isomorphism

$$\begin{aligned} \mathrm{d}\mathcal{I}_3 : \mathfrak{sl}(4,\mathbb{C}) &\to \mathfrak{so}(6,\mathbb{C}), \\ \dot{A} &\mapsto \dot{A} \otimes I_4 + I_4 \otimes \dot{A}, \end{aligned} \tag{21}$$

where I_4 denotes the identity map on $\mathbb{C}^4$ and the endomorphism $\dot{A} \otimes I_4 + I_4 \otimes \dot{A}$ is understood to be the restriction to $\Lambda^2\mathbb{C}^4 \subset \mathbb{C}^4 \otimes \mathbb{C}^4$.

Remark 10 If $\dot{A}$ has eigenvalues $\{\lambda_a\}_{a=1}^4$, then as a map on $\mathbb{C}^6 = \Lambda^2\mathbb{C}^4$ the image $\mathrm{d}\mathcal{I}_3(\dot{A})$ has eigenvalues $\{\lambda_a + \lambda_b \,|1 \le a < b \le 4\}$.

Restricted to $\mathbb{R}^6$, the quadratic form Q_6 has signature (3, 3) and thus $\mathcal{I}_3$ restricts to

$$\mathcal{I}_3 : \mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}_0(3,3), \tag{22}$$

with a corresponding restriction of (21). In particular, if $\dot{a} \in \mathfrak{sl}(4,\mathbb{R})$ is symmetric, trace-free, with entries a_{ij}, fixing a basis for $\mathbb{R}^6$ such that the orthogonal structure has the form $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, the map has the form

$$\mathrm{d}\mathcal{I}_3(\dot{a}) = \begin{bmatrix} 0 & \alpha \\ \alpha^t & 0 \end{bmatrix} \in \mathfrak{so}(3,3). \tag{23}$$

[3] Notice that $\alpha^t = -(-I \cdot \alpha^t \cdot I)$ so that in this case $\alpha^t = -\alpha^T$.

As in the previous case, the precise form of α depends on the orientation chosen for $\mathbb{R}^6$, i.e., on the identification $\Lambda^6\mathbb{R}^6 \simeq \mathbb{R}$, and a standard choice yields

$$\alpha = \begin{bmatrix} a_{13}+a_{24} & -a_{14}+a_{23} & a_{11}+a_{22} \\ -a_{12}+a_{34} & a_{11}+a_{33} & a_{14}+a_{23} \\ -a_{22}-a_{33} & a_{12}+a_{34} & -a_{13}+a_{24} \end{bmatrix}. \tag{24}$$

In all cases, the Pfaffian is given by $Pf(\mathrm{d}\mathcal{I}_3(a)) = \pm\det(\alpha)$, where the sign depends on the choice of orientation.

Remark 11 Fixing a maximal compact subgroup $\mathrm{SO}(4) \subset \mathrm{SL}(4,\mathbb{R})$, one obtains extra structure related to the presence of two inequivalent normal SO(3) subgroups, by virtue of which SO(4) fails to be simple. These subgroups have an important influence on the map induced by $\mathcal{I}_3$ on $\mathrm{SL}(4,\mathbb{R})$-Higgs bundles, described in more detail in Sect. 6.3, and considered also in Sect. 7.2.

5 The rank 2 isogeny and the Hitchin fibration

In this section, we describe the map corresponding to the isogeny $\mathcal{I}_2$ in Sect. 4.1 in terms of spectral data. After exploring the maps for the complex groups, we examine the extra conditions required to understand the corresponding maps for the split real forms.

5.1 $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$- and $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$-Higgs bundles

An $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$-Higgs bundle on Σ is defined by a pair of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles (E_i, Φ_i), for $i = 1, 2$. An $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$-Higgs bundle can be viewed as a pair of this type in which the bundles have oriented orthogonal structures and the Higgs fields are traceless and symmetric with respect to the orthogonal structures. Up to isomorphism, we may thus assume in this case that the bundles are of the form $E_i = N_i \oplus N_i^*$ where N_i is a line bundle of nonnegative degree, the orthogonal structure is defined by the isomorphism

$$q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : N_i \oplus N_i^* \to N_i^* \oplus N_i, \tag{25}$$

and the Higgs field is of the form

$$\Phi_i = \begin{pmatrix} 0 & \beta_i \\ \gamma_i & 0 \end{pmatrix}. \tag{26}$$

The characteristic polynomials for the Higgs fields define two spectral curves $\pi_i : S_i \to \Sigma$ in the total space of K as in Eq. (6). These are twofold covers of the Riemann surface, with equations

$$\eta^2 + a_i = 0,$$

for $a_i \in H^0(\Sigma, K^2)$ and η the tautological section of π^*K. Generically, the quadratic differentials a_i have simple zeros, and thus, by Bertini's theorem the curves S_i are generically smooth. As seen in Sect. 3.2, the spectral data associated with these $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles are completed by line bundles $L_i \in \mathrm{Prym}(S_i, \Sigma)$.

In the case of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$-Higgs bundles, by [25, Theorem 4.12], the line bundles are required to satisfy $L_i^2 \cong \mathcal{O}_{S_i}$. Note that, since $L_i \in \mathrm{Prym}(S_i, \Sigma)$ if and only if $\sigma^*L \cong L_i^*$, the conditions that $L_i \in \mathrm{Prym}(S_i, \Sigma)$ and $L_i^2 \cong \mathcal{O}$ are equivalent to the conditions that $\sigma^*L_i \cong L_i$ and $L_i^2 \cong \mathcal{O}$.

Remark 12 The $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles (E_i, Φ_i) have associated an integer invariant τ_{E_i} known as the Toledo invariant. This can be defined in several equivalent ways, including as the degree of the line bundle N_i or the Euler number of the SO(2)-principal bundle associated with $N_i \oplus N_i^*$. It can also be seen in the spectral data where it is detected by the action of the involution on the fibers of the line bundle at fixed points (see [26]). However, it is defined that the Toledo invariant satisfies a so-called Milnor–Wood inequality $|\tau_{E_i}| \leq 2g-2$.

5.2 $\mathrm{SO}(4,\mathbb{C})$-Higgs bundles and $\mathrm{SO}_0(2,2)$-Higgs bundles

An $SO_0(2,2)$-Higgs bundle can be described as an $\mathrm{SO}(4,\mathbb{C})$-Higgs bundle (E,Φ) where $E = W_1 \oplus W_2$ decomposes as the sum of two rank 2 holomorphic oriented orthogonal bundles, and

$$\Phi = \begin{pmatrix} 0 & \alpha \\ -\alpha^T & 0 \end{pmatrix}, \tag{27}$$

for $\alpha^T = q_2^{-1} \circ \alpha^* \circ q_1$ where α^{T} is the dual map and q_i is the orthogonal structure of W_i. Furthermore, Higgs bundles with structure group $\mathrm{SO}_0(2,2)$ have isomorphisms $\delta_i : \Lambda^2 W_i \cong \mathcal{O}$. As in the previous section, we may take the rank two orthogonal bundles to be of the form $W_1 = M_1 \oplus M_1^*$ and $W_2 = M_2 \oplus M_2^*$, for M_i line bundles on Σ with $\deg(M_i) \geq 0$, and with orthogonal structure as in (25).

As in Eq. (8), the Higgs field Φ in an $\mathrm{SO}(4,\mathbb{C})$-Higgs bundle defines a fourfold cover $\pi_4 : S_4 \to \Sigma$ with equation

$$P_4(\eta) := \det(\eta I - \Phi) = \eta^4 + b_1\eta^2 + b_2^2 = 0, \tag{28}$$

where b_2 is the Pfaffian, $b_i \in H^0(\Sigma, K^i)$, and η the tautological section of π^*K. The spectral data for an $SO_0(2,2)$-Higgs bundle are given by the spectral data $(\hat{S}_4, L)$ of the corresponding complex $\mathrm{SO}(4,\mathbb{C})$-Higgs bundle for which $L^2 \cong \mathcal{O}$ [25, Theorem 4.12]. Moreover, as in Sect. 3.4, the direct image of L in $\hat{S}_4/\hat{\sigma}$ defines two line bundles which induce W_i on Σ.

The group $\mathrm{SO}_0(2,2)$ is both a split real form of $\mathrm{SO}(4,\mathbb{C})$ and a group of Hermitian type. As a consequence of being a split real form, the Hitchin fibration admits a section which defines the Hitchin component in $\mathcal{M}_{\mathrm{SO}_0(2,2)}$. By virtue of the properties of groups of Hermitian type, the Higgs bundles carry Toledo invariants, i.e., discrete invariants which in the case of $\mathrm{SO}_0(2,2)$-Higgs bundles may be taken to be the degrees of the line bundles M_1 and M_2. The invariants are bounded by a Milnor–Wood type inequality which in this case is (e.g., see [5] or [23, Table C.2])

$$|\deg(M_i)| \leq 2g-2. \tag{29}$$

5.3 The induced map on the Higgs bundles

From Sect. 4.1 (see [1] for a detailed study of this), the map on $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$- Higgs bundles induced by $\mathcal{I}_2$ is

$$\mathcal{I}_2((E_1,\Phi_1),(E_2,\Phi_2)) = (E_1 \otimes E_2, \Phi_1 \otimes I + I \otimes \Phi_2), \tag{30}$$

where the orthogonal structure on $E_1 \otimes E_2$ is the tensor product of the symplectic structures ω_i on E_1 and E_2 (recall that $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Sp}(2,\mathbb{C})$).

Remark 13 Notice that the isomorphisms $\det(E_i) \simeq \mathcal{O}_\Sigma$ do not uniquely determine a trivialization $\delta : \det(E_1 \otimes E_2) \simeq \mathcal{O}_\Sigma$ compatible with the orthogonal structure on $E_1 \otimes E_2$. Indeed, if $\{e_i^1, e_i^2\}$ are local oriented frames for E_i satisfying $\omega_i(e_i^1, e_i^2) = 1$, then both $\{e_1^1 \otimes e_2^1, e_1^1 \otimes e_2^2, e_1^2 \otimes e_2^1, e_1^2 \otimes e_2^2\}$ and $\{e_1^1 \otimes e_2^1, e_1^2 \otimes e_2^1, e_1^1 \otimes e_2^2, e_1^2 \otimes e_2^2\}$ are orthonormal local frames for $E_1 \otimes E_2$ but they have opposite orientations. They determine the two inequivalent choices for $\delta : \det(E_1 \otimes E_2) \simeq \mathcal{O}_\Sigma$.

Following the notation of Sect. 5.1, the map $\mathcal{I}_2$ can be seen as

$$\begin{aligned}&\mathcal{I}_2((N_1 \oplus N_1^{-1}, \Phi_1), (N_2 \oplus N_2^{-1}, \Phi_2))\\ &\quad= \left((M_1 \oplus M_1^{-1}) \oplus (M_2 \oplus M_2^{-1}), \begin{pmatrix} 0 & \alpha \\ -\alpha^{\mathrm{T}} & 0 \end{pmatrix}\right),\end{aligned} \tag{31}$$

where $M_1 = N_1 \otimes N_2$ and $M_2 = N_1 \otimes N_2^{-1}$, and

$$\alpha = \begin{pmatrix} \beta_2 & \beta_1 \\ \gamma_1 & \gamma_2 \end{pmatrix}. \tag{32}$$

5.4 The induced map on spectral data

We shall first construct the spectral data $(\hat{S}_4, \mathcal{L})$ associated with the SO(4, $\mathbb{C}$)-Higgs bundle obtained via $\mathcal{I}_2$and then specialize to the spectral data associated with the split real forms SL(2, $\mathbb{R}$) × SL(2, $\mathbb{R}$) and $\mathrm{SO}_0(2, 2)$.

Proposition 14 *Let (S_i, L_i) be the spectral data corresponding to an* SL(2, $\mathbb{C}$) × SL(2, $\mathbb{C}$)-*Higgs bundle, where each S_i is defined by $\eta^2 + a_i = 0$ for $a_i \in H^0(\Sigma, K^2)$, and $L_i \in$* Prym*(S_i, Σ). Then, the pair $(\hat{S}_4, \mathcal{L})$ given by*

- *$\hat{S}_4 := S_1 \times_\Sigma S_2$ is the fiber product curve, and*
- *$\mathcal{L} = p_1^*(L_1) \otimes p_2^*(L_2)$ is the line bundle*

is the spectral data associated with an SO(4, $\mathbb{C}$)*-Higgs bundle.*

Remark 15 The total space of $K \oplus K$ can be identified with the fiber product $K \times_K K \subset K \times K$. The fiber product $S_1 \times_\Sigma S_2$ may thus be regarded as a subvariety of either $K \times_K K$ or of the total space of $K \oplus K$.

Proof The pair $(\hat{S}_4, \mathcal{L})$ shall be constructed as in diagram (3). The fiber product $\hat{S}_4$, as a curve in the total space of $K \oplus K$, is defined by the conditions

$$\eta_1^2 + a_1 = \eta_2^2 + a_2 = 0, \tag{33}$$

where (η_1, η_2) denotes the tautological section on $K \oplus K$. It follows that for a generic choice of (a_1, a_2) this curve is smooth. Moreover, if $S_4 \subset K$ denotes the image of $\hat{S}_4$ under the map

$$+ : K \oplus K \to K \tag{34}$$

given by fiberwise addition, then it is defined by the conditions

$$\eta = \eta_1 + \eta_2\ ;\ \eta_1^2 + a_1 = \eta_2^2 + a_2 = 0. \tag{35}$$

Hence, the fourfold cover S_4 is defined by the equation

$$\eta^4 + 2(a_1 + a_2)\eta^2 + (a_1 - a_2)^2 = 0, \tag{36}$$

which from Sect. 3.4 and Eq. (8), is the spectral curve of an $\mathrm{SO}(4,\mathbb{C})$-Higgs bundle. The curve S_4 is generically singular, with singularities over the zeros of $a_1 - a_2$, and by construction the map $+ : \hat{S}_4 \to S_4$ is an isomorphism on the smooth locus of S_4.

The involution $\sigma : \eta \mapsto -\eta$ which preserves S_1 and S_2 induces an involution (σ, σ) on $\hat{S}_4 = S_1 \times_\Sigma S_2$. When needed we shall denote the involution on S_1, S_2 by σ_i, for $i = 1, 2$ and on $\hat{S}_4$ by $\hat{\sigma}_4$. The fixed points of σ_i are the zeros of a_i, and thus, since the zeros of a_1 and a_2 are generically different, generically $\hat{\sigma}_4 = (\sigma_1, \sigma_2)$ does not have any fixed points. It is clear from (35) that $\hat{\sigma}_4$ descends to the involution $\eta \mapsto -\eta$ on the singular curve S_4, where it has fixed points at the branch locus.

In order to see that $\mathcal{L}$ is the spectral line bundle associated with an $\mathrm{SO}(4,\mathbb{C})$-Higgs bundle, one has to show that $\mathcal{L} \in \mathrm{Prym}(\hat{S}_4, \hat{S}_4/\hat{\sigma}_4)$. Since $L_i \in \mathrm{Prym}(S_i, \Sigma)$, one has that $\sigma_i^* L_i \cong L_i^*$ and so the line bundle $\mathcal{L} := p_1^*(L_1) \otimes p_2^*(L_2)$ is sent to its dual by the involution $\hat{\sigma}_4$. Hence, the line bundle $\mathcal{L}$ on $\hat{S}_4$ is in $\mathrm{Prym}(\hat{S}_4, \hat{S}_4/\hat{\sigma}_4)$ as required. □

Proposition 16 *The spectral data* $(\hat{S}_4, \mathcal{L})$ *induced by an* $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$*-Higgs bundle* $(E_1, \Phi_1), (E_2, \Phi_2)$*, as in Proposition 14, correspond to the spectral data of the* $\mathrm{SO}(4,\mathbb{C})$*-Higgs bundle* $\mathcal{I}_2[(E_1, \Phi_1), (E_2, \Phi_2)]$.

Proof As seen in Proposition 14, the spectral curve $\hat{S}_4$ is the curve associated with an $\mathrm{SO}(4,\mathbb{C})$-Higgs bundle. Furthermore, from Eq. (36), the curve is indeed the one associated with the Higgs bundle in the image of $(E_1, \Phi_1), (E_2, \Phi_2)$ through $\mathcal{I}_2$. With the notation of (39) below, in order to prove that the line bundle $\mathcal{L}$ is indued the spectral line bundle of the image $\mathrm{SO}(4,\mathbb{C})$-Higgs bundle, note that for any line bundles F_i on S_i one has

$$\pi_*(p_1^*(F_1) \otimes p_2^*(F_2)) = (\pi_1)_*(F_1) \otimes (\pi_2)_*(F_2).$$

Applying this to the line bundles L_i and using the relation (7) with $n = 2$, we get

$$\pi_*(\mathcal{L}) = (\pi_1)_*(L_1) \otimes (\pi_2)_*(L_2) = E_1 \otimes E_2 \otimes K^{-1},$$

whereas by (9) the vector bundle on Σ defined by $\mathcal{L}$ is

$$E = \pi_*(\mathcal{L} \otimes (K_{\hat{S}_4} \otimes \pi^* K^*)^{1/2}).$$

Recall that $K_{\hat{S}_4} \otimes \pi^* K^*$ corresponds to the ramification divisor $R \subset \hat{S}_4$, while the ramification divisors $R_i \subset S_i$ satisfy $[R_i] = \pi_i^* K$. It follows that

$$K_{\hat{S}_4} \otimes \pi^* K^* = [R] = p_1^*[R_1] \otimes p_2^*[R_2] = (\pi^* K)^2, \tag{37}$$

and hence that $E = \pi_*(\mathcal{L}) \otimes K = E_1 \otimes E_2$ as required.

5.5 The restriction to SL(2, $\mathbb{R}$) × SL(2, $\mathbb{R}$)

Let (S_i, L_i) be the spectral data of an $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle, for $i = 1, 2$. Then, one has that $\mathcal{L}^2 \cong p_1^2(L_1) \otimes p_2^2(L_2) \cong \mathcal{O}$, i.e.,

$$\mathcal{L} \in P_{\hat{\sigma}_4}[2] := \{M \in \mathrm{Prym}(\hat{S}_4, \hat{S}_4/\hat{\sigma}_4) \mid M^2 \cong \mathcal{O}\}. \tag{38}$$

Thus, as seen in Sect. 3.4 the line bundle $\mathcal{L}$ defines an $SO_0(2,2)$-Higgs bundle.

Since $\sigma_i^* L_i \simeq L_i^{-1} \simeq L_i$, it follows that $\hat{\sigma}_4^* \mathcal{L} \simeq \mathcal{L}^{-1} \simeq \mathcal{L}$. This means that under the projection $p : \hat{S}_4 \to \hat{S}_4/\hat{\sigma}_4$ the direct image sheaf $p_* \mathcal{L}$ splits as the sum of two line bundles $\mathcal{L}_\pm$, generated by $\hat{\sigma}_4$-invariant and anti-invariant local sections. The relation between the different covers of the Riemann surface and the line bundles on them is depicted in the following diagram:

$$
\begin{array}{ccc}
\mathcal{L} = p_1^* L_1 \otimes p_2^* L_2 & & p_* \mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- \\
\downarrow & & \downarrow \\
\hat{S}_4 = S_1 \times_{\Sigma} S_2 & \xrightarrow{\ p\ } & \hat{S}_4/\hat{\sigma}_4 \\
\downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \hat{\pi}} & \\
\Sigma & &
\end{array}
\tag{39}
$$

In particular, this implies that

$$\pi_* \mathcal{L} = (\hat{\pi})_* p_* \mathcal{L} = (\hat{\pi})_* (\mathcal{L}_+ \oplus \mathcal{L}_-). \tag{40}$$

Moreover, $\hat{S}_4$ is the normalization of the spectral curve $S_4 \subset \pi^* K$ and the involution $\hat{\sigma}_4$ on $\hat{S}_4$ corresponds to the involution $\eta \to -\eta$ on S_4. It follows that multiplication by the tautological section η interchanges $\mathcal{L}_+$ and $\mathcal{L}_-$, and thus that the Higgs field on $\pi_*(\mathcal{L})$ has the form as in (31). We thus get:

Proposition 17 *Consider* $((S_1, L_1), (S_2, L_2))$ *the spectral data for a point in* $\mathcal{M}_{\mathrm{SL}(2,\mathbb{R})\times \mathrm{SL}(2,\mathbb{R})}$ *represented by*

$$\left(\left(N_1 \oplus N_1^{-1}, \begin{bmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{bmatrix} \right), \left(N_2 \oplus N_2^{-1}, \begin{bmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{bmatrix} \right) \right),$$

and let $(\hat{S}_4, \mathcal{L})$ *be defined as in Proposition 14. Then* $(\hat{S}_4, \mathcal{L})$ *is the spectral data for the point in* $\mathcal{M}_{\mathrm{SO}_0(2,2)}$ *represented by*

$$\left((N_1 N_2 \oplus (N_1 N_2)^{-1}) \oplus N_1 N_2^{-1} \oplus (N_1^{-1} N_2)), \begin{bmatrix} 0 & \alpha \\ -\alpha^{\mathrm{T}} & 0 \end{bmatrix} \right),$$

where α *is as in Eq. (32).*

6 The rank 3 isogeny and the Hitchin fibration

We shall investigate now the induced map $\mathcal{I}_3 : \mathcal{M}_{\mathrm{SL}(4,C)} \to \mathcal{M}_{\mathrm{SO}(6,\mathbb{C})}$, and its restriction to the split real forms $\mathrm{SL}(4,\mathbb{R})$ and $\mathrm{SO}_0(3,3)$.

6.1 SL(4, ℂ)-Higgs bundles and SL(4, ℝ)-Higgs bundles

From Definition 4, an $\mathrm{SL}(4,\mathbb{R})$-Higgs bundle on Σ is a holomorphic $\mathrm{SO}(4,\mathbb{C})$-principal bundle together with a symmetric Higgs field. Equivalently it can be viewed as a pair (E, Φ) where E is an oriented holomorphic rank 4 orthogonal vector bundle, i.e., a vector bundle with a holomorphic symmetric non-degenerate bilinear paring Q, and a compatible isomorphism $\delta : \det(E) \simeq \mathcal{O}$, and the Higgs field $\Phi : E \to E \otimes K$ is traceless and symmetric with respect to Q.

Recall from Sect. 6.1 that the spectral curve for an $\mathrm{SL}(4,\mathbb{C})$-Higgs bundle (E, Φ) is a ramified fourfold cover $\pi : S \to \Sigma$ in the total space of K with equation

$$P_4(\eta) := \det(\eta I - \Phi) = \eta^4 + a_2 \eta^2 + a_3 \eta^3 + a_4 = 0, \tag{41}$$

for $a_i \in H^0(\Sigma, K^i)$ and η the tautological section of $\pi^* K$. For generic choices of $\{a_2, a_3, a_4\}$, the curve is smooth and has only the most generic ramification, i.e., in fibers over the branch locus there are two unramified points and one order two ramification point. The spectral data are completed by a line bundle $L \in \text{Prym}(S, \Sigma)$. For an $\text{SL}(4,\mathbb{R})$-Higgs bundle, from [25, Theorem 4.12] the spectral line bundle $L \in \text{Prym}(S, \Sigma)$ satisfies the extra condition $L^2 \cong \mathcal{O}$.

6.2 SO(6, $\mathbb{C}$)-Higgs bundles and $\text{SO}_0(3,3)$-Higgs bundles

An $SO_0(3,3)$-Higgs bundle can be described as $\text{SO}(6,\mathbb{C})$-Higgs bundle (E, Φ) where E decomposes as the sum of two rank 3 holomorphic oriented orthogonal bundles, say $E = W_1 \oplus W_2$ with orthogonal structures q_i and compatible isomorphisms $\delta_i : \Lambda^3 W_i \cong \mathcal{O}$. As seen in Sects. 3.3–3.4, for any $\text{SO}(6,\mathbb{C})$-Higgs bundle the Higgs field Φ defines a sixfold cover $\pi_6 : S_6 \to \Sigma$ with equation

$$P_6(\eta) := \det(\eta I - \Phi) = \eta^6 + b_1\eta^4 + b_2\eta^2 + b_3^2 = 0, \tag{42}$$

where b_3 is the Pfaffian and $b_i \in H^0(\Sigma, K^i)$. The spectral data of an $\text{SO}(6,\mathbb{C})$-Higgs bundle are then a pair $(\hat{S}_6, L_6)$ where $\hat{S}_6$ is the desingularization of S_6 and $L_6 \in \text{Prym}(\hat{S}_6, \hat{S}_6/\hat{\sigma})$ where $\hat{\sigma}$ is the (fixed-point-free) involution inherited from S_6 [14]. When $L_6^2 \cong \mathcal{O}$, the spectral data correspond to an $\text{SO}_0(3,3)$-Higgs bundle.

6.3 The induced map on Higgs bundles

Using the results in Sect. 4.2 for the induced action of $\mathcal{I}_3$ on vector bundles and Lie algebras, we get the map between complex Higgs bundles

$$\mathcal{I}_3(E, \Phi) = (\Lambda^2 E, \Phi \otimes I + I \otimes \Phi). \tag{43}$$

The orthogonal structure $Q : \Lambda^2 E \to \Lambda^2 E^*$ is induced by the combination of (19) and the trivialization of $\det(E)$. As in the case of the map defined by $\mathcal{I}_2$ (see Remark 13) an isomorphism $\delta : \det(\Lambda^2 E) \to \mathcal{O}_\Sigma$ such that δ^2 agrees with the trivialization of $(\det(\Lambda^2 E))^2$ determined by Q, is determined only up to a choice of sign. There are thus two (oppositely oriented) possible conventions for determining the $\text{SO}(6,\mathbb{C})$ structure on $\Lambda^2 E$. This choice plays a role in the map induced by $\mathcal{I}_3$ on the base of the Hitchin fibrations of the moduli spaces [see Eq. (73)].

The vector bundle E of an $\text{SL}(4,\mathbb{R})$-Higgs bundle (E, Φ) has an oriented orthogonal structure, i.e., an associated pair (q, ϵ) where q is a holomorphic orthogonal structure on E, and ϵ is a compatible isomorphism $\epsilon : \det(E) \simeq \mathcal{O}$ trivializing its determinant. The orthogonal structure induces an isomorphism (by abuse of notation, also denoted by q)

$$q : \Lambda^2 E \to \Lambda^2 E^*. \tag{44}$$

Using ϵ as the trivialization of $\det(E)$ required in the construction of Q yields an isomorphism

$$* = q^{-1} \cdot Q : \Lambda^2 E \to \Lambda^2 E, \tag{45}$$

which satisfies[4]

$$q(\alpha, \beta) = Q(\alpha, *\beta), \tag{46}$$

where q, Q are regarded as bilinear forms on $\Lambda^2 E$, and $\alpha, \beta \in \Lambda^2 E$. By using a local oriented orthonormal frame to compute $*$, it can be seen that $*$ satisfies $*^2 = I$. Taking the ± 1 eigenspaces of $*$ thus gives a decomposition

[4] We have denoted the involution by $*$ since when E is the cotangent bundle to a 4-manifold, the involution is precisely the Hodge star.

$$\Lambda^2 E = \Lambda^2_+ E \oplus \Lambda^2_- E. \tag{47}$$

The orthogonal structure on $\Lambda^2 E$ restricts to orthogonal structures on each summand, so that the structure group reduces to $\mathrm{SO}(3,\mathbb{C}) \times \mathrm{SO}(3,\mathbb{C})$.

Remark 18 The structure groups of the bundles $\Lambda^2 E$ and $\Lambda^2_+ E \oplus \Lambda^2_- E$ can be reduced to SO(4) and SO(3) × SO(3), respectively. The two copies of SO(3) are precisely the normal subgroups mentioned in Remark 11 by virtue of which SO(4) fails to be simple (see [10]) .

With respect to the reduction in (47), the Higgs field $\Phi \otimes I + I \otimes \Phi$ has the form in (27), where Φ and α are related as in Eq. (24) and $-\alpha^t$ is the orthogonal transpose. Denoting orthogonal structures on $\Lambda^2_\pm E$ by $q_\pm$, we thus get:

Proposition 19 *The isogeny $\mathcal{I}_3$ induces the following map between* $\mathrm{SL}(4,\mathbb{R})$*- Higgs bundles and* $\mathrm{SO}_0(3,3)$*-Higgs bundles:*

$$(E,\Phi) \mapsto \left(\Lambda^2_+ E \oplus \Lambda^2_- E, \begin{bmatrix} 0 & \alpha \\ -\alpha^{\mathrm{T}} & 0 \end{bmatrix}\right), \tag{48}$$

where if E has orthogonal structure q then the bundles $\Lambda^2_\pm E$ have orthogonal structures $q_\pm$, and Φ and α are related as in Eq. (23)*.*

Remark 20 The precise form of α depends on the choices of the orientations of $\Lambda^2 E$ and $\Lambda^2_\pm E$. Different choices will change the sign of $\det \alpha$, i.e., of the Pfaffian of $\begin{bmatrix} 0 & \alpha \\ -\alpha^{\mathrm{T}} & 0 \end{bmatrix}$.

6.4 The induced map on spectral data

Given $\mathrm{SL}(4,\mathbb{C})$-spectral data (S, L), with S defined by (41) and $L \in \mathrm{Prym}(S, \Sigma)$, we build $\mathrm{SO}(6,\mathbb{C})$-spectral data using a construction similar to the fiber product construction in Sect. 5.4, except in this case we take the product of (S, L) with itself, i.e., in diagram (3) we have $S_1 = S_2$ and $L_1 = L_2$. The resulting curve has both singularities and additional symmetries that are absent when S_1 and S_2 are different. Our construction takes both of these features into account in an essential way.

The curve $S \times_\Sigma S$ is a 16-fold cover of the Riemann surface Σ. Over a generic point in the Hitchin base, S is smooth and $S \times_\Sigma S$ has two smooth components, namely the diagonal $S_\Delta := \{(s,s) \in S \times_\Sigma S\}$ and another one which we denote by $(S \times_\Sigma S)_0$. The intersection of these components lies in fibers over the branch locus of the covering $\pi : S \to \Sigma$.

Viewing the curve $S \times_\Sigma S$ in the total space of $K \oplus K$, the involution $\tau : (x,y) \mapsto (y,x)$ interchanges the copies of S, and thus, the fixed point set of τ is S_Δ. The quotient map $\pi_\tau : (S \times_\Sigma S)_0 \to (S \times_\Sigma S)_0/\tau$ commutes with the projection onto Σ. It is an unramified double cover on $S \times_\Sigma S - S_\Delta$ but has ramification points in the fibers over the base locus of $\pi : S \to \Sigma$. Using the biholomorphism $Sym : (K \oplus K)/\tau \to K \oplus K^2$ given by

$$Sym : (x,y) \mapsto \left(\frac{x+y}{2}, xy\right), \tag{49}$$

we can view the quotient $(S \times_\Sigma S)_0/\tau$ as a curve in the total space of $K \oplus K^2$. We define

$$\hat{S}_6 := Sym((S \times_\Sigma S)_0/\tau), \tag{50}$$

and denote by $\hat{\pi}_\tau$ the composition of π_τ and Sym. A depiction of the relation between the above curves and projections is given in diagram (51) below. By abuse of notation, we denote by $p_i : (S \times_\Sigma S)_0 \to \Sigma$ the restrictions to $(S \times_\Sigma S)_0$ of the projection maps to the two factors of the full fiber product:

$$\begin{array}{ccccc} S & \overset{p_1}{\underset{p_2}{\Longleftarrow}} & (S \times_\Sigma S)_0 & \xrightarrow{\hat{\pi}_\tau} & \hat{S}_6 \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle \pi_0} & \swarrow{\scriptstyle \hat{\pi}_6} & \\ & & \Sigma & & \end{array} \tag{51}$$

Lemma 21 *For generic points in the Hitchin base, the ramification divisors R on S, R_0 on $(S \times_\Sigma S)_0$ and $\hat{R}_6$ on $\hat{S}_6$ for the projections π, $\hat{\pi}_\tau$ and $\hat{\pi}_6$, respectively, are related as follows:*

$$p_1^{-1}(R) + p_2^{-1}(R) = (\hat{\pi}_\tau)^{-1}(\hat{R}_6) + 2R_0. \tag{52}$$

Proof The ramification divisor for a covering $\pi : X \to Y$ is defined by $R = \Sigma_{y \in Y} R(y) \cdot y$ where the weights $R(y)$ are such that $R(y) + 1$ is the multiplicity of π at y. If S is a generic spectral curve and x is a point in the branch locus of $\pi : S \to \Sigma$, then we can write $\pi^{-1}(x) = \{y_1, y_2, y_3\}$, with $y_1 \in R$ (with weight 1) but π unramified at y_2, y_3. Then, the fiber of $(S \times_\Sigma S)_0/\tau$ over x consists of points $\{[y_1, y_1], [y_1, y_2], [y_1, y_3], [y_2, y_3]\}$. Of these, $Sym[y_1, y_2]$ and $Sym[y_1, y_3]$ land in $\hat{R}_6$, each with weight 1, while $\hat{\pi}_\tau^{-1} Sym([y_1, y_1]) = (y_1, y_1)$ is in R_0 (with weight 1). Notice now that in the fiber over x one has

$$\begin{aligned} &\pi_0^{-1}(x) \cap (p_1^{-1}(R) + p_2^{-1}(R)) \\ &\quad = 2(y_1, y_1) + (y_1, y_2) + (y_2, y_1) + (y_1, y_3) + (y_3, y_1) \\ &\quad = 2(y_1, y_1) + \hat{\pi}_\tau^{-1}(Sym([y_1, y_2]) + Sym([y_1, y_3])), \end{aligned}$$

as required.

Proposition 22 *The curve $\hat{S}_6$ is generically smooth and gives the canonical desingularization of its projection to K through the Sym map, which is the spectral curve of an* $\mathrm{SO}(6, \mathbb{C})$*-Higgs bundle.*

Proof In order to prove the proposition, one needs to show the following hold:

1. the curve $\hat{S}_6$ is generically smooth;
2. under the projection $q_1 : K \oplus K^2 \to K$ which on each fiber is given by $(u, v) \mapsto 2u$, the image $S_6 := q_1(\hat{S}_6)$ is a spectral curve defined by an equation of the form in (42) with

$$b_1 = 2a_2, b_2 = a_2^2 - 4a_4, b_3^2 = a_3^2; \tag{53}$$

3. the projection $q_1 : \hat{S}_6 \to S_6$ is an isomorphism away from the singularities of the curve S_6 at its intersection with the zero section of K.

To prove the above items, let $S \subset K$ be defined by the zero locus of (41). As a curve in the total space of $K \oplus K$, the fiber product $S \times_\Sigma S$ is defined by the conditions $P_4(\eta_1) = P_4(\eta_2) = 0$, where η_1, η_2 denote the tautological sections of the two summands in $K \oplus K$. After using the transformation *Sym* defined in (49) to realize the curve as a subvariety of $K \oplus K^2$, the component $\hat{S}_6$ is described locally (i.e., with respect to a trivialization of the bundles) as the zero locus of the map $\mathcal{F} : \mathbb{C}^3 \to \mathbb{C}^2$ given by $(z, u, v) \mapsto (8u^3 - 4uv + 2a_2(z)u + a_3(z), 8u^4 + 2a_2(z)u^2 - 8u^2v - a_2(z)v + a_3(z)u + v^2 + a_4(z))$. The projection q_1 onto the total space of K is given locally by $(z, u, v) \mapsto (z, 2u)$. Direct computation shows that away from $u = 0$, the map is a biholomorphism onto the curve defined by the equation

$$P_6(z,\eta) = \eta^6 + 2a_2(z)\eta^4 + (a_2(z)^2 - a_4(z))\eta^2 + a_3^2 = 0, \tag{54}$$

where η is the local fiber coordinate on K. This curve, which we denote by S_6, necessarily has singularities at the zeros of a_3, but is otherwise smooth for generic choice of a_2, a_3, a_4. For such choices, singularities in $\hat{S}_6$ can occur only at points where $u = 0$. Direct computation shows that the derivative of $\mathcal{F}$ has full rank at such points provided a_3 and $a_2^2 - a_4$ have no common zeros, proving the proposition.

Lemma 23 *The involution $\eta \to -\eta$ on S_6 lifts to the involution $\sigma : \hat{S}_6 \to \hat{S}_6$ which, away from the branch locus of $\pi : S \to \Sigma$, is given by the map*

$$Sym[y_i, y_j] \mapsto Sym[y_k, y_l],$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. When there is a ramification point, say y_1, the involution maps $Sym[y_1, y_1]$ to $Sym[y_2, y_3]$.

Proof From diagram (51), for $x \in \Sigma$ away from the base locus of $\pi : S \to \Sigma$, we can write $\pi^{-1}(x) = \{y_1, y_2, y_3, y_4\}$, and the coordinate η on S_6 has values $y_i + y_j$ for $i \neq j$. But $y_1 + y_2 + y_3 + y_4 = 0$ and hence $-(y_i + y_j) = (y_k + y_l)$ where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, i.e., $\eta \mapsto -\eta$ corresponds to the action of σ. Assuming that only the most generic type of ramification occurs, the computation is similar for $x \in \Sigma$ in the branch locus of π. □

To obtain the correspondence induced by $\mathcal{I}_3$ between spectral line bundles, we begin as in Sect. 5.4, but with $L_1 = L_2$, and consider

$$\mathcal{L} = p_1^*(L) \otimes p_2^*(L). \tag{55}$$

Since $\mathcal{L}$ is invariant under τ: $(x, y) \mapsto (y, x)$, its direct image on $\hat{S}_6$ decomposes as a sum of rank one locally free sheaves generated by the invariant and anti-invariant local sections, i.e.,

$$(\hat{\pi}_\tau)_*\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-. \tag{56}$$

Similarly

$$(\hat{\pi}_\tau)_*\mathcal{O}_{(S\times_\Sigma S)_0} = \mathcal{O}_{\hat{S}_6} \oplus T. \tag{57}$$

Moreover, if $[R_0]$ is the line bundle on $(S \times_\Sigma S)_0$ defined by the divisor R_0, then $(\hat{\pi}_\tau)^*T = [R_0]^{-1}$ (see, for example, [17, Lemma 3.1] or [7, p. 49]). We define

$$\mathcal{I}_3(L) := \mathcal{L}_- \otimes T^{-1}. \tag{58}$$

Remark 24 While we have defined $\mathcal{L}$ only on $(S \times_\Sigma S)_0 \subset S \times_\Sigma S$, this distinction disappears in $\mathcal{L}_-$. This is a consequence of the fact that anti-invariant local sections must vanish on the fixed points of τ, so that the sheaf on $(S \times_\Sigma S)/\tau$ generated by the anti-invariant sections has support only on $(S \times_\Sigma S)_0/\tau$.

Remark 25 Since $\mathcal{L}$ is clearly invariant under pullback by the involution τ and admits a lift of τ which acts as identity on fibers over all fixed points, it follows (see, for example, [19]) that $\mathcal{L} = \hat{\pi}_\tau^{-1}\mathcal{M}$ for some line bundle $\mathcal{M}$ on $\hat{S}_6$. We thus get

$$(\hat{\pi}_\tau)_*\mathcal{L} = (\hat{\pi}_\tau)_*((\hat{\pi}_\tau)^*\mathcal{M}) = \mathcal{M} \oplus (\mathcal{M} \otimes T),$$

with the summands $\mathcal{M}$ and $\mathcal{M} \otimes T$ generated by the τ-invariant and anti-invariant local sections of $(\hat{\pi}_\tau)_*(\hat{\pi}_\tau)^*\mathcal{M}$, respectively. With this choice of $\mathcal{M}$, we get

$$\mathcal{I}_3(L) = \mathcal{M}. \tag{59}$$

Proposition 26 *If* $L \in \mathrm{Prym}(S, \Sigma)$, *then* $\mathcal{I}_3(L) \in \mathrm{Prym}(\hat{S}_6, \hat{S}_6/\sigma)$.

Proof Let L be defined by the divisor D on S given by

$$D = \sum_{x\in\Sigma} \sum_{y\in\pi_0^{-1}(x)} D(y)\cdot y := \sum_{x\in\Sigma} D_x.$$

For $\mathcal{D}$ the divisor $p_1^*(D) + p_2^*(D)$ on $(S \times_\Sigma S)_0$, one has that $\mathcal{D}$ is in the linear system of $\mathcal{L}$. Moreover, when x is not in the branch locus of π_0, so that $\pi_0^{-1}(x)$ has four distinct points $\{y_1, y_2, y_3, y_4\}$, one has that

$$\begin{aligned} \mathcal{D}_x &:= \mathcal{D} \cap \pi_0^{-1}(x) \\ &= \hat{\pi}_\tau^*\Big(\sum_{i\neq j}(D(y_i) + D(y_j))\cdot Sym([y_i, y_j])\Big) \\ &:= \hat{\pi}_\tau^*\mathcal{C}_x. \end{aligned} \tag{60}$$

On the other hand, if x is a branch point, then under our genericity assumptions on S, we can assume that $\pi_0^{-1}(x)$ has one point (say y_1) where π_0 has multiplicity two, and two points (say y_2, y_3) where π_0 is unramified. Then, $\mathcal{D}_x = \hat{\pi}_\tau^*\mathcal{C}_x$ with

$$\begin{aligned} \mathcal{C}_x &= D(y_1)\cdot Sym[y_1, y_1] + (D(y_2) + D(y_3))\cdot Sym[y_2, y_3] \\ &\quad + \left[\sum_{i=2,3}(D(y_1) + 2D(y_i))\cdot Sym[y_1, y_i\right] \end{aligned} \tag{61}$$

where the factor 2 in the last term comes from the fact that the projection p_1 is ramified at the points (y_1, y_2) and (y_1, y_3), while p_2 is ramified at the points (y_2, y_1) and (y_3, y_1). Moreover, from Remark 25 one has that $\mathcal{L} = \hat{\pi}_\tau^*(\mathcal{M})$ and $\mathcal{I}_3(L) = \mathcal{M}$, and hence, $\mathcal{C}$ is defined by (60), and (61) is a divisor in the linear system of $\mathcal{I}_3(L)$. Notice that on fibers over branch points of π_0 one has $\sigma[y_1, y_1] = [y_2, y_3]$. Then, denoting the branch locus of π_0 by B, one has that

$$Nm_\sigma(\mathcal{C}_x) = (D(y_1) + D(y_2) + D(y_3) + D(y_4))\cdot \sum_{i=2}^{4}[[y_1, y_i]] \tag{62}$$

if $x \in \Sigma - B$; otherwise, if $x \in B$, the norm is

$$Nm_\sigma(\mathcal{C}_x) = (D(y_1) + D(y_2) + D(y_3))\cdot([[y_1, y_1]] + 2[[y_1, y_2]]),$$

where $[[y_1, y_2]]$ denotes the point in $\hat{S}_6/\sigma$ and Nm_σ is the norm map for the covering $\hat{S}_6 \to \hat{S}_6/\sigma$. Finally, note that if $L \in Prym(S, \Sigma)$, then we can pick D so that $Nm_{\pi_0}(D_x) = 0$ for all $x \in \Sigma$ and hence $Nm_\sigma(\mathcal{I}_3(L)) = Nm_\sigma(\mathcal{C}) = 0$, as required. □

We have shown that the map $(S, L) \mapsto (\hat{S}_6, \mathcal{I}_3(L))$ sends spectral data for an $\mathrm{SL}(4,\mathbb{C})$-Higgs bundle to $\mathrm{SO}(6,\mathbb{C})$-spectral data. We now show that this map is compatible with the map given by (43).

Proposition 27 *The spectral data* $(\hat{S}_6, \mathcal{I}_3(L))$ *induced by an* $\mathrm{SL}(4,\mathbb{C})$*-Higgs bundle* (E, Φ) *via* (50) *and* (58) *correspond to the spectral data of the* $\mathrm{SO}(6,\mathbb{C})$*-Higgs bundle* $\mathcal{I}_3[(E, \Phi)]$.

Proof By (43) the Higgs field in $\mathcal{I}_3[(E,\Phi)]$ is $\Phi_6 := \Phi\otimes I + I\otimes\Phi$, viewed as a map on $\Lambda^2 E\subset E\otimes E$. Let $\eta^4+a_2\eta^2+a_3\eta+a_4$ and $\eta^6+b_2\eta^4+b_4\eta^2+b_3^2$ be the characteristic polynomials for Φ and Φ_6, respectively. Then, a calculation based on Remark 10 shows that the coefficients are related by (53). It follows that the curve $\hat{S}_6$ is the spectral curve for the SO(6, $\mathbb{C}$)-Higgs bundle $\mathcal{I}_3[(E,\Phi)]$. Moreover, Proposition 22 then shows that Φ_6 can be recovered by pushing down (η,η^2), where η is tautological section of K.

The direct image $\pi_*(\mathcal{I}_3(L)\otimes[\hat{R}_6]^{1/2})$ acquires a special orthogonal structure in the usual way, with the trivialization of its determinant bundle determined by the Prym condition $\sigma^*(\mathcal{I}_3(L))\simeq\mathcal{I}_3(L)^*$. In order to see that $\pi_*(\mathcal{I}_3(L)\otimes[\hat{R}_6]^{1/2})=\Lambda^2E$, consider the (singular) full fiber product $S\times_\Sigma S$ and for any line bundle N on S let $\tilde{\mathcal{N}}=p_1^*(N)\otimes p_2^*(N)$, where now the projections are from the entire $S\times_\Sigma S$. Arguing as before, since $\tilde{\mathcal{N}}$ is invariant under the involution τ, it follows that its direct image under the projection $\pi_\tau : S\times_\Sigma S\to (S\times_\Sigma S)/\tau$ decomposes as the sum of rank one coherent sheaves $\tilde{\mathcal{N}}_-\oplus\tilde{\mathcal{N}}_+$. Though in principle these summands need not be locally free, their restriction to $(S\times_\Sigma S)_0$ is well behaved. Indeed, given the projections $\pi : S\to\Sigma$ and $\hat{\pi} : (S\times_\Sigma S)/\tau\to\Sigma$, one gets

$$\hat{\pi}_*(\tilde{\mathcal{N}}_+)\oplus\hat{\pi}_*(\tilde{\mathcal{N}}_-) = Sym^2(\pi_*(p_1^*(N)\otimes p_2^*(N)))\oplus\Lambda^2(\pi_*(p_1^*(N)\otimes p_2^*(N))).$$

But the left- and right-hand sides of this identification are both ±1-eigenspace decompositions for $\mathbb{Z}_2$-actions that are compatible with the projection map. We can thus identify $\hat{\pi}_*(\tilde{\mathcal{N}}_-)\simeq\Lambda^2(\pi_*(p_1^*(N)\otimes p_2^*(N)))$. Notice, furthermore, that the support of $\tilde{\mathcal{N}}_-$ is $(S\times_\Sigma S)_0$, so that $\tilde{\mathcal{N}}_-=\mathcal{N}_-$ where dropping the tilde on $\tilde{\mathcal{N}}$ denotes the restriction to $(S\times_\Sigma S)_0$. It follows that

$$\hat{\pi}_*\mathcal{N}_- = \Lambda^2(\pi_*N). \tag{63}$$

In particular, taking $N=L\otimes[R]^{1/2}$ where R is the ramification divisor on S, so that $\pi_*N=E$, it follows that (63) gives $\hat{\pi}_*\mathcal{N}_-=\Lambda^2E$.

It thus remains to show that $\mathcal{N}_-=\mathcal{I}_3(L)\otimes[\hat{R}_6]^{1/2}$. But by Lemma 21 and (52)

$$\begin{aligned}\mathcal{N} &= \mathcal{L}\otimes[p_1^{-1}(R)+p_2^{-1}(R)]^{1/2}\\ &= \pi_\tau^*(\mathcal{M}\otimes[\hat{R}_6]^{1/2}\otimes T^{-1}),\end{aligned}$$

where $\mathcal{M}$ is as in Remark 25 and T is as above. It follows that

$$\mathcal{N}_- = \mathcal{M}\otimes[\hat{R}_6]^{1/2}\otimes T^{-1}\otimes T = \mathcal{I}_3(L)\otimes[\hat{R}_6]^{1/2}, \tag{64}$$

as required.

We shall end this section with a discussion of the relation between our construction and the so-called trigonal construction of Recillas (see [8] or [22]). Given a smooth curve Σ, this construction relates a smooth fourfold cover of Σ to a sixfold cover with a fixed-point-free involution, whose quotient is thus a threefold smooth cover. Taking S as the fourfold cover, the curve $\mathbb{S} := Sym^{-1}(\hat{S}_6) = (S\times_\Sigma S)_0/\tau$ is precisely the corresponding sixfold cover with involution. This is most easily seen by considering the fibers of the covering maps onto Σ. At a regular fiber of S over a point $x\in\Sigma$ consisting of points $\{y_1(x), y_2(x), y_3(x), y_4(x)\}$, the fiber of $(S\times_\Sigma S)_0/\tau$ over x consists of the points corresponding to the unordered pairs $\{[y_1,y_2],[y_1,y_3],[y_1,y_4],[y_2,y_3],[y_2,y_4],[y_3,y_4]\}$ (where we have dropped the dependence on x to simplify the notation).

Considering $\pi : S\to\Sigma$ a smooth cover with only the most generic ramification, if x is in the branch locus and $\pi^{-1}(x)=\{y_1,y_2,y_3\}$ (as in the proof of Proposition 26), then the fiber of $\mathbb{S}$ consists of the unordered pairs $\{[y_1, y_1], [y_1, y_2], [y_1, y_3], [y_2, y_3]\}$. This relation between S and $\mathbb{S}$ can be regarded as a map from S to the third symmetric product of $\mathbb{S}$, mapping a point in S to the unordered pairs containing that point. This determines a correspondence defined by an effective divisor on $S \times \mathbb{S}$, but for the sake of comparison with our construction, we use the map *Sym* to replace $\mathbb{S}$ with $\hat{S}_6$.

Definition 28 Define the effective divisor $\Delta \subset S \times \hat{S}_6$ by

$$\Delta = \sum_{y \in S,\ [y,y'] \in \mathbb{S}} (y, Sym[y, y']). \tag{65}$$

Notice that Δ is contained in $S \times_\Sigma \hat{S}_6$ and is given explicitly by

$$\begin{aligned}\Delta = &\sum_{x \in \Sigma - B} \sum_{i=1}^{4} \sum_{j \neq i} \Big(y_i(x), Sym[y_i(x), y_j(x)] \Big) \\ &+ \sum_{x \in B} \left(\sum_{j=1}^{3} (y_1(x), Sym[y_1(x), y_j(x)]\Big) \right. \\ &\left. + \sum_{i=2}^{3} \sum_{j \neq i} \Big(y_i(x), Sym[y_i(x), y_j(x)] \Big) \right).\end{aligned}$$

Here the set $B \subset \Sigma$, as in (62), is the branch locus of π, and at $x \in B$ we label the points in the fiber so that π has degree 2 at $y_1(x)$.

Remark 29 The map

$$\begin{aligned} C : (S \times_\Sigma S)_0 &\rightarrow S \times \hat{S}_6 \\ (y_i(x), y_j(x)) &\mapsto \Big(y_i(x), Sym[y_i(x), y_j(x)] \Big), \end{aligned}$$

is injective, and its image is precisely the divisor Δ.

Using the correspondence defined by Δ, we may define

$$\begin{aligned} C_\Delta : Jac(S) &\rightarrow Jac(\hat{S}_6) \\ [D] &\mapsto [Nm_2(\pi_1^*(D) \cap \Delta)] \end{aligned}$$

for π_1 the projection $S \times \hat{S}_6 \rightarrow S$, and the norm map $Nm_2 : Jac(S \times \hat{S}_6) \rightarrow Jac(\hat{S}_6)$.

Proposition 30 *The map C_Δ restricts to a map between Prym varieties, i.e.,*

$$C_\Delta : Prym(S, \Sigma) \rightarrow Prym(\hat{S}_6, \hat{S}_6/\sigma).$$

This map agrees with the map defined by virtue of Proposition 26, i.e., with $L \mapsto \mathcal{I}_3(L)$ given by (58).

Proof Let $L \in Jac(S)$ be defined by divisor D on S. With $\mathcal{C}$ as in the proof of Proposition 26, it follows from (60) and (61) that

$$\mathcal{C} = Nm_2(\pi_1^*(D) \cap \Delta),$$

and hence (as in the proof of Proposition 26) that $\mathcal{C}$ is in the linear system for $\mathcal{I}_3(L)$. It thus follows from Proposition 26 that C_Δ defines a map between the indicated Prym varieties. □

6.5 The restriction to the split real form SL(4, $\mathbb{R}$)

We now examine the consequences of imposing the additional condition $L^2 = \mathcal{O}_S$ on the spectral data (S, L) described in Sect. 6.4.

Proposition 31 *Under the assumptions and notation of Proposition 26, if $L^2 \simeq \mathcal{O}_S$, then $\mathcal{I}_3(L)$ is a point of order two in $Jac(\hat{S}_6)$.*

Proof The map $L \mapsto \mathcal{I}_3(L)$ defined by (58) is a group homomorphism between Jacobians. Thus, in particular, if $L^2 \simeq \mathcal{O}_S$ then $\mathcal{I}_3(L)^2 = \mathcal{I}_3(L^2) = \mathcal{I}_3(\mathcal{O}_S) = \mathcal{O}_{\hat{S}_6}$. □

It follows (see [25, Theorem 4.12]) that our construction maps spectral data for SL(4, $\mathbb{R}$)-Higgs bundles to spectral data for $\mathrm{SO}_0(3,3)$-Higgs bundles. As in the rank two case discussed in Sect. 5.5, if $\mathcal{I}_3(L)$ is a point of order two in $Prym(\hat{S}_6, \hat{S}_6/\sigma)$, then it is invariant under the involution σ. Under the projection $p : \hat{S}_6 \to \hat{S}_6/\sigma$, the direct image sheaf $p_*\mathcal{I}_3 L$ thus splits as the sum of two line bundles $\mathcal{I}_3(L)_\pm$, generated by σ-invariant and anti-invariant local sections. In particular,

$$\hat{\pi}_{6*}\mathcal{I}_3(L) = \pi_{\sigma *}\mathcal{I}_3(L)_+ \oplus \pi_{\sigma *}\mathcal{I}_3(L)_-. \tag{66}$$

We get a diagram similar to (39):

$$\begin{array}{ccc}
\mathcal{I}_3(L) & & p_*(\mathcal{I}_3(L)) = \mathcal{I}_3(L)_+ \oplus \mathcal{I}_3(L)_- \\
\downarrow & & \downarrow \\
\hat{S}_6 & \xrightarrow{\ p\ } & \hat{S}_6/\sigma \\
\downarrow{\scriptstyle \hat{\pi}_6} & \swarrow{\scriptstyle \pi_\sigma} & \\
\Sigma & &
\end{array} \tag{67}$$

Recall from Proposition 27 that $\hat{S}_6$ is the normalization of the spectral curve $S_6 \subset \pi^*K$ and that the involution σ on $\hat{S}_6$ corresponds to the involution $\eta \mapsto -\eta$ on S_6. It follows that multiplication by the tautological section η interchanges $\mathcal{I}_3(L)_-$ and $\mathcal{I}_3(L)_+$, and thus that the Higgs field on $\hat{\pi}_{6*}\mathcal{I}_3(L)$ has the form as in (48). Combined with Proposition 19, we thus get:

Proposition 32 *Let (S, L) be the spectral data for a point in $\mathcal{M}_{\mathrm{SL}(4,\mathbb{R})}$ represented by a Higgs bundle (E, Φ) with orthogonal structure q on E, and isomorphism $\delta : \det E \simeq \mathcal{O}_\Sigma$. Let $(\hat{S}_6, \mathcal{I}_3(L))$ be defined as (50) and (58), i.e., $\hat{S}_6 := Sym((S \times_\Sigma S)_0/\tau)$ and $\mathcal{I}_3(L) = \mathcal{L}_- \otimes T^{-1}$.*
Then $(\hat{S}_6, \mathcal{I}_3(L))$ is the spectral data for the point in $\mathcal{M}_{\mathrm{SO}_0(3,3)}$ represented by

$$\left(\Lambda^2_+ E \oplus \Lambda^2_- E, \begin{pmatrix} 0 & \alpha \\ -\alpha^{\mathrm{T}} & 0 \end{pmatrix}\right), \tag{68}$$

where the bundles have oriented orthogonal structures $(q_\pm, \delta_\pm)$ as in Sect. 6.3, and α is as in (48).

7 Maps between moduli spaces and Hitchin fibrations

Thus far, we have examined the maps induced by the isogenies on individual Higgs bundles and their spectral data. In this section, we collect together some remarks about the induced maps on the corresponding moduli spaces and on their Hitchin fibrations. We note that the maps on Higgs bundles (given in Sect. 5.3 and Proposition 19) are defined for all Higgs bundles, but do not obviously preserve stability properties. On the other hand, the maps on spectral data (see Propositions 14 and 27) automatically preserve stability but apply only to generic points in the moduli spaces—where the stability condition is vacuous. Throughout this section, we shall limit our study to the dense open sets in the moduli spaces which exclude the non-generic fibers of their Hitchin fibrations, for which stability is automatically obtained (see Remark 7). We denote these sets by $\tilde{\mathcal{M}}_G \subset \mathcal{M}_G$.

Note that for the groups in the isogenies studied in this paper, the special linear groups can be identified as the spin groups for the special orthogonal groups. The induced map $\tilde{\mathcal{M}}_{\mathrm{Spin}(2n,\mathbb{C})} \to \tilde{\mathcal{M}}_{\mathrm{SO}(2n,\mathbb{C})}$ is a finite map. For any n, the moduli space $\mathcal{M}_{\mathrm{SO}(2n,\mathbb{C})}$ has two components corresponding to the two possible values for the second Stiefel–Whitney class of an $\mathrm{SO}(2n,\mathbb{C})$-principal bundle (see for example [16]). In contrast, the underlying holomorphic bundles for the Higgs bundles in $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})}$ and $\mathcal{M}_{\mathrm{SL}(4,\mathbb{C})}$ have just one topological type, and the moduli spaces are connected. The images of the maps $\mathcal{I}_2$ and $\mathcal{I}_3$ thus see just one of the components in $\mathcal{M}_{\mathrm{SO}(4,\mathbb{C})}$ or $\mathcal{M}_{\mathrm{SO}(6,\mathbb{C})}$. Indeed, this can be understood from the point of view of the surface group representations corresponding to the Higgs bundles via non-abelian Hodge theory. From this point of view, the component in the image of the map contains precisely the representations in $\mathrm{SO}(2n,\mathbb{C})$ which lift to $\mathrm{Spin}(2n, C)$. We therefore see only the Higgs bundles in which the underlying holomorphic bundle has $w_2 = 0$.

The situation is more nuanced for the restriction of the maps to the moduli spaces for the split real forms. In this case, the underlying holomorphic bundles have more complicated topology than in the case of the Higgs bundles for the complex groups. Moreover, fixing the topological type of the bundle does not ensure connectedness of the components. What remains true is that the images of the maps $\mathcal{I}_2$ and $\mathcal{I}_3$ contain only those components of the moduli spaces in which the Higgs bundles correspond to representations which lift to the appropriate spin group.

In terms of Hitchin fibrations, one should note that for each $n = 2, 3$, the bases of the fibrations of $\mathrm{Spin}(2n,\mathbb{C})$ and $\mathrm{SO}(2n,\mathbb{C})$-Higgs bundles are the same. In the case of $n = 2$, the base is $H^0(\Sigma, K^2)\oplus H^0(\Sigma, K^2)$, and for $n = 3$ it is $H^0(\Sigma, K^2)\oplus H^0(\Sigma, K^3)\oplus H^0(\Sigma, K^4)$. In order to understand the maps induced on these bases, it is necessary to understand exactly the relation between coordinates of a point in the base and the coefficients in the defining equation for the spectral curve. This is completely straightforward for $\mathrm{SL}(n,\mathbb{C})$, where the two coincide, but less so in the case of $\mathrm{SO}(2n,\mathbb{C})$ where the relation is complicated by the role of the Pfaffian. In fact, the maps $S \mapsto \hat{S}_{2n}$ (for $n = 2, 3$) do not unambiguously descend to the base of the fibration. The ambiguity stems from the fact that the induced orthogonal structures on $E_1 \otimes E_2$ or $\Lambda^2 E$ do not have a canonical orientation. The choice of orientation corresponds on the one hand to a choice of trivialization on the determinant bundles, and on the other hand to a choice of sign in the Pfaffian.

Remark 33 One should note that whist not done here, the isogenies could also be understood through the language of Cameral covers introduced in [8].

7.1 The isogeny $\mathcal{I}_2$ on moduli spaces

Once the isomorphism $\delta : E_1 \otimes E_2 \simeq \mathcal{O}_\Sigma$ is fixed, the map (30) determines a map on the base of the Hitchin fibrations for $\mathcal{M}_{\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})}$ and $\mathcal{M}_{\mathrm{SO}(4,\mathbb{C})}$. The explicit form of the map depends on the generators chosen for the rings of invariant polynomials. Taking the coefficients of the characteristic equation $\det(\varphi - \eta I)$ as generators, it follows from (36) that this map is given on the generic points in the base by

$$\mathcal{I}_2 : H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^2) \to H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^2)$$
$$(a_1, a_2) \mapsto (2(a_1 + a_2), \pm(a_1 - a_2)), \tag{69}$$

where the sign in the last component is determined by the isomorphism δ. For example, following the notation of Remark 13, if $\{e_1^1 \otimes e_2^1, e_1^1 \otimes e_2^2, e_1^2 \otimes e_2^1, e_1^2 \otimes e_2^2\}$ is the positively oriented local frame, then

$$\mathrm{Pf}(\Phi_1 \otimes I + I \otimes \Phi_2) = a_2 - a_1$$

where $a_i = \det(\Phi_i)$, so the second component in (69) is $-(a_1 - a_2)$.

Remark 34 As explained at the beginning of Sect. 4, the relation between spectral curves and thus the map (69) can be understood heuristically from the relation between the eigenvalues of the Φ_i and those of $\Phi_1 \otimes I + I \otimes \Phi_2$. By Remark 9, at least at smooth unramified points, the equations for the spectral curves S_i are given by

$$0 = \eta^2 + a_i = (\eta - \eta_1^{(i)})(\eta + \eta_1^{(i)}), \text{ for } i = 1, 2,$$

and the equation for the spectral curve S_4 is

$$\begin{aligned} 0 &= \eta^4 + b_2\eta^2 + b_3^2 \\ &= \prod (\eta \pm \eta_1^{(1)} \pm \eta_1^{(2)}) \\ &= (\eta^2 - (\eta_1^{(1)} + \eta_1^{(2)})^2)(\eta^2 - (\eta_1^{(1)} - \eta_1^{(2)})^2), \end{aligned}$$

from which the relations between (b_1, b_2, b_3^2) and (a_1, a_2) can be deduced.

As described in Sect. 3.4, the vector bundle of an $\mathrm{SO}_0(2,2)$-Higgs bundles may be expressed as $(M_1 \oplus M_1^*) \oplus (M_2 \oplus M_2^*)$, and thus, they are labeled by two integer topological invariants, say (c_1, c_2), corresponding to the degrees of the line bundles M_1 and M_2 (or, equivalently by classes in $\pi_1(\mathrm{SO}(2,\mathbb{C})) \cong \mathbb{Z}$). The moduli space $\mathcal{M}_{\mathrm{SO}_0(2,2)}$ is thus a disjoint union of (possibly disconnected or empty) subspaces

$$\mathcal{M}_{\mathrm{SO}_0(2,2)} = \bigsqcup_{c_1,c_2} \mathcal{M}^{c_1c_2}_{\mathrm{SO}_0(2,2)}. \tag{70}$$

The moduli space $\mathcal{M}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})}$ is similarly a disjoint union of (possibly disconnected or empty) subspaces

$$\mathcal{M}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})} = \bigsqcup_{d_1,d_2} \mathcal{M}^{d_1d_2}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})}, \tag{71}$$

for d_1, d_2 the degrees of the line bundles defining the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$-Higgs bundles. As seen in [13], the components are non-empty if and only if $|d_i| \le g - 1$, inequalities which are known as the the Milnor–Wood bounds.

Proposition 35 *The map* $\mathcal{I}_2 : \tilde{\mathcal{M}}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})} \to \tilde{\mathcal{M}}_{\mathrm{SO}_0(2,2)}$ *is a* 2^{2g+1}*-fold covering onto the components satisfying* $c_1 = c_2 \bmod 2$ *and* $|c_i| \le 2g - 2$, *for* $i = 1, 2$. *The map restricts to maps*

$$\mathcal{I}_2 : \tilde{\mathcal{M}}^{d_1d_2}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})} \to \tilde{\mathcal{M}}^{c_1c_2}_{\mathrm{SO}_0(2,2)} \tag{72}$$

for $c_1 = d_1 + d_2$ *and* $c_2 = d_1 - d_2$.

Proof The mod 2 congruence condition follows from the fact that any $\mathrm{SO}_0(2,2)$-Higgs bundle as in (5.1) in the image of $\mathcal{I}_2$ has

$$\deg(M_1) = \deg(N_1) + \deg(N_2),$$
$$\deg(M_2) = \deg(N_1) - \deg(N_2),$$

where N_1, N_2 are line bundles defining $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles (see Sect. 5.3). The bounds on $|c_i|$ follow from the Milnor–Wood bound for $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles. The rest of the proposition follows from the observation that the preimage under $\mathcal{I}_2$ for any Higgs bundle of the form (5.1) consists of all $\mathrm{SL}(2,\mathbb{R})\times \mathrm{SL}(2,\mathbb{R})$-Higgs bundles defined by (L_1, β_1, γ_1), and (L_2, β_2, γ_2) with $L_1^2 = M_1M_2$ and $L_2^2 = M_1M_2^{-1}$. If $\deg(M_1) = \deg(M_2)$ mod 2, then there are 2^{2g} solutions for L_1 and L_2. □

While the moduli space of $\mathrm{SL}(4,\mathbb{R})$-Higgs bundles has 2^{2g} Hitchin components, the one for $\mathrm{SO}_0(3,3)$-Higgs bundles has just one Hitchin component. From the analysis of topological invariants, which are constant on connected components, one has $4+1=5$ components, 4 coming from the 4 pairs of w_2's characterizing $\mathrm{SO}(3)\times\mathrm{SO}(3)$ bundles.

Proposition 36 *The isogeny $\mathcal{I}_2$ between moduli spaces of Higgs bundles takes the 2^{2g} Hitchin components to the one Hitchin component, and the other 2 components to the two components (possibly disconnected) where the two w_2's are the same.*

As seen before, the map $\mathcal{I}_2$ constructed in Sect. 5 is surjective onto some of the components of $\mathcal{M}_{\mathrm{SO}_0(2,2)}$. The components in the image correspond to those components in the representation variety $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{SO}_0(2,2))$ for which the representations lift to $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$.

We shall denote by $\mathcal{M}^0_{\mathrm{SO}_0(2,2)}$ the union of components of $\mathcal{M}_{\mathrm{SO}_0(2,2)}$ obtained through (30) and (32) with $\deg(M_1) = \deg(M_2)$ mod 2. Equivalently, let $\mathrm{Rep}^0(\pi_1(\Sigma), \mathrm{SO}_0(2,2))$ be the union of components of $\mathrm{Rep}(\pi_1(\Sigma), \mathrm{SO}_0(2,2))$ which correspond to the components in $\mathcal{M}_0(\mathrm{SO}_0(2,2)$.

Corollary 37 *The structure group of an* $\mathrm{SO}_0(2,2)$*-Higgs bundle lifts to* $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ *if and only if the* $\mathrm{SO}_0(2,2)$*-Higgs bundle lies in* $\mathcal{M}^0_{\mathrm{SO}_0(2,2)}$*. Equivalently, a reductive surface group representation into* $\mathrm{SO}_0(2,2)$ *lifts to a representation into* $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ *if and only if the representation lies* $\mathrm{Rep}^0(\pi_1(\Sigma), \mathrm{SO}_0(2,2))$.

Remark 38 By realizing $SO_0(2,2)$-Higgs bundles in terms of rank 2-Higgs bundles, one can understand the monodromy action studied in [2] for rank 4 Higgs bundles in terms of monodromy of lower-rank Higgs bundles. Indeed, taking b_1 and b_2 as in Eq. (36) one recovers the rank 4 monodromy as a product of actions coming from the rank 2-Hitchin systems.

7.2 The isogeny $\mathcal{I}_3$ on moduli spaces

To fully specify the map induced by $\mathcal{I}_3$ [as in (43)] on $\mathcal{M}_{\mathrm{SL}(4,\mathbb{C})}$, one needs to specify the trivialization $\delta : \det(\Lambda^2(E)) \simeq \mathcal{O}_\Sigma$. Then, using the coefficients of the characteristic equation $\det(\varphi - \eta I)$ as generators for the rings of invariant polynomials, it follows from (53) that the map on the generic points of the Hitchin base is

$$\begin{aligned} \mathcal{I}_3 : \bigoplus_{i=2}^{4} H^0(\Sigma, K^i) &\to \bigoplus_{i=2}^{4} H^0(\Sigma, K^i) \\ (a_2, a_3, a_4) &\mapsto (2a_2, \pm a_3, a_2^2 - 4a_2), \end{aligned} \tag{73}$$

where the sign in the second component is determined by the isomorphism δ.

Remark 39 For both n=2,3 the variety $Prym(\hat{S}_{2n}, \hat{S}_{2n}/_{\sigma})$) has two components. In the case $n = 3$, given $\hat{S}_6$ the spectral curve defined by $(2a_2, a_2^2 - 4a_2, a_3^2)$, the two components occur in the fibers over both points $(2a_2, \pm a_3, a_2^2 - 4a_2)$ in the Hitchin base. However, on each fiber only one of the components is in the image of the map induced by $\mathcal{I}_3$. A similar phenomenon occurs for $n = 2$.

Remark 40 The map (73) can be understood heuristically from eigenvalue considerations in the same way as (69), i.e., as explained in Remark 34.

As a consequence of its special orthogonal structure, the vector bundle in an SL(4, $\mathbb{R}$)-Higgs bundle is classified topologically by a second Stiefel–Whitney class. The moduli space $\mathcal{M}_{\mathrm{SL}(4,\mathbb{R})}$ thus decomposes into components (not necessarily connected) labeled by this $\mathbb{Z}_2$-valued invariant. The moduli space $\mathcal{M}_{SO_0(3,3)}$ likewise has a decomposition into components labeled by a pair of $\mathbb{Z}_2$-valued invariants corresponding to the second Stiefel–Whitney classes of the two rank three bundles. Using superscripts to indicate these characteristic classes, we can thus write

$$\begin{aligned} \mathcal{M}_{\mathrm{SL}(4,\mathbb{R})} &= \coprod_{a \,\in\, H^2(\Sigma,\mathbb{Z}_2)} \mathcal{M}^a_{\mathrm{SL}(4,\mathbb{R})}; \\ \mathcal{M}_{\mathrm{SO}_0(3,3)} &= \coprod_{b_i \,\in\, H^2(\Sigma,\mathbb{Z}_2)} \mathcal{M}^{(b_1,b_2)}_{\mathrm{SO}_0(3,3)}. \end{aligned}$$

Proposition 41 *The map* $\mathcal{I}_3 : \tilde{\mathcal{M}}_{\mathrm{SL}(4,\mathbb{R})} \to \tilde{\mathcal{M}}_{\mathrm{SO}_0(3,3)}$ *is a* 2^{2g}*-fold covering onto the components satisfying* $b_1 = b_2$ mod 2. *For* $a = w_2(E)$ *the orthogonal rank 4 bundle E, and* $b = w_2(\Lambda^2(E)$, *the map restricts to*

$$\mathcal{I}_3 : \tilde{\mathcal{M}}^a_{\mathrm{SL}(4,\mathbb{R})} \to \tilde{\mathcal{M}}^{(b,b)}_{\mathrm{SO}_0(3,3)}. \tag{74}$$

Proof It follows from Proposition 19 that the image of $\tilde{\mathcal{M}}^a_{\mathrm{SL}(4,\mathbb{R})}$ lies in the component $\tilde{\mathcal{M}}^{(b_1,b_2)}_{\mathrm{SO}_0(3,3)}$, where $b_1 = w_2(\Lambda^2_+(E))$ and $b_2 = w_2(\Lambda^2_-(E))$ for some SO(4, $\mathbb{C}$) vector bundle E with $w_2(E) = a$. Moreover, from [11, Proposition 1.8] it follows that $w_2(\Lambda^2(E)_\pm) = w_2(\Lambda^2(E))$, so in particular $b_1 = b_2$. However, any pair of SO(3, $\mathbb{C}$)-bundles with the same second Stiefel–Whitney class arises in this way, i.e., as $\Lambda^2(E)_\pm$ where E is an SO(4, $\mathbb{C}$) bundle. Finally, if $(\Lambda^2 E, \Phi \otimes I + I \otimes \Phi)$ represents a point in the image of $\mathcal{I}_3$, then the 2^{2g} preimages come from twisting E by any point of order two in $Jac(\Sigma)$. □

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References

1. Aparicio Arroyo, M.: The geometry of SO(p, q)-higgs bundles. Ph.D. Thesis, Universidad de Salamanca (2009)
2. Baraglia D., Schaposnik L.P.: Monodromy of rank 2 twisted Hitchin systems and real character varieties. (2015). arXiv:1506.00372
3. Beauville, A., Narasimhan, M.S., Ramanan, S.: Spectral curves and the generalised theta divisor. J. Reine Angew. Math. **398**, 169–179 (1989)
4. Bradlow, S., Garcia-Prada, O., Gothen, P.B.: Surface group representations and $U(p, q)$-bundles. J. Differ. Geom. **64**, 111–170 (2003)
5. Bradlow, S., Garcia-Prada, O., Gothen, P.B.: Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces. Geom. Dedicata **122**, 185–213 (2006)
6. Cartan, E.: Les groupes réeles simples finis et continus. Ann. Sci. École Norm. Super. **31**, 263–355 (1914)
7. Catanese, F.: Homological algebra and algebraic surfaces. In: Kollár, J., Lazarsfeld, R., Morrison, D.R. (eds.) Algebraic geometry–Santa Cruz 1995. Proceedings of Symposia on Pure Mathematics, vol. 62, pp. 3–56. American Mathematical Society, Providence, RI (1997)
8. Donagi, R.: Spectral covers. MSRI Ser. **28**, 65–86 (1995)
9. Donagi, R.: The tetragonal construction. Bull. Am. Math. Soc. **4**(2), 181–185 (1981)
10. Gothen P.B.: The topology of Higgs bundle moduli spaces. Ph.D. thesis, University of Warwick (1995)
11. Grove, K., Ziller, W.: Lifting group actions and non-negative curvature. Trans. Am. Math. Soc. **363**, 2865–2890 (2011)
12. Helgason, S.: Differential Geometry, Lie Groups and Symmetric Spaces, vol. 80. Academic Press, San Diego (1978)
13. Hitchin, N.J.: The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. **55**, 3 (1987)
14. Hitchin, N.J.: Stable bundles and integrable systems. Duke Math. J. **54**(1), 91–114 (1987)
15. Hitchin, N.J.: Lie groups and Teichmüller space. Topology **31**(3), 449–473 (1992)
16. Hitchin, N.J.: Langlands duality and G2 spectral curves. Q. J. Math. **58**, 319–344 (2007)
17. Manetti, M.: Iterated double covers and connected components of moduli spaces. Topology **36**, 745–764 (1996)
18. Mumford, D.: Theta characteristics of an algebraic curve. Ann. Sci. École Norm. Sup. (4) **4**, 181–192 (1971)
19. Nevins, T.: Descent of coherent sheaves and complexes to geometric invariant theory quotients. J. Algebra **320**(6), 2481–2495 (2008)
20. Procesi, C.: Lie Groups: An Approach Through Invariants and Representations. Springer, Berlin (2007)
21. Ramanan, S.: Orthogonal and spin bundles over hyperelliptic curves. Proc. Indian Acad. Sci. Math. Sci. **90**(2), 151–166 (1981)
22. Recillas, S.: Jacobians of curves with a g^1_4 are Prym varieties of trigonal curves. Bol. Soc. Mat. Mex. **19**, 9–13 (1974)
23. Rubio Nunez R.: Higgs bundles and Hermitian symmetric spaces Ph.D. Thesis, UAM (2012)
24. Schaposnik, L.P.: Monodromy of the SL_2 Hitchin fibration. Int. J. Math. **24**, 2 (2013). arXiv:1111.2550
25. Schaposnik L.P.: Spectral data for G-Higgs bundles, D.Phil. Thesis, University of Oxford (2013). arXiv.1408.0333
26. Schaposnik, L.P.: Spectral data for $U(m, m)$-Higgs bundles. Int. Math. Res. Not. **2015**(11), 3486–3498 (2015). arXiv:1307.4419
27. Schaposnik L.P.: An introduction to spectral data for Higgs bundles. (2014). arXiv:1408.0333
28. Simpson, C.T.: Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization. J. Am. Math. Soc. **1**, 867–918 (1988)