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Power and loyalty defined by proximity to influential relations

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Abstract

This paper examines a simple definition of power as a composite centrality being the composition of eigenvector centrality and edge betweeness. Various centralities related to the composition are compared on social and collaboration networks. A derived defection score for social fission scenarios is introduced and is demonstrated in Zachary's Karate club to predict the sole defection in terms of network measures rather than psychological factors. In a network of political power in Mexico across various periods, the two definitions of power serve to shed light on a political power transition between two groups.

Keywords: Composite centrality; Power; Loyalty; Eigenvector; Betweeness; Fission

Introduction

Networks are often modeled as a *graph*, which consists of set of nodes (V) and edges (E), such that $E \subseteq V \times V$. If E is a symmetric relation, then G is called an *undirected* graph. A *network centrality* is a function defined on V which assigns importance to nodes according to certain criteria.

Various feedback centralities have been introduced (Seeley [1], Hubbell [2], Katz [3], Bonacich [4]), which share the common objective of measuring a node's importance while taking into account the importance of its neighbors. The simplified form of a feedback centrality termed eigenvector centrality is based on the Perron-Frobenius theorem which ensures that for a strongly connected graph, the leading eigenvector of the adjacency matrix contains only real positive values ([5]). Let $X = (x_1 \ldots x_n)$ be the eigenvector of the largest eigenvalue of the adjacency matrix A_G of G, and λ_1 is the largest eigenvalue. Then, the eigenvector centrality of node i is $C_{EV}(i) = \frac{1}{\lambda_1} x_i$. Informally, C_{EV} will find a set of nodes which are more densely connected (clique-like) than other subsets of V. A node with a high C_{EV} score would have relatively more edges between its neighbors.

Betweeness centrality, which was introduced by Freeman in [6] and Anthonisse in [7], measures the proportion of shortest paths passing by a given node. Formally, let $\sigma_{s,t}$ be the number of shortest paths between nodes s,t, and $\sigma_{s,t}(v)$ be the number of shortest paths between s,t that pass through v; then, the betweeness of v is defined as $C_B(v) = \sum_{s \neq v} \sum_{t \neq v} \frac{\sigma_{s,t}(v)}{\sigma_{s,t}}$. In [7], betweeness is also defined for edges, for an edge $e \in E$, $C_{EB}(e) = \sum_{s \in v} \sum_{t \in v} \frac{\sigma_{s,t}(e)}{\sigma_{s,t}}$. In a social network, an edge with high betweeness would mean that the relation between the represented actors is important in the sense that it is expected to be used more by other actors in the network. Edge betweeness has also been used to detect community structure ([8]).

Definitions and properties

Composite centralities have been suggested based on statistical measures ([9]); here, the natural composition is taken. Let C_1 be a node centrality, and C_2 be an edge centrality whose values are non negative. The fact that C_1 is node based and C_2 is edge based suggests a natural function composition, define:

$$A_{C_2}\left[i,j\right] = \begin{cases} C_2((i,j)) & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases} \tag{1}$$

And notate $C_1(C_2)(v)$ as the value for v when C_1 is computed on A_{C_2}. A matrix is said to be *irreducible* if its interpretation as a graph adjacency matrix produces a strongly connected graph. If G is an edge weighted graph, it may be that $C_{EB}(e) = 0$ for $e \in E$, while for non-weighted graphs, this is not the case; since every edge would be on the shortest path between its endpoints. Thus, for a weighted graph, using Equation 1 may produce a reducible matrix, since some edges may have zero betweeness.

Proposition 0.1. *Let G be an positive edge weighted undirected connected graph, then $A_{C_{EB}}$ is irreducible.*

Proof. Let $u, v \in V$, since G is connected, there exists at least one shortest path P_{uv} connecting u and v. From the definition of C_{EB}, for any edge $e \in P_{uv}$, $C_{EB}(e) > 0$. Therefore, in the graph defined by $A_{C_{EB}}$, there exists a positively weighted path connecting u and v. □

Notate $C_{EVB}(v) = C_{EV}(C_{EB})(v)$. From 0.1, C_{EVB} is well defined, as the Perron-Frobenius theorem holds the same way as for C_{EV}. In this case, it is assumed that high edge betweeness indicates a potentially important relation, and that an actor is more powerful if it participates in important relations, either directly, or its neighbors have important relations between themselves.

An artificial example

A simple example in Figure 1 demonstrates that C_{EVB} may differ from both C_{EV},C_B or any linear combination of them. The example consists of two small complete graphs which are connected by one node (node 11) and another node (node 12) connected to node 11. Clearly, node 12 has zero C_B and low C_{EV}, but it has the second highest C_{EVB}. Node 12 is accessible only via node 11, the most powerful node in all three measures, so if we assume that all nodes are initially accessed at a similar rate, the relation between 11 and 12 will be the most used, while the nodes within the cliques would have many relations that are used only between the two endpoints (see Figure 1). This shows that C_{EVB} may detect a 'behind the scenes' player like node 12, while C_{EV} and C_B would assign it low scores.

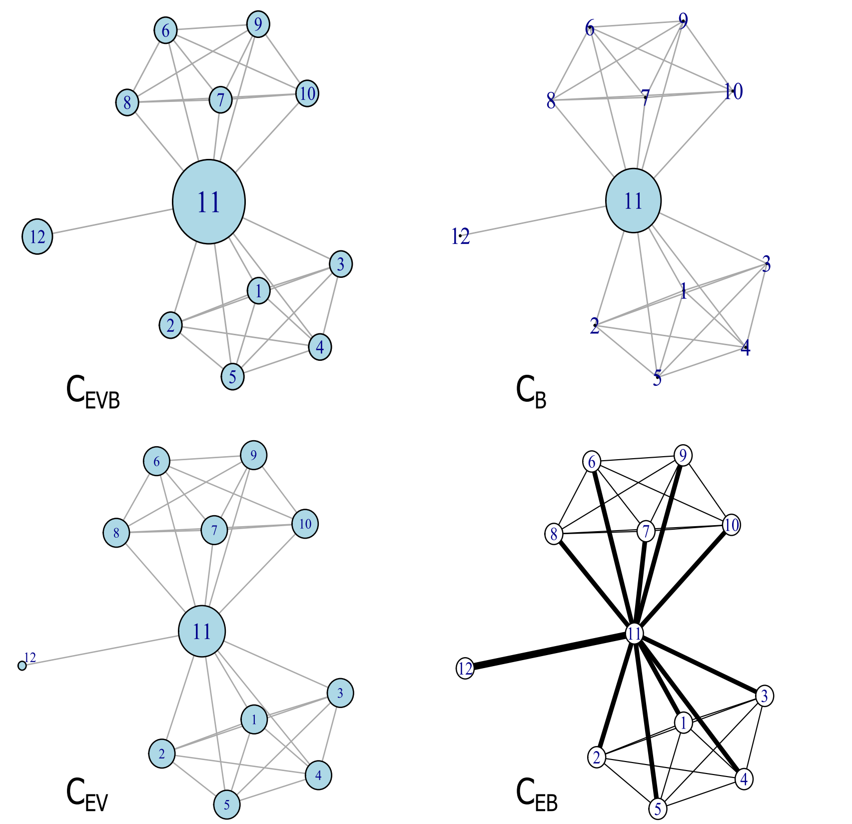

Figure 1 A simple example.

Eigenvectors in a weighted graph and scaling behaviour

The justification of using C_{EV} on an edge weighted network is, as explained by Newman in [10] , if $X = (x_1 \dots x_n)$ is the leading eigenvector, then

$$C_{EV}(i) = x_i = \frac{1}{\lambda_1} \sum_j A_{ij} x_j \tag{2}$$

hence multiplying the weight of an edge by a positive factor will adjust the contribution of the neighbour incident on that edge to the eigenvector centralities of its incident nodes by the same factor, i.e. if the weight of (v,u) is 3 then the contribution of u to $C_{EV}(v)$ is multiplied by a factor of 3. Thus, calculating the eigenvector centrality of an edge weighted network would score nodes according to the weighted density of their neighborhood.

An informal scaling argument regarding C_{EVB} is shown as follows. It is proven in [11] that for a graph $G = (V, E)$ and any node $v \in V$,

$$C_B(v) = \sum_{(u,v) \in E} C_{EB}((u, v)) - (n - 1) \tag{3}$$

Furthermore, it is numerically demonstrated in [12] that if G is a node degree scale free network with exponent between 2 and 3, then C_B follows a power law distribution with exponent approximately 2.2. So, if C_{EV} scales 'nicely' in relation to a node degree power law exponent, that would mean that the row sums of the original adjacency matrix are related to the scaling behaviour of C_{EV}. Since node betweeness is distributed as a power law as mentioned, then Equation 3 implies that C_{EVB} will scale in a similar way in relation to C_B, as the row sums in the edge betweeness weighted adjacency matrix are proportional to C_B. Indeed, visually inspecting the scatterplots of log-transformed C_B and C_{EVB} (Figure 2) demonstrates that for nodes that do have zero C_B may still have significant values of C_{EVB} in a similar way as in the artificial example; as for nodes that do not have zero C_B, there is a linear or 'cone'-shaped relation , which provides some evidence of a power scaling relation between C_B and C_{EVB}.

Predicting loyalty in a fission scenario

Let $V = S_1 \cup S_2$ be a disjoint partition of V. Assume w.l.o.g that $v \in S_1$. Let $X = (x_1, ..., x_n)$ be the leading eigenvector, then the contribution of S_2 to $C_{EVB}(v)$ is:

$$C_{EVB}^{S_2}(v) = \sum_{(v,v_i)\in E, v_i \in S_2} C_{EB}((v, v_i))x_i \quad (4)$$

By the definition of C_{EVB} as power, Equation 4 describes the proportion of power of v that comes from direct links to the opposing group members. In a social fission situation, it may be the case that members of one group defect to the other as in [13]. Motivated by defection prediction, define the *defection score* as:

$$D_{EVB}(v) = C_{EVB}^{S_2}(v) - C_{EVB}^{S_1}(v) \quad (5)$$

For a node v, $D_{EVB}(v)$ is simply the difference between the power of v that comes from links to the opposing group and the power that comes from links to its own group. It is hypothesized that a high positive D_{EVB} would mean a higher temptation to defect, while a more negative D_{EVB} would mean a greater tendency to stay put.

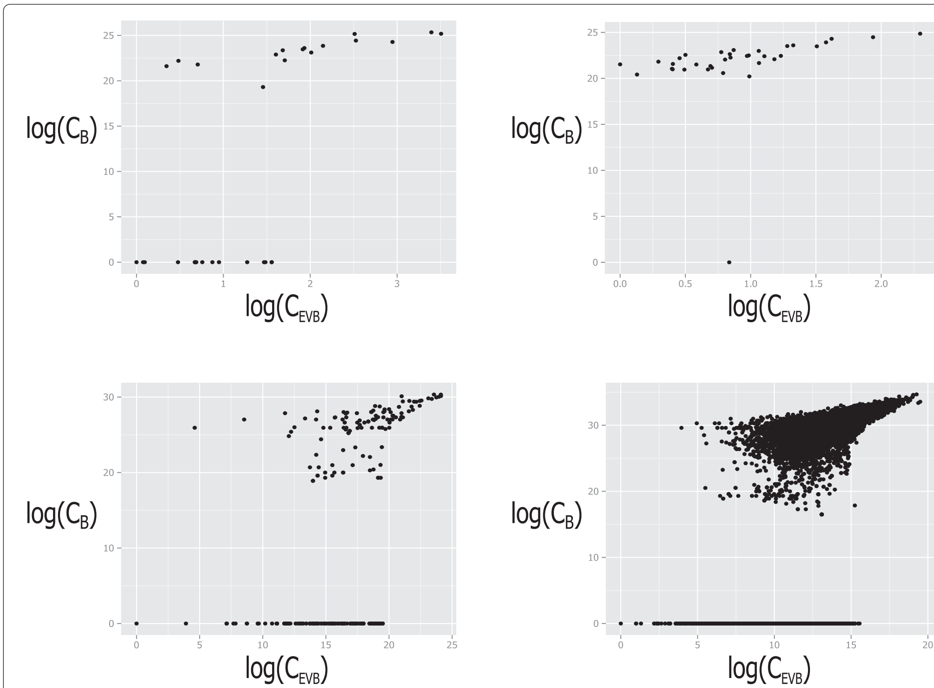

Figure 2 The Log/log relation of C_B and C_{EVB}. Clockwise from top left : Zachary's Karate club, Mexican political elite, Astrophysics collaboration and Newmans network science collaboration network. Negligible values were added to C_B to allow log of zero values. Both functions are translated to the origin $(0, 0)$.

Computational complexity

The complexity of computing a composite function as defined here is simply the sum of the complexities of the underlying functions. An algorithm of $O(|V||E|)$ for betweeness is described in [14]. Eigenvector centrality requires only the largest eigenvalue and the corresponding eigenvector. In practice, this is solvable in $O(|V| + |E|)$ using an ARPACK eigenvector solver. Thus, the expected overall time is the same as for edge betweeness. The computational complexity for $D_{EVB}(v)$ is $O(|V|)$ if $C_{EVB}(v)$ is already computed.

Case studies

Several social networks are examined, two 'friendship' networks: Zachary's Karate club ([13]) and a network of the Mexican political elite in the twentieth century. In addition, two larger collaboration networks are studied, a collaboration network of researchers in Network Science (NS) taken from [15] and the collaboration network of preprints on the astrophysics archive at www.arxiv.org, 1995-1999, as compiled by Newman [16]. As can be seen in Table 1, C_{EVB} and C_B are more correlated in the friendship networks than the collaboration networks.

A network describing social fission

Zachary's Karate club is one of the earliest social networks studied as a graph ([13]). The network consists of 34 actors whose common activity is a Karate club, edges are weighted by the level of acquaintance shared by actors beyond the club. The club underwent fission during the period of observation due to a long conflict between the club administrator and the Karate instructor. Zachary's original analysis was based on network flow and minimal cuts, where the edge weights represented capacity. In this case, the reciprocal of the weights are taken, as the edge weights represent distance and not flow capacity. As seen in Figure 3, C_{EVB} differs from C_{EV} and C_B with regard to key players; for instance, actor 34 is reduced in C_{EVB} in comparison to other actors, while actors 3 and 9 have a relative increase in C_{EVB}. By inspecting C_{EB} visually in Figure 3, it indeed seems that actors 3 and 9 are better located within the network regarding proximity to edges with high betweeness. Zachary's flow analysis managed to model and predict the group affiliation before and after the fissure with near perfection save one case.

Zachary's original explanation [13] was psychological, based on the temporal circumstance of individual 9:

'This can be explained by noting that he was only three weeks away from a test for black belt (master status) when the split in the club occurred. Had he joined the officers' club he would have had to give up his rank and begin again in a new style of karate with a white (beginner's) belt, since the officers had decided to change the style of karate practiced in their new club'.

Table 1 Spearman rank correlation scores compared for various networks

Network	$\|V\|$	$\|E\|$	$C_{EVB} \mid C_B$	$C_{EVB} \mid C_{EV}$	$C_B \mid C_{EV}$
Karate club	34	78	0.762	0.479	0.398
Mexican politicians	35	117	0.703	0.607	0.739
NS collaboration	374	914	0.427	0.504	0.049
Astro-ph	14,845	119,652	0.604	0.786	0.431

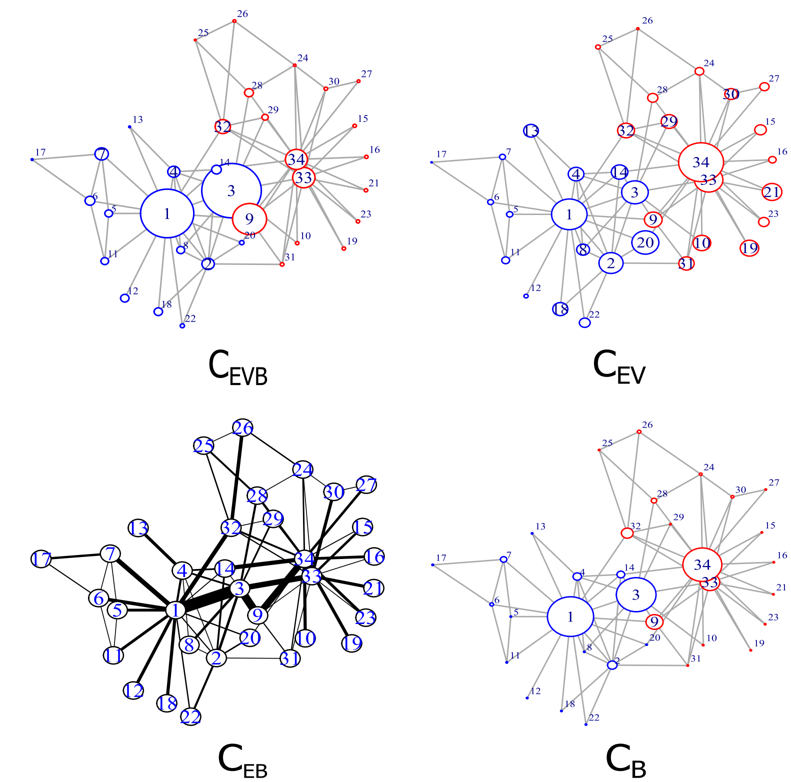

Figure 3 The Karate club. Four centralities in Zachary's network. Colors of nodes (blue or red) designate faction membership, while the larger the node is, the higher the centrality score. For edge betweeness (lower left), thicker edges have higher scores.

Here, an additional factor may be observed based on D_{EVB} scores. D_{EV} is defined in a similar way to D_{EVB}, using the adjacency matrix of G without computing edge betweeness, and as visible in Figure 4, D_{EV} predicts that actor 10 would have the highest incentive to defect while D_{EVB} predicts actor 9 for defection (as indeed took place). In addition, D_{EVB} predicts that actors 1 and 34, which are the leaders of the factions, would have the greatest tendency to stay put while D_{EV} makes no such prediction. Unfortunately, no data exists as to the possible dilemmas of other actors such as nodes 10 and 33.

Collaboration networks

The NS network constructed by Newman ([15]) consists of 1,589 nodes; here, the largest connected component is studied, consisting of 379 nodes. Edges are weighted by collaboration strength as defined in [16], so reciprocals are taken here to represent distance between collaborators. The three centralities are shown in Figure 5, in which the five nodes with the highest scores are identified by their ID (specific names are available in [15] for readers interested). The occasional 'local' nature of C_{EV} is apparent in Figure 5, nodes on one sole branch of the network receive the highest scores due to a higher density of edges. In this case, C_{EV} may show isolation rather than power, a group with many links between themselves on an isolated branch. Examining the differences between C_{EVB} and C_B, it is clear that actors 51, 52 and 95 score well in both measures, but actor 26 loses power in C_{EVB}, while actor 4 drops to the 19th place in the C_{EVB} scores.

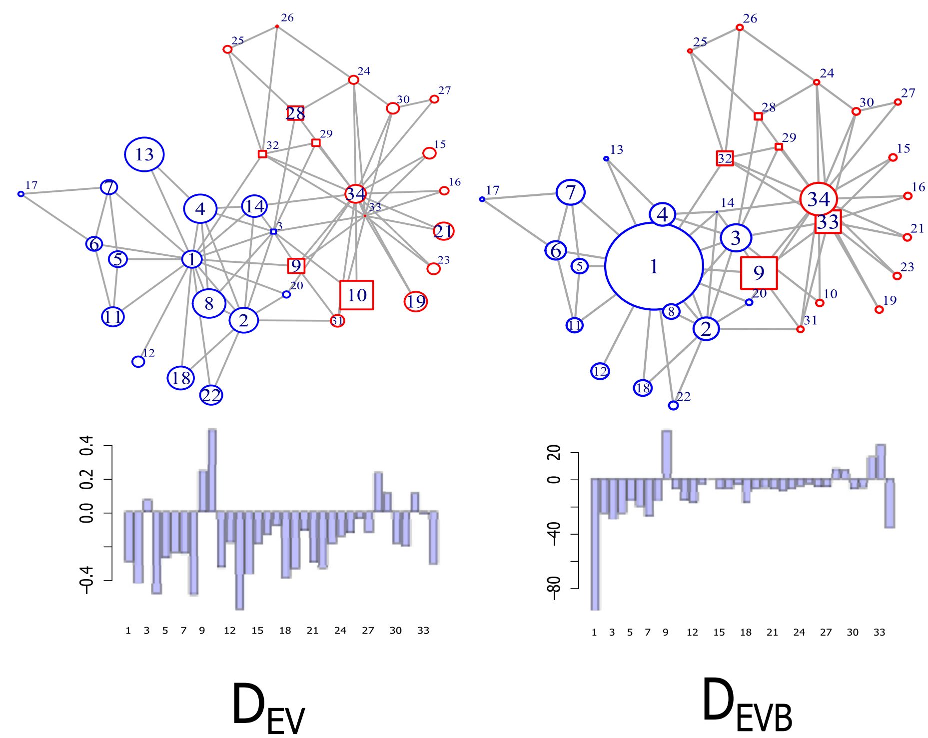

Figure 4 Loyalty or defection. Predicting the incentive to defect. Larger squares mean a higher positive defection score, while larger circles mean a more negative score, which represents the tendency to stay within the original group.

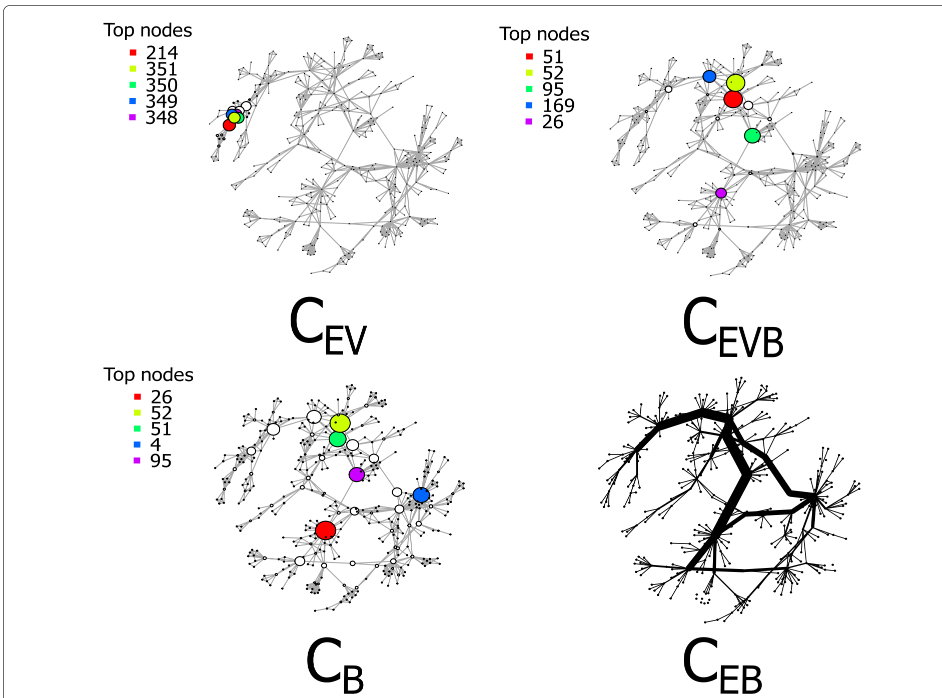

Figure 5 A collaboration network of network scientists. Newman's NS network. The ID's of the five nodes with the highest scores in each case are listed in descending order.

As with the Karate club, the reason why actor 4 loses power according to C_{EVB} is clearer by looking at Figure 5, the edges with higher betweeness form a path through the 'middle' while actor 4 is located on a lower scored subsidiary of the sub-network of edges with high betweeness. So, although actor 4 has high betweeness as a node centrality, C_{EVB} is reduced due to a lower scoring edge betweeness neighborhood.

A transition of political power

A network of the Mexican political elite was described in [17] and compiled in [18]. The network consists of the core of the political elite and their collaborators across a time period stretching from the early to late twentieth century. The edges of the network are unweighted and represent close ties. During the time period examined, the PRI (Partido Revolucionario Institucional) was continuously in power, however there was an internal struggle between two main groups within the party, politicians associated with the military against 'civilian' politicians. During the period, there was a transition of power from the former to the latter. In this context, since the network spans most of the twentieth century, C_{EV} would represent the amount of connections surrounding a politician during the height of his political activity but as already demonstrated in the NS network (Figure 5) that is not necessarily the same as a high C_{EVB} score. On the other hand, C_{EVB} would indicate the proximity of a politician to the relations that are expected to be significant throughout the era; therefore, C_{EVB} is interpreted as political power. The defection scores, in this context, are interpreted as the level of political collaboration with members of the other side. In Figure 6, C_{EV} and C_{EVB} are examined in order to understand if both perspectives can illuminate the power transition purely by examining the network. In Figure 6, it can be seen that node 12 has the highest C_{EVB} and C_{EV} score. Node 12 represents Miguel Alemán Valdés, the 46th Mexican president whose reign marked the transition from military associated power to more 'civilian' rule. Interestingly, he also has the highest D_{EVB} and D_{EV} scores, meaning that the most powerful politician in the network (highest C_{EVB} score) collaborated closely with 'civilian' politicians, both from a 'local' viewpoint (high D_{EV}) and from a 'global power' viewpoint (high D_{EVB}). Indeed, in 1952, he was succeeded by node 18, Adolfo Ruiz Cortines, a 'civilian' politician, which signified the beginning of the new era. A different observation from the D_{EVB} chart is that a high level of collaboration with the opposing side is more expressed than in D_{EV} when family ties are present. For instance, node 34 is Miguel Alemán Velasco who is the son of Miguel Alemán Valdés, and node 14 is Ramón Beteta who was the brother of node 13, major general Ignacio Beteta, a close associate of node 10, the powerful Lázaro Cárdenas, and both 34 and 14 score high on D_{EVB}. To conclude, the point of political power transition is visible in Figure 6, and the idea that family ties may precede group affiliation in political power sharing is visible in D_{EVB}.

Conclusions

The composition C_{EVB} was shown to be well defined, and it was shown to differ in several aspects from C_{EV} and C_B in case studies. A node defection score based on C_{EVB} and C_{EV} was defined for two-fission situations, and D_{EVB} was shown to better predict the sole defection in Zachary's study than a similar defection score based purely on C_{EV}. A significance threshold for D_{EVB} could be useful (scores below the threshold would mean no defection) and may be worthwhile of further research. The empirical analysis suggests

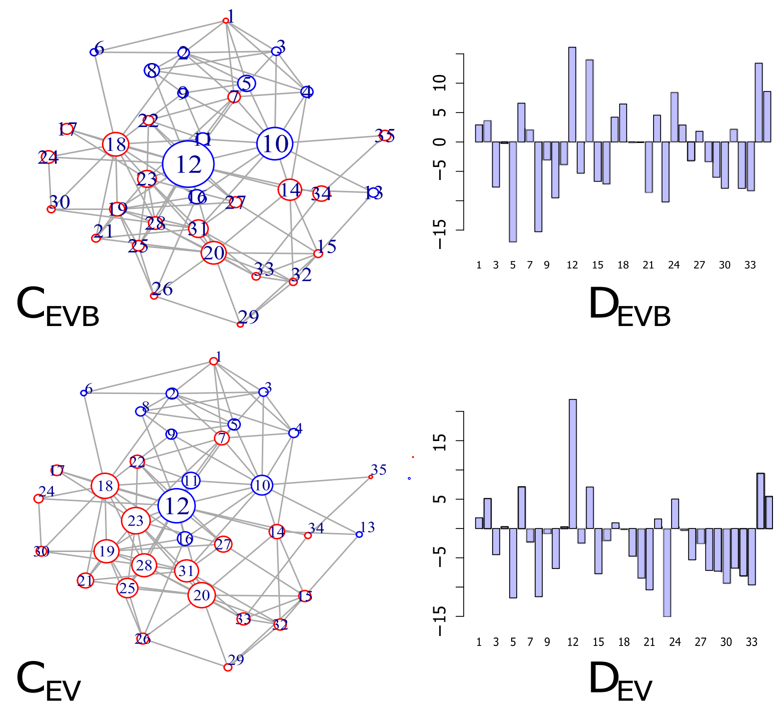

Figure 6 An analysis of a network of the Mexican elite in the twentieth century. Viewing the different measures side by side enables to pinpoint the individual whose reign marked the power transition from 'militarists' to 'civilians'.

that C_{EVB} balances the local properties of eigenvector centrality with the global properties of betweeness, giving a different perspective on power distribution. C_{EVB} in combination with C_{EV} and the defection scores were demonstrated to be useful tools in the analysis of the transition and sharing of power in twentieth century Mexican politics. Finally, the possibility of modeling *k*-fission scenarios (using a more general defection score) is a natural expansion but would need considerable supporting empirical data as to the behaviour of individuals in such situations.

Data accessibility

Newman's NS network is available at [19], and Zachary's Karate club data was accessed through [20].

Competing interests

The author declares that he has no competing interests.

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