A self-adaptive extragradient method for fixed-point and pseudomonotone equilibrium problems in Hadamard spaces

Kazeem Olalekan Aremu[1,2], Lateef Olakunle Jolaoso[1,3] and Olawale Kazeem Oyewole[4*]

*Correspondence:
oyewoleolawalekazeem@
gmail.com;
oyewoleok@campus.tehnion.ac.il
[4]Technion-Israel Institute of
Technology, Haifa, Israel
Full list of author information is
available at the end of the article

Abstract

In this work, we study a self-adaptive extragradient algorithm for approximating a common solution of a pseudomonotone equilibrium problem and fixed-point problem for multivalued nonexpansive mapping in Hadamard spaces. Our proposed algorithm is designed in such a way that its step size does not require knowledge of the Lipschitz-like constants of the bifunction. Under some appropriate conditions, we establish the strong convergence of the algorithm without prior knowledge of the Lipschitz constants. Furthermore, we provide a numerical experiment to demonstrate the efficiency of our algorithm. This result extends and complements recent results in the literature.

MSC: 65K15; 47J25; 65J15; 90C33

Keywords: Extragradient method; Equilibrium problems; Common fixed point; Pseudomontone; Hadamard spaces

1 Introduction

In this article, we consider the problem of approximating a common solution of an equilibrium problem (EP) and a fixed-point problem for a multivalued mapping in a nonlinear space. Let C be a nonempty, closed, and convex subset of a metric space X. A multivalued nonlinear mapping $T : X \to 2^X$ is said to have a fixed point $x \in X$ if $x \in Tx$. We denote the fixed-point set of T by $F(T)$, i.e., $F(T) = \{x \in X : x \in Tx\}$. Let $f : C \times C \to \mathbb{R}$ be a bifunction, the EP is to find a point $x^* \in C$ such that

$$f(x^*, y) \ge 0, \quad \forall y \in C, \tag{1.1}$$

where $f(x,x) = 0$ and $f(x, \cdot)$ is convex on C. The set of solutions of (1.1) is denoted by $\mathrm{EP}(f, C)$. The EP was first introduced in the linear setting by Ky Fan [13] and was later developed by Muu and Oettli [35] (see also Blum and Oettli [3]). The problem (1.1) is known to cover other important mathematical problems such as the minimization problem, the variational inequality problem, the saddle-point problem, Nash equilibrium involving noncooperative games, the fixed-point problem, etc. (see [3, 35]). It is well known

that many real-life problems are characterized by phenomena that can be modeled as nonlinear, nonconvex, continuous optimization problems. Approximating the solution of such problems may actually be complicated in the framework of linear spaces. Dedieu et al. [7] (see also Li and Wang [32]) noted that these problems can be overcome in the nonlinear space settings. For this reason, Colao et al. [6] extended problem (1.1) to nonlinear spaces (in particular, a Riemannian manifold) and later in 2019 Khatibzadeh and Mohebbi [27] also studied EP in Hadamard spaces. These pioneering works have increased the interest of researchers in solving EP (1.1) in nonlinear spaces; see [1, 19, 28, 31]. We remark here that the aforementioned works were in the case when the bifunction f in (1.1) is monotone. Usually, the iterative methods employed in this instance adopt the popular regularization technique associated with a strongly monotone subproblem. Hoewever, in the case when f in (1.1) is nonmonotone, the regularization subproblem ceases to be strongly monotone and the iterative method becomes complex.

It is well known that pseudomonotone operators are obvious generalizations of the monotone operators and many real-life EPs can be modeled with pseudomonotone bifunctions. One of the notable methods used to solve the nonmonotone (in particular, the pseudomonotone) EP is the extragradient algorithm (EA). The EA was introduced by Koperlevich [30] and later redefined by Trans et al. [40]. Recently, Khammahawong et al. [26] employed an EA to approximate the solution of a strongly pseudomonotone EP in a Hadamard (complete Riemannian) manifold. Their proposed algorithm is as follows: Given $x_0, y_0 \in C$, the sequence $\{y_n\}$ is generated as:

$$
\begin{cases}
x_{n+1} \in \arg\min_{y \in C}\{f(y_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y)\}, \\
y_{n+1} \in \arg\min_{y \in C}\{f(y_n, y) + \frac{1}{2\lambda_n}d^2(x_{n+1}, y)\},
\end{cases}
\tag{1.2}
$$

where $\{\lambda_n\}$ is a positive sequence and f is a strongly pseudomonotone bifunction. They obtained the convergence result of (1.2) under some mild conditions. We note that algorithm (1.2) has been extensively studied in both Hilbert and Banach spaces, see [37–39] and other references therein. To obtain the convergence of the sequence generated by (1.2) (and in the work of Khatibzadeh and Mohebbi [27] in Hadamard spaces) there is need to know in advance the estimates of the Lipschitz-like constants of the bifunction. These are usually not easy to derive when the structure of the bifunction is not simple and may affect the efficiency of the algorithm.

In 2021, Iusem and Mohebbi [18] introduced two EA with a linesearch technique to solve (1.2) with a pseudomontone bifunction in Hadamard spaces. Their proposed algorithms are as follows:

Algorithm 1.1 (Extragradient Algorithm with Linesearch (EAL))
Initialization: Choose $x_0 \in C \subset X$. Take $\delta \in (0,1), \hat{\lambda}, \bar{\lambda}$ satisfying $0 < \hat{\lambda} \le \bar{\lambda}$, and $\lambda_n \subseteq [\hat{\lambda}, \bar{\lambda}]$.
Iterative Step: Given x_n, define

$$
z_n \in \arg\min\left\{f(x_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y) : y \in C\right\}.
\tag{1.3}
$$

If $x_n = z_n$ stop. Otherwise, let

$$
l(n) = \min\left\{l \ge 0 : \lambda_n f(y_l, x_n) - \lambda_n f(y_l, z_n) \ge \frac{\delta}{2}d^2(z_n, x_n)\right\},
$$

where

$$y_l = 2^{-l}z_n \oplus \left(1 - 2^{-l}\right)x_n.$$

Take

$$\alpha_n := 2^{-l(n)},$$

$$y_n = \alpha_n z_n \oplus (1 - \alpha_n)x_n,$$

$$w_n = P_{H_n}(x_n),$$

where

$$H_n = \left\{y \in X : f(y_n, y) \le 0\right\}.$$

Then,

$$x_{n+1} = P_C w_n.$$

Iusem and Mohebbi [18] obtained the Δ-convergence of *EAL* to an element in EP(f, C). In addition, they proposed another algorithm to ensure strong convergence that was more desirable than the Δ-convergence in *EAL*. Precisely, the algorithm is as follows:

Algorithm 1.2 (Halpern Extragradient Algorithm with Linesearch (HEAL))
Initialization: Choose $x_0 \in C$, $u \in X$. Consider a sequence $\{\gamma_n\} \subset (0, 1)$ such that $\lim_{n \to \infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$.
Iterative Step: Given x_n, define

$$z_n \in \arg\min\left\{f(x_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y) : y \in C\right\}. \tag{1.4}$$

If $x_n = z_n$ stop. Otherwise, let

$$l(n) = \min\left\{l \ge 0 : \lambda_n f(y_l, x_n) - \lambda_n f(y_l, z_n) \ge \frac{\delta}{2}d^2(z_n, x_n)\right\},$$

where

$$y_l = 2^{-l}z_n \oplus \left(1 - 2^{-l}\right)x_n.$$

Take

$$\alpha_n := 2^{-l(n)},$$

$$y_n = \alpha_n z_n \oplus (1 - \alpha_n)x_n,$$

$$w_n = P_{H_n}(x_n),$$

where

$$H_n = \left\{ y \in X : f(y_n, y) \leq 0 \right\}.$$

Then,

$$v_n = \gamma_n u \oplus (1 - \gamma_n) w_n,$$

$$x_{n+1} = P_C v_n.$$

It is worth noting that Algorithm 1.1 and Algorithm 1.2 eliminate the challenges observed in (1.2) and [27] by introducing a linesearch technique. However, the presence of linesearch techniques in the algorithms to select step sizes will require more computation of nested iterations, which will be time consuming.

Motivated by the work of Iusem and Mohebbi [18] and other works in this direction, we introduce a self-adaptive extragradient algorithm in Hadamard spaces that does not need to have prior knowledge of Lipschitz-like constants or require a linesearch technique for execution. Our algorithm converges strongly to a common solution of the EP and the fixed point of a multivalued nonexpansive mapping. Furthermore, we give a numerical example in a Hadamard space to demonstrate the efficiency of our method. Our result extends and complements other recent results in this direction. Our contributions in this work are briefly highlighted as follows:

(i) Our work extends step-adaptive algorithms for pseudomonotone EP from linear spaces (see, for example, [4, 14–16, 21, 23]) to nonlinear spaces.

(ii) Our work improves the works of Khatibzadeh and Mohebbi [27], also that of Iusem and Mohebbi [18]. That is, our algorithm neither depends on the Lipschitz constants nor uses a linesearch technique.

2 Preliminaries

In this section, we present some notations, known definitions, and useful results that will be needed in the proof of our main result. Throughout this work, we use the notations "$\rightharpoonup$" and "$\rightarrow$" to denote Δ-convergence and strong convergence, respectively.

Let X be a metric space and $x, y \in X$. A geodesic from x to y is a map (or a curve) c from the closed interval $[0, d(x, y)] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. The image of c is called a geodesic segment joining x to y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset C of a geodesic space X is said to be convex, if for any two points $x, y \in C$, the geodesic joining x and y is contained in C, that is, if $c : [0, d(x, y)] \to X$ is a geodesic such that $x = c(0)$ and $y = c(d(x, y))$, then $c(t) \in C \ \forall t \in [0, d(x, y)]$. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) with unparameterized geodesic segments between each pair of vertices. For any geodesic triangle there is a comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$, such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$, for $i, j \in \{1, 2, 3\}$. A geodesic space X is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its

comparison triangle $\bar{\Delta}$. If Δ and $\bar{\Delta}$ are geodesic and comparison triangles in X, respectively, then Δ is said to satisfy the CAT(0) inequality for all points x, y of Δ and $\bar{x}, \bar{y}$ of $\bar{\Delta}$ if

$$d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.1}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \le \frac{1}{2} d^2(x, y) + \frac{1}{2} d^2(x, z) - \frac{1}{4} d(y, z). \tag{2.2}$$

Berg and Nikolaev [2] introduced the notion of quasilinearization in a CAT(0) space as follows: Let a pair $(a, b) \in X \times X$ denoted by $\overrightarrow{ab}$, be called a vector. Then, the quasilinearization map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} \big(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \big), \quad \text{for all } a, b, c, d \in X. \tag{2.3}$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, for all $a, b, c, d, e \in X$. Furthermore, a geodesic space X is said to satisfy the Cauchy–Schwartz inequality, if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a, b) d(c, d)$$

for all $a, b, c, d \in X$. It is well known that a geodesically connected space is a CAT(0) space if and only if it satisfies the Cauchy–Schwartz inequality [12]. Also, it is known that complete CAT(0) spaces are called Hadamard spaces.

Let $\{x_n\}$ be a bounded sequence in a metric space X and $r(\cdot, \{x_n\}) : X \to [0, \infty)$ be a continuous functional defined by $r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\}$, while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we say that x is the Δ-limit of $\{x_n\}$ (see [11, 29]). The notion of Δ-convergence in metric spaces was introduced and studied by Lim [33], and it is known as the analog of the notion of weak convergence in Banach spaces.

Let X be a geodesic convex metric space and A be a nonempty subset of X. A subset A is called proximal (see [9]), if for each $x \in X$ there exists $a \in A$ such that

$$\text{dist}(x, A) = \inf\big\{d(x, a) : a \in A\big\}.$$

It is well known that if A is proximal, then A is closed. We denote the family of all nonempty proximal subsets of X by $P(X)$ and the family of closed and bounded subsets of X by $CB(X)$, respectively. If A and B are nonempty subsets of X, then the Hausdorff metric H on $P(X)$ is defined by

$$H(A, B) = \max\Big\{\sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A)\Big\}, \quad \forall A, B \in P(X).$$

Let $T : X \to CB(X)$ be a multivalued mapping. A point $x \in X$ is called a strict fixed point of T if $Tx = \{x\}$. In this case, T is said to satisfy the end point condition and denote the set of end points of the mapping T as $E(T)$. The mapping T is said to be multivalued nonexpansive, if

$$H(Tx, Ty) \leq d(x, y) \quad \text{for all } x, y \in X.$$

Remark 2.1 Let $T : C \to CB(C)$ be a multivalued mapping. If $p \in C$ is an end point of the mapping T, then p is also a fixed point of T. That is, $E(T) \subseteq F(T)$, in fact equality holds if T is single valued. However, the converse may not hold.

Example 2.2 Let $X = \mathbb{R}$ and $C = \{x : 0 \leq x \leq 1\}$ with the usual metric. For each $x \in C$, let $T : C \to CB(X)$ be defined as $Tx = [0, x^n]$, $1 \leq n < \infty$. It is obvious that T is nonexpansive with $E(T) = \{0\}$ and $F(T) = \{0, 1\}$.

Definition 2.3 Let X be a Hadamard space. A multivalued nonlinear mapping $T : X \to 2^X$ is said to be demiclosed if for any bounded sequence $\{x_n\}$ in X such that $\Delta - \lim_{n \to \infty} x_n = x^*$ and $\lim_{n \to \infty} d(x_n, z_n) = 0$, (where $z_n \in Tx_n$) we have that $x^* \in F(T)$.

Lemma 2.4 ([17]) *Let X be a metric space and A, B are nonempty subsets in $P(X)$. Then, for all $a \in A$, there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Definition 2.5 Let X be a Hadamard space. A function $h : X \to (-\infty, \infty]$ is said to be
   (i) convex, if

$$h\big(\lambda x \oplus (1 - \lambda)y\big) \leq \lambda h(x) + (1 - \lambda)h(y), \quad \forall x, y \in X, \lambda \in (0, 1),$$

   (ii) lower semicontinuous (or upper semicontinuous) at a point $x \in C$, if

$$h(x) \leq \liminf_{n \to \infty} h(x_n) \qquad \left(\text{or } h(x) \geq \limsup_{n \to \infty} h(x_n)\right), \tag{2.4}$$

   for each sequence $\{x_n\}$ in C such that $\lim_{n \to \infty} x_n = x$,
   (iii) lower semicontinuous (or upper semicontinuous) on C, if it is lower semicontinuous (or upper semicontinuous) at every point in C.
The convex programming associated with the proper, convex, and lower semicontinuous function h for all $\lambda > 0$ is given by:

$$\arg\min_{x \in X}\left(h(x) + \frac{1}{2\lambda}d^2(x, y)\right) \tag{2.5}$$

for all $y \in X$.

Remark 2.6 ([24]) The subproblem (2.5) is well defined for all $\lambda > 0$.

Definition 2.7 Let C be a nonempty, closed, and convex subset of a Hadamard space X. A bifunction $f : C \times C \to \mathbb{R}$ is said to be

(i) monotone, if

$$f(x,y) + f(y,x) \leq 0, \quad \forall x, y \in C;$$

(ii) pseudomonotone if

$$f(x,y) \geq 0 \quad \Rightarrow \quad f(y,x) \leq 0, \quad \forall x, y \in C.$$

It is well known that monotone bifunctions are pseudomonotone, but the converse is not true in general (see, for instance [20, 22]).

Definition 2.8 Let X be a Hadamard space. A bifunction f is said to satisfy the Lipschitz-like continuity if there exist two constants $c_1, c_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1 d^2(x,y) - c_2 d^2(y,z), \quad \forall x, y, z \in X.$$

To solve the EP, the following assumptions are important and necessary on the bifunction f in establishing our result:

(A1) $f(x, \cdot) : X \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in X$,
(A2) $f(\cdot, y)$ is Δ-upper semicontinuous for all $x \in X$,
(A3) f satisfies the Lipschitz-type continuity condition,
(A4) f is pseudomonotone.

Definition 2.9 Let C be a nonempty, closed, and convex subset of a Hadamard space X. The metric projection is a mapping $P_C : X \to C$ that assigns to each $x \in X$, the unique point $P_C x \in C$ such that

$$d(x, P_C x) = \inf\{d(x,y) : y \in C\}.$$

Lemma 2.10 ([12]) *Every bounded sequence in a Hadamard space has a Δ-convergent subsequence.*

Lemma 2.11 *Let X be a Hadamard space, $x, y, z \in X$, and $t, s \in [0, 1]$. Then,*
(i) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$ (see [12]).
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$ (see [12]).
(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2 d^2(x,z) + (1-t)^2 d^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ (see [8]).

Lemma 2.12 ([25]) *Let X be a Hadamard space, $\{x_n\}$ be a sequence in X, and $x \in X$. Then, $\{x_n\}$ Δ-converges to x if and only if*

$$\limsup_{n \to \infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \leq 0, \quad \forall y \in X.$$

Lemma 2.13 ([8]) *Let C be a nonempty, convex subset of a Hadamard space X, $x \in X$ and $u \in C$. Then, $u = P_C x$ if and only if $\langle \overrightarrow{ux}, \overrightarrow{yu} \rangle \leq 0$, for all $y \in C$.*

Lemma 2.14 ([10]) *Let X be a Hadamard space and $T : X \to X$ be a nonexpansive mapping. Then, T is Δ-demiclosed.*

Lemma 2.15 ([36]) *Let X be a Hadamard space and $\{x_n\}$ be a sequence in X. If there exists a nonempty subset C in which*

(i) $\lim_{n \to \infty} d(x_n, z)$ *exists for every $z \in C$, and*

(ii) *if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ that is Δ-convergent to x, then $x \in C$,*

then, there is a $p \in C$ such that $\{x_n\}$ is Δ-convergent to p in X.

Lemma 2.16 ([41]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\limsup_{n \to \infty} \delta_n \leq 0$,

(iii) $\gamma_n \geq 0 (n \geq 0)$, $\Sigma_{n=0}^{\infty}\gamma_n < \infty$.

Then, $\lim_{n \to \infty} a_n = 0$.

Lemma 2.17 ([34]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1}$ $\forall j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad and \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

3 Main results

In this section, we present our algorithm and its convergence analysis for approximating the solutions of EP and the fixed point of a multivalued nonexpansive mapping in Hadamard spaces. In the following, let C be a nonempty, closed, convex subset of a Hadamard space $X, f : X \times X \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4) and let $T : C \to CB(X)$ be a multivalued nonexpansive mapping such that $Tx^* = \{x^*\}$. Suppose the solution set of the aforementioned problems is

$$\Gamma := \mathrm{EP}(f, C) \cap F(T) \neq \emptyset.$$

Now, we present our algorithm as follows:

Algorithm 3.1 (Self-Adaptive Extragradient Algorithm (SAEA))

Initialization: Choose $u, x_0 \in C$, $n \geq 0$, $\lambda_n > 0$, $\mu \in (0,1)$.

Iterative steps: Take $\{\alpha_n\} \subseteq (0,1)$ such that $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\beta_n\} \subseteq (0,1)$ such that $\liminf_{n \to \infty}(1 - \alpha_n)\beta_n > 0$. Given the nth iterate, compute the $(n+1)$th iterate via the following procedure:

Step 1: Compute

$$y_n = \arg\min\left\{f(x_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y) : y \in C\right\}. \tag{3.1}$$

If $x_n = y_n$, then stop. Otherwise go to Step 2.

Step 2: Compute

$$w_n = \arg\min\left\{f(y_n, y) + \frac{1}{2\lambda_n}d^2(x_n, y) : y \in C\right\}. \tag{3.2}$$

Step 3: Compute

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)\left[\beta_n h_n \oplus (1 - \beta_n)w_n\right],$$

where $h_n \in Tw_n$.

Step 4: Evaluate

$$\lambda_{n+1} = \begin{cases} \min\{\lambda_n, \frac{\mu[d^2(x_n,y_n)+d^2(w_n,y_n)]}{2[f(x_n,w_n)-f(x_n,y_n)-f(y_n,w_n)]_+}\}, \\ \quad \text{if } f(x_n, w_n) - f(x_n, y_n) - f(y_n, w_n) > 0, \\ \lambda_0, \quad \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n + 1$ and go back to Step 1.

We begin with the following lemma that is crucial to the nonincreasing monotonicity of sequence (3.3). The lemma has been proved by many authors in the framework of Hilbert and Banach spaces (see [22, 42] and other references therein). We state the lemma here in a Hadamard-space setting and give the proof for completeness.

Lemma 3.2 *The sequence* $\{\lambda_n\}$ *defined by* (3.3) *is monotonically nonincreasing and bounded and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \left\{\frac{\mu}{2\max\{c_1, c_2\}}, \lambda_1\right\}.$$

Proof It is obvious that the sequence $\{\lambda_n\}$ is monotonic nonincreasing. By the Lipschitz-like property of f, we obtain that

$$\frac{[d^2(x_n,y_n)+d^2(w_n,y_n)]}{2[f(x_n,w_n)-f(x_n,y_n)-f(y_n,w_n)]} \geq \frac{[d^2(x_n,y_n)+d^2(w_n,y_n)]}{2[c_1 d^2(x_n,y_n)+c_2 d^2 f(y_n,w_n)]} \geq \frac{\mu}{2\max\{c_1,c_2\}}.$$

Hence, the sequence $\{\lambda_n\}$ is nonincreasing and has the lower bound $\frac{\mu}{2\max\{c_1,c_2\}}$. Hence, the limit $\lim_{n\to\infty} \lambda_n = \lambda > 0$ exists. $\quad\square$

The following lemma is vital in proving the convergence of our proposed Algorithm 3.1.

Lemma 3.3 *Assume the bifunction* f *satisfies Assumption* (A1). *Suppose* $\{w_n\}$, $\{x_n\}$, *and* $\{y_n\}$ *are generated as in Algorithm* 3.1 *and* $y \in C$. *Then,*

$$\frac{1}{2}\left[d^2(w_n, x_n) - d^2(x_n, y) + d^2(w_n, y)\right] \leq \lambda_n\left[f(x_n, y) - f(x_n, w_n)\right]. \tag{3.4}$$

Proof Take $y \in C$. Since w_n is a solution of (3.2), let $v_n = ty \oplus (1-t)w_n$ such that $t \in [0, 1)$. Then, from Lemma 2.11(i), we have

$$
\begin{aligned}
f(x_n, w_n) &+ \frac{1}{2\lambda_n} d^2(w_n, x_n) \\
&\leq f(x_n, y) + \frac{1}{2\lambda_n} d^2(y, x_n) \\
&\leq f\big(x_n, ty \oplus (1-t)w_n\big) + \frac{1}{2\lambda_n} d^2\big(ty \oplus (1-t)w_n, x_n\big) \\
&\leq tf(x_n, y) + (1-t)f(x_n, w_n) \\
&\quad + \frac{1}{2\lambda_n}\big\{td^2(y, x_n) + (1-t)d^2(w_n, x_n) - t(1-t)d^2(y, w_n)\big\}.
\end{aligned} \tag{3.5}
$$

The inequality (3.5) can be reduced to

$$
\frac{1}{2\lambda_n}\big\{d^2(w_n, x_n) - d^2(y, x_n) - (1-t)d^2(y, w_n)\big\} \leq f(x_n, y) - f(x_n, w_n). \tag{3.6}
$$

If $t \to 1^-$ in (3.6), we have

$$
\frac{1}{2\lambda_n}\big\{d^2(w_n, x_n) - d^2(y, x_n) - d^2(y, w_n)\big\} \leq f(x_n, y) - f(x_n, w_n),
$$

which implies that

$$
\frac{1}{2}\big\{d^2(w_n, x_n) - d^2(y, x_n) - d^2(y, w_n)\big\} \leq \lambda_n\big[f(x_n, y) - f(x_n, w_n)\big]. \tag{3.7}
$$
$\square$

Lemma 3.4 *Suppose $\{w_n\}$, $\{x_n\}$, and $\{y_n\}$ are sequences generated by Algorithm 3.1. Then,*

$$
d^2(w_n, x^*) \leq d^2(x_n, x^*) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)d^2(x_n, y_n) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)d^2(w_n, y_n).
$$

Proof From (3.2), we have

$$
f(x_n, w_n) - f(x_n, y_n) - f(y_n, w_n) \leq \frac{\mu}{2\lambda_{n+1}}\big[d^2(x_n, y_n) + d^2(w_n, y_n)\big]. \tag{3.8}
$$

Since $\lambda_n > 0$, we obtain from (3.8) that

$$
\lambda_n f(x_n, w_n) \leq \lambda_n\big[f(x_n, y_n) + f(y_n, w_n)\big] + \frac{\mu\lambda_n}{2\lambda_{n+1}}\big[d^2(x_n, y_n) + d^2(w_n, y_n)\big]. \tag{3.9}
$$

This implies from (3.6) and the quasilinearization properties that

$$
\langle \overrightarrow{w_n y}, \overrightarrow{x_n w_n}\rangle \geq \lambda_n\big[f(x_n, w_n) - f(x_n, y)\big]. \tag{3.10}
$$

Thus, from (3.9) and (3.10), we have

$$\langle \overrightarrow{w_n y}, \overrightarrow{x_n w_n} \rangle \geq \lambda_n \big[f(x_n, w_n) - f(x_n, y) \big] - \left(1 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \right) d^2(x_n, y_n)$$

$$- \left(1 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \right) d^2(w_n, y_n). \tag{3.11}$$

From (3.2) and Remark 2.6, we can obtain that

$$\lambda_n \big[f(x_n, w_n) - f(x_n, y_n) \big] \geq \langle \overrightarrow{y_n x_n}, \overrightarrow{y_n w_n} \rangle. \tag{3.12}$$

Hence, from (3.11) and (3.12), we have

$$\langle \overrightarrow{w_n y}, \overrightarrow{x_n w_n} \rangle \geq \langle \overrightarrow{y_n x_n}, \overrightarrow{y_n w_n} \rangle - \left(1 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \right) d^2(x_n, y_n) - \left(1 - \frac{\mu \lambda_n}{2\lambda_{n+1}} \right) d^2(w_n, y_n),$$

which implies that

$$2\langle \overrightarrow{y_n x_n}, \overrightarrow{y_n w_n} \rangle \geq 2\langle \overrightarrow{y_n x_n}, \overrightarrow{y_n w_n} \rangle - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) d^2(x_n, y_n) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) d^2(w_n, y_n). \tag{3.13}$$

The following facts are obvious from the quasilinearization properties

$$\begin{cases} 2\langle \overrightarrow{y_n x_n}, \overrightarrow{y_n w_n} \rangle = \{ d^2(y_n, w_n) + d^2(x_n, y_n) - d^2(x_n, w_n) \}, \\ 2\langle \overrightarrow{x_n w_n}, \overrightarrow{w_n y} \rangle = \{ d^2(x_n, y) - d^2(x_n, w_n) - d^2(w_n, y) \}. \end{cases} \tag{3.14}$$

Hence, from (3.13) and (3.14), we obtain

$$d^2(w_n, y) \leq d^2(x_n, y) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) d^2(x_n, y_n) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) d^2(w_n, y_n). \tag{3.15}$$

For each $x^* \in \Gamma$, by Assumption (A4) on f and $f(x^*, y_n) \geq 0$, it implies that $f(y_n, x^*) \leq 0$. Hence, if $x^* = y$ in (3.15) we have

$$d^2(w_n, x^*) \leq d^2(x_n, x^*) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) d^2(x_n, y_n) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) d^2(w_n, y_n). \tag{3.16}$$

$\square$

Theorem 3.5 *Let C be a nonempty, closed, and convex subset of a Hadamard space X. Suppose that $f : C \times C \to \mathbb{R}$ is a bifunction satisfying conditions (A1)–(A4) and $T : X \to CB(X)$ is a multivalued nonexpansive mapping such that $Tx^* = \{x^*\}$. Suppose that the solution set $\Gamma \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $\hat{u} = P_\Gamma \hat{u}$.*

Proof We first show that the sequence $\{x_n\}$ is bounded. Let $\kappa \in (1 - \mu)$ be some fixed number. From Lemma 3.2

$$\lim_{n \to \infty} \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}} \right) = 1 - \mu > \kappa > 0.$$

Thus, there exists $n \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) > \kappa > 0, \quad \forall n \in \mathbb{N}.$$

This implies that

$$d^2\left(w_n, x^*\right) \le d^2\left(x_n, x^*\right) - \kappa\left(d^2(x_n, y_n) + d^2(w_n, y_n)\right). \tag{3.17}$$

Let $x^* \in \Gamma$ and from Algorithm 3.1, Lemma 2.11(i), (3.17), and the multivalued nonexpansivity of T we obtain

$$
\begin{aligned}
d\left(x_{n+1}, x^*\right) &= d\left(\alpha_n u \oplus (1-\alpha_n)\left(\beta_n h_n \oplus (1-\beta_n)w_n\right), x^*\right) \\
&\le \alpha_n d\left(u, x^*\right) + (1-\alpha_n)d\left(\beta_n h_n \oplus (1-\beta_n)w_n, x^*\right) \\
&\le \alpha_n d\left(u, x^*\right) + (1-\alpha_n)\left[\beta_n d\left(h_n, x^*\right) + (1-\beta_n)d\left(w_n, x^*\right)\right] \\
&\le \alpha_n d\left(u, x^*\right) + (1-\alpha_n)\left[\beta_n H\left(Tw_n, Tx^*\right) + (1-\beta_n)d\left(w_n, x^*\right)\right] \\
&= \alpha_n d\left(u, x^*\right) + (1-\alpha_n)d\left(w_n, x^*\right) \\
&\le \alpha_n d\left(u, x^*\right) + (1-\alpha_n)d\left(x_n, x^*\right) \\
&\le \max\left\{d\left(u, x^*\right), d\left(x_n, x^*\right)\right\} \\
&\quad \vdots \\
&\le \max\left\{d\left(u, x^*\right), d\left(x_1, x^*\right)\right\}. \tag{3.18}
\end{aligned}
$$

Therefore, $\{x_n\}$ is bounded. It follows also that $\{w_n\}$ and $\{y_n\}$ are bounded.

From Algorithm 3.1, Lemma 2.11(ii), Lemma 3.4, and (3.17), we obtain that

$$
\begin{aligned}
&d^2\left(x_{n+1}, x^*\right) \\
&= d^2\left(\alpha_n u \oplus (1-\alpha_n)\left(\beta_n h_n \oplus (1-\beta_n,)w_n\right), x^*\right) \\
&\le \alpha_n d^2\left(u, x^*\right) + (1-\alpha_n)d^2\left(\beta_n h_n \oplus (1-\beta_n,)w_n, x^*\right) \\
&\quad - \alpha_n(1-\alpha_n)d^2\left(u, \beta_n h_n \oplus (1-\beta_n,)w_n\right) \\
&\le \alpha_n d^2\left(u, x^*\right) + (1-\alpha_n)\left[\beta_n d^2\left(h_n, x^*\right) + (1-\beta_n,)d^2\left(w_n, x^*\right) - \beta_n(1-\beta_n)d^2(h_n, w_n)\right] \\
&\quad - \alpha_n(1-\alpha_n)\left[\beta_n d^2(u, h_n) + (1-\beta_n,)d^2(u, w_n) - \beta_n(1-\beta_n)d^2(h_n, w_n)\right] \\
&\le \alpha_n d^2\left(u, x^*\right) \\
&\quad + (1-\alpha_n)\left[\beta_n H^2\left(Tw_n, Tx^*\right) + (1-\beta_n,)d^2\left(w_n, x^*\right) - \beta_n(1-\beta_n)d^2(h_n, w_n)\right] \\
&\quad - \alpha_n(1-\alpha_n)\left[\beta_n d^2(u, h_n) + (1-\beta_n,)d^2(u, w_n)\right] - \beta_n(1-\beta_n)d^2(h_n, w_n)] \\
&\le \alpha_n d^2\left(u, x^*\right) + (1-\alpha_n)\left[d^2\left(w_n, x^*\right) - \beta_n(1-\beta_n)d^2(h_n, w_n)\right] \\
&\le \alpha_n d^2\left(u, x^*\right) \\
&\quad + (1-\alpha_n)\left[d^2\left(x_n, x^*\right) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)d^2(x_n, y_n) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)d^2(w_n, y_n)\right] \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)d^2(h_n, w_n). \tag{3.19}
\end{aligned}
$$

This implies from (3.19) that

$$d^2(h_n, w_n) \leq \frac{\alpha_n d^2(u, x^*)}{(1 - \alpha_n)\beta_n(1 - \beta_n)} + \frac{[d^2(x_n, x^*) - d^2(x_{n+1}, x^*)]}{(1 - \alpha_n)\beta_n(1 - \beta_n)}. \tag{3.20}$$

We next divide the rest of the proof into two cases:

Case 1: Assume that $\{d^2(x_n, x^*)\}$ is a monotone nondecreasing sequence. Then, $\{d^2(x_n, x^*)\}$ is convergent and

$$\lim_{n \to \infty} \left(d^2(x_n, x^*) - d^2(x_{n+1}, x^*)\right) = 0. \tag{3.21}$$

Then, by this fact and the condition on α_n, we obtain that

$$\lim_{n \to \infty} d^2(h_n, w_n) = 0. \tag{3.22}$$

From (3.19) we have

$$(1 - \alpha_n)\left[\left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) d^2(x_n, y_n) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) d^2(w_n, y_n)\right]$$
$$\leq \alpha_n d^2(u, x^*) + d^2(x_n, x^*) - d^2(x_{n+1}, x^*).$$

This implies that

$$\lim_{n \to \infty} d^2(x_n, y_n) = d^2(w_n, y_n) = 0. \tag{3.23}$$

We obtain from (3.23) that

$$d(x_n, w_n) \leq d(x_n, y_n) + d(y_n, w_n) \longrightarrow 0. \tag{3.24}$$

Also, from (3.22) and (3.23) we obtain that

$$d(h_n, x_n) \leq d(h_n, w_n) + d(w_n, x_n) \longrightarrow 0. \tag{3.25}$$

Again, from (3.22), (3.24), and (3.25), we have

$$\text{dist}(w_n, Tw_n) \leq d(w_n, x_n) + d(x_n, h_n) + \text{dist}(h_n, Tw_n)$$
$$\leq d(w_n, x_n) + d(x_n, h_n) + d(h_n, w_n) \longrightarrow 0. \tag{3.26}$$

By the nonexpansivity of T, (3.25), and (3.26), we obtain

$$\text{dist}(x_n, Tx_n) \leq d(x_n, w_n) + \text{dist}(w_n, Tw_n) + H(Tw_n, Tx_n)$$
$$\leq 2d(x_n, w_n) + \text{dist}(w_n, Tw_n) \longrightarrow 0. \tag{3.27}$$

From Algorithm 3.1 and Lemma 2.11(i), we obtain

$$d(x_{n+1}, x_n) \leq \alpha_n d(u, x_n) + (1 - \alpha_n)\beta_n d(h_n, x_n) + (1 - \alpha_n)(1 - \beta_n)d(w_n, x_n).$$

Hence, from (3.24), (3.25), and the condition on α_n, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.28}$$

Since $\{x_n\}$ is bounded, by Lemma 2.10 there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} x_{n_k} = z$ for some $z \in C$. Then, it follows from (3.25) and the demiclosedness property of T that $z \in F(T)$.

From Algorithm 3.1, w_n solves the subproblem (3.2). By letting $v = tw_n \oplus (1 - t)y$ such that $t \in [0, 1)$ and $y \in C$, we have

$$\begin{aligned}
f(y_n, & w_n) + \frac{1}{2\lambda_n} d^2(x_n, w_n) \\
&\leq f(y_n, v) + \frac{1}{2\lambda_n} d^2(x_n, v) \\
&\leq f\big(y_n, tw_n \oplus (1-t)y\big) + \frac{1}{2\lambda_n} d^2\big(x_n, tw_n \oplus (1-t)y\big) \\
&\leq t f(y_n, w_n) + (1-t)f(y_n, y) + \frac{1}{2\lambda_n}\big\{ td^2(x_n, w_n) + (1-t)d^2(x_n, y) \\
&\quad - t(1-t)d^2(w_n, y) \big\}.
\end{aligned} \tag{3.29}$$

By a similar approach as in (3.8)–(3.10), we have from (3.29) that

$$f(y_n, w_n) - f(y_n, y) \leq \frac{1}{2\lambda_n}\big\{ d^2(x_n, y) - d^2(x_n, w_n) - td^2(w_n, y) \big\}. \tag{3.30}$$

If $t \to 1^-$, we obtain that

$$f(y_n, w_n) - f(y_n, y) \leq \frac{1}{2\lambda_n}\big\{ d^2(x_n, y) - d^2(x_n, w_n) - d^2(w_n, y) \big\}. \tag{3.31}$$

This implies from (3.31) that

$$f(y_n, y) \geq \frac{1}{2\lambda_n}\big\{ d^2(x_n, w_n) + d^2(w_n, y) - d^2(x_n, y) \big\} + f(y_n, w_n), \tag{3.32}$$

which by quasilinearization properties is equivalent to

$$f(y_n, y) \geq \frac{1}{\lambda_n} \langle \overrightarrow{w_n y}, \overrightarrow{x_n w_n} \rangle + f(y_n, w_n). \tag{3.33}$$

Thus, from (3.23), (3.24), and $\lambda_n > 0$ we have that $f(y_n, y) \geq 0$. Since $\{x_n\}$ is Δ-convergent to z, by the fact that $f(y_n, y) \geq 0$ and Assumption (A3), we conclude that $f(z, y) \geq 0$. Thus, $z \in \text{EP}(f, C)$. Hence, $z \in \Gamma$. Now, let $s_n = \beta_n h_n \oplus (1 - \beta_n)w_n$, then by Lemma 2.11(i), (3.24), and (3.25) we have that

$$d(s_n, x_n) \leq \beta_n d(h_n, x_n) + (1 - \beta_n)d(w_n, x_n) \longrightarrow 0. \tag{3.34}$$

Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that is Δ-convergent to $z \in \Gamma$ such that

$$\limsup_{n \to \infty} \langle \overrightarrow{u\hat{u}}, \overrightarrow{x_n\hat{u}} \rangle = \lim_{k \to \infty} \langle \overrightarrow{u\hat{u}}, \overrightarrow{x_{n_k}\hat{u}} \rangle = \langle \overrightarrow{u\hat{u}}, \overrightarrow{z\hat{u}} \rangle.$$

Then, by Lemma 2.13, we obtain

$$\limsup_{n \to \infty} \langle \overrightarrow{u\hat{u}}, \overrightarrow{x_n\hat{u}} \rangle = \langle \overrightarrow{u\hat{u}}, \overrightarrow{z\hat{u}} \rangle \le 0. \tag{3.35}$$

Furthermore, by quasilinearization properties, we have

$$\langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle = \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n x_n} \rangle + \langle \overrightarrow{u\hat{u}}, \overrightarrow{x_n\hat{u}} \rangle$$

$$\le d(u, \hat{u}) d(s_n, x_n) + \langle \overrightarrow{u\hat{u}}, \overrightarrow{x_n\hat{u}} \rangle. \tag{3.36}$$

Thus, from (3.34) and (3.35), we obtain that

$$\limsup_{n \to \infty} \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle \le 0. \tag{3.37}$$

Also, by the condition on α_n and inequality (3.37), we obtain

$$\limsup_{n \to \infty} \left[\alpha_n d^2(u, \hat{u}) + 2(1 - \alpha_n) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle \right] \le 0. \tag{3.38}$$

Furthermore, we obtain from Algorithm 3.1, (3.18), Lemma 3.4, and Lemma 2.11(iii) that

$$d^2(x_{n+1}, \hat{u}) \le \alpha_n^2 d^2(u, \hat{u}) + (1 - \alpha_n)^2 d^2(s_n, \hat{u}) + 2\alpha_n(1 - \alpha_n) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle$$

$$\le \alpha_n^2 d^2(u, \hat{u}) + (1 - \alpha_n)^2 \left[\beta_n d^2(h_n, \hat{u}) + (1 - \beta_n) d^2(w_n, \hat{u}) \right]$$

$$+ 2\alpha_n(1 - \alpha_n) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle$$

$$\le \alpha_n^2 d^2(u, \hat{u}) + (1 - \alpha_n)^2 \left[\beta_n H^2(Tw_n, T\hat{u}) + (1 - \beta_n) d^2(w_n, \hat{u}) \right]$$

$$+ 2\alpha_n(1 - \alpha_n) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle$$

$$\le \alpha_n^2 d^2(u, \hat{u}) + (1 - \alpha_n)^2 d^2(w_n, \hat{u}) + 2\alpha_n(1 - \alpha_n) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle$$

$$\le \alpha_n^2 d^2(u, \hat{u}) + (1 - \alpha_n)^2 d^2(x_n, \hat{u}) + 2\alpha_n(1 - \alpha_n) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle$$

$$\le (1 - \alpha_n) d^2(x_n, \hat{u}) + \alpha_n \left(\alpha_n d^2(u, \hat{u}) + 2(1 - \alpha_n) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle \right). \tag{3.39}$$

Therefore, from (3.38) and (3.39), we conclude by Lemma 2.16 that $\{x_n\}$ converges strongly to $\hat{u} = P_\Gamma \hat{u}$.

Case 2: Suppose that $\{d^2(x_n, \hat{u})\}$ is not a monotone decreasing sequence, then there exists a subsequence $\{d^2(x_{n_k}, \hat{u})\}$ of $\{d^2(x_n, \hat{u})\}$ such that $d^2(x_{n_k}, \hat{u}) \le d^2(x_{n_k+1}, \hat{u})$, $\forall k \in \mathbb{N}$. Then, by Lemma 2.17, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and

$$d^2(x_{m_k}, \hat{u}) < d^2(x_{m_k+1}, \hat{u}) \quad \text{and} \quad d^2(x_k, \hat{u}) < d^2(x_{k+1}, \hat{u}), \quad k \in \mathbb{N}. \tag{3.40}$$

Thus, from Algorithm 3.1, Lemma 2.11(i), and (3.18), we have

$$0 \leq \lim_{k \to \infty} \left(d^2(x_{m_k+1}, \hat{u}) - d^2(x_{m_k}, \hat{u}) \right)$$

$$\leq \limsup_{n \to \infty} \left(d^2(x_{n+1}, \hat{u}) - d^2(x_n, \hat{u}) \right)$$

$$\leq \limsup_{n \to \infty} \left(\alpha_n d^2(u, \hat{u}) + (1 - \alpha_n) d^2(s_n, \hat{u}) - d^2(x_n, \hat{u}) \right)$$

$$\leq \limsup_{n \to \infty} \left(\alpha_n d^2(u, \hat{u}) + (1 - \alpha_n) \left[\beta_n d^2(h_n, \hat{u}) + (1 - \beta_n) d^2(w_n, \hat{u}) \right] - d^2(x_n, \hat{u}) \right)$$

$$\leq \limsup_{n \to \infty} \left(\alpha_n d^2(u, \hat{u}) + (1 - \alpha_n) \left[\beta_n H^2(Tw_n, T\hat{u}) + (1 - \beta_n) d^2(w_n, \hat{u}) \right] - d^2(x_n, \hat{u}) \right)$$

$$\leq \limsup_{n \to \infty} \left(\alpha_n d^2(u, \hat{u}) + (1 - \alpha_n) d^2(w_n, \hat{u}) - d^2(x_n, \hat{u}) \right)$$

$$\leq \limsup_{n \to \infty} \alpha_n \left(d^2(u, \hat{u}) - d^2(x_n, \hat{u}) \right) = 0.$$

This implies that

$$\lim_{k \to \infty} \left(d\left(x_{m_k+1}, x^*\right) - d\left(x_{m_k}, x^*\right) \right) = 0.$$

From (3.34), this implies that $\lim_{k \to \infty} d(s_{m_k}, x_{m_k}) = 0$. Hence, by this fact and $\alpha_{m_k} \to 0$, we have

$$d(x_{m_k+1}, x_{m_k}) \leq \alpha_{m_k} d(u, x_{m_k}) + (1 - \alpha_{n_k}) d(s_{m_k}, x_{nk}) \longrightarrow 0. \tag{3.41}$$

Following the same argument as in Case 1, we obtain

$$\lim_{k \to \infty} \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_{m_k}\hat{u}} \rangle \leq 0$$

and

$$\lim_{k \to \infty} \left[\alpha_{m_k} d^2(u, \hat{u}) + 2(1 - \alpha_{m_k}) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_{m_k}\hat{u}} \rangle \right] \leq 0. \tag{3.42}$$

Hence, from (3.39), we obtain

$$d^2(x_{m_k+1}, \hat{u}) \leq (1 - \alpha_{m_k}) d^2(x_{m_k}, \hat{u}) + \alpha_{m_k} \left(\alpha_{m_k} d^2(u, \hat{u}) + 2(1 - \alpha_{m_k}) \langle \overrightarrow{u\hat{u}}, \overrightarrow{s_n\hat{u}} \rangle \right).$$

In addition, from (3.40), we have that

$$d^2(x_{m_k}, \hat{u}) \leq d^2(x_{m_k+1}, \hat{u}),$$

which implies that

$$\lim_{k \to \infty} d^2(x_{m_k}, \hat{u}) = 0.$$

Thus, from Cases 1 and 2, we conclude that $\{x_n\}$ converges strongly to $\hat{u} = P_\Gamma \hat{u}$. This completes the proof. □

4 Numerical example

In this section, we present a numerical experiment to demonstrate the performance of our method. All codes were written in MATLAB 2020 on a Dell Core i5 PC.

Example 4.1 Let $Y := \{(x, e^x) : x \in \mathbb{R}\}$ and $X_n := \{(n, y) : y \geq e^n\}$ for each $n \in \mathbb{Z}$. Set $X := Y \cup \bigcup_{n \in \mathbb{Z}} X_n$ equipped with a metric $d : X \times X \to [0, \infty)$, defined for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$ by

$$d(x, y) = \begin{cases} \int_{x_1}^{y_1} \|\dot{\gamma}(t)\|_2 \, dt + |x_2 - e^{x_1}| + |y_2 - e^{y_1}| & \text{if } x_1 \neq y_1, \\ |x_2 - y_2| & \text{if } x_1 = y_1, \end{cases} \tag{4.1}$$

where $\dot{\gamma}$ is the derivative of the curve $\gamma : \mathbb{R} \to X$ given as $\gamma(t) := (t, e^t)$ for each $t \in \mathbb{R}$ (see [5]). Then, (X, d) is a Hadamard space.

Now, let $T : X \to CB(X)$ be defined by $Tx = \{(-x_1, e^{-x_1}), (0, 0)\}$ for all $x = (x_1, x_2) \in X$. Clearly, $F(T) = \{(0, 0)\}$ and also satisfies the end-point condition. We check that T is nonexpansive. Indeed, for each $(x_1, x_2), (y_1, y_2) \in X$, we have

$$\text{dist}\big((x_1, e^{x_1}), Ty\big) = \inf\big\{d\big((x_1, e^{x_1}), (-y_1, e^{-y_1})\big), d\big((0, 0), (x_1, e^{x_1})\big)\big\}.$$

However,

$$d\big((x_1, e^{x_1}), (-y_1, e^{-y_1})\big) = \begin{cases} \int_{-y_1}^{x_1} \|\dot{\gamma}(t)\|_2 \, dt + |e^{-y_1} - e^{-y_1}| + |e^{x_1} - e^{x_1}| & \text{if } x_1 \neq y_1, \\ |e^{-y_1} - e^{x_1}| & \text{if } x_1 = y_1 \end{cases}$$

$$= \begin{cases} \int_{-y_1}^{x_1} \|\dot{\gamma}(t)\|_2 \, dt & \text{if } x_1 \neq y_1, \\ |e^{-y_1} - e^{x_1}| & \text{if } x_1 = y_1, \end{cases}$$

and

$$d\big((0, 0), (x_1, e^{x_1})\big) = \begin{cases} \int_0^{x_1} \|\dot{\gamma}(t)\|_2 \, dt + |0 - e^0| + |e^{x_1} - e^{x_1}| & \text{if } x_1 \neq 0, \\ e^{x_1} & \text{if } x_1 = 0 \end{cases}$$

$$= \begin{cases} \int_0^{x_1} \|\dot{\gamma}(t)\|_2 \, dt + 1 & \text{if } x_1 \neq 0, \\ 1 & \text{if } x_1 = 0. \end{cases}$$

Therefore,

$$\text{dist}\big((x_1, e^{x_1}), Ty\big) = d\big((x_1, e^{x_1}), (-y_1, e^{-y_1})\big).$$

Also,

$$\begin{aligned} \text{dist}\big((0, 0), Ty\big) &= \inf\big\{d\big((0, 0), (-y_1, e^{-y_1})\big), d\big((0, 0), (0, 0)\big)\big\} \\ &= d\big((0, 0), (0, 0)\big). \end{aligned}$$

Similarly,

$$\text{dist}\big((y_1, e^{y_1}), Tx\big) = \inf\big\{d\big((y_1, e^{y_1}), (-x_1, e^{-x_1})\big), d\big((0,0), (y_1, e^{y_1})\big)\big\}$$
$$= d\big((y_1, e^{y_1}), (-x_1, e^{-x_1})\big)$$

and

$$\text{dist}\big((0,0), Tx\big) = \inf\big\{d\big((0,0), (-x_1, e^{-x_1})\big), d\big((0,0), (0,0)\big)\big\}$$
$$= d\big((0,0), (0,0)\big).$$

Hence,

$$H(Tx, Ty) = \max\Big\{\sup_{a \in Tx} \text{dist}(a, Ty), \sup_{b \in Ty} \text{dist}(b, Tx)\Big\}$$
$$= \max\big\{\sup\big\{d\big((-x_1, e^{-x_1}), (-y_1, e^{-y_1})\big), d\big((0,0), (0,0)\big)\big\},$$
$$\sup\big\{d\big((-y_1, e^{-y_1}), (-x_1, e^{-x_1})\big), d\big((0,0), (0,0)\big)\big\}\big\}$$
$$= d\big((-x_1, e^{-x_1}), (-y_1, e^{-y_1})\big)$$
$$= \begin{cases} \int_{-x_1}^{-y_1} \|\dot{\gamma}(t)\|_2 \, dt & \text{if } x_1 \neq y_1 (x_1 > y_1), \\ |e^{-x_1} - e^{-y_1}| & \text{if } x_1 = y_1 \end{cases}$$
$$= \begin{cases} \int_{-y_1}^{-x_1} \|\dot{\gamma}(t)\|_2 \, dt & \text{if } x_1 \neq y_1 (x_1 < y_1), \\ |e^{-y_1} - e^{-x_1}| & \text{if } x_1 = y_1 \end{cases}$$
$$\leq d(x, y).$$

Therefore, T is a multivalued nonexpansive mapping.

Furthermore, let $P(n, \mathbb{R})$ be the space of $(n \times n)$ positive symmetric definite matrices endowed with the Riemannian metric

$$\langle A, B \rangle_E := Tr\big(E^{-1} A E^{-1} B\big),$$

for all $A, B \in T_E(P(n, \mathbb{R}))$ and every $E \in P(n, \mathbb{R})$. The pair $(P(n, \mathbb{R}), \langle A, B \rangle_E)$ is a Hadamard space (see [27]). Let $\mathbb{R}^+$ be the set of positive real numbers. Now, consider the space $P(n, \mathbb{R})$ such that $n = 1$ with an inner product $\langle a, b \rangle_\lambda = \frac{1}{\lambda^2} ab$ for $\lambda > 0$ and $a, b \in T_\lambda \mathbb{R}^+ = \mathbb{R}$. Let (X, d) be a metric space with $X = \mathbb{R}^+$ and $d : X \times X \to \mathbb{R}$ be defined by

$$d(a, b) = |\ln a - \ln b|,$$

with the geodesic between $a, b \in X$ defined as $\gamma(\kappa) = a(\frac{b}{a})^\kappa$. Therefore, the pair (X, d) is a CAT(0) space with the geodesic between a and b given as

$$\ln \gamma(\kappa) = \ln a \left(\frac{b}{a}\right)^\kappa = \ln a + \kappa(\ln b - \ln a) = (1 - \kappa) \ln a + \kappa \ln b. \tag{4.2}$$

Now, let $f : X \times X \to \mathbb{R}$ be bifunctions defined by $f(x,y) = \ln x(\ln \frac{y}{x})$. From (4.2), we have that

$$f(x, \gamma(\kappa)) = \ln x \left(\ln \frac{\gamma(\kappa)}{x} \right) = (1 - \kappa) \ln x \left(\ln \frac{a}{x} \right) + \kappa \ln x \left(\ln \frac{b}{x} \right)$$
$$= (1 - \kappa)f(x,a) + \kappa f(x,b).$$

Clearly, the bifunction f satisfies assumptions (A1) and (A2). Next, we show that f satisfies assumption (A3). Let $x, y, z \in X$, then

$$f(x,y) + f(y,z) - f(x,z) = \ln x \left(\ln \frac{y}{x} \right) + \ln y \left(\ln \frac{z}{y} \right) - \ln x \left(\ln \frac{z}{x} \right) \tag{4.3}$$

$$= \ln x \left(\ln \frac{y}{x} - \ln \frac{z}{x} \right) + \ln y \left(\ln \frac{z}{y} \right) \tag{4.4}$$

$$= \ln x \left(\ln \frac{y}{z} \right) - \ln y \left(\ln \frac{y}{z} \right) \tag{4.5}$$

$$= |\ln x - \ln y||\ln y - \ln z| \tag{4.6}$$

$$= d(x,y)d(y,z) \tag{4.7}$$

$$\geq -\frac{1}{2}d(x,y)d(y,z) \tag{4.8}$$

$$\geq -\frac{1}{2}d(x,y) - \frac{1}{2}d(y,z). $$

Hence, f satisfies the Lipschitz-type condition with Lipschitz constants $c_1 = c_2 = \frac{1}{2}$. Moreover,

$$f(x,y) + f(y,x) = \ln x \left(\ln \frac{y}{x} \right) + \ln y \left(\ln \frac{x}{y} \right) \tag{4.9}$$

$$= \ln x \left(\ln \frac{y}{x} \right) - \ln y \left(\ln \frac{y}{x} \right) \tag{4.10}$$

$$= \left(\ln \frac{x}{y} \right) \left(\ln \frac{y}{x} \right) \tag{4.11}$$

$$= - \left(\ln \frac{x}{y} \right)^2 \leq 0.$$

Hence, f is monotone (and thus pseudomonotone).

For the sake of numerical computation, we choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{2n}{5n+3}$, $\lambda_0 = 0.9$, $\mu = 0.6$, $u = \frac{\sqrt{3}}{3}$; for Algorithm EA, we choose $y_0 = \frac{1}{33}$, $\lambda_n = \frac{1}{2c_1}$; for Algorithm EAL, we take $\delta = 0.4$, $\lambda_n = \frac{1}{2c_1}$, $\alpha_n = \frac{1}{n+1}$, and in addition for Algorithm HEAL, we take $\beta_n = \frac{2n}{5n+5}$. We compute the algorithms for three different points. The stopping criteria used for the algorithms is Err $= \|x_n - y_n\| < 10^{-6}$. The numerical results are shown in Table 1 and Figs. 1–3. The numerical computation shows that our proposed algorithm successfully approximates the common solution of the pseudomonotone equilibrium problem and the fixed point of a nonexpansive mapping. Furthermore, it performs better than the other methods in the literature in terms of number of iterations and CPU time taken for the computation.

Table 1 Computational result for Example 4.1

|		Case I ($x_0 = \sqrt{3}$)	Case II ($x_0 = 5$)	Case III ($x_0 = 11$)
SAEA algorithm	Iter.	4	5	7
	Time (s)	0.0190	0.0157	0.0328
EA algorithm	Iter.	24	37	24
	Time (s)	0.0911	0.1591	0.0880
EAL algorithm	Iter.	37	29	42
	Time (s)	0.1562	0.0775	0.1664
HEAL algorithm	Iter.	11	10	10
	Time (s)	0.0548	0.0289	0.0518
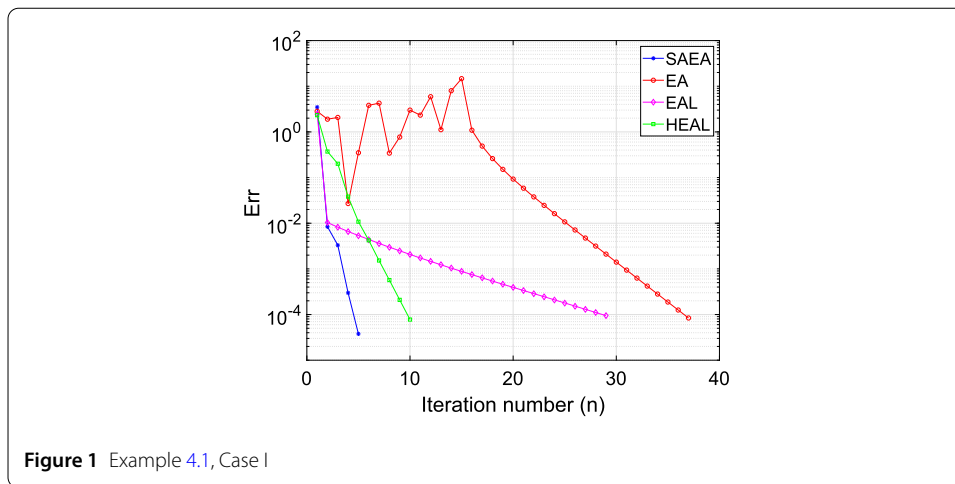
Figure 1 Example 4.1, Case I
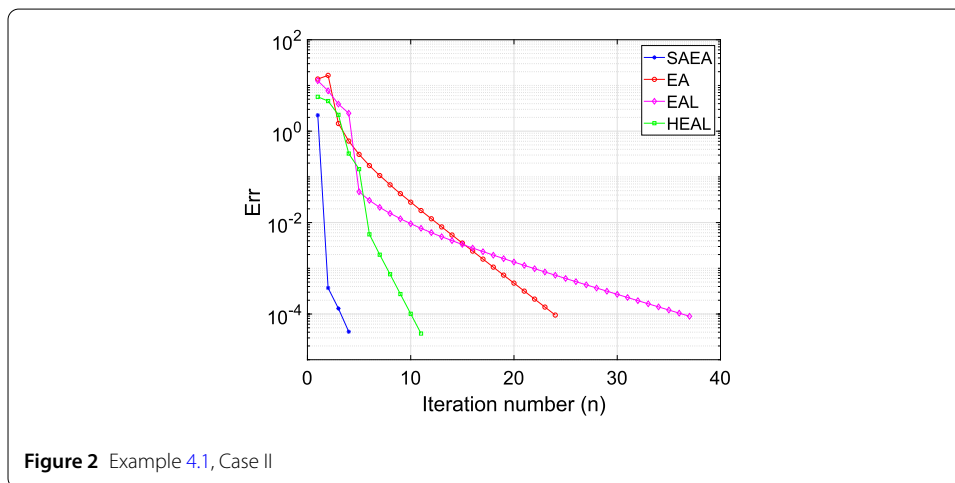
Figure 2 Example 4.1, Case II

5 Conclusion

In this paper, we studied a self-adaptive extragradient algorithm for approximating a common solution of a pseudomonotone equilibrium problem and fixed-point problem for a multivalued nonexpansive mapping in Hadamard spaces. We proposed an algorithm and obtained strong convergence without prior knowledge of the Lipschitz constants of the pseudomonotone bifunction. Furthermore, we provide a numerical experiment to demonstrate the efficiency of our algorithm. Our result extends and complements recent results in the literature.
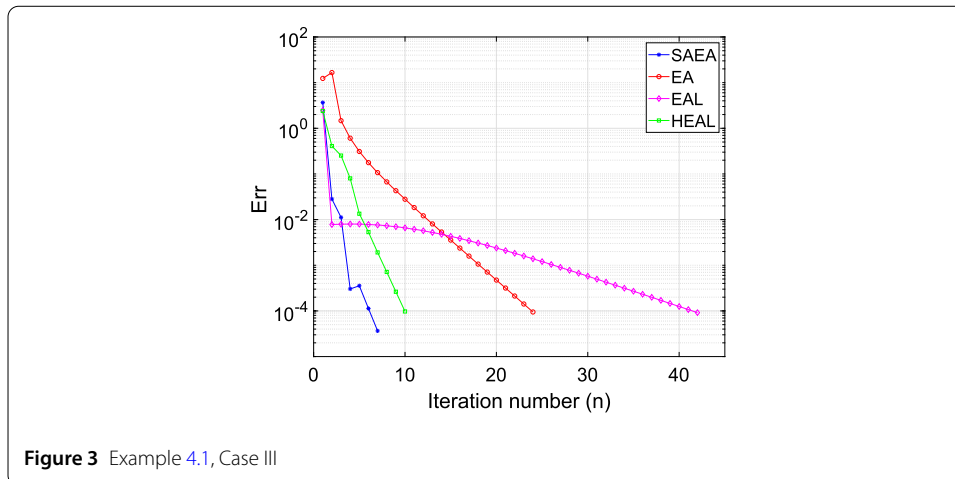
Figure 3 Example 4.1, Case III

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
K.O. wrote the main manuscript, L.O. prepared the figures, and O.K. read through the prove. All authors reviewed the manuscript.

Author details
[1]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, P.O. Box 94 Medunsa 0204, Ga-Rankuwa, South Africa. [2]Department of Mathematics, Usmanu Danfodiyo University Sokoto, P.M.B. 2346, Sokoto, Sokoto state, Nigeria. [3]School of Mathematical Sciences, University of Southampton, Southampton, SO171 BJ, United Kingdom. [4]Technion-Israel Institute of Technology, Haifa, Israel.

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