Wei et al. SpringerPlus (2016) 5:1710
DOI 10.1186/s40064-016-3358-z

SpringerPlus

RESEARCH Open Access

Jacobi spectral collocation method for the approximate solution of multidimensional nonlinear Volterra integral equation

Yunxia Wei[1*], Yanping Chen[2], Xiulian Shi[3] and Yuanyuan Zhang[4]

*Correspondence: yunxiawei@126.com
[1] College of Mathematic and Information Science, Shandong Institute of Business and Technology, Yantai 264005, China
Full list of author information is available at the end of the article

Abstract

We present in this paper the convergence properties of Jacobi spectral collocation method when used to approximate the solution of multidimensional nonlinear Volterra integral equation. The solution is sufficiently smooth while the source function and the kernel function are smooth. We choose the Jacobi–Gauss points associated with the multidimensional Jacobi weight function $\omega(\mathbf{x}) = \Pi_{i=1}^{d}(1-x_i)^{\alpha}(1+x_i)^{\beta},\ -1 < \alpha, \beta < \frac{1}{d} - \frac{1}{2}$ (d denotes the space dimensions) as the collocation points. The error analysis in L^{∞}-norm and L^2_{ω}-norm theoretically justifies the exponential convergence of spectral collocation method in multidimensional space. We give two numerical examples in order to illustrate the validity of the proposed Jacobi spectral collocation method.

Keywords: Multidimensional nonlinear Volterra integral equation, Jacobi collocation discretization, Multidimensional Gauss quadrature formula, Error estimates

Mathematics Subject Classification: 65R20, 45J05, 65N12

Background

We observe that there are many numerical approaches for solving one-dimensional Volterra integral equation, such as Runge–Kutta method (Brunner 1984; Yuan and Tang 1990), polynomial collocation method (Brunner 1986; Brunner et al. 2001; Brunner and Tang 1989), multistep method (Mckee 1979; Houwen and Riele 1985), hp-discontinuous Galerkin method (Brunner and Schötzau 2006) and Taylor series method (Goldfine 1977). The spectral collocation method is the most popular form of the spectral methods among practitioners. It is convenient to implement for one-dimensional problems and generally leads to satisfactory results an long as the problems possess sufficient smoothness. In the literature (Tang et al. 2008), the authors proposed a Legendre spectral collocation method for Volterra integral equation with a regular kernel in one-dimensional space. Subsequently, Chen and Tang (2009, 2010), Chen et al. (2013), developed the spectral collocation method for one-dimensional weakly singular Volterra integral equation. The proofs of the convergence properties of spectral collocation method for

Volterra integro-differential equation with a single spatial variable are given in Wei and Chen (2012a, b, 2013, 2014). Nevertheless, to the best of our knowledge, there have been no works regarding the theoretical analysis of the spectral approximation for multidimensional Volterra integral equation (Atdev and Ashirov 1977; Beesack 1985; Pachpatte 2011; Suryanarayana 1972), even for the case with smooth kernel.

We shall extend to several space dimensions the approximation results in Tang et al. (2008) for a single spatial variable. The expansion of Jacobi will be considered. We will be concerned with Sobolev-type norms that are most frequently applied to the convergence analysis of spectral methods. We get the discrete scheme by using multidimensional Gauss quadrature formula for the integral term. We will provide a rigorous verification of the exponential decay of the errors for approximate solution.

We study the multidimensional nonlinear Volterra integral equation of the form

$$\begin{aligned} &y(t_1,t_2,\ldots,t_d)+\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_d}K(t_1,s_1,t_2,s_2,\ldots,t_d,s_d,y(s_1,s_2,\ldots,s_d))\\ &\quad ds_d\ldots ds_2ds_1=g(t_1,t_2,\ldots,t_d),\quad t_i\in[0,T_i],\quad i=1,2,\ldots,d, \end{aligned}\tag{1}$$

by the Jacobi spectral collocation method. Here, $g:[0,T_1]\times[0,T_2]\times\cdots\times[0,T_d]\to R$ and $K:D\times R\to R$ (where $D:=\{(t_1,s_1,t_2,s_2,\ldots,t_d,s_d):0\le s_i\le t_i\le T_i, i=1,2,\ldots,d\}$) are given smooth functions. If the given functions are smooth on their respective domains, the solution y is also the smooth function (see Brunner 2004). This fact will be the standing point of this paper.

Discretization scheme

We consider now the domain $\Omega=(-1,1)^d$ and we denote an element of $\mathbb{R}^d$ by $\mathbf{x}=(x_1,x_2,\ldots,x_d)$. Let $-1<\alpha,\beta<\frac{1}{d}-\frac{1}{2}$, if $\omega=\omega(\mathbf{x})=\Pi_{i=1}^{d}(1-x_i)^{\alpha}(1+x_i)^{\beta}$ denotes a d-dimensional Jacobi weight function on Ω, we denote by $L^2_\omega(\Omega)$ the space of the measurable functions $u:\Omega\to\mathbb{R}$ such that $\int_\Omega|u(\mathbf{x})|^2\omega(\mathbf{x})d\mathbf{x}<+\infty$. It is a Banach space for the norm

$$\|u\|_{L^2_\omega(\Omega)}=\left(\int_\Omega|u(\mathbf{x})|^2\omega(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{2}}.$$

The space $L^2_\omega(\Omega)$ is a Hilbert space for the inner product

$$(u,v)_\omega=\int_\Omega u(\mathbf{x})v(\mathbf{x})\omega(\mathbf{x})d\mathbf{x}.$$

$L^\infty(\Omega)$ is the Banach space of the measurable functions $u:\Omega\to\mathbb{R}$ that are bounded outside a set of measure zero, equipped with the norm

$$\|u\|_{L^\infty(\Omega)}=ess\ sup_{\mathbf{x}\in\Omega}|u(\mathbf{x})|.$$

Given a multi-index $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_d)$ of nonnegative integers, we set

$$|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_d$$

and

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}}.$$

We define $H_\omega^m(\Omega) = \{v \in L_\omega^2(\Omega):$ for each nonnegative multi-index α with $|\alpha| \le m$, the distributional derivative $D^\alpha v$ belongs to $L_\omega^2(\Omega)\}$. This is a Hilbert space for the inner product

$$(u, v)_{m,\omega} = \sum_{|\alpha| \le m} \int_\Omega D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x}) \omega(\mathbf{x}) d\mathbf{x},$$

which induces the norm

$$\|v\|_{H_\omega^m(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha v\|_{L_\omega^2(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Let $\{\tilde{x}_j, 0 \le j \le N\}$ denote the Jacobi Gauss points on the one-dimensional interval $(-1, 1)$ (see Canuto et al. 2006; Shen and Tang 2006). We now consider multidimensional Jacobi interpolation. Let $\mathbb{P}_N(\Omega)$ be the space of all algebraic polynomials of degree up to N in each variable x_i for $i = 1, 2, \ldots, d$. Let us introduce the Jacobi Gauss points in Ω:

$$\tilde{\mathbf{x}}_\mathbf{j} = (\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) \text{ for } \mathbf{j} = (j_1, j_2, \ldots, j_d) \in \mathbb{N}^d, \quad \|\mathbf{j}\| = \max_{1 \le i \le d} j_i \le N,$$

and denote by I_N the interpolation operator at these points, i.e., for each continuous function u, $I_N u \in \mathbb{P}_N$ satisfies

$$(I_N u)(\tilde{\mathbf{x}}_\mathbf{j}) = u(\tilde{\mathbf{x}}_\mathbf{j}) \text{ for all } \mathbf{j} \in \mathbb{N}^d, \quad \|\mathbf{j}\| \le N.$$

We can represent $I_N u$ as follows:

$$I_N u(\mathbf{x}) = \sum_{\|\mathbf{j}\| \le N} u(\tilde{\mathbf{x}}_\mathbf{j}) F_\mathbf{j}(\mathbf{x}),$$

where $F_\mathbf{j}(\mathbf{x}) = F_{j_1}(x_1) F_{j_2}(x_2) \ldots F_{j_d}(x_d)$, $\{F_j\}_{j=0}^N$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{\tilde{x}_j\}_{j=0}^N$. The multidimensional Jacobi Gauss quadrature formula is

$$\int_\Omega f(\mathbf{x}) d\mathbf{x} \approx \sum_{\|\mathbf{j}\| \le N} f(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) \omega_{j_1} \omega_{j_2} \ldots \omega_{j_d}. \tag{2}$$

We use the variable transformations $t_i = \frac{T_i}{2}(1 + x_i)$, $x_i \in [-1, 1]$ and $s_i = \frac{T_i}{2}(1 + \tau_i)$, $\tau_i \in [-1, x_i]$, $i = 1, 2, \ldots, d$ to rewrite (1) as follows

$$u(x_1, x_2, \ldots, x_d) + \int_{-1}^{x_1} \int_{-1}^{x_2} \cdots \int_{-1}^{x_d} \hat{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d,$$
$$u(\tau_1, \tau_2, \ldots, \tau_d)) d\tau_d \ldots d\tau_2 d\tau_1 = f(x_1, x_2, \ldots, x_d). \tag{3}$$

Here,

$$f(x_1,x_2,\ldots,x_d)=g\left(\frac{T_1}{2}(1+x_1),\frac{T_2}{2}(1+x_2),\ldots,\frac{T_d}{2}(1+x_d)\right),$$
$$\hat{K}(x_1,\tau_1,x_2,\tau_2,\ldots,x_d,\tau_d,u)=\frac{T_1}{2}\frac{T_2}{2}\cdots\frac{T_d}{2}K\left(\frac{T_1}{2}(1+x_1),\frac{T_1}{2}(1+\tau_1),\frac{T_2}{2}(1+x_2),\right.$$
$$\left.\frac{T_2}{2}(1+\tau_2),\ldots,\frac{T_d}{2}(1+x_d),\frac{T_d}{2}(1+\tau_d),u\right),$$

and $u(x_1,x_2,\ldots,x_d)=y\left(\frac{T_1}{2}(1+x_1),\frac{T_2}{2}(1+x_2),\ldots,\frac{T_d}{2}(1+x_d)\right)$ is the smooth solution of problem (3).

Firstly, Eq. (3) holds at the collocation points $\tilde{\mathbf{x}}_{\mathbf{j}}=(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d})$ on Ω, i.e.,

$$u(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d})+\int_{-1}^{\tilde{x}_{j_1}}\int_{-1}^{\tilde{x}_{j_2}}\cdots\int_{-1}^{\tilde{x}_{j_d}}\hat{K}(\tilde{x}_{j_1},\tau_1,\tilde{x}_{j_2},\tau_2,\ldots,\tilde{x}_{j_d},\tau_d,u(\tau_1,\tau_2,\ldots,\tau_d))$$
$$d\tau_d\cdots d\tau_2 d\tau_1=f(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d}). \tag{4}$$

In order to obtain high order accuracy for the problem (4), we transfer the integral domain $[-1,\tilde{x}_{j_1}]\times[-1,\tilde{x}_{j_2}]\cdots\times[-1,\tilde{x}_{j_d}]$ to a fixed interval $\bar{\Omega}$

$$u(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d})+\int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1}\tilde{K}(\tilde{x}_{j_1},\tau_1(\tilde{x}_{j_1},\theta_1),\tilde{x}_{j_2},\tau_2(\tilde{x}_{j_2},\theta_2),\ldots,\tilde{x}_{j_d},$$
$$\tau_d(\tilde{x}_{j_d},\theta_d),u(\tau_1(\tilde{x}_{j_1},\theta_1),\tau_2(\tilde{x}_{j_2},\theta_2),\ldots,\tau_d(\tilde{x}_{j_d},\theta_d)))d\theta_d\cdots d\theta_2 d\theta_1=f(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d}), \tag{5}$$

by using the following transformation

$$\tau_i=\tau_i(\tilde{x}_{j_i},\theta_i)=\frac{1+\tilde{x}_{j_i}}{2}\theta_i+\frac{\tilde{x}_{j_i}-1}{2},\quad i=1,2,\ldots,d, \tag{6}$$

where

$$\tilde{K}(\tilde{x}_{j_1},\tau_1,\tilde{x}_{j_2},\tau_2,\ldots,\tilde{x}_{j_d},\tau_d,u)=\frac{1+\tilde{x}_{j_1}}{2}\frac{1+\tilde{x}_{j_2}}{2}\cdots\frac{1+\tilde{x}_{j_d}}{2}\hat{K}(\tilde{x}_{j_1},\tau_1,\tilde{x}_{j_2},\tau_2,\ldots,\tilde{x}_{j_d},\tau_d,u).$$

Next, let $u_{j_1j_2\cdots j_d}$ be the approximation of the function value $u(\tilde{\mathbf{x}}_{\mathbf{j}})$ and use Legendre Gauss quadrature formula, (5) becomes

$$u_{j_1j_2\cdots j_d}+\sum_{\|\mathbf{k}\|\le N}\tilde{K}(\tilde{x}_{j_1},\tau_1(\tilde{x}_{j_1},\theta_{k_1}),\tilde{x}_{j_2},\tau_2(\tilde{x}_{j_2},\theta_{k_2}),\ldots,\tilde{x}_{j_d},\tau_d(\tilde{x}_{j_d},\theta_{k_d}),$$
$$u(\tau_1(\tilde{x}_{j_1},\theta_{k_1}),\tau_2(\tilde{x}_{j_2},\theta_{k_2}),\ldots,\tau_d(\tilde{x}_{j_d},\theta_{k_d})))\omega_{k_1}\omega_{k_2}\ldots\omega_{k_d}=f(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d}). \tag{7}$$

Here, $\{\theta_{\mathbf{k}},\|\mathbf{k}\|\le N\}$ denotes the Legendre Gauss points on the multidimensional space Ω and $\{\omega_{\mathbf{k}},\|\mathbf{k}\|\le N\}$ denotes the corresponding weights. Let $u_N(x_1,x_2,\ldots,x_d)=\sum_{\|\mathbf{i}\|\le N}u_{i_1i_2\ldots i_d}F_{i_1}(x_1)F_{i_2}(x_2)\ldots F_{i_d}(x_d)$. Now, we use u_N to approximate the solution u. Then, the Jacobi spectral collocation method is to seek u_N such that $u_{i_1i_2\cdots i_d}$ satisfy the following collocation equation:

$$
\begin{aligned}
&u_{j_1 j_2 \cdots j_d} + \sum_{\|\mathbf{k}\| \le N} \tilde{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_1}, \theta_{k_1}), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_2}, \theta_{k_2}), \ldots, \tilde{x}_{j_d}, \tau_d(\tilde{x}_{j_d}, \theta_{k_d}), \\
&\quad \sum_{\|\mathbf{i}\| \le N} u_{i_1 i_2 \ldots i_d} F_{i_1}(\tau_1(\tilde{x}_{j_1}, \theta_{k_1})) F_{i_2}(\tau_2(\tilde{x}_{j_2}, \theta_{k_2})) \ldots F_{i_d}(\tau_d(\tilde{x}_{j_d}, \theta_{k_d}))) \omega_{k_1} \omega_{k_2} \ldots \omega_{k_d} \\
&\quad = f(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}).
\end{aligned} \tag{8}
$$

We can get the values of $u_{i_1 i_2 \cdots i_d}$ by solving (8) and obtain the expressions of $u_N(\mathbf{x})$ accordingly.

Let the error function of the solution be written as $e_u(\mathbf{x}) := u(\mathbf{x}) - u_N(\mathbf{x})$. Since the exact solution of the problem (1) can be written as $y(\mathbf{t}) = u(\mathbf{x})$ ($t_i = \frac{T_i}{2}(1 + x_i)$, $t_i \in [0, T_i]$, $x_i \in [-1, 1]$), we can define its approximate solution $y_N(\mathbf{t}) = u_N(\mathbf{x})$. Then the corresponding error function satisfy

$$
\varepsilon_y(\mathbf{t}) := y(\mathbf{t}) - y_N(\mathbf{t}) = e_u(\mathbf{x}) = e_u\left(\frac{2}{T_1} t_1 - 1, \frac{2}{T_2} t_2 - 1, \ldots, \frac{2}{T_d} t_d - 1\right).
$$

Remark In our work, we let the multidimensional Jacobi weight function $\omega(\mathbf{x}) = \Pi_{i=1}^{d} (1 - x_i)^{\alpha} (1 + x_i)^{\beta}$, $-1 < \alpha, \beta < \frac{1}{d} - \frac{1}{2}$. So $\omega(x) = (1 - x)^{\alpha} (1 + x)^{\beta}$, $-1 < \alpha, \beta < \frac{1}{2}$ for $d = 1$. In Tang et al. (2008), the authors choose $\alpha = \beta = 0$.

Some lemmas

The following result can be found in Canuto et al. (2006).

Lemma 1 *Assume that Gauss quadrature formula is used to integrate the product $u\phi$, where $u \in H^m(\Omega)$ for some $m > \frac{d}{2}$ and $\phi \in \mathbb{P}_N(\Omega)$. Then there exists a constant C independent of N such that*

$$
|(u, \phi) - (u, \phi)_N| \le C N^{-m} |u|_{H^{m;N}(\Omega)} \|\phi\|_{L^2(\Omega)}, \tag{9}
$$

where $(\cdot, \cdot)$ represents the continuous inner product in $L^2(\Omega)$ space and

$$
(u, \phi)_N = \sum_{\|\mathbf{j}\| \le N} u(\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_d}) \phi(\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_d}) \omega_{j_1} \omega_{j_2} \ldots \omega_{j_d}.
$$

The seminorm is defined as

$$
|u|_{H^{m;N}(\Omega)} = \left(\sum_{k=\min(m, N+1)}^{m} \sum_{i=1}^{d} \left\| \frac{\partial^k u}{\partial x_i^k} \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
$$

Note that only pure derivatives in each spatial direction appear in this expression.

From Fedotov (2004), we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the Jacobi-Gauss points.

Lemma 2 *Let $\|I_N\|_\infty := \max_{\mathbf{x} \in \bar{\Omega}} \sum_{\|\mathbf{k}\| \le N} |F_{k_1}(x_1) F_{k_2}(x_2) \cdots F_{k_d}(x_d)|$, we have*

$$\|I_N\|_\infty = \begin{cases} \mathcal{O}\big((\log N)^d\big), & \text{if } -1<\alpha,\beta\le -\frac{1}{2}, \\ \mathcal{O}\Big((N^{\max(\alpha,\beta)+\frac{1}{2}})^d\Big), & \text{if } -\frac{1}{2}<\alpha,\beta<\frac{1}{d}-\frac{1}{2}, \\ \mathcal{O}\Big((N^{\alpha+\frac{1}{2}})^d\Big), & \text{if } -1<\beta\le -\frac{1}{2},\ -\frac{1}{2}<\alpha<\frac{1}{d}-\frac{1}{2}, \\ \mathcal{O}\Big((N^{\beta+\frac{1}{2}})^d\Big), & \text{if } -1<\alpha\le -\frac{1}{2},\ -\frac{1}{2}<\beta<\frac{1}{d}-\frac{1}{2}. \end{cases} \tag{10}$$

Lemma 3 *Assume that $u(\mathbf{x})\in H^m_\omega(\Omega)$ for $m>\frac{d}{2}$ and denote $(I_N u)(\mathbf{x})$ its interpolation polynomial associated with the multidimensional Jacobi Gauss points $\{\tilde{\mathbf{x}}_{\mathbf{j}}, \|\mathbf{j}\|\le N\}$. Then the following estimates hold*

$$\|u-I_N u\|_{L^2_\omega(\Omega)} \le CN^{-m}|u|_{H^{m;N}_\omega(\Omega)}, \tag{11}$$

$$\|u-I_N u\|_{L^\infty(\Omega)} \le CN^{d+2-m}|u|_{H^{m;N}_\omega(\Omega)}. \tag{12}$$

Proof The inequality (11) can be found in Canuto et al. (2006). We now prove (12). From Canuto et al. (2006), we have

$$\|u-I_N u\|_{H^l_\omega(\Omega)} \le CN^{2l-m}|u|_{H^{m;N}_\omega(\Omega)}, \quad 0\le l\le m.$$

We know that $H^l_\omega(\Omega)$ is embedded in $C(\bar{\Omega})$ for $l>\frac{d}{2}$, namely,

$$\begin{aligned} \|u-I_N u\|_{L^\infty(\Omega)} &\le C\|u-I_N u\|_{H^l_\omega(\Omega)} \le CN^{2l-m}|u|_{H^{m;N}_\omega(\Omega)} \\ &\le \begin{cases} CN^{d+2-m}|u|_{H^{m;N}_\omega(\Omega)}, & \text{when } d \text{ is an even number}, \\ CN^{d+1-m}|u|_{H^{m;N}_\omega(\Omega)}, & \text{when } d \text{ is an odd number}. \end{cases} \\ &\le CN^{d+2-m}|u|_{H^{m;N}_\omega(\Omega)}. \end{aligned}$$

□

The following Gronwall Lemma, whose proof can be found in Headley (1974), will be essential for establishing our main results.

Lemma 4 *Suppose $M\ge 0$, a nonnegative integrable function $E(\mathbf{x})$ satisfies*

$$\begin{aligned} E(x_1,x_2,\ldots,x_d) &\le M\int_{-1}^{x_1}\int_{-1}^{x_2}\cdots\int_{-1}^{x_d} E(\tau_1,\tau_2,\ldots,\tau_d)d\tau_d\ldots d\tau_2 d\tau_1 \\ &\quad + G(x_1,x_2,\ldots,x_d), \quad (x_1,x_2,\ldots,x_d)\in\Omega, \end{aligned}$$

where $G(\mathbf{x})$ is also an integrable function, we have

$$\|E\|_{L^2_\omega(\Omega)} \le C\|G\|_{L^2_\omega(\Omega)}, \tag{13}$$

$$\|E\|_{L^\infty(\Omega)} \le C\|G\|_{L^\infty(\Omega)}. \tag{14}$$

From Theorem 1 in Nevai (1984), we have the following mean convergence result of Lagrange interpolation based at the multidimensional Jacobi-Gauss points.

Lemma 5 *For every bounded function $v(\mathbf{x})$, there exists a constant C independent of v such that*

$$\sup_N \left\| \sum_{\|\mathbf{j}\| \le N} v(\tilde{\mathbf{x}}_\mathbf{j}) F_\mathbf{j}(\mathbf{x}) \right\|_{L^2_\omega(\Omega)} \le C \max_{\mathbf{x}\in\bar{\Omega}} |v(\mathbf{x})|. \tag{15}$$

For $r \ge 0$ and $\kappa \in (0,1)$, $\mathcal{C}^{r,\kappa}(\bar{\Omega})$ will denote the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the norm:

$$\|v\|_{C^{r,\kappa}(\bar{\Omega})} = \max_{|\alpha|\le r} \max_{\mathbf{x}\in\bar{\Omega}} \left| \frac{\partial^{|\alpha|} v(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}} \right| \\ + \max_{|\alpha|\le r} \sup_{\mathbf{x}' \ne \mathbf{x}'' \in \bar{\Omega}} \left| \frac{\frac{\partial^{|\alpha|} v(\mathbf{x}')}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}} - \frac{\partial^{|\alpha|} v(\mathbf{x}'')}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}}{\left((x_1' - x_1'')^2 + (x_2' - x_2'')^2 + \cdots + (x_d' - x_d'')^2\right)^{\frac{\kappa}{2}}} \right|.$$

We shall make use of a result of Ragozin (1970, (1971) *in the following lemma.*

Lemma 6 *For nonnegative integer r and $\kappa \in (0,1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in \mathcal{C}^{r,\kappa}(\bar{\Omega})$, there exists a polynomial function $\mathcal{T}_N v \in \mathbb{P}_N$ such that*

$$\|v - \mathcal{T}_N v\|_{L^\infty(\Omega)} \le C_{r,\kappa} N^{-(r+\kappa)} \|v\|_{\mathcal{C}^{r,\kappa}(\bar{\Omega})}, \tag{16}$$

Actually, $\mathcal{T}_N$ is a linear operator from $\mathcal{C}^{r,\kappa}(\bar{\Omega})$ into $\mathbb{P}_N$.

Lemma 7 *Assume there are constants $L_0, L_1, L_2, \ldots, L_d$ such that*

$$|\hat{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_1) - \hat{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_2)| \le L_0 |v_1 - v_2|, \\ |\hat{K}_{x_i}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_1) - \hat{K}_{x_i}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_2)| \le L_i |v_1 - v_2|, \\ i = 1, 2, \ldots, d.$$

Let M_{v_1,v_2} be defined by

$$M_{v_1,v_2}(\mathbf{x}) = \int_{-1}^{x_1} \int_{-1}^{x_2} \cdots \int_{-1}^{x_d} [\hat{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_1(\tau_1, \tau_2, \ldots, \tau_d)) \\ - \hat{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_2(\tau_1, \tau_2, \ldots, \tau_d))] d\tau_d \ldots d\tau_2 d\tau_1. \tag{17}$$

Then, for any $\kappa \in (0,1)$ and $v_1, v_2 \in \mathcal{C}(\bar{\Omega})$, there exists a positive constant $C \sim L_0, L_1, L_2, \ldots, L_d$ such that

$$\frac{|M_{v_1,v_2}(\mathbf{x}') - M_{v_1,v_2}(\mathbf{x}'')|}{\left((x_1' - x_1'')^2 + (x_2' - x_2'')^2 + \cdots + (x_d' - x_d'')^2\right)^{\frac{\kappa}{2}}} \le C \max_{\mathbf{x}\in\bar{\Omega}} |v_1(\mathbf{x}) - v_2(\mathbf{x})|, \tag{18}$$

for any $\mathbf{x}', \mathbf{x}'' \in \bar{\Omega}$ and $\mathbf{x}' \ne \mathbf{x}''$. This implies that

$$\|M_{\nu_1,\nu_2}\|_{C^{0,\kappa}(\bar{\Omega})} \leq C \max_{\mathbf{x}\in\bar{\Omega}} |\nu_1(\mathbf{x}) - \nu_2(\mathbf{x})|. \tag{19}$$

Proof For ease of exposition, and without essential loss of generality, we will proof this lemma for $d = 2$ and assume $x_1'' < x_1'$, $x_2'' < x_2'$,

$$\begin{aligned}
&|M_{\nu_1,\nu_2}(\mathbf{x}') - M_{\nu_1,\nu_2}(\mathbf{x}'')| \\
&= \left| \int_{-1}^{x_1'} \int_{-1}^{x_2'} [\hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_2 d\tau_1 \right. \\
&\quad \left. - \int_{-1}^{x_1''} \int_{-1}^{x_2''} [\hat{K}(x_1'', \tau_1, x_2'', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1'', \tau_1, x_2'', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_2 d\tau_1 \right| \\
&\leq E_1 + E_2.
\end{aligned} \tag{20}$$

Here,

$$\begin{aligned}
E_1 &= \left| \int_{-1}^{x_1'} \int_{-1}^{x_2'} [\hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_2 d\tau_1 \right. \\
&\quad \left. - \int_{-1}^{x_1'} \int_{-1}^{x_2''} [\hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_2 d\tau_1 \right| \\
&\leq \int_{-1}^{x_1'} \int_{x_2''}^{x_2'} |\hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_2(\tau_1, \tau_2))| d\tau_2 d\tau_1 \\
&\leq CL_0 \|\nu_1 - \nu_2\|_{L^\infty(\Omega)} (x_2' - x_2'') \\
&\leq CL_0 \|\nu_1 - \nu_2\|_{L^\infty(\Omega)} (x_2' - x_2'')^{1+\kappa} (x_2' - x_2'')^{-\kappa} \\
&\leq C \|\nu_1 - \nu_2\|_{L^\infty(\Omega)} [(x_1' - x_1'')^2 + (x_2' - x_2'')^2]^{-\frac{\kappa}{2}},
\end{aligned} \tag{21}$$

$$\begin{aligned}
E_2 &= \left| \int_{-1}^{x_1'} \int_{-1}^{x_2''} [\hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_2 d\tau_1 \right. \\
&\quad \left. - \int_{-1}^{x_1''} \int_{-1}^{x_2''} [\hat{K}(x_1'', \tau_1, x_2'', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1'', \tau_1, x_2'', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_2 d\tau_1 \right| \\
&\leq \int_{-1}^{x_2''} (P_1 + P_2 + P_3) d\tau_2 \leq C \max_{\mathbf{x}\in\bar{\Omega}} (P_1 + P_2 + P_3),
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
P_1 &= \left| \int_{-1}^{x_1'} [\hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_1 \right. \\
&\quad \left. - \int_{-1}^{x_1'} [\hat{K}(x_1', \tau_1, x_2'', \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2'', \tau_2, \nu_2(\tau_1, \tau_2))] d\tau_1 \right| \\
&= \left| \int_{-1}^{x_1'} [\hat{K}_{x_2}(x_1', \tau_1, \xi, \tau_2, \nu_1(\tau_1, \tau_2)) - \hat{K}_{x_2}(x_1', \tau_1, \xi, \tau_2, \nu_2(\tau_1, \tau_2))] (x_2' - x_2'') d\tau_1 \right| \\
&\leq CL_2 \|\nu_1 - \nu_2\|_{L^\infty(\Omega)} (x_2' - x_2'') \\
&\leq C \|\nu_1 - \nu_2\|_{L^\infty(\Omega)} (x_2' - x_2'')^{1+\kappa} (x_2' - x_2'')^{-\kappa} \\
&\leq C \|\nu_1 - \nu_2\|_{L^\infty(\Omega)} [(x_1' - x_1'')^2 + (x_2' - x_2'')^2]^{-\frac{\kappa}{2}}, \qquad \exists\, \xi \in (x_2'', x_2').
\end{aligned} \tag{23}$$

similarly,

$$
\begin{aligned}
P_2 = &\left| \int_{-1}^{x_1'} [\hat{K}(x_1', \tau_1, x_2'', \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x_2'', \tau_2, v_2(\tau_1, \tau_2))] d\tau_1 \right. \\
&\left. - \int_{-1}^{x_1'} [\hat{K}(x_1'', \tau_1, x_2'', \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1'', \tau_1, x_2'', \tau_2, v_2(\tau_1, \tau_2))] d\tau_1 \right| \\
= &\left| \int_{-1}^{x_1'} [\hat{K}_{x_1}(\eta, \tau_1, x_2'', \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}_{x_1}(\eta, \tau_1, x_2'', \tau_2, v_2(\tau_1, \tau_2))](x_1' - x_1'') d\tau_1 \right| \\
\leq &\, CL_1 \|v_1 - v_2\|_{L^\infty(\Omega)} (x_1' - x_1'') \\
\leq &\, C\|v_1 - v_2\|_{L^\infty(\Omega)} [(x_1' - x_1'')^2 + (x_2' - x_2'')^2]^{-\frac{\kappa}{2}}, \quad \exists\, \eta \in (x_1'', x_1').
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
P_3 = &\left| \int_{-1}^{x_1'} [\hat{K}(x_1'', \tau_1, x_2'', \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1'', \tau_1, x_2'', \tau_2, v_2(\tau_1, \tau_2))] d\tau_1 \right. \\
&\left. - \int_{-1}^{x_1''} [\hat{K}(x_1'', \tau_1, x_2'', \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1'', \tau_1, x_2'', \tau_2, v_2(\tau_1, \tau_2))] d\tau_1 \right| \\
\leq &\int_{x_1''}^{x_1'} L_0 |v_1 - v_2| d\tau_1 \\
\leq &\, C\|v_1 - v_2\|_{L^\infty(\Omega)} (x_1' - x_1'')^{1+\kappa} (x_1' - x_1'')^{-\kappa} \\
\leq &\, C\|v_1 - v_2\|_{L^\infty(\Omega)} [(x_1' - x_1'')^2 + (x_2' - x_2'')^2]^{-\frac{\kappa}{2}}.
\end{aligned} \tag{25}
$$

The estimate (18) for $d = 2$ is obtained by combining (20)–(24). □

Error estimates

Theorem 1 *Let $u(\mathbf{x})$ be the exact solution of the multidimensional nonlinear Volterra integral equation (3), which is smooth. $u_N(\mathbf{x})$ is the approximate solution, i.e., $u(\mathbf{x}) \approx u_N(\mathbf{x})$. Assume that*

$$
\begin{aligned}
&\left| \frac{\partial^k}{\partial \theta_i^k} \tilde{K}(x_1, \theta_1, x_2, \theta_2, \ldots, x_d, \theta_d, v_1) - \frac{\partial^k}{\partial \theta_i^k} \tilde{K}(x_1, \theta_1, x_2, \theta_2, \ldots, x_d, \theta_d, v_2) \right| \\
&\leq L_{ik} |v_1 - v_2|, \quad i = 1, 2, \ldots, d; \quad k = 1, 2, \ldots, m, \\
&L = \max_{1 \leq i \leq d, 1 \leq k \leq m} L_{ik}.
\end{aligned}
$$

Then there is a constant C such that the errors satisfy for $m > d + 2$,

$$
||u - u_N||_{L^\infty(\Omega)} \leq CN^{-m}
\begin{cases}
(\log N)^d K^* + N^{d+2} |u|_{H_\omega^{m;N}(\Omega)}, & \text{if } -1 < \alpha, \beta \leq -\frac{1}{2}, \\
\left(N^{\max(\alpha,\beta)+\frac{1}{2}}\right)^d K^* + N^{d+2} |u|_{H_\omega^{m;N}(\Omega)}, & \text{if } -\frac{1}{2} < \alpha, \beta < \frac{1}{d} - \frac{1}{2}, \\
\left(N^{\alpha+\frac{1}{2}}\right)^d K^* + N^{d+2} |u|_{H_\omega^{m;N}(\Omega)}, & \text{if } -1 < \beta \leq -\frac{1}{2},\ -\frac{1}{2} < \alpha < \frac{1}{d} - \frac{1}{2}, \\
\left(N^{\beta+\frac{1}{2}}\right)^d K^* + N^{d+2} |u|_{H_\omega^{m;N}(\Omega)}, & \text{if } -1 < \alpha \leq -\frac{1}{2},\ -\frac{1}{2} < \beta < \frac{1}{d} - \frac{1}{2}.
\end{cases} \tag{26}
$$

where

$$K^* = \max_{\|\mathbf{j}\|\le N} |\tilde{K}(\tilde{x}_{j_1}, \theta_1, \tilde{x}_{j_2}, \theta_2, \ldots, \tilde{x}_{j_d}, \theta_d, u(\theta_1, \theta_2, \ldots, \theta_d))|_{H^{m;N}(\Omega)},$$
$$C \sim d, L, L_0, L_1, L_2, \ldots, L_d.$$

Proof We subtract (8) from (5) to get the error equation

$$\begin{aligned}&u(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) - u_{j_1 j_2 \ldots j_d} + \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1}[\tilde{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_1}, \theta_1), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_2}, \theta_2), \ldots, \tilde{x}_{j_d},\\ &\tau_d(\tilde{x}_{j_d}, \theta_d), u(\tau_1(\tilde{x}_{j_1}, \theta_1), \tau_2(\tilde{x}_{j_2}, \theta_2), \ldots, \tau_d(\tilde{x}_{j_d}, \theta_d))) - \tilde{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_1}, \theta_1), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_2}, \theta_2), \ldots, \tilde{x}_{j_d},\\ &\tau_d(\tilde{x}_{j_d}, \theta_d), u_N(\tau_1(\tilde{x}_{j_1}, \theta_1), \tau_2(\tilde{x}_{j_2}, \theta_2), \ldots, \tau_d(\tilde{x}_{j_d}, \theta_d)))]d\theta_d \cdots d\theta_2 d\theta_1 = I(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}),\end{aligned}$$

where

$$\begin{aligned}I(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) = &\sum_{\|\mathbf{k}\|\le N} \tilde{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_1}, \theta_{k_1}), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_2}, \theta_{k_2}), \ldots, \tilde{x}_{j_d},\\ &\tau_d(\tilde{x}_{j_d}, \theta_{k_d}), u_N(\tau_1(\tilde{x}_{j_1}, \theta_{k_1}), \tau_2(\tilde{x}_{j_2}, \theta_{k_2}), \ldots, \tau_d(\tilde{x}_{j_d}, \theta_{k_d})))\omega_{k_1}\omega_{k_2}\ldots\omega_{k_d}\\ &- \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1}\tilde{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_1}, \theta_1), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_2}, \theta_2), \ldots, \tilde{x}_{j_d}, \tau_d(\tilde{x}_{j_d}, \theta_d),\\ &u_N(\tau_1(\tilde{x}_{j_1}, \theta_1), \tau_2(\tilde{x}_{j_2}, \theta_2), \ldots, \tau_d(\tilde{x}_{j_d}, \theta_d)))d\theta_d \ldots d\theta_2 d\theta_1.\end{aligned}$$

Using the variable transformation (6), we have

$$\begin{aligned}&u(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) - u_{j_1 j_2 \ldots j_d}\\ &\quad + \int_{-1}^{\tilde{x}_{j_1}}\int_{-1}^{\tilde{x}_{j_2}}\cdots\int_{-1}^{\tilde{x}_{j_d}}[\hat{K}(\tilde{x}_{j_1}, \tau_1, \tilde{x}_{j_2}, \tau_2, \ldots, \tilde{x}_{j_d}, \tau_d, u(\tau_1, \tau_2, \ldots, \tau_d))\\ &\quad - \hat{K}(\tilde{x}_{j_1}, \tau_1, \tilde{x}_{j_2}, \tau_2, \ldots, \tilde{x}_{j_d}, \tau_d, u_N(\tau_1, \tau_2, \ldots, \tau_d))]d\tau_d \ldots d\tau_2 d\tau_1\\ &= I(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}).\end{aligned} \tag{27}$$

Multiplying $F_{j_1}(x_1)F_{j_2}(x_2)\ldots F_{j_d}(x_d)$ on both sides of Eq. (27) and summing up $\|\mathbf{j}\| \le N$ yield

$$\begin{aligned}&e_u(x_1, x_2, \ldots, x_d) + \int_{-1}^{x_1}\int_{-1}^{x_2}\cdots\int_{-1}^{x_d}[\hat{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, u(\tau_1, \tau_2, \ldots, \tau_d))\\ &- \hat{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, u_N(\tau_1, \tau_2, \ldots, \tau_d))]d\tau_d \ldots d\tau_2 d\tau_1 = J_1(x_1, x_2, \ldots, x_d)\\ &+ J_2(x_1, x_2, \ldots, x_d) + J_3(x_1, x_2, \ldots, x_d).\end{aligned} \tag{28}$$

Consequently,

$$\begin{aligned}|e_u(x_1, x_2, \ldots, x_d)| \le &L_0\int_{-1}^{x_1}\int_{-1}^{x_2}\cdots\int_{-1}^{x_d}|e_u(\tau_1, \tau_2, \ldots, \tau_d)|d\tau_d \ldots d\tau_2 d\tau_1\\ &+ |J_1(x_1, x_2, \ldots, x_d)| + |J_2(x_1, x_2, \ldots, x_d)| + |J_3(x_1, x_2, \ldots, x_d)|,\end{aligned} \tag{29}$$

where

$$J_1(\mathbf{x}) = \sum_{\|\mathbf{j}\|\le N} I(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d})F_{j_1}(x_1)F_{j_2}(x_2)\ldots F_{j_d}(x_d),$$
$$J_2(\mathbf{x}) = u(x_1,x_2,\ldots,x_d) - (I_N u)(x_1,x_2,\ldots,x_d),$$
$$\begin{aligned}J_3(\mathbf{x}) = &\int_{-1}^{x_1}\int_{-1}^{x_2}\cdots\int_{-1}^{x_d}[\hat{K}(x_1,\tau_1,x_2,\tau_2,\ldots,x_d,\tau_d,u(\tau_1,\tau_2,\ldots,\tau_d))\\ &- \hat{K}(x_1,\tau_1,x_2,\tau_2,\ldots,x_d,\tau_d,u_N(\tau_1,\tau_2,\ldots,\tau_d))]d\tau_d\ldots d\tau_2 d\tau_1\\ &- I_N\int_{-1}^{x_1}\int_{-1}^{x_2}\cdots\int_{-1}^{x_d}[\hat{K}(x_1,\tau_1,x_2,\tau_2,\ldots,x_d,\tau_d,u(\tau_1,\tau_2,\ldots,\tau_d))\\ &- \hat{K}(x_1,\tau_1,x_2,\tau_2,\ldots,x_d,\tau_d,u_N(\tau_1,\tau_2,\ldots,\tau_d))]d\tau_d\ldots d\tau_2 d\tau_1.\end{aligned}$$

It follows from the Gronwall inequality in Lemma 4 that

$$\|e_u\|_{L_\infty(\Omega)} \le C\big(\|J_1\|_{L_\infty(\Omega)} + \|J_2\|_{L_\infty(\Omega)} + \|J_3\|_{L_\infty(\Omega)}\big). \tag{30}$$

Using (9) and (10), we have

$$\begin{aligned}||J_1||_{L_\infty(\Omega)} &\le C\|I_N\|_\infty\left(\max_{\|\mathbf{j}\|\le N}|I(\tilde{x}_{j_1},\tilde{x}_{j_2},\ldots,\tilde{x}_{j_d})|\right)\\ &\le C\|I_N\|_\infty N^{-m}|\tilde{K}(\tilde{x}_{j_1},\theta_1,\tilde{x}_{j_2},\theta_2,\ldots,\tilde{x}_{j_d},\theta_d,u_N(\theta_1,\theta_2,\ldots,\theta_d))|_{H^{m;N}(\Omega)},\\ &\le C\|I_N\|_\infty N^{-m}(K^* + |\tilde{K}(\tilde{x}_{j_1},\theta_1,\tilde{x}_{j_2},\theta_2,\ldots,\tilde{x}_{j_d},\theta_d,u_N(\theta_1,\theta_2,\ldots,\theta_d))\\ &\quad - \tilde{K}(\tilde{x}_{j_1},\theta_1,\tilde{x}_{j_2},\theta_2,\ldots,\tilde{x}_{j_d},\theta_d,u(\theta_1,\theta_2,\ldots,\theta_d))|_{H^{m;N}(\Omega)}).\end{aligned} \tag{31}$$

A straightforward computation shows that

$$\begin{aligned}&|\tilde{K}(\tilde{x}_{j_1},\theta_1,\tilde{x}_{j_2},\theta_2,\ldots,\tilde{x}_{j_d},\theta_d,u_N(\theta_1,\theta_2,\ldots,\theta_d))\\ &\quad - \tilde{K}(\tilde{x}_{j_1},\theta_1,\tilde{x}_{j_2},\theta_2,\ldots,\tilde{x}_{j_d},\theta_d,u(\theta_1,\theta_2,\ldots,\theta_d))|_{H^{m;N}(\Omega)}\\ &\quad\le \left(\sum_{k=1}^{m}\sum_{i=1}^{d}\|\frac{\partial^k}{\partial\theta_i^k}\tilde{K}(\tilde{x}_{j_1},\theta_1,\tilde{x}_{j_2},\theta_2,\ldots,\tilde{x}_{j_d},\theta_d,u_N(\theta_1,\theta_2,\ldots,\theta_d))\right.\\ &\quad\left. -\frac{\partial^k}{\partial\theta_i^k}\tilde{K}(\tilde{x}_{j_1},\theta_1,\tilde{x}_{j_2},\theta_2,\ldots,\tilde{x}_{j_d},\theta_d,u(\theta_1,\theta_2,\ldots,\theta_d))\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\\ &\quad\le \left(\sum_{k=1}^{m}\sum_{i=1}^{d}L_{ik}\|u_N-u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\\ &\quad\le L^{\frac{m\times d}{2}}\|u_N-u\|_{L^2(\Omega)} \le C\|e_u\|_{L^\infty(\Omega)}.\end{aligned} \tag{32}$$

Due to Lemma 3,

$$\|J_2\|_{L^\infty(\Omega)} \le CN^{d+2-m}|u|_{H_\omega^{m;N}(\Omega)}. \tag{33}$$

By virtue of Lemmas 6 and 7,

$$
\begin{aligned}
\|J_3\|_{L^\infty(\Omega)} &= \|(I-I_N)M_{u,u_N}\|_{L^\infty(\Omega)} \\
&= \|(I-I_N)(M_{u,u_N}-\mathcal{T}_N M_{u,u_N})\|_{L^\infty(\Omega)} \\
&\le (1+\|I_N\|_\infty)\|M_{u,u_N}-\mathcal{T}_N M_{u,u_N}\|_{L^\infty(\Omega)} \\
&\le C\|I_N\|_\infty N^{-\kappa}\|M_{u,u_N}\|_{C^{0,\kappa}(\bar{\Omega})} \\
&\le C\|I_N\|_\infty N^{-\kappa}\|e_u\|_{L^\infty(\Omega)} \\
&\begin{cases}
C(\log N)^d N^{-\kappa}\|e_u\|_{L^\infty(\Omega)}, & \text{if } -1<\alpha,\beta\le -\frac{1}{2}, \\
C(N^{\max(\alpha,\beta)+\frac{1}{2}})^d N^{-\kappa}\|e_u\|_{L^\infty(\Omega)}, & \text{if } -\frac{1}{2}<\alpha,\beta<\frac{1}{d}-\frac{1}{2}, \\
C(N^{\alpha+\frac{1}{2}})^d N^{-\kappa}\|e_u\|_{L^\infty(\Omega)}, & \text{if } -1<\beta\le -\frac{1}{2},\ -\frac{1}{2}<\alpha<\frac{1}{d}-\frac{1}{2}, \\
C(N^{\beta+\frac{1}{2}})^d N^{-\kappa}\|e_u\|_{L^\infty(\Omega)}, & \text{if } -1<\alpha\le -\frac{1}{2},\ -\frac{1}{2}<\beta<\frac{1}{d}-\frac{1}{2}.
\end{cases}
\end{aligned}
\tag{34}
$$

We now obtain the estimate for $\|e_u\|_{L^\infty(\Omega)}$ by using (30)–(34),

$$
\|e_u\|_{L^\infty(\Omega)} \le CN^{-m}\Big(\|I_N\|_\infty K^* + N^{d+2}|u|_{H^{m;N}_\omega(\Omega)}\Big),
$$

where in last step we have used the following assumption,

$$
\begin{cases}
0<\kappa<1, & \text{if } -1<\alpha,\beta\le -\frac{1}{2}, \\
(\max(\alpha,\beta)+\frac{1}{2})^d<\kappa<1, & \text{if } -\frac{1}{2}<\alpha,\beta<\frac{1}{d}-\frac{1}{2}, \\
(\alpha+\frac{1}{2})^d<\kappa<1, & \text{if } -1<\beta\le -\frac{1}{2},\ -\frac{1}{2}<\alpha<\frac{1}{d}-\frac{1}{2}, \\
(\beta+\frac{1}{2})^d<\kappa<1, & \text{if } -1<\alpha\le -\frac{1}{2},\ -\frac{1}{2}<\beta<\frac{1}{d}-\frac{1}{2}.
\end{cases}
\tag{35}
$$

This completes the proof of the theorem. □

Theorem 2 *If the hypotheses given in Theorem 1 hold and κ satisfies (35), then*

$$
\|u-u_N\|_{L^2_\omega(\Omega)} \le CN^{-m}
\begin{cases}
(1+N^{-\kappa}(\log N)^d)K^* + (1+N^{d+2-\kappa})|u|_{H^{m;N}_\omega(\Omega)}, \\
\qquad \text{if } -1<\alpha,\beta\le -\frac{1}{2}, \\
\left(1+N^{-\kappa}\left(N^{\max(\alpha,\beta)+\frac{1}{2}}\right)^d\right)K^* + (1+N^{d+2-\kappa})|u|_{H^{m;N}_\omega(\Omega)}, \\
\qquad \text{if } -\frac{1}{2}<\alpha,\beta<\frac{1}{d}-\frac{1}{2}, \\
(1+N^{-\kappa}\left(N^{\alpha+\frac{1}{2}})^d\right)K^* + (1+N^{d+2-\kappa})|u|_{H^{m;N}_\omega(\Omega)}, \\
\qquad \text{if } -1<\beta\le -\frac{1}{2},\ -\frac{1}{2}<\alpha<\frac{1}{d}-\frac{1}{2}, \\
\left(1+N^{-\kappa}\left(N^{\beta+\frac{1}{2}}\right)^d\right)K^* + (1+N^{d+2-\kappa})|u|_{H^{m;N}_\omega(\Omega)}, \\
\qquad \text{if } -1<\alpha\le -\frac{1}{2},\ -\frac{1}{2}<\beta<\frac{1}{d}-\frac{1}{2}.
\end{cases}
\tag{36}
$$

Proof By using (28) and Gronwall inequality in Lemma 4, we obtain that

$$
\|e_u\|_{L^2_\omega(\Omega)} \le C\Big(\|J_1\|_{L^2_\omega(\Omega)} + \|J_2\|_{L^2_\omega(\Omega)} + \|J_3\|_{L^2_\omega(\Omega)}\Big).
\tag{37}
$$

Using Lemmas 1, 5 and (32) we have for

$$
\|J_1\|_{L^2_\omega(\Omega)} \le C\max_{\mathbf{x}\in\bar{\Omega}}|I(\mathbf{x})| \le CN^{-m}(K^* + \|e_u\|_{L^2_\omega(\Omega)}).
\tag{38}
$$

Due to Lemma 3,

$$\|J_2\|_{L^2_\omega(\Omega)} \leq CN^{-m}|u|_{H^{m;N}_\omega(\Omega)}. \tag{39}$$

By virtue of Lemmas 6 and 7,

$$\begin{aligned}\|J_3\|_{L^2_\omega(\Omega)} &= \|(I-I_N)M_{u,u_N}\|_{L^2_\omega(\Omega)}\\ &= \|(I-I_N)(M_{u,u_N}-\mathcal{T}_N M_{u,u_N})\|_{L^2_\omega(\Omega)}\\ &\leq \|M_{u,u_N}-\mathcal{T}_N M_{u,u_N}\|_{L^2_\omega(\Omega)} + \|I_N(M_{u,u_N}-\mathcal{T}_N M_{u,u_N})\|_{L^2_\omega(\Omega)}\\ &\leq C\|M_{u,u_N}-\mathcal{T}_N M_{u,u_N}\|_{L^\infty(\Omega)}\\ &\leq CN^{-\kappa}\|e_u\|_{L^\infty(\Omega)}\\ &\leq CN^{-m-\kappa}\left(\|I_N\|_\infty K^* + N^{d+2}|u|_{H^{m;N}_\omega(\Omega)}\right).\end{aligned} \tag{40}$$

The desired estimate (36) is obtained by combining (37)–(40) and using the same technique as in the proof of Theorem 1. □

Numerical results

We give two numerical examples to confirm our analysis. To examine the accuracy of the results, L^2_ω and L^∞ errors are employed to assess the efficiency of the method. All the calculations are supported by the software Matlab.

Example 1 We consider the following two-dimensional Volterra integral equation

$$\begin{aligned}&u(x,y) + \int_{-1}^{x}\int_{-1}^{y}\cos(x+y)e^{\frac{\xi\eta}{2}}u(\xi,\eta)d\eta d\xi\\ &\quad= e^{-\frac{xy}{2}} + \cos(x+y)(x+1)(y+1).\end{aligned} \tag{41}$$

The corresponding exact solution is given by $u(x,y)=e^{-\frac{xy}{2}}$. We select $\alpha=-\frac{2}{3}$, $\beta=-\frac{1}{2}$. Table 1 shows the errors $\|u-u_N\|_{L^2_\omega(\Omega)}$ and $\|u-u_N\|_{L^\infty(\Omega)}$ obtained by using the spectral collocation method described above. Furthermore, the numerical results are plotted for $2 \leq N \leq 12$ in Fig. 1. It is observed that the desired exponential rate of convergence is obtained.

Example 2 Consider the equation with

Table 1 The errors $\|u-u_N\|_{L^2_\omega(\Omega)}$ and $\|u-u_N\|_{L^\infty(\Omega)}$

N	**2**	**4**	**6**
L^∞-error	9.3273e−003	3.3409e−005	5.1698e−008
L^2_ω-error	1.8151e−003	1.4154e−006	1.0899e−009
N	**8**	**10**	**12**
L^∞-error	4.5534e−011	6.5281e−014	6.7390e−014
L^2_ω-error	6.0859e−013	1.3022e−013	1.3124e−013

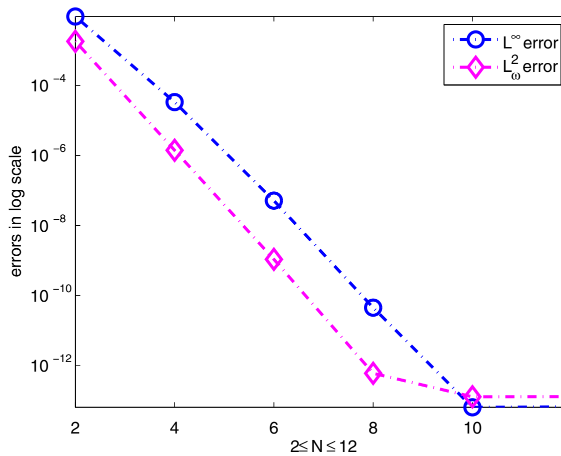

Fig. 1 The errors $u - u_N$ versus the number of collocation points in L^∞ and L^2_ω norms

$$\begin{aligned} v(x,y) &+ \int_{-1}^{x}\int_{-1}^{y} cos(x+\xi)v(\xi,\eta)d\eta d\xi \\ &= sin(x+y) - \frac{1}{4}sin(3x+y) + \frac{1}{4}sin(x+y-2) - \frac{1}{2}(x+1)cos(x-y) \\ &\quad + \frac{1}{2}(x+1)cos(x+1) + \frac{1}{4}sin(3x-1) - \frac{1}{4}sin(x-3). \end{aligned} \tag{42}$$

The corresponding exact solution is given by $v(x,y) = sin(x+y)$. We select $\alpha = -\frac{2}{3}$, $\beta = -\frac{3}{4}$. Table 2 shows the errors $\|v - v_N\|_{L^2_\omega(\Omega)}$ and $\|v - v_N\|_{L^\infty(\Omega)}$. The numerical results are plotted for $2 \le N \le 12$ in Fig. 2.

Conclusions

In this paper, we proposed a spectral collocation method based on Jacobi orthogonal polynomials to obtain approximate solution for multidimensional nonlinear Volterra integral equation. The most important contribution of this work is that we are able to

Table 2 The errors $\|v - v_N\|_{L^2_\omega(\Omega)}$ and $\|v - v_N\|_{L^\infty(\Omega)}$

N	**2**	**4**	**6**
L^∞-error	1.0746e−001	1.2307e−003	7.3430e−006
L^2_ω-error	4.7972e−002	3.5805e−004	1.3925e−006
N	**8**	**10**	**12**
L^∞-error	2.5031e−008	5.6992e−011	1.2992e−013
L^2_ω-error	3.6864e−009	6.7815e−012	7.9865e−014

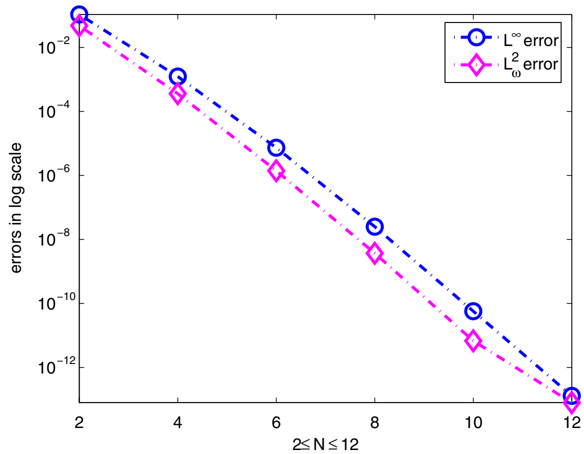

Fig. 2 The errors $v - v_N$ versus the number of collocation points in L^∞ and L^2_ω norms

demonstrate rigorously that the errors of spectral approximations decay exponentially in both $L^\infty(\Omega)$ norm and $L^2_\omega(\Omega)$ norm on d-dimensional space, which is a desired feature for a spectral method.

Authors' contributions

YW and YC carried out the spectral collocation method studies, performed the error analysis and drafted the manuscript. XS participated in the numerical experiments. YZ helped to draft the manuscript. All authors read and approved the final manuscript.

Author details

[1] College of Mathematic and Information Science, Shandong Institute of Business and Technology, Yantai 264005, China. [2] School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China. [3] School of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, China. [4] Department of Mathematics and Information Science, Yantai University, Yantai 264005, China.

Acknowledgements

This work is supported by National Natural Science Foundation of China (11401347, 91430104, 11271145, 61401255, 11426193).

Competing interests

The authors declare that they have no competing interests.

Received: 6 June 2016 Accepted: 22 September 2016
Published online: 04 October 2016

References

Atdev S, Ashirov S (1977) Solutions of multidimensional nonlinear Volterra operator equations. Ukr Math J 29:437–442

Beesack PR (1985) Systems of multidimensional Volterra integral equations and inequalities. Nonlinear Anal Theory 9:1451–1486

Brunner H (1984) Implicit Runge–Kutta methods of optimal order for Volterra integro-differential equations. Math Comput 42:95–109

Brunner H (1986) Polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels. IMA J Numer Anal 6:221–239

Brunner H (2004) Collocation methods for Volterra integral and related functional equations. Cambridge University Press, Cambridge

Brunner H, Tang T (1989) Polynomial spline collocation methods for the nonlinear Basset equation. Comput Math Appl 18:449–457

Brunner H, Schötzau D (2006) hp-discontinuous Galerkin time-stepping for Volterra integrodifferential equations. SIAM J Numer Anal 44:224–245

Brunner H, Pedas A, Vainikko G (2001) Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels. SIAM J Numer Anal 39:957–982

Canuto C, Hussaini MY, Quarteroni A et al (2006) Spectral methods fundamentals in single domains. Springer, Berlin

Chen Y, Tang T (2009) Spectral methods for weakly singular Volterra integral equations with smooth solutions. J Comput Appl Math 233:938–950
Chen Y, Tang T (2010) Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel. Math Comput 79:147–167
Chen Y, Li X, Tang T (2013) A note on Jacobi spectral-collocation methods for weakly singular Volterra integral equations with smooth solutions. J Comput Math 31:47–56
Fedotov AI (2004) Lebesgue constant estimation in multidimensional Sobolev space. J Math 14:25–32
Goldfine A (1977) Taylor series methods for the solution of Volterra integral and integro-differential equations. Math Comput 31:691–707
Headley VB (1974) A multidimensional nonlinear Gronwall inequality. J Math Anal Appl 47:250–255
Mckee S (1979) Cyclic multistep methods for solving Volterra integro-differential equations. SIAM J Numer Anal 16:106–114
Nevai P (1984) Mean convergence of Lagrange interpolation. Trans Am Math Soc 282:669–698
Pachpatte BG (2011) Multidimensional integral equations and inequalities. Springer, Berlin 9
Ragozin DL (1970) Polynomial approximation on compact manifolds and homogeneous spaces. Trans Am Math Soc 150:41–53
Ragozin DL (1971) Constructive polynomial approximation on spheres and projective spaces. Trans Am Math Soc 162:157–170
Shen J, Tang T (2006) Spectral and high-order methods with applications. Science Press, Beijing
Suryanarayana MB (1972) On multidimensional integral equations of Volterra type. Pac J Math 41:809–828
Tang T, Xu X, Chen J (2008) On spectral methods for Volterra integral equations and the convergence analysis. J Comput Math 26:825–837
van der Houwen PJ, te Riele HJJ (1985) Linear multistep methods for Volterra integral and integro-differential equations. Math Comput 45:439–461
Wei YX, Chen Y (2012a) Convergence analysis of the spectral methods for weakly singular Volterra integro-differential equations with smooth solutions. Adv Appl Math Mech 4:1–20
Wei YX, Chen Y (2012b) Legendre spectral collocation methods for pantograph Volterra delay-integro-differential equation. J Sci Comput 53:672–688
Wei YX, Chen Y (2013) A spectral method for neutral Volterra integro-differential equation with weakly singular kernel. Numer Math Theory Methods Appl 6:424–446
Wei YX, Chen Y (2014) Legendre spectral collocation method for neutral and high-order Volterra integro-differential equation. Appl Numer Math 81:15–29
Yuan W, Tang T (1990) The numerical analysis of implicit Runge–Kutta methods for a certain nonlinear integro-differential equation. Math Comput 54:155–168