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Unsteady non-Newtonian fluid flow with heat transfer and Tresca's friction boundary conditions

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Abstract

We consider an unsteady non-isothermal flow problem for a general class of non-Newtonian fluids. More precisely the stress tensor follows a power law of parameter p, namely $\sigma = 2\mu(\theta, \upsilon, \|D(\upsilon)\|)\|D(\upsilon)\|^{p-2}D(\upsilon) - \pi\,\mathrm{Id}$ where θ is the temperature, π is the pressure, υ is the velocity, and $D(\upsilon)$ is the strain rate tensor of the fluid. The problem is then described by a non-stationary p-Laplacian Stokes system coupled to an L^1-parabolic equation describing thermal effects in the fluid. We also assume that the velocity field satisfies non-standard threshold slip-adhesion boundary conditions reminiscent of Tresca's friction law for solids. First, we consider an approximate problem (P_δ), where the L^1 coupling term in the heat equation is replaced by a bounded one depending on a small parameter $0 < \delta \ll 1$, and we establish the existence of a solution to (P_δ) by using a fixed point technique. Then we prove the convergence of the approximate solutions to a solution to our original fluid flow/heat transfer problem as δ tends to zero.

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1 Introduction

Motivated by lubrication or injection/extrusion industrial processes, we consider in this paper an unsteady incompressible non-isothermal flow problem with non-linear boundary conditions of friction type for a general class of non-Newtonian fluids. More precisely, we assume that the stress tensor is given by

$$\sigma = 2\mu\big(\theta, \upsilon, \|D(\upsilon)\|\big)\|D(\upsilon)\|^{p-2}D(\upsilon) - \pi\,\mathrm{Id}_{\mathbb{R}^3}, \tag{1.1}$$

where μ is a given mapping, θ is the temperature, π is the pressure, υ is the velocity, $D(\upsilon)$ is the strain rate tensor, and $p \in (1, +\infty)$ is a real parameter.

When $p > 2$, this non-linear power-law models the behavior of dilatant (or shear thickening) fluids like colloidal fluids, while the case $p \in (1, 2)$ gives a description of pseudo-plastic (or shear thinning) fluids like molten polymers [1, 3, 25, 30, 33, 37].

When $p = 2$, the relationship between the stress tensor, the strain rate tensor and the pressure is still non-linear since the viscosity mapping μ depends on θ, υ, and $|D(\upsilon)|$, and we obtain a constitutive law that allows considering non-Newtonian fluids like oils [22].

Let us mention that (1.1) corresponds to a quasi-linear version of the Newton constitutive law and is also called the generalized Newtonian fluid model.

Several experimental studies have shown that such complex fluids exhibit a non-standard behavior at the boundary with threshold slip-adhesion phenomena reminiscent of Tresca's friction law for solids [4, 13, 21, 26, 27, 35]. The first existence results for this kind of boundary conditions have been obtained by H. Fujita in [14–19, 31, 32] for stationary Newtonian Stokes flows and developed later on for steady and unsteady Newtonian fluid flows [7, 8, 23, 24, 36].

The case of stationary non-Newtonian fluids satisfying the general power law (1.1) is considered in [9], and thermal effects lead to a coupled fluid flow/heat transfer problem.

This paper aims to extend this result to the non-stationary case. More precisely, we consider the fluid flow domain

$$\Omega = \{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, 0 < x_3 < h(x')\},$$

where ω is a non-empty bounded domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary, and h is a Lipschitz continuous function bounded from above and below by some positive real numbers.

The conservation of mass and momentum and the energy conservation law yield the following p-Laplacian Stokes system:

$$\begin{cases} \frac{\partial \upsilon}{\partial t} - 2\,\mathrm{div}(\mu(\theta, \upsilon, \|D(\upsilon)\|)\|D(\upsilon)\|^{p-2}D(\upsilon)) + \nabla\pi = f \quad \text{in } (0,T) \times \Omega, \\ \mathrm{div}(\upsilon) = 0 \quad \text{in } (0,T) \times \Omega, \end{cases} \tag{1.2}$$

where $(0, T)$ is a non-trivial time interval, and f describes the external forces coupled to the following heat equation:

$$c\frac{\partial\theta}{\partial t} - \mathrm{div}(K\nabla\theta) = 2\mu(\theta, \upsilon, \|D(\upsilon)\|)\|D(\upsilon)\|^p + r(\theta) \quad \text{in } (0,T) \times \Omega, \tag{1.3}$$

where c is the heat capacity, K is the thermal conductivity tensor, and r is a real function.

Let us observe that we take into account only two kinds of coupling effects in this description of the fluid flow/heat transfer problem. More precisely, in (1.2), the viscosity mapping μ depends on the temperature, and in (1.3), the right-hand side contains the heat source term $2\mu(\theta, \upsilon, \|D(\upsilon)\|)\|D(\upsilon)\|^p$, which describes the heat generation due to inner friction. Indeed, in this first attempt for an existence result generalizing [9] to the non-stationary case, we focus in this paper on the difficulty due to the L^1 right-hand side in (1.3), which will be dealt with through the truncation technique introduced by L. Boccardo and T. Gallouët in [6].

Hence, we choose to neglect convective effects both in the momentum equation and in the heat equation. A more complete description, including the convective terms, could be handled with the same proof strategy (with more technicalities) under some restrictive conditions on the value of the parameter p and some compatibility conditions between the regularity properties of the fluid velocity and temperature fields. Similarly, we choose to neglect the semilinear temperature-dependent term modeling the buoyancy force in the momentum equation.

We decompose the boundary of Ω as $\partial\Omega = \Gamma_0 \cup \Gamma_L \cup \Gamma_1$ with

$$\Gamma_0 = \{(x', x_3) \in \overline{\Omega} : x_3 = 0\}, \qquad \Gamma_1 = \{(x', x_3) \in \overline{\Omega} : x_3 = h(x')\},$$

and Γ_L is the lateral part of $\partial\Omega$. We introduce a function $g : \partial\Omega \to \mathbb{R}^3$ such that

$$\int_{\Gamma_L} g \cdot n \, dY = 0, \qquad g = 0 \quad \text{on } \Gamma_1, \qquad g \cdot n = 0 \quad \text{on } \Gamma_0, \qquad g \neq 0 \quad \text{on } \Gamma_L, \tag{1.4}$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal vector to $\partial\Omega$, and $g \cdot n$ denotes the Euclidean inner product of the vectors g and n in $\mathbb{R}^3$. We define by $\upsilon_n = \upsilon \cdot n$ and $\upsilon_\tau = \upsilon - \upsilon_n n$ the normal and the tangential velocities on $\partial\Omega$. The normal and tangential components of the stress vector on $\partial\Omega$ are given by σ_n and σ_τ with

$$\sigma_n = \sum_{i,j=1}^{3} \sigma_{ij} n_j n_i, \qquad \sigma_\tau = \left(\sum_{j=1}^{3} \sigma_{ij} n_j - \sigma_n n_i \right)_{1 \le i \le 3}.$$

As usual in lubrication or extrusion/injection problems, the upper part of the boundary is a fixed wall, while the lower part is a moving device. Hence, we assume that the fluid is subjected to the non-homogeneous Dirichlet boundary conditions on $\Gamma_1 \cup \Gamma_L$ and to non-linear slip boundary conditions of friction type on Γ_0, i.e.,

$$\upsilon = 0 \quad \text{on } (0, T) \times \Gamma_1, \qquad \upsilon = g\xi \quad \text{on } (0, T) \times \Gamma_L, \tag{1.5}$$

where ξ is a function depending only on the time variable such that

$$\xi(0) = 1 \tag{1.6}$$

and

$$\upsilon_n = 0 \quad \text{on } (0, T) \times \Gamma_0 \quad \text{(slip condition)}, \tag{1.7}$$

$$\left.\begin{array}{lll} \|\sigma_\tau\| = k & \Rightarrow & \exists \lambda \ge 0 \quad \upsilon_\tau = s - \lambda \sigma_\tau, \\ \|\sigma_\tau\| < k & \Rightarrow & \upsilon_\tau = s \end{array}\right\} \quad \text{on } (0, T) \times \Gamma_0 \quad \text{(Tresca's law)}, \tag{1.8}$$

where s is the sliding velocity of the lower part of the boundary, and k is the positive friction threshold.

Moreover, we assume the mixed Dirichlet–Neumann boundary conditions on $\Gamma_1 \cup \Gamma_L$ and Γ_0 for the temperature, i.e.,

$$\theta = 0 \quad \text{on } (0, T) \times (\Gamma_1 \cup \Gamma_L), \qquad (K\nabla\theta) \cdot n = \theta^b \quad \text{on } (0, T) \times \Gamma_0, \tag{1.9}$$

where θ^b is a given heat flux on Γ_0.

The paper is organized as follows: In Sect. 2, we introduce the functional framework and derive the mathematical formulation of the problem as a non-linear parabolic variational inequality for the velocity and pressure fields coupled to a non-linear parabolic equation for the temperature. By observing that the right-hand side of the heat equation belongs to $L^1(0,T;L^1(\Omega))$, we introduce in Sect. 3 an approximate problem (P_δ), where the L^1 coupling term is replaced by a bounded one depending on a small parameter $0 < \delta \ll 1$, and we establish the existence of a solution to (P_δ) by using a fixed point technique. Finally, in Sect. 4, we prove that the approximate solutions $(\upsilon_\delta, \pi_\delta, \theta_\delta)$ converge to a solution to our original fluid flow/heat transfer problem as δ tends to zero.

2 Mathematical formulation of the problem

Throughout the paper, we will denote by $\mathbf{X}$ the functional space X^3.

In order to describe the fluid flow problem, we introduce the following subspaces of $\mathbf{W}^{1,p}(\Omega)$:

$$V^p_{\Gamma_1} = \{\varphi \in \mathbf{W}^{1,p}(\Omega); \varphi = 0 \text{ on } \Gamma_1\},$$
$$V^p_0 = \{\varphi \in \mathbf{W}^{1,p}(\Omega); \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \text{ and } \varphi \cdot n = 0 \text{ on } \Gamma_0\}$$

and

$$V^p_{0.\mathrm{div}} = \{\varphi \in V^p_0; \operatorname{div}(\varphi) = 0 \text{ in } \Omega\}$$

for all $p > 1$ endowed with the norm

$$\|\upsilon\|_{1.p} = \left(\int_\Omega \|\nabla\upsilon\|^p \, dx\right)^{1/p}.$$

By using the convexity of the mapping $z \mapsto z^p$ on $\mathbb{R}^+_*$, we obtain

$$\left(\int_\Omega \|D(u)\|^p \, dx\right)^{1/p} = \|D(u)\|_{(L^p(\Omega))^{3\times 3}} \le \|u\|_{1.p} \quad \forall u \in \mathbf{W}^{1,p}(\Omega), \tag{2.1}$$

where $D(u) = (d_{ij}(u))_{1\le i,j\le 3} = (\frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}))_{1\le i,j\le 3}$, and with Korn's inequality [38], we have

$$\left(\int_\Omega \|D(u)\|^p \, dx\right)^{1/p} = \|D(u)\|_{(L^p(\Omega))^{3\times 3}} \ge C_{\mathrm{Korn},p}\|u\|_{1.p} \quad \forall u \in V^p_{\Gamma_1}, \tag{2.2}$$

where $C_{\mathrm{Korn},p} > 0$. Moreover, let $\mathcal{Y} = \{\psi \in \mathbf{L}^2(\Omega); \operatorname{div}(\psi) \in L^2(\Omega)\}$ endowed with its canonical norm

$$\|\psi\|_{\mathcal{Y}} = \left(\|\psi\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}(\psi)\|^2_{L^2(\Omega)}\right)^{1/2} \quad \forall \psi \in \mathcal{Y},$$

and let H be its subspace given by $H = \{\psi \in \mathbf{L}^2(\Omega); \operatorname{div}(\psi) = 0 \text{ in } \Omega, \psi \cdot n = 0 \text{ on } \partial\Omega\}$. Owing to that $\mathbf{W}^{1,p}(\Omega)$ is continuously embedded into $\mathbf{L}^2(\Omega)$ if and only if $p \ge \frac{6}{5}$ in a 3D setting and that the space $\mathcal{V} = \{\Phi \in (\mathcal{D}(\Omega))^3; \operatorname{div}(\Phi) = 0 \text{ in } \Omega\}$ is dense in H (see Chap. 1, Theorem 2.8 in [20]), we obtain that the embedding of $V^p_{0.\mathrm{div}}$ into H is continuous and dense if and only if $p \geq \frac{6}{5}$, and $(V^p_{0.\mathrm{div}}, H, (V^p_{0.\mathrm{div}})')$ is a Gelfand triplet. Let us also recall that the trace operator is compact from $\mathbf{W}^{1,p}(\Omega)$ into $\mathbf{L}^p(\partial\Omega)$ for all $p > 1$ (see Theorem 1.23 in [29]), which will allow us to deal with the boundary friction term (see (2.13) and the definition of the mapping J).

Similarly, we let

$$W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega) = \{\varphi \in W^{1,q}(\Omega) : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L\}$$

for all $q > 1$ endowed with the norm

$$\|\varphi\|_{W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega)} = \left(\int_\Omega \|\nabla\varphi\|^q \, dx \right)^{1/q}.$$

Let us now introduce the assumptions on the data.

The heat capacity and the thermal conductivity tensor satisfy

$$c \in W^{1,q'}(\Omega), \qquad K \in \left(L^\infty(\Omega)\right)^{3\times 3} \tag{2.3}$$

$$\begin{aligned} &\text{there exists } (c_0, c_1) \in \mathbb{R}^2 \text{ such that} \\ &0 < c_0 \leq c(x) \leq c_1 \quad \text{for a.e. } x \in \Omega, \end{aligned} \tag{2.4}$$

where $q' = \frac{q}{q-1}$ is the conjugate number of q, and

$$\begin{aligned} &\text{there exists } k_0 > 0 \text{ such that } \sum_{i,j=1}^{3} K_{ij}(x)\gamma_i\gamma_j \geq k_0 \sum_{i=1}^{3} |\gamma_i|^2 \\ &\text{for all } \gamma = (\gamma_i)_{1\leq i\leq 3} \in \mathbb{R}^3 \text{ for a.e. } x \in \Omega. \end{aligned} \tag{2.5}$$

We also assume that

$$\text{the mapping } r : \mathbb{R} \to \mathbb{R} \text{ is continuous,} \tag{2.6}$$

and

$$\text{there exists } r_1 \in \mathbb{R} \text{ such that } \left|r(z)\right| \leq r_1 \quad \text{for all } z \in \mathbb{R}. \tag{2.7}$$

The viscosity mapping $\mu : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$(o, e, d) \mapsto \mu(o, e, d) \text{ is continuous on } \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+, \tag{2.8}$$

$$d \mapsto \mu(\cdot, \cdot, d) \text{ is monotone increasing on } \mathbb{R}_+, \tag{2.9}$$

$$\begin{aligned} &\text{there exists } (\mu_0, \mu_1) \in \mathbb{R}^2 \text{ such that} \\ &0 < \mu_0 \leq \mu(o, e, d) \leq \mu_1 \quad \text{for all } (o, e, d) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+, \end{aligned} \tag{2.10}$$

and we define $\mathcal{F} : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ by

$$\begin{cases} \mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 2\mu(\lambda_0, \lambda_1, \|\lambda_2\|)\|\lambda_2\|^{p-2}\lambda_2 & \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3\times 3}}, \\ \mathcal{F}(\lambda_0, \lambda_1, \lambda_2) = 0_{\mathbb{R}^{3\times 3}} & \text{otherwise.} \end{cases} \tag{2.11}$$

With (2.10), we obtain immediately

$$\left|\mathcal{F}(\lambda_0,\lambda_1,\lambda_2)\right| \le 2\mu_1\|\lambda_2\|^{p-1} \quad \forall(\lambda_0,\lambda_1,\lambda_2)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^{3\times 3}.$$

Let $\tilde{p}>1$ and $p>1$ such that $\tilde{p}-p+1>0$. Then for any $\theta\in L^{\tilde{q}}(0,T;L^q(\Omega))$ with $\tilde{q}\ge 1$ and $q\ge 1$, and for any $u\in L^{\tilde{p}}(0,T;\mathbf{W}^{1,p}(\Omega))$, we have

$$\mathcal{F}\big(\theta,u,D(u)\big)\in L^{\frac{\tilde{p}}{p-1}}\big(0,T;\big(L^{p'}(\Omega)\big)^{3\times 3}\big),$$

where $p'=\frac{p}{p-1}$ is the conjugate number of p. Hence,

$$\mathcal{F}\big(\theta,u,D(u)\big):D(\overline{\varphi})\in L^1\big(0,T;L^1(\Omega)\big)$$

for all $\overline{\varphi}\in L^{\frac{\tilde{p}}{\tilde{p}-p+1}}(0,T;\mathbf{W}^{1,p}(\Omega))$, and the right-hand side of the heat equation is well defined in $L^1(0,T;L^1(\Omega))$ if and only if $\tilde{p}\ge\frac{\tilde{p}}{\tilde{p}-p+1}$, i.e., $\tilde{p}\ge p$. Then we may expect $\theta\in L^q(0,T;W^1_{\Gamma_1\cup\Gamma_L}(\Omega))$ with $1<q<\frac{5}{4}$ (see [6]).

Let us introduce the operator $\mathcal{A}:L^{\tilde{p}}(0,T;V^p_{\Gamma_1})\to(L^{\tilde{p}}(0,T;V^p_{\Gamma_1}))'$ defined by

$$\big[\mathcal{A}(u),\overline{\varphi}\big]=\int_0^T\int_\Omega\mathcal{F}\big(\theta,u,D(u)\big):D(\overline{\varphi})\,dx\,dt \quad \forall(u,\overline{\varphi})\in\big(L^{\tilde{p}}\big(0,T;V^p_{\Gamma_1}\big)\big)^2,$$

where $[\cdot,\cdot]$ denotes the duality product between the space $L^{\tilde{p}}(0,T;V^p_{\Gamma_1})$ and its dual $(L^{\tilde{p}}(0,T;V^p_{\Gamma_1}))'$. With (2.10), we have

$$\begin{aligned}\big|\big[\mathcal{A}(u),\overline{\varphi}\big]\big| &\le 2\mu_1\big\|D(u)\big\|^{p-1}_{L^{\tilde{p}}(0,T;L^p(\Omega))}\big\|D(\overline{\varphi})\big\|_{L^{\frac{\tilde{p}}{\tilde{p}-p+1}}(0,T;L^p(\Omega))}\\ &\le 2\mu_1 T^{\frac{\tilde{p}-p}{\tilde{p}}}\|u\|^{p-1}_{L^{\tilde{p}}(0,T;V^p_{\Gamma_1})}\|\overline{\varphi}\|_{L^{\tilde{p}}(0,T;V^p_{\Gamma_1})}\end{aligned}$$

and

$$\big[\mathcal{A}(u),u-\overline{\varphi}\big]\ge 2(C_{\mathrm{Korn},p})^p\mu_0\|u\|^p_{L^p(0,T;V^p_{\Gamma_1})}-2\mu_1\|u\|^{p-1}_{L^{\tilde{p}}(0,T;V^p_{\Gamma_1})}\|\overline{\varphi}\|_{L^{\frac{\tilde{p}}{\tilde{p}-p+1}}(0,T;V^p_{\Gamma_1})}$$

for any $(u,\overline{\varphi})\in(L^{\tilde{p}}(0,T;V^p_{\Gamma_1}))^2$. It follows that $\mathcal{A}$ is a bounded operator and is coercive when $\tilde{p}=p$.

Hence, from now on, we will assume that $\tilde{p}=p\ge\frac{6}{5}$ and $q\in(1,\frac{5}{4})$.

Let $\upsilon^0\in\mathbf{W}^{1,p}(\Omega)$ be a given initial velocity such that

$$\begin{aligned}&\operatorname{div}\big(\upsilon^0\big)=0 \quad \text{in }\Omega, \qquad \upsilon^0=0 \quad \text{on }\Gamma_1, \qquad \upsilon^0=g \quad \text{on }\Gamma_L,\\ &\upsilon^0\cdot n=0 \quad \text{on }\Gamma_0.\end{aligned} \tag{2.12}$$

In order to deal with homogeneous boundary conditions on $(0,T)\times(\Gamma_1\cup\Gamma_L)$, we set $\overline{\upsilon}=\upsilon-\upsilon^0\xi$. We obtain the following weak formulation of the problem:

Problem (P) Let c, K, r, and μ satisfy (2.3)–(2.10). Let $f\in L^{p'}(0,T;\mathbf{L}^2(\Omega))$, $k\in L^{p'}(0,T;L^{p'}_+(\Gamma_0))$, $s\in L^p(0,T;\mathbf{L}^p(\Gamma_0))$, $\xi\in W^{1,p'}(0,T)$ satisfying (1.6), $\theta^b\in L^1((0,T)\times\omega)$, $\theta^0\in L^1(\Omega)$ and $\upsilon^0\in\mathbf{W}^{1,p}(\Omega)$ satisfying (2.12).

Find $\theta \in L^q(0,T;W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega))$, $\overline{\upsilon} \in C([0,T];\mathbf{L}^2(\Omega)) \cap L^p(0,T;V^p_{0.\,\mathrm{div}})$ with $\frac{\partial\overline{\upsilon}}{\partial t} \in L^{p'}(0,T;(V^p_{0.\,\mathrm{div}})')$ and $\pi \in H^{-1}(0,T;L^{p'}_0(\Omega))$ satisfying the following parabolic variational coupled problem:

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}(\overline{\upsilon},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)},\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\theta,\overline{\upsilon}+\upsilon^0\xi, D\big(\overline{\upsilon}+\upsilon^0\xi\big)\big) : D(\tilde{\vartheta})\zeta\,dx\,dt \\
&\quad - \left\langle \int_\Omega \pi\,\mathrm{div}(\tilde{\vartheta})\,dx,\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + J(\overline{\upsilon}+\tilde{\vartheta}\zeta) - J(\overline{\upsilon}) \\
&\geq \int_0^T \left(f+\upsilon^0\frac{\partial\xi}{\partial t},\tilde{\vartheta}\right)_{\mathbf{L}^2(\Omega)}\zeta\,dt \quad \forall\tilde{\vartheta}\in V^p_0, \forall\zeta\in\mathcal{D}(0,T)
\end{aligned} \tag{2.13}$$

with the initial condition

$$\overline{\upsilon}(0,\cdot) = \upsilon^0 - \upsilon^0\xi(0) = 0 \quad \text{in } \Omega \tag{2.14}$$

and

$$\begin{aligned}
&-\int_0^T \int_\Omega c\theta w\widetilde{\zeta}'\,dx\,dt + \int_0^T \int_\Omega (K\nabla\theta)\cdot\nabla w\widetilde{\zeta}\,dx\,dt \\
&= \int_0^T \int_\Omega \big(\mathcal{F}\big(\theta,\overline{\upsilon}+\upsilon^0\xi, D\big(\overline{\upsilon}+\upsilon^0\xi\big)\big) : D\big(\overline{\upsilon}+\upsilon^0\xi\big)\big) w\widetilde{\zeta}\,dx\,dt \\
&\quad + \int_0^T \int_\Omega r(\theta)w\widetilde{\zeta}\,dx\,dt + \int_0^T \int_\omega \theta^b w\widetilde{\zeta}\,dx'\,dt + \int_\Omega c\theta^0 w\widetilde{\zeta}(0)\,dx \\
&\forall w\in W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega), \forall\widetilde{\zeta}\in C^\infty\big([0,T]\big) \text{ s.t. } \widetilde{\zeta}(T)=0,
\end{aligned} \tag{2.15}$$

where

$$J: \quad \begin{cases} L^p(0,T;V^p_0) \to \mathbb{R}, \\ \overline{\varphi} \mapsto \int_0^T \int_{\Gamma_0} k|\overline{\varphi}-\tilde{s}|\,dx\,dt, \qquad \tilde{s} = s - (\upsilon^0)_\tau\xi \end{cases}$$

and $\langle\cdot,\cdot\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}$ and $(\cdot,\cdot)_{\mathbf{L}^2(\Omega)}$ (respectively $(\cdot,\cdot)_{L^2(\Omega)}$) denote the duality product between $\mathcal{D}(0,T)$ and $\mathcal{D}'(0,T)$ and the inner product in $\mathbf{L}^2(\Omega)$ (respectively $L^2(\Omega)$).

For this non-linear fluid flow/heat transfer problem, a natural proof strategy applies a fixed point technique. Indeed, for any given temperature field $\theta \in L^{\tilde{q}}(0,T;L^q(\Omega))$ with $\tilde{q} \geq 1$ and $q \geq 1$, the fluid flow problem (2.13)–(2.14) admits a solution [10, 11]. Moreover, we know that parabolic problems with L^1 data given by

$$\begin{cases} \frac{\partial\theta}{\partial t} - \mathrm{div}(a(x,\nabla\theta)) = g \quad \text{in } \Omega\times(0,T), \\ \theta = 0 \quad \text{in } \partial\Omega\times(0,T) \end{cases} \tag{2.16}$$

with $g \in L^1(0,T;L^1(\Omega))$ and the coercivity property $a(x,\gamma)\cdot\gamma \geq \alpha_a\|\gamma\|^2$ ($\alpha_a > 0$) for all $\gamma \in \mathbb{R}^3$ and for almost every $x \in \Omega$ admit a solution $\theta \in L^q(0,T;W^{1,q}_0(\Omega))$ with $1 < q < \frac{5}{4}$

(see [6]). For a given fluid velocity $\upsilon = \overline{\upsilon} + \upsilon^0\xi \in L^p(0,T;V^p_{\Gamma_1})$, we can not apply this result directly to the heat transfer problem since we consider a nonconstant heat capacity and mixed Dirichlet–Neumann boundary conditions, but we may still expect the existence of a solution $\theta \in L^q(0,T;W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega))$. Nevertheless, in both cases, the proofs of existence rely on compactness arguments and uniqueness of the solution to the flow problem (2.13)–(2.14) for a given temperature is not ensured, nor the uniqueness of the solution to L^1-parabolic problems (2.16).

In order to cope with this difficulty, we will consider in Sect. 3 an auxiliary approximate fluid flow/heat transfer problem (P_δ), where the L^1 right-hand side in the heat equation (1.3) is replaced by a bounded one depending on a small parameter $0 < \delta \ll 1$, and we will prove the existence of a solution $(\overline{\upsilon}_\delta, \pi_\delta, \theta_\delta)$ to (P_δ) by using a fixed point argument. Then in Sect. 4, we will prove that the sequence $(\overline{\upsilon}_\delta, \pi_\delta, \theta_\delta)_{\delta>0}$ converges to a solution of problem (P).

3 The approximate problem (P_δ)

For any $\delta > 0$, we consider the following approximate problem:

Problem (P_δ) Let c, K, r, and μ satisfy (2.3)–(2.10). Let $f \in L^{p'}(0,T;\mathbf{L}^2(\Omega))$, $k \in L^{p'}(0,T;L^{p'}_+(\Gamma_0))$, $s \in L^p(0,T;\mathbf{L}^p(\Gamma_0))$, $\xi \in W^{1,p'}(0,T)$ satisfying (1.6), $\theta^b \in L^1((0,T)\times\omega)$, $\theta^0 \in L^1(\Omega)$ and $\upsilon^0 \in \mathbf{W}^{1,p}(\Omega)$ satisfying (2.12).

Find $\theta_\delta \in W^{1,2}((0,T)\times\Omega) \cap L^\infty(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega)) \cap C^0([0,T];L^2(\Omega))$, $\overline{\upsilon}_\delta \in C([0,T];\mathbf{L}^2(\Omega)) \cap L^p(0,T;V^p_{0.\mathrm{div}})$ with $\frac{\partial\overline{\upsilon}_\delta}{\partial t} \in L^{p'}(0,T;(V^p_{0.\mathrm{div}})')$ and $\pi_\delta \in H^{-1}(0,T;L^{p'}_0(\Omega))$, satisfying the following parabolic variational coupled problem:

$$
\begin{aligned}
&\left\langle \frac{\partial}{\partial t}(\overline{\upsilon}_\delta, \tilde{\vartheta})_{\mathbf{L}^2(\Omega)}, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\theta_\delta, \overline{\upsilon}_\delta + \upsilon^0\xi, D(\overline{\upsilon}_\delta + \upsilon^0\xi)\big) : D(\tilde{\vartheta})\zeta \,dx\,dt \\
&\quad - \left\langle \int_\Omega \pi_\delta \operatorname{div}(\tilde{\vartheta})\,dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + J(\overline{\upsilon}_\delta + \tilde{\vartheta}\zeta) - J(\overline{\upsilon}_\delta) \\
&\geq \int_0^T \left(f + \upsilon^0 \frac{\partial\xi}{\partial t}, \tilde{\vartheta} \right)_{\mathbf{L}^2(\Omega)} \zeta\,dt \quad \forall\tilde{\vartheta} \in V^p_0, \forall\zeta \in \mathcal{D}(0,T)
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
&\int_0^T \int_\Omega c \frac{\partial\theta_\delta}{\partial t} w\tilde{\zeta}\,dx\,dt + \int_0^T \int_\Omega (K\nabla\theta_\delta)\cdot\nabla w\tilde{\zeta}\,dx\,dt \\
&\quad = \int_0^T \int_\Omega g_\delta(\theta_\delta, \overline{\upsilon}_\delta) w\tilde{\zeta}\,dx\,dt + \int_0^T \int_\Omega r(\theta_\delta) w\tilde{\zeta}\,dx\,dt \\
&\quad\quad + \int_0^T \int_\omega \theta^b_\delta w\tilde{\zeta}\,dx'\,dt \quad \forall w \in W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega), \forall\tilde{\zeta} \in \mathcal{D}(0,T)
\end{aligned}
\tag{3.2}
$$

with the initial conditions

$$
\overline{\upsilon}_\delta(0,\cdot) = 0 \quad \text{in } \Omega
\tag{3.3}
$$

and

$$\theta_\delta(0,\cdot) = \theta_\delta^0 \quad \text{in } \Omega, \tag{3.4}$$

where

$$g_\delta(\theta_\delta, \overline{\upsilon}_\delta) = \frac{2\mu(\theta_\delta, \overline{\upsilon}_\delta + \upsilon^0\xi, |D(\overline{\upsilon}_\delta + \upsilon^0\xi)|)|D(\overline{\upsilon}_\delta + \upsilon^0\xi)|^p}{1 + 2\delta\mu(\theta_\delta, \overline{\upsilon}_\delta + \upsilon^0\xi, |D(\overline{\upsilon}_\delta + \upsilon^0\xi)|)|D(\overline{\upsilon}_\delta + \upsilon^0\xi)|^p} \tag{3.5}$$

and θ_δ^b and θ_δ^0 are chosen as smooth approximations of θ^b and θ^0, respectively, i.e., $\theta_\delta^b \in \mathcal{D}((0,T)\times\omega)$ and $\theta_\delta^0 \in \mathcal{D}(\Omega)$ such that

$$\|\theta_\delta^b - \theta^b\|_{L^1((0,T)\times\omega)} \leq \delta, \qquad \|\theta_\delta^0 - \theta^0\|_{L^1(\Omega)} \leq \delta. \tag{3.6}$$

In order to prove the existence of a solution to (P_δ), we apply a fixed point technique. As a first step, we consider the fully decoupled fluid flow and heat transfer problems $(P^{\text{flow}}_{(\tilde{u},\tilde{\theta})})$ and $(P^{\text{heat}}_{\delta,(\tilde{u},\tilde{\theta})})$:

Problem $(P^{\text{flow}}_{(\tilde{u},\tilde{\theta})})$ Let $(\tilde{u},\tilde{\theta})$ be given in $L^p(0,T;\mathbf{L}^p(\Omega)) \times L^{\tilde{q}_1}(0,T;L^{\tilde{q}_2}(\Omega))$ with $\tilde{q}_1 > 1$ and $\tilde{q}_2 > 1$. Find $\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} \in C([0,T];\mathbf{L}^2(\Omega)) \cap L^p(0,T;V^p_{0.\text{div}})$ with $\frac{\partial \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}}{\partial t} \in L^{p'}(0,T;(V^p_{0.\text{div}})')$ and $\pi_{(\tilde{u},\tilde{\theta})} \in H^{-1}(0,T;L_0^{p'}(\Omega))$, satisfying the following parabolic variational inequality

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}, \tilde{\vartheta})_{\mathbf{L}^2(\Omega)}, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\tilde{\theta}, \tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) : D(\tilde{\vartheta})\zeta \, dx\, dt \\
&\quad - \left\langle \int_\Omega \pi_{(\tilde{u},\tilde{\theta})} \operatorname{div}(\tilde{\vartheta})\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + J(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \tilde{\vartheta}\zeta) - J(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \\
&\geq \int_0^T \left(f + \upsilon^0 \frac{\partial \xi}{\partial t}, \tilde{\vartheta} \right)_{\mathbf{L}^2(\Omega)} \zeta \, dt \quad \forall \tilde{\vartheta} \in V_0^p, \forall \zeta \in \mathcal{D}(0,T)
\end{aligned} \tag{3.7}$$

with the initial condition

$$\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}(0,\cdot) = 0 \quad \text{in } \Omega \tag{3.8}$$

and

Problem $(P^{\text{heat}}_{\delta,(\tilde{u},\tilde{\theta})})$ Let $(\tilde{u},\tilde{\theta})$ be given in $L^p(0,T;V^p_{0.\text{div}}) \times L^{\tilde{q}_1}(0,T;L^{\tilde{q}_2}(\Omega))$ with $\tilde{q}_1 > 1$ and $\tilde{q}_2 > 1$. Find $\theta_{\delta,(\tilde{u},\tilde{\theta})} \in W^{1,2}((0,T)\times\Omega) \cap L^\infty(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega)) \cap C^0([0,T];L^2(\Omega))$, satisfying the following parabolic variational equality:

$$\begin{aligned}
&\int_0^T \int_\Omega c \frac{\partial \theta_{\delta,(\tilde{u},\tilde{\theta})}}{\partial t} w\tilde{\zeta} \, dx\, dt + \int_0^T \int_\Omega (K\nabla\theta_{\delta,(\tilde{u},\tilde{\theta})}) \cdot \nabla w\tilde{\zeta} \, dx\, dt \\
&\quad = \int_0^T \int_\Omega g_\delta(\tilde{\theta},\tilde{u}) w\tilde{\zeta}\, dx\, dt + \int_0^T \int_\Omega r(\tilde{\theta}) w\tilde{\zeta} \, dx\, dt \\
&\qquad + \int_0^T \int_\omega \theta_\delta^b w\tilde{\zeta}\, dx'\, dt \quad \forall w \in W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega), \forall \tilde{\zeta} \in \mathcal{D}(0,T)
\end{aligned} \tag{3.9}$$

with the initial condition

$$\theta_{\delta,(\tilde{u},\tilde{\theta})}(0,\cdot) = \theta_\delta^0 \quad \text{in } \Omega. \tag{3.10}$$

Remark 3.1 Let us emphasize that problem $(P^{\text{flow}}_{(\tilde{u},\tilde{\theta})})$ does not depend on δ.

Then we have

Proposition 3.2 (Existence and uniqueness result for $(P^{\text{flow}}_{(\tilde{u},\tilde{\theta})})$) *Let $(\tilde{u},\tilde{\theta})$ be given in $L^p(0,T;\mathbf{L}^p(\Omega)) \times L^{\tilde{q}_1}(0,T;L^{\tilde{q}_2}(\Omega))$ with $\tilde{q}_1 > 1$ and $\tilde{q}_2 > 1$. Let μ satisfy (2.8)–(2.10), $f \in L^{p'}(0,T;\mathbf{L}^2(\Omega))$, $k \in L^{p'}(0,T;L_+^{p'}(\Gamma_0))$, $s \in L^p(0,T;\mathbf{L}^p(\Gamma_0))$, $\xi \in W^{1,p'}(0,T)$ satisfying (1.6), and $\upsilon^0 \in \mathbf{W}^{1,p}(\Omega)$ satisfying (2.12). Then problem $(P^{\text{flow}}_{(\tilde{u},\tilde{\theta})})$ admits a unique solution. Moreover, there exists a constant C^{flow}, independent of $(\tilde{\theta},\tilde{u})$, such that*

$$\|\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}\|_{L^p(0,T;V^p_{0.\,\text{div}})} \le C^{\text{flow}}, \tag{3.11}$$

$$\|\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \le C^{\text{flow}} \tag{3.12}$$

and

$$\left\| \frac{\partial \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}}{\partial t} \right\|_{L^{p'}(0,T;(V^p_{0.\,\text{div}})')} \le C^{\text{flow}}. \tag{3.13}$$

Proof The existence and uniqueness of a solution to the problem $(P^{\text{flow}}_{(\tilde{u},\tilde{\theta})})$ is an immediate consequence of Theorem 3.1 in [10] when $p \in [6/5,2)$ and Theorem 3.1, Theorem 4.1 and Remark 4.1 in [11] when $p \ge 2$. Let us prove now (3.11) and (3.13). For any $\overline{\varphi} = \tilde{\vartheta}\zeta$ with $\tilde{\vartheta} \in V^p_{0.\,\text{div}}$ and $\zeta \in \mathcal{D}(0,T)$, we have

$$\begin{aligned}
&\underbrace{\left\langle \frac{\partial}{\partial t}(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)},\zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}}_{=\int_0^T < \frac{\partial \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}}{\partial t},\overline{\varphi} >_{(V^p_{0.\,\text{div}})',V^p_{0.\,\text{div}}}\,dt} \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\tilde{\theta},\tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) : D(\overline{\varphi})\,dx\,dt \\
&\quad + J(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \overline{\varphi}) - J(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \ge \int_0^T \left(f + \upsilon^0 \frac{\partial \xi}{\partial t}, \overline{\varphi} \right)_{\mathbf{L}^2(\Omega)} dt.
\end{aligned} \tag{3.14}$$

By density of $\mathcal{D}(0,T) \otimes V^p_{0.\,\text{div}}$ into $L^p(0,T;V^p_{0.\,\text{div}})$, the same inequality is true for any $\overline{\varphi} \in L^p(0,T;V^p_{0.\,\text{div}})$.

Let us choose $\overline{\varphi} = -\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}\mathbf{1}_{|[0,t]}$, where $t \in (0,T]$, and $\mathbf{1}_{[0,t]}$ is the indicatrix function of the time interval $[0,t]$. With (3.14), we have

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}}{\partial t}, \overline{\upsilon}_{(\tilde{u},\tilde{\theta})} \right\rangle_{(V^p_{0.\,\text{div}})',V^p_{0.\,\text{div}}} d\tilde{t} \\
&\quad + \int_0^t \int_\Omega \mathcal{F}\big(\tilde{\theta},\tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) : D(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,d\tilde{t} \\
&\le \int_0^t \left(f + \upsilon^0 \frac{\partial \xi}{\partial t}, \overline{\upsilon}_{(\tilde{u},\tilde{\theta})} \right)_{\mathbf{L}^2(\Omega)} d\tilde{t} + \int_0^t \int_{\Gamma_0} k|\tilde{s}|\,dx\,d\tilde{t}.
\end{aligned}$$

With (2.10) and (2.1)–(2.2), we get

$$\begin{aligned}
&\int_0^t \int_\Omega \mathcal{F}\big(\tilde\theta, \tilde u + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde u,\tilde\theta)} + \upsilon^0\xi\big)\big) : D\big(\overline{\upsilon}_{(\tilde u,\tilde\theta)}\big)\,dx\,d\tilde t \\
&\quad\geq 2(C_{\mathrm{Korn},p})^p \mu_0 \int_0^t \big\|\overline{\upsilon}_{(\tilde u,\tilde\theta)} + \upsilon^0\xi\big\|_{1,p}^p\,d\tilde t \\
&\qquad - 2\mu_1 \int_0^t \big\|\upsilon^0\xi\big\|_{1,p}\big\|\overline{\upsilon}_{(\tilde u,\tilde\theta)} + \upsilon^0\xi\big\|_{1,p}^{p-1}\,d\tilde t \\
&\quad\geq 2(C_{\mathrm{Korn},p})^p \mu_0 \big| \|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,t;V^p_{0.\mathrm{div}})} - \big\|\upsilon^0\xi\big\|_{L^p(0,t;V^p_{\Gamma_1})}\big|^p \\
&\qquad - 2\mu_1 \big(\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,t;V^p_{0.\mathrm{div}})} + \big\|\upsilon^0\xi\big\|_{L^p(0,t;V^p_{\Gamma_1})}\big)^{p-1} \big\|\upsilon^0\xi\big\|_{L^p(0,t;V^p_{\Gamma_1})}.
\end{aligned}$$

Since $f + \upsilon^0 \frac{\partial\xi}{\partial t} \in L^{p'}(0,T;\mathbf{L}^2(\Omega))$, we obtain

$$\begin{aligned}
&\frac{1}{2}\big\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}(t)\big\|^2_{\mathbf{L}^2(\Omega)} \\
&\qquad + 2(C_{\mathrm{Korn},p})^p \mu_0 \big| \|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,t;V^p_{0.\mathrm{div}})} - \big\|\upsilon^0\xi\big\|_{L^p(0,t;V^p_{\Gamma_1})}\big|^p \\
&\quad\leq \widetilde{C}\bigg\|f + \upsilon^0\frac{\partial\xi}{\partial t}\bigg\|_{L^{p'}(0,t;\mathbf{L}^2(\Omega))} \|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,t;V^p_{0.\mathrm{div}})} + \int_0^t \int_{\Gamma_0} k|\tilde s|\,dx\,d\tilde t \\
&\qquad + 2\mu_1 \big(\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,t;V^p_{0.\mathrm{div}})} + \big\|\upsilon^0\xi\big\|_{L^p(0,t;V^p_{\Gamma_1})}\big)^{p-1} \big\|\upsilon^0\xi\big\|_{L^p(0,t;V^p_{\Gamma_1})},
\end{aligned} \tag{3.15}$$

where $\widetilde{C}$ denotes the norm of the continuous injection of V^p_0 into $\mathbf{L}^2(\Omega)$.

Let us consider first $t = T$. If $\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,T;V^p_{0.\mathrm{div}})} \neq 0$, it follows that

$$\begin{aligned}
&2(C_{\mathrm{Korn},p})^p \mu_0 \left| 1 - \frac{\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,T;V^p_{0.\mathrm{div}})}} \right|^p \\
&\quad\leq \widetilde{C}\bigg\|f + \upsilon^0\frac{\partial\xi}{\partial t}\bigg\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))} \|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|^{1-p}_{L^p(0,T;V^p_{0.\mathrm{div}})} + \frac{J(0)}{\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|^p_{L^p(0,T;V^p_{0.\mathrm{div}})}} \\
&\qquad + 2\mu_1 \left(1 + \frac{\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,T;V^p_{0.\mathrm{div}})}}\right)^{p-1} \frac{\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,T;V^p_{0.\mathrm{div}})}}.
\end{aligned}$$

However, the mapping

$$\begin{aligned}
z \mapsto\ & 2(C_{\mathrm{Korn},p})^p \mu_0 \left| 1 - \frac{\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{z} \right|^p - \widetilde{C}\bigg\|f + \upsilon^0\frac{\partial\xi}{\partial t}\bigg\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))} z^{1-p} \\
&- \frac{J(0)}{z^p} - 2\mu_1\left(1 + \frac{\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{z}\right)^{p-1} \frac{\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}}{z}
\end{aligned}$$

admits $2(C_{\mathrm{Korn},p})^p\mu_0 > 0$ as a limit when z tends to $+\infty$. Thus, there exists a real number $C > 0$, independent of $(\tilde u, \tilde\theta)$, such that

$$\|\overline{\upsilon}_{(\tilde u,\tilde\theta)}\|_{L^p(0,T;V^p_{0.\mathrm{div}})} \leq C, \tag{3.16}$$

which yields (3.11). Going back to (3.15), we obtain (3.12).

Let us choose now $\overline{\varphi} = \pm\tilde{\vartheta}\zeta$ with $\tilde{\vartheta} \in V^p_{0.\mathrm{div}}$ and $\zeta \in \mathcal{D}(0,T)$ in (3.14). We obtain

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial \overline{v}_{(\tilde{u},\tilde{\theta})}}{\partial t}, \pm\tilde{\vartheta}\zeta \right\rangle_{(V^p_{0.\mathrm{div}})', V^p_{0.\mathrm{div}}} dt \\ &\quad + \int_0^T \int_\Omega \mathcal{F}\big(\tilde{\theta}, \tilde{u} + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big) : D(\pm\tilde{\vartheta}\zeta)\,dx\,dt \\ &\quad + J(\overline{v}_{(\tilde{u},\tilde{\theta})} \pm \tilde{\vartheta}\zeta) - J(\overline{v}_{(\tilde{u},\tilde{\theta})}) \geq \int_0^T \left(f + v^0 \frac{\partial \xi}{\partial t}, \pm\tilde{\vartheta}\zeta \right)_{\mathbf{L}^2(\Omega)} dt. \end{aligned} \tag{3.17}$$

But

$$\big|J(\overline{v}_{(\tilde{u},\tilde{\theta})} \pm \tilde{\vartheta}\zeta) - J(\overline{v}_{(\tilde{u},\tilde{\theta})})\big| \leq \int_0^T \int_{\Gamma_0} k|\tilde{\vartheta}\zeta|\,dx\,dt$$

and, recalling that $k \in L^{p'}(0,T; L^{p'}_+(\Gamma_0))$, we get

$$\begin{aligned} &\big|J(\overline{v}_{(\tilde{u},\tilde{\theta})} \pm \tilde{\vartheta}\zeta) - J(\overline{v}_{(\tilde{u},\tilde{\theta})})\big| \\ &\quad \leq \|\gamma_p\|_{\mathcal{L}(\mathbf{W}^{1,p}(\Omega),\mathbf{L}^p(\partial\Omega))} \|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))} \|\tilde{\vartheta}\zeta\|_{L^p(0,T;V^p_{0.\mathrm{div}})}, \end{aligned}$$

where γ_p denotes the trace operator from $\mathbf{W}^{1,p}(\Omega)$ into $\mathbf{L}^p(\partial\Omega)$.

On the other hand,

$$\begin{aligned} &\left| \int_0^T \int_\Omega \mathcal{F}\big(\tilde{\theta}, \tilde{u} + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big) : D(\tilde{\vartheta}\zeta)\,dx\,dt \right| \\ &\quad \leq 2\mu_1 \big\| D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big) \big\|^{p-1}_{L^p(0,T;(L^p(\Omega))^{3\times3})} \big\| D(\tilde{\vartheta}\zeta) \big\|_{L^p(0,T;(L^p(\Omega))^{3\times3})}, \end{aligned}$$

and with (3.16), we obtain

$$\begin{aligned} &\left| \int_0^T \int_\Omega \mathcal{F}\big(\tilde{\theta}, \tilde{u} + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big) : D(\tilde{\vartheta}\zeta)\,dx\,dt \right| \\ &\quad \leq \big(2\mu_1\big(C + \|v^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\big)^{p-1}\big) \|\tilde{\vartheta}\zeta\|_{L^p(0,T;V^p_{0.\mathrm{div}})}. \end{aligned}$$

Going back to (3.17), we obtain

$$\begin{aligned} &\left| \int_0^T \left\langle \frac{\partial \overline{v}_{(\tilde{u},\tilde{\theta})}}{\partial t}, \tilde{\vartheta}\zeta \right\rangle_{(V^p_{0.\mathrm{div}})', V^p_{0.\mathrm{div}}} dt \right| \\ &\quad \leq \|\gamma_p\|_{\mathcal{L}(\mathbf{W}^{1,p}(\Omega),\mathbf{L}^p(\partial\Omega))} \|k\|_{L^{p'}(0,T;L^{p'}(\Gamma_0))} \|\tilde{\vartheta}\zeta\|_{L^p(0,T;V^p_{0.\mathrm{div}})} \\ &\qquad + \widetilde{C} \left\| f + v^0 \frac{\partial \xi}{\partial t} \right\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))} \|\tilde{\vartheta}\zeta\|_{L^p(0,T;V^p_{0.\mathrm{div}})} \\ &\qquad + \big(2\mu_1\big(C + \|v^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\big)^{p-1}\big) \|\tilde{\vartheta}\zeta\|_{L^p(0,T;V^p_{0.\mathrm{div}})}, \end{aligned}$$

which yields (3.13). □

Proposition 3.3 (Existence and uniqueness result for $(P^{\text{heat}}_{\delta,(\tilde u,\tilde\theta)})$) *Let $(\tilde u,\tilde\theta)$ be given in $L^p(0,T;V^p_{0.\text{div}})\times L^{\tilde q_1}(0,T;L^{\tilde q_2}(\Omega))$ with $\tilde q_1>1$ and $\tilde q_2>1$. Let c, K, r, and μ satisfy (2.3)–(2.10). Let $\xi\in W^{1,p'}(0,T)$ satisfying (1.6), $\theta^b_\delta\in\mathcal D((0,T)\times\omega)$, $\theta^0_\delta\in\mathcal D(\Omega)$ and $\upsilon^0\in\mathbf W^{1,p}(\Omega)$ satisfying (2.12). Then problem $(P^{\text{heat}}_{\delta,(\tilde u,\tilde\theta)})$ admits a unique solution. Moreover, there exists a constant C^{heat}, depending only on the data K and c, such that*

$$\left\|\frac{\partial\theta_{\delta,(\tilde u,\tilde\theta)}}{\partial t}\right\|^2_{L^2(0,T;L^2(\Omega))}+\|\nabla\theta_{\delta,(\tilde u,\tilde\theta)}\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$\le C^{\text{heat}}\big(\|g_\delta(\tilde\theta,\tilde u)+r(\tilde\theta)\|^2_{L^2(0,T;L^2(\Omega))}+\|\theta^0_\delta\|^2_{H^1(\Omega)}+\|\theta^b_\delta\|^2_{W^{1,2}(0,T;L^2(\omega))}\big).$$

Proof The result is straightforward with the Galerkin method. The details are left to the reader. □

Owing to the definition of $g_\delta(\tilde\theta,\tilde u)$ and the uniform boundedness of the mapping r, we obtain immediately

Corollary 3.4 *Under the previous assumptions, there exists a constant C^{heat}_δ, depending only on δ, $\|\theta^0_\delta\|_{H^1(\Omega)}$ and $\|\theta^b_\delta\|_{W^{1,2}(0,T;L^2(\omega))}$ such that*

$$\left\|\frac{\partial\theta_{\delta,(\tilde u,\tilde\theta)}}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}+\|\nabla\theta_{\delta,(\tilde u,\tilde\theta)}\|_{L^\infty(0,T;L^2(\Omega))}\le C^{\text{heat}}_\delta. \tag{3.18}$$

Proof By using (2.7) and (3.5), we obtain

$$\|g_\delta(\tilde\theta,\tilde u)+r(\tilde\theta)\|^2_{L^2(0,T;L^2(\Omega))}$$
$$\le 2\int_0^T\int_\Omega\big(g_\delta(\tilde\theta,\tilde u)\big)^2\,dx\,dt+2\int_0^T\int_\Omega\big(r(\tilde\theta)\big)^2\,dx\,dt$$
$$\le 2\left(\frac{1}{\delta^2}+r_1^2\right)\text{meas}(\Omega)T,$$

which allows us to conclude. □

It follows that, for any $(\tilde u,\tilde\theta)\in L^p(0,T;\mathbf L^p(\Omega))\times L^2(0,T;L^2(\Omega))$, and for any $\delta>0$, we may define $(\overline\upsilon_{(\tilde u,\tilde\theta)},\pi_{(\tilde u,\tilde\theta)})\in L^p(0,T;V^p_{0.\text{div}})\times L^{p'}(0,T;L^{p'}_0(\Omega))$ as the unique solution of $(P^{\text{flow}}_{(\tilde u,\tilde\theta)})$ and $\theta_{\delta,(\overline\upsilon_{(\tilde u,\tilde\theta)},\tilde\theta)}\in W^{1,2}((0,T)\times\Omega)$ as the unique solution of $(P^{\text{heat}}_{\delta,(\overline\upsilon_{(\tilde u,\tilde\theta)},\tilde\theta)})$. Then we define the mapping $T_\delta: L^p(0,T;\mathbf L^p(\Omega))\times L^2(0,T;L^2(\Omega))\to L^p(0,T;\mathbf L^p(\Omega))\times L^2(0,T;L^2(\Omega))$ by

$$T_\delta(\tilde u,\tilde\theta)=(\overline\upsilon_{(\tilde u,\tilde\theta)},\theta_{\delta,(\overline\upsilon_{(\tilde u,\tilde\theta)},\tilde\theta)}),$$

and we will prove that T_δ admits a fixed point.

Theorem 3.5 *Let $\delta>0$. Let the mappings c, K, r, and μ satisfy (2.3)–(2.10), $f\in L^{p'}(0,T;\mathbf L^2(\Omega))$, $k\in L^{p'}(0,T;L^{p'}_+(\Gamma_0))$, $s\in L^p(0,T;\mathbf L^p(\Gamma_0))$, $\xi\in W^{1,p'}(0,T)$ satisfying (1.6), $\theta^0_\delta\in\mathcal D(\Omega)$, $\theta^b_\delta\in\mathcal D((0,T)\times\omega)$ and $\upsilon^0\in\mathbf W^{1,p}(\Omega)$ satisfying (2.12). Then the mapping T_δ admits a fixed point in $L^p(0,T;\mathbf L^p(\Omega))\times L^2(0,T;L^2(\Omega))$.*

Proof Let $(\tilde{u},\tilde{\theta}) \in L^p(0,T;\mathbf{L}^p(\Omega)) \times L^2(0,T;L^2(\Omega))$. With the previous estimates, we have

$$\|\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}\|_{L^p(0,T;V^p_{0.\mathrm{div}})} \le C^{\mathrm{flow}}, \qquad \|\theta_{\delta,(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})}\|_{L^\infty(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega))} \le C^{\mathrm{heat}}_\delta.$$

Thus, $T_\delta(\mathcal{C}) \subset \mathcal{C}$ with $\mathcal{C} = \overline{B}_{L^p(0,T;\mathbf{L}^p(\Omega))}(0, C_p C^{\mathrm{flow}}) \times \overline{B}_{L^2(0,T;L^2(\Omega))}(0, \sqrt{T} C_2 C^{\mathrm{heat}}_\delta)$, where C_p and C_2 denote Poincaré's constant on $V^p_{0.\mathrm{div}}$ and on $W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega)$, respectively. Moreover, the estimates (3.13) and (3.18) imply that $T_\delta(\mathcal{C})$ is bounded in the space $W^{1,p'}(0,T;(V^p_{0.\mathrm{div}})') \times W^{1,2}(0,T;L^2(\Omega))$, and we may conclude that $T_\delta(\mathcal{C})$ is relatively compact in $L^p(0,T;\mathbf{L}^p(\Omega)) \times L^2(0,T;L^2(\Omega))$.

Let us now prove that T_δ is continuous. Let $(\tilde{u}_n,\tilde{\theta}_n)_{n\ge1}$ be a sequence, which converges strongly in $L^p(0,T;\mathbf{L}^p(\Omega)) \times L^2(0,T;L^2(\Omega))$ to $(\tilde{u},\tilde{\theta})$, and let us define

$$T_\delta(\tilde{u}_n,\tilde{\theta}_n) = (\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}, \theta_{\delta,(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)}) \quad \text{for all } n \ge 1$$

and

$$T_\delta(\tilde{u},\tilde{\theta}) = (\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}, \theta_{\delta,(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})}).$$

We have to prove that the sequence $(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}, \theta_{\delta,(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})_{n\ge1}$ converges strongly to $(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}, \theta_{(\overline{\upsilon}_{\delta,(\tilde{u},\tilde{\theta})},\tilde{\theta})})$ in $L^p(0,T;\mathbf{L}^p(\Omega)) \times L^2(0,T;L^2(\Omega))$.

The sequence $(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}, \theta_{\delta,(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})_{n\ge1}$ satisfies the estimates (3.11)–(3.13) and (3.18), so it admits strongly converging subsequences in $L^p(0,T;\mathbf{L}^p(\Omega)) \times L^2(0,T;L^2(\Omega))$.

Let us consider such a subsequence still denoted $(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}, \theta_{\delta,(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})_{n\ge1}$.

With (3.14), we have

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial \overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}}{\partial t}, \overline{\varphi} - \overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} \right\rangle_{(V^p_{0.\mathrm{div}})', V^p_{0.\mathrm{div}}} dt \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\tilde{\theta}_n, \tilde{u}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} + \upsilon^0\xi\big)\big) : D(\overline{\varphi} - \overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)})\,dx\,dt \\
&\quad + J(\overline{\varphi}) - J(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}) \\
&\ge \int_0^T \left(f + \upsilon^0 \frac{\partial \xi}{\partial t}, \overline{\varphi} - \overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} \right)_{\mathbf{L}^2(\Omega)} dt \quad \forall \overline{\varphi} \in L^p\big(0,T;V^p_{0.\mathrm{div}}\big)
\end{aligned} \tag{3.19}$$

for all $n \ge 1$, we also have

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}}{\partial t}, \overline{\varphi} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})} \right\rangle_{(V^p_{0.\mathrm{div}})', V^p_{0.\mathrm{div}}} dt \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\tilde{\theta}, \tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) : D(\overline{\varphi} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,dt \\
&\quad + J(\overline{\varphi}) - J(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \\
&\ge \int_0^T \left(f + \upsilon^0 \frac{\partial \xi}{\partial t}, \overline{\varphi} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})} \right)_{\mathbf{L}^2(\Omega)} dt \quad \forall \overline{\varphi} \in L^p\big(0,T;V^p_{0.\mathrm{div}}\big).
\end{aligned} \tag{3.20}$$

We choose $\overline{\varphi} = \overline{v}_{(\tilde{u},\tilde{\theta})}$ in (3.19) then $\overline{\varphi} = \overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)}$ in (3.20) and add the two inequalities. We obtain:

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{v}_{(\tilde{u},\tilde{\theta})})}{\partial t}, \overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{v}_{(\tilde{u},\tilde{\theta})} \right\rangle_{(V^p_{0.\mathrm{div}})', V^p_{0.\mathrm{div}}} dt \\
&\quad + \int_0^T \int_\Omega \big(\mathcal{F}\big(\tilde{\theta}_n, \tilde{u}_n + v^0\xi, D\big(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} + v^0\xi\big)\big) \\
&\quad - \mathcal{F}\big(\tilde{\theta}, \tilde{u} + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big)\big) : D(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{v}_{(\tilde{u},\tilde{\theta})})\,dx\,dt \le 0
\end{aligned}$$

and thus,

$$\begin{aligned}
&\frac{1}{2}\|\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{v}_{(\tilde{u},\tilde{\theta})}\|^2_{\mathbf{L}^2(\Omega)}(T) \\
&\qquad + \int_0^T \int_\Omega \big(\mathcal{F}\big(\tilde{\theta}_n, \tilde{u}_n + v^0\xi, D\big(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} + v^0\xi\big)\big) \\
&\qquad - \mathcal{F}\big(\tilde{\theta}_n, \tilde{u}_n + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big)\big) : D(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{v}_{(\tilde{u},\tilde{\theta})})\,dx\,dt \\
&\quad \le \int_0^T \int_\Omega \big(\mathcal{F}\big(\tilde{\theta}, \tilde{u} + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big) \\
&\qquad - \mathcal{F}\big(\tilde{\theta}_n, \tilde{u}_n + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big)\big) : D(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{v}_{(\tilde{u},\tilde{\theta})})\,dx\,dt.
\end{aligned}$$

We decompose $\mathcal{F}$ as $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ with

$$\mathcal{F}_1(\lambda_2) = \mu_0\|\lambda_2\|^{p-2}\lambda_2 \quad \text{if } \lambda_2 \ne 0_{\mathbb{R}^{3\times 3}}, \qquad \mathcal{F}_1(\lambda_2) = 0_{\mathbb{R}^{3\times 3}} \quad \text{otherwise}$$

and

$$\begin{cases} \mathcal{F}_2(\lambda_0,\lambda_1,\lambda_2) = 2\overline{\mu}(\lambda_0,\lambda_1,\|\lambda_2\|)\|\lambda_2\|^{p-2}\lambda_2 & \text{if } \lambda_2 \ne 0_{\mathbb{R}^{3\times 3}}, \\ \mathcal{F}_2(\lambda_0,\lambda_1,\lambda_2) = 0_{\mathbb{R}^{3\times 3}} & \text{otherwise}, \end{cases}$$

where $\overline{\mu} = \mu - \frac{\mu_0}{2}$. By observing that $\overline{\mu}$ satisfies

$$\begin{aligned}
&d \mapsto \overline{\mu}(\cdot,\cdot,d) \text{ is monotone increasing on } \mathbb{R}_+, \\
&0 < \frac{\mu_0}{2} \le \overline{\mu}(o,e,d) \le \mu_1 - \frac{\mu_0}{2} \quad \text{for all } (o,e,d) \in \mathbb{R}\times\mathbb{R}^3\times\mathbb{R}_+,
\end{aligned}$$

we infer from Lemma 1 in [9] that $\lambda_2 \mapsto \mathcal{F}_2(\cdot,\cdot,\lambda_2)$ is monotone in $\mathbb{R}^{3\times 3}$. Hence,

$$\begin{aligned}
&\int_0^T \int_\Omega \big(\mathcal{F}_2\big(\tilde{\theta}_n, \tilde{u}_n + v^0\xi, D\big(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} + v^0\xi\big)\big) \\
&\quad - \mathcal{F}_2\big(\tilde{\theta}_n, \tilde{u}_n + v^0\xi, D\big(\overline{v}_{(\tilde{u},\tilde{\theta})} + v^0\xi\big)\big)\big) : D(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{v}_{(\tilde{u},\tilde{\theta})})\,dx\,dt \ge 0
\end{aligned}$$

and thus,

$$\begin{aligned}
&\frac{1}{2}\|\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}\|^2_{\mathbf{L}^2(\Omega)}(T) \\
&\quad + \int_0^T \int_\Omega \big(\mathcal{F}_1\big(D\big(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} + \upsilon^0\xi\big)\big) - \mathcal{F}_1\big(D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big)\big) \\
&\quad : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,dt \\
&\le \int_0^T \int_\Omega \big(\mathcal{F}_2\big(\tilde{\theta}, \tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) \\
&\quad - \mathcal{F}_2\big(\tilde{\theta}_n, \tilde{u}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big)\big) : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,dt.
\end{aligned} \tag{3.21}$$

Now we distinguish two cases.

Case 1: $p \in [6/5, 2)$

With some algebraic computations, we can prove that

$$\big(\|\lambda\| + \|\lambda'\|\big)^{2-p}\big(\mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda')\big) : (\lambda - \lambda') \ge \mu_0(p-1)\|\lambda - \lambda'\|^2,$$

which yields

$$\big(\|\lambda\|^p + \|\lambda'\|^p\big)^{\frac{2-p}{2}}\big(\big(\mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda')\big) : (\lambda - \lambda')\big)^{\frac{p}{2}} \ge \frac{(\mu_0(p-1))^{\frac{p}{2}}}{2^{\frac{(p-1)(2-p)}{2}}}\|\lambda - \lambda'\|^p$$

for all $(\lambda, \lambda') \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ (see, for instance, Theorem 4.1 in [10]).

It follows that

$$\begin{aligned}
&\frac{(\mu_0(p-1))^{\frac{p}{2}}}{2^{\frac{(p-1)(2-p)}{2}}} \int_0^T \int_\Omega \|D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\|^p\,dx\,dt \\
&\le \bigg(\int_0^T \int_\Omega \big(\mathcal{F}_1\big(D\big(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} + \upsilon^0\xi\big)\big) - \mathcal{F}_1\big(D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big)\big) \\
&\quad : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,dt\bigg)^{\frac{p}{2}} \\
&\quad \times \bigg(\int_0^T \int_\Omega \big(\|D\big(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} + \upsilon^0\xi\big)\|^p + \|D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\|^p\big)\,dx\,dt\bigg)^{\frac{2-p}{2}} \\
&\le \bigg(\int_0^T \int_\Omega \big(\mathcal{F}_1\big(D\big(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} + \upsilon^0\xi\big)\big) - \mathcal{F}_1\big(D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big)\big) \\
&\quad : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,dt\bigg)^{\frac{p}{2}} \\
&\quad \times \big(\|\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} + \upsilon^0\xi\|^p_{L^p(0,T;V^p_{\Gamma_1})} + \|\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\|^p_{L^p(0,T;V^p_{\Gamma_1})}\big)^{\frac{2-p}{2}}.
\end{aligned}$$

Thus, with (3.11), we get

$$\begin{aligned}
&\frac{(\mu_0(p-1))^{\frac{p}{2}}}{2^{\frac{(p-1)(2-p)}{2}}}\left\|D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\right\|^p_{L^p(0,T;(L^p(\Omega))^{3\times 3})}\\
&\leq \left(\int_0^T\int_\Omega \left(\mathcal{F}_1\left(D\left(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}+\upsilon^0\xi\right)\right)-\mathcal{F}_1\left(D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right)\right.\\
&\qquad \left. : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,dt\right)^{\frac{p}{2}}\\
&\qquad \times \left(2\left(C^{\text{flow}}+\left\|\upsilon^0\xi\right\|_{L^p(0,T;V^p_{\Gamma_1})}\right)^p\right)^{\frac{2-p}{2}},
\end{aligned}$$

and finally,

$$\begin{aligned}
&\frac{\mu_0(p-1)}{2^{(2-p)}(C^{\text{flow}}+\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})})^{(2-p)}}\left\|D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\right\|^2_{L^p(0,T;(L^p(\Omega))^{3\times 3})}\\
&\leq \int_0^T\int_\Omega \left(\mathcal{F}_1\left(D\left(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}+\upsilon^0\xi\right)\right)-\mathcal{F}_1\left(D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right)\\
&\qquad : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)})\,dx\,dt.
\end{aligned}$$

Going back to (3.21), we obtain

$$\begin{aligned}
&\frac{\mu_0(p-1)}{2^{(2-p)}(C^{\text{flow}}+\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})})^{(2-p)}}\left\|D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\right\|^2_{L^p(0,T;(L^p(\Omega))^{3\times 3})}\\
&\leq \int_0^T\int_\Omega \left(\mathcal{F}_2\left(\tilde{\theta},\tilde{u}+\upsilon^0\xi,D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right.\\
&\qquad \left.-\mathcal{F}_2\left(\tilde{\theta}_n,\tilde{u}_n+\upsilon^0\xi,D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right) : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\,dx\,dt\\
&\leq \left\|\mathcal{F}_2\left(\tilde{\theta},\tilde{u}+\upsilon^0\xi,D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right.\\
&\qquad \left.-\mathcal{F}_2\left(\tilde{\theta}_n,\tilde{u}_n+\upsilon^0\xi,D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})}\\
&\qquad \times \left\|D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\right\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})},
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{\mu_0(p-1)}{2^{(2-p)}(C^{\text{flow}}+\|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})})^{(2-p)}}\left\|D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}-\overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\right\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})}\\
&\leq \left\|\mathcal{F}_2\left(\tilde{\theta},\tilde{u}+\upsilon^0\xi,D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right.\\
&\qquad \left.-\mathcal{F}_2\left(\tilde{\theta}_n,\tilde{u}_n+\upsilon^0\xi,D\left(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}+\upsilon^0\xi\right)\right)\right\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})}.
\end{aligned}\tag{3.22}$$

Case 2: $p \in [2,+\infty)$

Once again, with some algebraic computations, we get

$$\left(\mathcal{F}_1(\lambda)-\mathcal{F}_1(\lambda')\right):\left(\lambda-\lambda'\right)\geq \frac{\mu_0}{2^{p-1}}\left\|\lambda-\lambda'\right\|^p$$

for all $(\lambda, \lambda') \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ (see, for instance, Theorem 4.1 in [11]). Going back to (3.21),

$$\begin{aligned}
&\frac{\mu_0}{2^{p-1}} \left\| D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \right\|^p_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\quad \le \int_0^T \int_\Omega \big(\mathcal{F}_2\big(\tilde{\theta}, \tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) - \mathcal{F}_2\big(\tilde{\theta}_n, \tilde{u}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big)\big) \\
&\qquad : D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})})\, dx\, dt \\
&\quad \le \big\| \mathcal{F}_2\big(\tilde{\theta}, \tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) \\
&\qquad - \mathcal{F}_2\big(\tilde{\theta}_n, \tilde{u}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) \big\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})} \\
&\qquad \times \left\| D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \right\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\mu_0}{2^{p-1}} \left\| D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \right\|^{p-1}_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\quad \le \big\| \mathcal{F}_2\big(\tilde{\theta}, \tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) \\
&\qquad - \mathcal{F}_2\big(\tilde{\theta}_n, \tilde{u}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) \big\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})}.
\end{aligned} \tag{3.23}$$

By possibly extracting a subsequence still denoted $(\tilde{u}_n, \tilde{\theta}_n)_{n\ge 1}$, we have

$$\tilde{u}_n \longrightarrow \tilde{u}, \qquad \tilde{\theta}_n \longrightarrow \tilde{\theta} \quad \text{a.e. in } (0,T)\times\Omega.$$

By using the continuity and boundedness assumptions (2.8) and (2.10) for the mapping μ, we infer with Lebesgue's theorem that

$$\begin{aligned}
&\mathcal{F}_2\big(\tilde{\theta}_n, \tilde{u}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) \longrightarrow \mathcal{F}_2\big(\tilde{\theta}, \tilde{u} + \upsilon^0\xi, D\big(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})} + \upsilon^0\xi\big)\big) \\
&\quad \text{strongly in } L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3}).
\end{aligned}$$

With (3.22) when $p \in [6/5, 2)$ or (3.23) when $p \in [2, +\infty)$, we obtain

$$\left\| D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} - \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \right\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \longrightarrow 0,$$

and with Korn's inequality, we infer that

$$\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} \longrightarrow \overline{\upsilon}_{(\tilde{u},\tilde{\theta})} \quad \text{strongly in } L^p(0,T;V^p_{0.\mathrm{div}}) \text{ and in } L^p(0,T;\mathbf{L}^p(\Omega)).$$

By possibly extracting another subsequence still denoted $(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)})_{n\ge 1}$, we have

$$D(\overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}) \longrightarrow D(\overline{\upsilon}_{(\tilde{u},\tilde{\theta})}), \qquad \overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)} \longrightarrow \overline{\upsilon}_{(\tilde{u},\tilde{\theta})} \quad \text{a.e. in } (0,T)\times\Omega.$$

By using the continuity and boundedness properties of the mappings μ and r, we obtain

$$g_\delta(\tilde{\theta}_n, \overline{\upsilon}_{(\tilde{u}_n,\tilde{\theta}_n)}) \longrightarrow g_\delta(\tilde{\theta}, \overline{\upsilon}_{(\tilde{u},\tilde{\theta})}) \quad \text{strongly in } L^2(0,T;L^2(\Omega))$$

and

$$r(\tilde{\theta}_n) \longrightarrow r(\tilde{\theta}) \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Moreover, $(\theta_{\delta,(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})_{n\geq 1}$ satisfies the estimate (3.18). It follows that there exists $\theta_\delta^* \in W^{1,2}((0,T)\times\Omega)\cap L^2(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega))$ such that, by possibly extracting a subsequence still denoted $(\theta_{\delta,(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})_{n\geq 1}$, we have

$$\theta_{\delta,(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)} \longrightarrow \theta_\delta^* \quad \text{weakly in } W^{1,2}((0,T)\times\Omega)$$
$$\text{and weakly* in } L^\infty(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega))$$

and with Simon's lemma [34]

$$\theta_{\delta,(\overline{v}_{\delta,(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)} \longrightarrow \theta_\delta^* \quad \text{strongly in } C^0([0,T];L^2(\Omega)) \text{ and in } L^2(0,T;L^2(\Omega)).$$

Hence, we may pass to the limit in all the terms of $(P^{\text{heat}}_{\delta,(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})$ and obtain that θ_δ^* is the solution of $(P^{\text{heat}}_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})})$. By uniqueness of the solution of $(P^{\text{heat}}_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})})$, we infer that $\theta_\delta^* = \theta_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})}$.

Finally, recalling that $(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\theta_{\delta,(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})_{n\geq 1}$ satisfies the estimates (3.11)–(3.13) and (3.18), we infer that the whole sequence $(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\theta_{\delta,(\overline{v}_{(\tilde{u}_n,\tilde{\theta}_n)},\tilde{\theta}_n)})_{n\geq 1}$ converges strongly to $(\overline{v}_{(\tilde{u},\tilde{\theta})},\theta_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})})$ in the space $L^p(0,T;\mathbf{L}^p(\Omega))\times L^2(0,T;L^2(\Omega))$, which allows us to conclude. □

As a corollary, we obtain an existence result for the approximate problem (P_δ).

Corollary 3.6 *Let* $\delta > 0$. *Let the mappings* c, K, r, *and* μ *satisfy* (2.3)–(2.10), $f \in L^{p'}(0,T;\mathbf{L}^2(\Omega))$, $k \in L^{p'}(0,T;L^{p'}_+(\Gamma_0))$, $s \in L^p(0,T;\mathbf{L}^p(\Gamma_0))$, $\xi \in W^{1,p'}(0,T)$ *satisfying* (1.6), $\theta_\delta^0 \in \mathcal{D}(\Omega)$, $\theta_\delta^b \in \mathcal{D}((0,T)\times\omega)$ *and* $v^0 \in \mathbf{W}^{1,p}(\Omega)$ *satisfying* (2.12). *Then problem* (P_δ) *admits a solution.*

Proof With the previous theorem, we know that the mapping T_δ admits a fixed point, i.e., there exists $(\tilde{u},\tilde{\theta}) \in L^p(0,T;\mathbf{L}^p(\Omega))\times L^2(0,T;L^2(\Omega))$ such that

$$T_\delta(\tilde{u},\tilde{\theta}) = (\overline{v}_{(\tilde{u},\tilde{\theta})},\theta_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})}) = (\tilde{u},\tilde{\theta}).$$

By the definition of T_δ, we infer that $\tilde{u} = \overline{v}_{(\tilde{u},\tilde{\theta})} \in C([0,T];\mathbf{L}^2(\Omega))\cap L^p(0,T;V^p_{0.\text{div}})$ with $\frac{\partial \tilde{u}}{\partial t} = \frac{\partial \overline{v}_{(\tilde{u},\tilde{\theta})}}{\partial t} \in L^{p'}(0,T;(V^p_{0.\text{div}})')$, and there exists $\pi_{(\tilde{u},\tilde{\theta})} \in H^{-1}(0,T;L^{p'}_0(\Omega))$ such that $(\tilde{u},\pi_{(\tilde{u},\tilde{\theta})}) = (\overline{v}_{(\tilde{u},\tilde{\theta})},\pi_{(\tilde{u},\tilde{\theta})})$ is solution of $(P^{\text{flow}}_{(\tilde{u},\tilde{\theta})})$, while $\tilde{\theta} = \theta_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})} \in W^{1,2}((0,T)\times\Omega)\cap L^\infty(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega))\cap C([0,T];L^2(\Omega))$ is a solution of the problem $(P^{\text{heat}}_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})}) = (P^{\text{heat}}_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\theta_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})})})$. Hence, the triplet $(\overline{v}_{(\tilde{u},\tilde{\theta})},\pi_{(\tilde{u},\tilde{\theta})},\theta_{\delta,(\overline{v}_{(\tilde{u},\tilde{\theta})},\tilde{\theta})})$ is a solution of (P_δ). Aubin and Simon □

4 Existence result for the coupled fluid flow / heat transfer problem (*P*)

In order to prove the existence of a solution to the coupled fluid flow/heat transfer problem (P), we consider a sequence of approximate solutions and prove its convergence to a triplet $(\overline{v},\pi,\theta)$, satisfying (2.13)–(2.15).

More precisely, let $\delta_n = \frac{1}{n}$ for all $n \geq 1$ and $(\overline{v}_n, \pi_n, \theta_n)$ be a solution of the approximate problem (P_{δ_n}). For all $n \geq 1$, we have $\theta_n \in W^{1,2}((0,T) \times \Omega) \cap L^\infty(0,T; W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega)) \cap C^0([0,T]; L^2(\Omega))$, $\overline{v}_n \in C([0,T]; \mathbf{L}^2(\Omega)) \cap L^p(0,T; V^p_{0.\mathrm{div}})$ with $\frac{\partial \overline{v}_n}{\partial t} \in L^{p'}(0,T; (V^p_{0.\mathrm{div}})')$ and $\pi_n \in H^{-1}(0,T; L^{p'}_0(\Omega))$ such that

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}(\overline{v}_n, \tilde{\vartheta})_{\mathbf{L}^2(\Omega)}, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\theta_n, \overline{v}_n + v^0 \xi, D(\overline{v}_n + v^0 \xi)\big) : D(\tilde{\vartheta}) \zeta \, dx \, dt \\
&\quad - \left\langle \int_\Omega \pi_n \operatorname{div}(\tilde{\vartheta}) \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} + J(\overline{v}_n + \tilde{\vartheta} \zeta) - J(\overline{v}_n) \\
&\geq \int_0^T \left(f + v^0 \frac{\partial \xi}{\partial t}, \tilde{\vartheta} \right)_{\mathbf{L}^2(\Omega)} \zeta \, dt \quad \forall \tilde{\vartheta} \in V^p_0, \forall \zeta \in \mathcal{D}(0,T)
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\int_0^T \int_\Omega c \frac{\partial \theta_n}{\partial t} w \widetilde{\zeta} \, dx \, dt + \int_0^T \int_\Omega (K \nabla \theta_n) \cdot \nabla w \widetilde{\zeta} \, dx \, dt \\
&= \int_0^T \int_\Omega g_{\delta_n}(\theta_n, \overline{v}_n) w \widetilde{\zeta} \, dx \, dt + \int_0^T \int_\Omega r(\theta_n) w \widetilde{\zeta} \, dx \, dt \\
&\quad + \int_0^T \int_\omega \theta^b_{\delta_n} w \widetilde{\zeta} \, dx' \, dt \quad \forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega), \forall \widetilde{\zeta} \in \mathcal{D}(0,T)
\end{aligned} \tag{4.2}$$

with the initial conditions

$$\overline{v}_n(0, \cdot) = 0 \quad \text{in } \Omega \tag{4.3}$$

and

$$\theta_n(0, \cdot) = \theta^0_{\delta_n} \quad \text{in } \Omega. \tag{4.4}$$

Owing to that $\theta_n \in W^{1,2}((0,T) \times \Omega) \cap L^\infty(0,T; W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega)) \cap C^0([0,T]; L^2(\Omega))$ the variational equality (4.2) remains true for all $\widetilde{\zeta} \in L^2(0,T; \mathbb{R})$. Let us choose $\widetilde{\zeta} \in C^\infty([0,T])$ such that $\widetilde{\zeta}(T) = 0$: with integration by part with respect to the time-variable, we obtain

$$\begin{aligned}
&- \int_0^T \int_\Omega c \theta_n w \widetilde{\zeta}' \, dx \, dt + \int_0^T \int_\Omega (K \nabla \theta_n) \cdot \nabla w \widetilde{\zeta} \, dx \, dt \\
&= \int_0^T \int_\Omega g_{\delta_n}(\theta_n, \overline{v}_n) w \widetilde{\zeta} \, dx \, dt + \int_0^T \int_\Omega r(\theta_n) w \widetilde{\zeta} \, dx \, dt \\
&\quad + \int_0^T \int_\omega \theta^b_{\delta_n} w \widetilde{\zeta} \, dx' \, dt + \int_\Omega c \theta^0_{\delta_n} w \widetilde{\zeta}(0) \, dx \\
&\forall w \in W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega), \quad \forall \widetilde{\zeta} \in C^\infty\big([0,T]\big) \text{ s.t. } \widetilde{\zeta}(T) = 0.
\end{aligned} \tag{4.5}$$

Let us establish first some a priori estimates for $(\overline{v}_n, \pi_n)_{n \geq 1}$.

Proposition 4.1 (A priori estimates of $(\overline{\upsilon}_n, \pi_n)_{n\geq 1}$) *Let μ satisfy (2.8)–(2.10), $f \in L^{p'}(0,T;\mathbf{L}^2(\Omega))$, $k \in L^{p'}(0,T;L^{p'}_+(\Gamma_0))$, $s \in L^p(0,T;\mathbf{L}^p(\Gamma_0))$, $\xi \in W^{1,p'}(0,T)$ satisfying (1.6), and $\upsilon^0 \in \mathbf{W}^{1,p}(\Omega)$ satisfying (2.12). Then there exists a constant $\widetilde{C}^{\text{flow}}$, independent of n, such that for all $n \geq 1$:*

$$\|\overline{\upsilon}_n\|_{L^p(0,T;V^p_{0.\operatorname{div}})} \leq \widetilde{C}^{\text{flow}}, \tag{4.6}$$

$$\|\overline{\upsilon}_n\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq \widetilde{C}^{\text{flow}}, \tag{4.7}$$

$$\left\|\frac{\partial \overline{\upsilon}_n}{\partial t}\right\|_{L^{p'}(0,T;(V^p_{0.\operatorname{div}})')} \leq \widetilde{C}^{\text{flow}} \tag{4.8}$$

and

$$\|\pi_n\|_{H^{-1}(0,T;L^{p'}_0(\Omega))} \leq \widetilde{C}^{\text{flow}}. \tag{4.9}$$

Proof By using the same computations as in Proposition 3.2, we obtain immediately (4.6)–(4.8).

Now let us prove (4.9). We choose $\overline{\varphi} = \pm\tilde{\vartheta}\zeta$ with $\tilde{\vartheta} \in \mathbf{W}^{1,p}_0(\Omega)$ and $\zeta \in \mathcal{D}(0,T)$ in (4.1). We obtain

$$\begin{aligned}
&\left\langle \int_\Omega \pi_n \operatorname{div}(\tilde{\vartheta})\,dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad = -\int_0^T (\overline{\upsilon}_n, \tilde{\vartheta})_{\mathbf{L}^2(\Omega)} \zeta'\,dt \\
&\qquad + \int_0^T \int_\Omega \mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\overline{\upsilon}_n + \upsilon^0\xi)\big) : D(\tilde{\vartheta})\zeta\,dx\,dt \\
&\qquad - \int_0^T \left(f + \upsilon^0\frac{\partial\xi}{\partial t}, \tilde{\vartheta}\right)_{\mathbf{L}^2(\Omega)} \zeta\,dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\left\langle \int_\Omega \pi_n \operatorname{div}(\tilde{\vartheta})\,dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\right| \\
&\quad \leq \sqrt{T}\|\overline{\upsilon}_n\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \|\tilde{\vartheta}\zeta'\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \\
&\qquad + 2\mu_1\big(\|\overline{\upsilon}_n\|_{L^p(0,T;V^p_{0.\operatorname{div}})} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\big)^{p-1} \|\tilde{\vartheta}\zeta\|_{L^p(0,T;V^p_0)} \\
&\qquad + \left\|f + \upsilon^0\frac{\partial\xi}{\partial t}\right\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))} \|\tilde{\vartheta}\zeta\|_{L^p(0,T;\mathbf{L}^2(\Omega))}
\end{aligned}$$

and with (4.6)–(4.8), we get

$$\begin{aligned}
&\left|\left\langle \int_\Omega \pi_n \operatorname{div}(\tilde{\vartheta})\,dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\right| \\
&\quad \leq \Big(\sqrt{T}\widetilde{C}\widetilde{C}^{\text{flow}} + 2\mu_1 C_\infty T^{1/p}\big(\widetilde{C}^{\text{flow}} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\big)^{p-1} \\
&\qquad + \widetilde{C}C_\infty T^{1/p}\left\|f + \upsilon^0\frac{\partial\xi}{\partial t}\right\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))}\Big) \|\tilde{\vartheta}\zeta\|_{H^1_0(0,T;V^p_0)},
\end{aligned}$$

where $\widetilde{C}$ denotes the norm of the continuous injection of V_0^p into $\mathbf{L}^2(\Omega)$, and C_∞ is the norm of the continuous injection of $H^1(0,T;\mathbb{R})$ into $L^\infty(0,T;\mathbb{R})$.

Moreover, for any $p>1$, there exists a linear and continuous operator $P_p : L_0^p(\Omega) \to \mathbf{W}_0^{1,p}(\Omega)$ such that

$$\operatorname{div}\bigl(P_p(\varpi)\bigr) = \varpi \quad \forall \varpi \in L_0^p(\Omega)$$

(see Corollary 3.1 in [2]). It follows that for any $\varpi \in L_0^p(\Omega)$ and $\zeta \in \mathcal{D}(0,T)$, we have

$$\begin{aligned}
&\left| \left\langle \int_\Omega \pi_n \varpi \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \right| \\
&\quad \le \|P_p\|_{\mathcal{L}(L_0^p(\Omega),\mathbf{W}_0^{1,p}(\Omega))} \Bigl(\sqrt{T}\widetilde{C}\widetilde{C}^{\mathrm{flow}} + 2\mu_1 C_\infty T^{1/p} \bigl(\widetilde{C}^{\mathrm{flow}} + \bigl\| \upsilon^0 \xi \bigr\|_{L^p(0,T;V_{\Gamma_1}^p)} \bigr)^{p-1} \\
&\qquad + \widetilde{C} C_\infty T^{1/p} \left\| f + \upsilon^0 \frac{\partial \xi}{\partial t} \right\|_{L^{p'}(0,T;\mathbf{L}^2(\Omega))} \Bigr) \|\varpi \zeta\|_{H_0^1(0,T;L^p(\Omega))}.
\end{aligned}$$

Hence, there exists a real number $C' > 0$, independent of n, such that for all $n \ge 1$, we have

$$\left| \left\langle \int_\Omega \pi_n \varpi \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \right| \le C' \|\varpi \zeta\|_{H_0^1(0,T;L^p(\Omega))} \quad \forall \varpi \in L_0^p(\Omega), \forall \zeta \in \mathcal{D}(0,T).$$

Furthermore, for any $\varpi^* \in L^p(\Omega)$, we may define $\varpi \in L_0^p(\Omega)$ by

$$\varpi = \varpi^* - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \varpi^* \, dx.$$

We have $\|\varpi\|_{L^p(\Omega)} \le 2\|\varpi^*\|_{L^p(\Omega)}$, and since $\pi_n \in H^{-1}(0,T;L_0^{p'}(\Omega))$, we have

$$\begin{aligned}
&\left\langle \int_\Omega \pi_n \left(\varpi^* - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \varpi^* \, dx \right) dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad = \left\langle \int_\Omega \pi_n \varpi^* \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\qquad - \frac{1}{\operatorname{meas}(\Omega)} \left(\int_\Omega \varpi^* \, dx \right) \left\langle \int_\Omega \pi_n \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad = \left\langle \int_\Omega \pi_n \varpi^* \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \left\langle \int_\Omega \pi_n \varpi^* \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \right| \\
&\quad = \left| \left\langle \int_\Omega \pi_n \varpi \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \right| \\
&\quad \le 2C' \bigl\| \varpi^* \zeta \bigr\|_{H_0^1(0,T;L^p(\Omega))} \quad \forall \varpi^* \in L^p(\Omega), \forall \zeta \in \mathcal{D}(0,T).
\end{aligned}$$

Hence, by possibly modifying $\widetilde{C}^{\text{flow}}$, we have

$$\left|\left\langle \int_\Omega \pi_n \varpi^* \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}\right| \le \widetilde{C}^{\text{flow}} \|\varpi^* \zeta\|_{H_0^1(0,T;L^p(\Omega))}$$
$$\forall \varpi^* \in L^p(\Omega), \forall \zeta \in \mathcal{D}(0,T),$$

and we may conclude by using the density of $\mathcal{D}(0,T) \otimes L^p(\Omega)$ into $H_0^1(0,T;L^p(\Omega))$. □

Let us now obtain some a priori estimates for $(\theta_n)_{n\ge 1}$.

Proposition 4.2 (Further a priori estimates for θ_n) *Let $q \in (1, \frac{5}{4})$. Under the previous assumptions, there exists a constant $C_q^{\text{heat}} > 0$, independent of n, such that*

$$\|\theta_n\|_{L^q(0,T;W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega))} \le C_q^{\text{heat}} \tag{4.10}$$

for all $n \ge 1$.

Proof Let $\widetilde{w} = w\widetilde{\zeta}$ with $w \in W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega)$ and $\zeta \in \mathcal{D}(0,T)$. With (4.2), we have

$$\begin{aligned} &\int_0^T \int_\Omega c\frac{\partial \theta_n}{\partial t}\widetilde{w}\, dx\, dt + \int_0^T \int_\Omega (K\nabla\theta_n)\cdot\nabla\widetilde{w}\, dx\, dt \\ &\quad = \int_0^T \int_\Omega g_{\delta_n}(\theta_n, \overline{v}_n)\widetilde{w}\, dx\, dt + \int_0^T \int_\Omega r(\theta_n)\widetilde{w}\, dx\, dt + \int_0^T \int_\omega \theta^b_{\delta_n}\widetilde{w}\, dx'\, dt \end{aligned} \tag{4.11}$$

and by density of $\mathcal{D}(0,T) \otimes W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega)$ into $L^2(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega))$ the same equality holds for all $\widetilde{w} \in L^2(0,T;W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega))$.

In order to obtain (4.10), we perform the same kind of computations as in [6]. More precisely, for all $m \ge 1$ let $\Psi_0^m : \mathbb{R} \to \mathbb{R}$ and $\Psi_0 : \mathbb{R} \to \mathbb{R}$ be given by

$$\Psi_0^m(s) = \begin{cases} 0 & \text{if } s \ge 1 + \frac{1}{m}, \\ 1 - m(s-1) & \text{if } 1 < s < 1 + \frac{1}{m}, \\ 1 & \text{if } -1 \le s \le 1, \\ 1 + m(s+1) & \text{if } -1 - \frac{1}{m} < s < -1, \\ 0 & \text{if } s \le -1 - \frac{1}{m}, \end{cases} \qquad \Psi_0(s) = \begin{cases} 0 & \text{if } s > 1, \\ 1 & \text{if } -1 \le s \le 1, \\ 0 & \text{if } s < -1 \end{cases}$$

and let

$$\psi_0^m(s) = \int_0^s \Psi_0^m(z)\, dz, \qquad \psi_0(s) = \int_0^s \Psi_0(z)\, dz \quad \text{for all } s \in \mathbb{R},$$
$$\Phi_0^m(s) = \int_0^s \psi_0^m(z)\, dz, \qquad \Phi_0(s) = \int_0^s \psi_0(z)\, dz \quad \text{for all } s \in \mathbb{R}.$$

We choose $\widetilde{w} = \psi_0^m(\theta_n)\mathbf{1}_{[0,t]}$, where $\mathbf{1}_{[0,t]}$ is the indicatrix function of the time-interval $[0,t]$ with $t \in (0,T]$, in (4.11). We obtain

$$\begin{aligned}
&\int_\Omega c(x)\Phi_0^m\big(\theta_n(t,x)\big)\,dx \\
&\qquad + \int_0^t \int_\Omega \Psi_0^m\big(\theta_n(s,x)\big)\big(K(x)\nabla\theta_n(s,x)\big)\cdot\nabla\theta_n(s,x)\,dx\,ds \\
&= \int_0^t \int_\Omega g_{\delta_n}(\theta_n,\overline{v}_n)\psi_0^m(\theta_n)\,dx\,ds + \int_0^t \int_\Omega r(\theta_n)\psi_0^m(\theta_n)\,dx\,ds \\
&\qquad + \int_0^t \int_\omega \theta_{\delta_n}^b \psi_0^m(\theta_n)\,dx\,ds + \int_\Omega c(x)\Phi_0^m\big(\theta_{\delta_n}^0\big)\,dx.
\end{aligned}$$

By observing that $|\Psi_0^m(s)| \le 1$, $|\psi_0^m(s)| \le 2$ and $|\Phi_0^m(s)| \le 2|s|$ for all $s \in \mathbb{R}$ and for all $m \ge 1$, we may pass to the limit as m tends to $+\infty$ by using Lebesgue's theorem, and we get

$$\begin{aligned}
&\int_\Omega c(x)\Phi_0\big(\theta_n(t,x)\big)\,dx \\
&\qquad + \int_0^t \int_\Omega \Psi_0\big(\theta_n(s,x)\big)\big(K(x)\nabla\theta_n(s,x)\big)\cdot\nabla\theta_n(s,x)\,dx\,ds \\
&= \int_0^t \int_\Omega g_{\delta_n}(\theta_n,\overline{v}_n)\psi_0(\theta_n)\,dx\,ds + \int_0^t \int_\Omega r(\theta_n)\psi_0(\theta_n)\,dx\,ds \\
&\qquad + \int_0^t \int_\omega \theta_{\delta_n}^b \psi_0(\theta_n)\,dx'\,ds + \int_\Omega c(x)\Phi_0\big(\theta_{\delta_n}^0\big)\,dx.
\end{aligned}$$

Furthermore, $|s| - \frac{1}{2} \le \Phi_0(s) \le |s|$ and $|\psi_0(s)| \le 1$ for all $s \in \mathbb{R}$. Thus,

$$\begin{aligned}
&c_0 \int_\Omega \big|\theta_n(t,x)\big|\,dx + k_0 \int_0^t \int_\Omega \underbrace{\Psi_0\big(\theta_n(s,x)\big)\big\|\nabla\theta_n(s,x)\big\|^2}_{\ge 0}\,dx\,ds \\
&\le \big\|g_{\delta_n}(\theta_n,\overline{v}_n)\big\|_{L^1(0,T;L^1(\Omega))} + \big\|r(\theta_n)\big\|_{L^1(0,T;L^1(\Omega))} + \big\|\theta_{\delta_n}^b\big\|_{L^1((0,T)\times\omega)} \\
&\qquad + \frac{1}{2}\|c\|_{L^1(\Omega)} + \big\|c\theta_{\delta_n}^0\big\|_{L^1(\Omega)}
\end{aligned} \tag{4.12}$$

for all $t \in [0,T]$. Next, for all $k \ge 1$ and for all $m \ge 1$, we define $\Psi_k^m : \mathbb{R} \to \mathbb{R}$ and $\Psi_k : \mathbb{R} \to \mathbb{R}$ by

$$\Psi_k^m(s) = \begin{cases}
0 & \text{if } s \ge k+1+\frac{1}{m}, \\
1 - m(s-(k+1)) & \text{if } k+1 < s < k+1+\frac{1}{m}, \\
1 & \text{if } k+\frac{1}{m} \le s \le k+1, \\
m(s-k) & \text{if } k < s < k+\frac{1}{m}, \\
0 & \text{if } -k \le s \le k, \\
-m(s+k) & \text{if } -k-\frac{1}{m} < s < -k, \\
1 & \text{if } -k-1 \le s \le -k-\frac{1}{m}, \\
1 + m(s-(-k-1)) & \text{if } -k-1-\frac{1}{m} < s < -k-1, \\
0 & \text{if } s \le -k-1-\frac{1}{m},
\end{cases}$$

$$\Psi_k(s) = \begin{cases} 0 & \text{if } s > k+1, \\ 1 & \text{if } k < s \le k+1, \\ 0 & \text{if } -k \le s \le k, \\ 1 & \text{if } -k-1 \le s < -k, \\ 0 & \text{if } s < -k-1 \end{cases}$$

and let

$$\psi_k^m(s) = \int_0^s \Psi_k^m(z)\,dz, \qquad \psi_k(s) = \int_0^s \Psi_k(z)\,dz \quad \text{for all } s \in \mathbb{R},$$
$$\Phi_k^m(s) = \int_0^s \psi_k^m(z)\,dz, \qquad \Phi_k(s) = \int_0^s \psi_k(z)\,dz \quad \text{for all } s \in \mathbb{R}.$$

Now we choose $\widetilde{w} = \psi_k^m(\theta_n)$ in (4.11) with $k \ge 0$ and $m \ge 1$. We get

$$\begin{aligned} &\int_\Omega c(x)\Phi_k^m\big(\theta_n(t,x)\big)\,dx \\ &\qquad + \int_0^t \int_\Omega \Psi_k^m\big(\theta_n(s,x)\big)\big(K(x)\nabla\theta_n(s,x)\big)\cdot\nabla\theta_n(s,x)\,dx\,ds \\ &= \int_0^t \int_\Omega g_{\delta_n}(\theta_n,\overline{v}_n)\psi_k^m(\theta_n)\,dx\,ds + \int_0^t \int_\Omega r(\theta_n)\psi_k^m(\theta_n)\,dx\,ds \\ &\qquad + \int_0^t \int_\omega \theta_{\delta_n}^b \psi_k^m(\theta_n)\,dx'\,ds + \int_\Omega c(x)\Phi_k^m\big(\theta_{\delta_n}^0\big)\,dx. \end{aligned}$$

We may observe that $|\Psi_k^m(s)| \le 1$, $|\psi_k^m(s)| \le 2$, and $|\Phi_k^m(s)| \le 2|s|$ for all $s \in \mathbb{R}$, for all $k \ge 0$ and for all $m \ge 1$. By passing to the limit as m tends to $+\infty$, we obtain

$$\begin{aligned} &\int_\Omega c(x)\Phi_k\big(\theta_n(t,x)\big)\,dx \\ &\qquad + \int_0^t \int_\Omega \Psi_k\big(\theta_n(s,x)\big)\big(K(x)\nabla\theta_n(s,x)\big)\cdot\nabla\theta_n(s,x)\,dx\,ds \\ &= \int_0^t \int_\Omega g_{\delta_n}(\theta_n,\overline{v}_n)\psi_k(\theta_n)\,dx\,ds + \int_0^t \int_\Omega r(\theta_n)\psi_k(\theta_n)\,dx\,ds \\ &\qquad + \int_0^t \int_\omega \theta_{\delta_n}^b \psi_k(\theta_n)\,dx'\,ds + \int_\Omega c(x)\Phi_k\big(\theta_{\delta_n}^0\big)\,dx. \end{aligned}$$

Obviously $0 \le \Phi_k(s) \le |s|$ and $|\psi_k(s)| \le 1$ for all $s \in \mathbb{R}$ and for all $k \ge 0$. Hence,

$$\begin{aligned} &k_0 \int_0^T \int_\Omega \Psi_k(\theta_n)\|\nabla\theta_n\|^2\,dx\,dt \\ &\quad \le \big\|g_{\delta_n}(\theta_n,\overline{v}_n)\big\|_{L^1(0,T;L^1(\Omega))} \\ &\qquad + \big\|r(\theta_n)\big\|_{L^1(0,T;L^1(\Omega))} + \big\|\theta_{\delta_n}^b\big\|_{L^1((0,T)\times\omega)} + \big\|c\theta_{\delta_n}^0\big\|_{L^1(\Omega)}. \end{aligned} \tag{4.13}$$

Let us define C_{δ_n} as

$$C_{\delta_n} = \left\| g_{\delta_n}(\theta_n, \overline{\upsilon}_n) \right\|_{L^1(0,T;L^1(\Omega))} + \left\| r(\theta_n) \right\|_{L^1(0,T;L^1(\Omega))} + \left\| \theta^b_{\delta_n} \right\|_{L^1((0,T)\times\omega)} + \frac{1}{2}\|c\|_{L^1(\Omega)} + \left\| c\theta^0_{\delta_n} \right\|_{L^1(\Omega)}$$

and

$$Q_0 = \left\{(t,x) \in (0,T) \times \Omega; \left|\theta_n(t,x)\right| \le 1\right\},$$
$$Q_k = \left\{(t,x) \in (0,T) \times \Omega; k < \left|\theta_n(t,x)\right| \le k+1\right\} \quad \text{for all } k \in \mathbb{N}^*.$$

With (4.13), we have

$$\int_{Q_k} \|\nabla\theta_n\|^2 \, dx\, dt \le \frac{C_{\delta_n}}{k_0} \quad \forall k \in \mathbb{N}.$$

Since $q \in (1, \frac{5}{4})$, we may use Hölder's inequality, which yields

$$\int_{Q_k} \|\nabla\theta_n\|^q \, dx\, dt \le \left(\int_{Q_k} \|\nabla\theta_n\|^2 \, dx\, dt\right)^{q/2} \left(\operatorname{meas}(Q_k)\right)^{1-\frac{q}{2}} \le \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \left(\operatorname{meas}(Q_k)\right)^{\frac{2-q}{2}} \quad \forall k \in \mathbb{N}.$$

Then

$$\int_0^T \int_\Omega \|\nabla\theta_n\|^q \, dx\, dt = \int_{\bigcup_{k\ge 0} Q_k} \|\nabla\theta_n\|^q \, dx\, dt \le \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \left(T \operatorname{meas}(\Omega)\right)^{\frac{2-q}{2}} + \sum_{k\ge 1} \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \frac{1}{k^{r\frac{2-q}{2}}} \left(\int_{Q_k} |\theta_n|^r \, dx\, dt\right)^{\frac{2-q}{2}}$$

and using again Hölder's inequality

$$\int_0^T \int_\Omega \|\nabla\theta_n\|^q \, dx\, dt \le \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \left(T \operatorname{meas}(\Omega)\right)^{\frac{2-q}{2}} + \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \left(\sum_{k\ge 1} \frac{1}{k^{r\frac{2-q}{q}}}\right)^{q/2} \left(\int_0^T \int_\Omega |\theta_n|^r \, dx\, dt\right)^{\frac{2-q}{2}} \tag{4.14}$$

for all $r \ge 1$ such that $r\frac{2-q}{q} > 1$. Next, we choose $r = \frac{4}{3}q$. It follows that $\frac{q^*(1-r)}{1-q^*} = q$, where $q^* = \frac{3q}{3-q}$ and

$$\int_0^T \int_\Omega |\theta_n|^r \, dx\, dt \le \int_0^T \left(\int_\Omega |\theta_n| \, dx\right)^{\frac{r}{4}} \left(\int_\Omega |\theta_n|^{q^*} \, dx\right)^{\frac{3r}{4q^*}} dt.$$

Then (4.12) implies that

$$\begin{aligned}\|\theta_n\|^r_{L^r(0,T;L^r(\Omega))} &= \int_0^T \int_\Omega |\theta_n|^r \,dx\,dt \\ &\le \left(\frac{C_{\delta_n}}{c_0}\right)^{\frac{r}{4}} \int_0^T \left(\int_\Omega |\theta_n|^{q^*}\,dx\right)^{\frac{q}{q^*}} dt = \left(\frac{C_{\delta_n}}{c_0}\right)^{\frac{r}{4}} \|\theta_n\|^q_{L^q(0,T;L^{q^*}(\Omega))}.\end{aligned}$$

Going back to (4.14),

$$\begin{aligned}\int_0^T \|\theta_n\|^q_{W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega)}\,dt &= \int_0^T \int_\Omega \|\nabla\theta_n\|^q \,dx\,dt \\ &\le \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \big(T\,\mathrm{meas}(\Omega)\big)^{\frac{2-q}{2}} \\ &\quad + \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \left(\sum_{k\ge1} \frac{1}{k^{r\frac{2-q}{q}}}\right)^{q/2} \left(\frac{C_{\delta_n}}{c_0}\right)^{\frac{r}{4}\frac{2-q}{2}} \|\theta_n\|^{q\frac{2-q}{2}}_{L^q(0,T;L^{q^*}(\Omega))}.\end{aligned} \tag{4.15}$$

Owing to that $W^{1,q}(\Omega)$ is continuously embedded into $L^{q^*}(\Omega)$, we obtain

$$\begin{aligned}\|\theta_n\|^q_{L^q(0,T;L^{q^*}(\Omega))} &= \int_0^T \|\theta_n\|^q_{L^{q^*}(\Omega)}\,dt \\ &\le (C'_q)^q \left(\left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \big(T\,\mathrm{meas}(\Omega)\big)^{\frac{2-q}{2}}\right. \\ &\quad \left. + \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \left(\sum_{k\ge1} \frac{1}{k^{r\frac{2-q}{q}}}\right)^{q/2} \left(\frac{C_{\delta_n}}{c_0}\right)^{\frac{r}{4}\frac{2-q}{2}} \|\theta_n\|^{q\frac{2-q}{2}}_{L^q(0,T;L^{q^*}(\Omega))}\right),\end{aligned}$$

where C'_q is the norm of the identity mapping from $W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega)$ into $L^{q^*}(\Omega)$. By observing that $\frac{2-q}{2} < 1$ and recalling that $r\frac{2-q}{q} = \frac{4}{3}(2-q) > 1$, we obtain

$$\begin{aligned}\|\theta_n\|^q_{L^q(0,T;L^{q^*}(\Omega))} &\le \max\left(1, (C'_q)^2 \left(\left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \big(T\,\mathrm{meas}(\Omega)\big)^{\frac{2-q}{2}}\right.\right. \\ &\quad \left.\left. + \left(\frac{C_{\delta_n}}{k_0}\right)^{q/2} \left(\sum_{k\ge1} \frac{1}{k^{\frac{4}{3}(2-q)}}\right)^{q/2} \left(\frac{C_{\delta_n}}{c_0}\right)^{\frac{q}{3}\frac{2-q}{2}}\right)^{2/q}\right).\end{aligned} \tag{4.16}$$

Finally, recalling the definition of g_{δ_n} and using (2.4) and (2.7), we obtain

$$\begin{aligned}C_{\delta_n} &= \left\|\frac{2\mu(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, |D(\overline{\upsilon}_n + \upsilon^0\xi)|)|D(\overline{\upsilon}_n + \upsilon^0\xi)|^p}{1 + 2\delta_n\mu(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, |D(\overline{\upsilon}_n + \upsilon^0\xi)|)|D(\overline{\upsilon}_n + \upsilon^0\xi)|^p}\right\|_{L^1(0,T;L^1(\Omega))} \\ &\quad + \big\|r(\theta_n)\big\|_{L^1(0,T;L^1(\Omega))} + \big\|\theta^b_{\delta_n}\big\|_{L^1((0,T)\times\omega)} + \frac{1}{2}\|c\|_{L^1(\Omega)} + \big\|c\theta^0_{\delta_n}\big\|_{L^1(\Omega)} \\ &\le 2\mu_1 \big\|D(\overline{\upsilon}_n + \upsilon^0\xi)\big\|^p_{L^p(0,T;(L^p(\Omega))^{3\times3})} + \left(r_1 + \frac{c_1}{2}\right)\mathrm{meas}(\Omega)T \\ &\quad + \big\|\theta^b\big\|_{L^1((0,T)\times\omega)} + c_1\big\|\theta^0\big\|_{L^1(\Omega)} + \delta_n(1+c_1) \\ &\le 2\mu_1\big(\|\overline{\upsilon}_n\|_{L^p(0,T;V^p_{0.\mathrm{div}})} + \big\|\upsilon^0\xi\big\|_{L^p(0,T;V^p_{\Gamma_1})}\big)^p + \left(r_1 + \frac{c_1}{2}\right)\mathrm{meas}(\Omega)T\end{aligned}$$

$$+ \|\theta^b\|_{L^1((0,T)\times\omega)} + c_1\|\theta^0\|_{L^1(\Omega)} + \frac{1}{n}(1+c_1)$$

and with (4.6)

$$C_{\delta_n} \le 2\mu_1\big(\widetilde{C}^{\text{flow}} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\big)^p + \left(r_1 + \frac{c_1}{2}\right)\text{meas}(\Omega)T$$
$$+ \|\theta^b\|_{L^1((0,T)\times\omega)} + c_1\|\theta^0\|_{L^1(\Omega)} + (1+c_1)$$

for all $n \ge 1$. Then (4.15)–(4.16) allows us to conclude. □

As a corollary, we also obtain an estimate for $\frac{\partial\theta_n}{\partial t}$.

Proposition 4.3 (A priori estimate for $\frac{\partial\theta_n}{\partial t}$) *Let $q \in (1, \frac{5}{4})$. Under the previous assumptions, we have the following estimate*:

$$\left\|\frac{\partial\theta_n}{\partial t}\right\|_{L^1(0,T;(W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega))')}$$
$$\le \frac{1}{c_0}C_{q'}\big(2\mu_1\big(\widetilde{C}^{\text{flow}} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\big)^p + r_1\,\text{meas}(\Omega)T + \|\theta^b\|_{L^1((0,T)\times\omega)} + 1\big)$$
$$+ \frac{T^{1/q'}}{c_0^2}\|K\|_{(L^\infty(\Omega))^{3\times3}}C_q^{\text{heat}}\big(c_0 + C_{q'}\|\nabla c\|_{\mathbf{L}^{q'}(\Omega)}\big),$$

where $C_{q'}$ is the norm of the continuous injection of $W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega)$ into $C^0(\overline{\Omega})$.

Proof Since $q \in (1, \frac{5}{4})$, we have $q' = \frac{q}{q-1} > 5$, and $W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega)$ is continuously embedded into $W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega)$ and into $C^0(\overline{\Omega})$. Moreover, owing to that $c \in W^{1,q'}(\Omega)$, we also have $\frac{w}{c} \in W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega) \subset W^{1,2}_{\Gamma_1\cup\Gamma_L}(\Omega)$ for all $w \in W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega)$. Let $\psi = w\widetilde{\zeta}$ with $w \in W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega)$ and $\widetilde{\zeta} \in \mathcal{D}(0,T)$. With $\widetilde{w} = \frac{\psi}{c} = \frac{w}{c}\zeta$ in (4.11), we get

$$\left|\int_0^T\int_\Omega \frac{\partial\theta_n}{\partial t}\psi\,dx\,dt\right|$$
$$\le \left|\int_0^T\int_\Omega \frac{1}{c}(K\nabla\theta_n)\cdot\nabla\psi\,dx\,dt\right|$$
$$+ \left|\int_0^T\int_\Omega \frac{\psi}{c^2}(K\nabla\theta_n)\cdot\nabla c\,dx\,dt\right| + \left|\int_0^T\int_\Omega \frac{1}{c}\big(g_{\delta_n}(\theta_n,\overline{\upsilon}_n) + r(\theta_n)\big)\psi\,dx\,dt\right|$$
$$+ \left|\int_0^T\int_\omega \frac{1}{c}\theta^b_{\delta_n}\psi\,dx'\,dt\right|.$$

Recalling that $\theta_n \in W^{1,2}((0,T)\times\Omega)$, the same equality remains true for all $\psi \in L^2(0,T; W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega))$. Let us now consider $\psi \in L^\infty(0,T; W^{1,q'}_{\Gamma_1\cup\Gamma_L}(\Omega))$. We obtain

$$\left|\int_0^T\int_\Omega \frac{\partial\theta_n}{\partial t}\psi\,dx\,dt\right|$$
$$\le \frac{T^{1/q'}}{c_0}\|K\|_{(L^\infty(\Omega))^{3\times3}}\|\nabla\theta_n\|_{L^q(0,T;\mathbf{L}^q(\Omega))}\|\nabla\psi\|_{L^\infty(0,T;\mathbf{L}^{q'}(\Omega))}$$

$$
\begin{aligned}
&+ \frac{T^{1/q'}}{c_0^2} \|K\|_{(L^\infty(\Omega))^{3\times 3}} \|\nabla \theta_n\|_{L^q(0,T;\mathbf{L}^q(\Omega))} \|\psi\|_{L^\infty(0,T;C^0(\overline{\Omega}))} \|\nabla c\|_{\mathbf{L}^{q'}(\Omega)} \\
&+ \frac{1}{c_0} \left(\left\| g_{\delta_n}(\theta_n, \overline{\upsilon}_n) + r(\theta_n) \right\|_{L^1(0,T;L^1(\Omega))} + \left\| \theta^b_{\delta_n} \right\|_{L^1((0,T)\times\omega)} \right) \|\psi\|_{L^\infty(0,T;C^0(\overline{\Omega}))} \\
\le{}& \frac{T^{1/q'}}{c_0} \|K\|_{(L^\infty(\Omega))^{3\times 3}} \|\nabla \theta_n\|_{L^q(0,T;\mathbf{L}^q(\Omega))} \|\nabla \psi\|_{L^\infty(0,T;\mathbf{L}^{q'}(\Omega))} \\
&+ \frac{T^{1/q'}}{c_0^2} \|K\|_{(L^\infty(\Omega))^{3\times 3}} \|\nabla \theta_n\|_{L^q(0,T;\mathbf{L}^q(\Omega))} \|\psi\|_{L^\infty(0,T;C^0(\overline{\Omega}))} \|\nabla c\|_{\mathbf{L}^{q'}(\Omega)} \\
&+ \frac{1}{c_0} \left(2\mu_1 \left(\widetilde{C}^{\text{flow}} + \left\| \upsilon^0 \xi \right\|_{L^p(0,T;V^p_{\Gamma_1})} \right)^p + r_1 \operatorname{meas}(\Omega) T + \left\| \theta^b \right\|_{L^1((0,T)\times\omega)} + 1 \right) \\
&\times \|\psi\|_{L^\infty(0,T;C^0(\overline{\Omega}))}
\end{aligned}
$$

which allows us to conclude. □

It follows that, by possibly extracting a subsequence still denoted $(\overline{\upsilon}_n, \pi_n, \theta_n)_{n\ge 1}$, there exists a triplet $(\overline{\upsilon}, \pi, \theta)$ such that

$$
\begin{aligned}
\overline{\upsilon}_n \longrightarrow \overline{\upsilon} \quad & \text{weakly in } L^p(0,T;V^p_{0.\operatorname{div}}) \text{ and} \\
& \text{weakly* in } L^\infty(0,T;\mathbf{L}^2(\Omega)),
\end{aligned} \tag{4.17}
$$

$$
\frac{\partial \overline{\upsilon}_n}{\partial t} \longrightarrow \frac{\partial \overline{\upsilon}}{\partial t} \quad \text{weakly in } L^{p'}(0,T;(V^p_{0.\operatorname{div}})'), \tag{4.18}
$$

$$
\pi_n \longrightarrow \pi \quad \text{weakly* in } H^{-1}(0,T;L^{p'}_0(\Omega)), \tag{4.19}
$$

and

$$
\theta_n \longrightarrow \theta \quad \text{weakly in } L^q(0,T;W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega)). \tag{4.20}
$$

Moreover, with the Aubin and Simon lemmas [29, 34], by possibly extracting another subsequence still denoted $(\overline{\upsilon}_n, \pi_n, \theta_n)_{n\ge 1}$, we have

$$
\begin{aligned}
\overline{\upsilon}_n \longrightarrow \overline{\upsilon} \quad & \text{strongly in } C^0([0,T];H) \\
& \text{and in } L^p(0,T;\mathbf{L}^p(\Omega)),
\end{aligned} \tag{4.21}
$$

where we recall that $H = \{\psi \in \mathbf{L}^2(\Omega); \operatorname{div}(\psi) = 0 \text{ in } \Omega, \psi.n = 0 \text{ on } \partial\Omega\}$, and with Aubin's generalized lemma [29], we also have

$$
\theta_n \longrightarrow \theta \quad \text{strongly in } L^q(0,T;L^q(\Omega)). \tag{4.22}
$$

We infer that

$$
\overline{\upsilon}_n(0,\cdot) \longrightarrow \overline{\upsilon}(0,\cdot) = 0 \quad \text{strongly in } \mathbf{L}^2(\Omega). \tag{4.23}
$$

Furthermore, by possibly extracting again a subsequence still denoted $(\overline{\upsilon}_n, \pi_n, \theta_n)_{n\ge 1}$, we have

$$
\overline{\upsilon}_n \longrightarrow \overline{\upsilon}, \qquad \theta_n \longrightarrow \theta \quad \text{a.e. in } (0,T)\times\Omega. \tag{4.24}
$$

These convergence properties do not allow us to pass directly to the limit as n tends to $+\infty$, and we need a better convergence result for $(\overline{\upsilon}_n)_{n\geq 1}$ to overcome the difficulty due to the non-linearity of the mapping $\mathcal{F}$.

Proposition 4.4 *Under the previous assumptions,* $(D(\overline{\upsilon}_n))_{n\geq 1}$ *converges strongly to* $D(\overline{\upsilon})$ *in* $L^p(0,T;(L^p(\Omega))^{3\times 3})$.

Proof Let $(\hat{\upsilon},\hat{\pi})$ be the unique solution of the problem $(P^{\text{flow}}_{(\overline{\upsilon},\theta)})$. We have

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial \hat{\upsilon}}{\partial t}, \overline{\varphi} - \hat{\upsilon} \right\rangle_{(V^p_{0.\,\text{div}})', V^p_{0.\,\text{div}}} dt \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\theta, \overline{\upsilon} + \upsilon^0 \xi, D\big(\hat{\upsilon} + \upsilon^0 \xi\big)\big) : D(\overline{\varphi} - \hat{\upsilon})\, dx\, dt \\
&\quad + J(\overline{\varphi}) - J(\hat{\upsilon}) \geq \int_0^T \left(f + \upsilon^0 \frac{\partial \xi}{\partial t}, \overline{\varphi} - \hat{\upsilon} \right)_{\mathbf{L}^2(\Omega)} dt \quad \forall \overline{\varphi} \in L^p\big(0,T;V^p_{0.\,\text{div}}\big).
\end{aligned} \tag{4.25}$$

For all $n \geq 1$, we also have

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial \overline{\upsilon}_n}{\partial t}, \overline{\varphi} - \overline{\upsilon}_n \right\rangle_{(V^p_{0.\,\text{div}})', V^p_{0.\,\text{div}}} dt \\
&\quad + \int_0^T \int_\Omega \mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0 \xi, D\big(\overline{\upsilon}_n + \upsilon^0 \xi\big)\big) : D(\overline{\varphi} - \overline{\upsilon}_n)\, dx\, dt \\
&\quad + J(\overline{\varphi}) - J(\overline{\upsilon}_n) \geq \int_0^T \left(f + \upsilon^0 \frac{\partial \xi}{\partial t}, \overline{\varphi} - \overline{\upsilon}_n \right)_{\mathbf{L}^2(\Omega)} dt \quad \forall \overline{\varphi} \in L^p\big(0,T;V^p_{0.\,\text{div}}\big).
\end{aligned} \tag{4.26}$$

Let us choose $\overline{\varphi} = \hat{\upsilon}$ in (4.26) and $\overline{\varphi} = \overline{\upsilon}_n$ in (4.25). By adding the two inequalities, we obtain:

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial (\overline{\upsilon}_n - \hat{\upsilon})}{\partial t}, \overline{\upsilon}_n - \hat{\upsilon} \right\rangle_{(V^p_{0.\,\text{div}})', V^p_{0.\,\text{div}}} dt \\
&\quad + \int_0^T \int_\Omega \big(\mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0 \xi, D\big(\overline{\upsilon}_n + \upsilon^0 \xi\big)\big) \\
&\quad - \mathcal{F}\big(\theta, \overline{\upsilon} + \upsilon^0 \xi, D\big(\hat{\upsilon} + \upsilon^0 \xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\, dx\, dt \leq 0
\end{aligned}$$

and thus,

$$\begin{aligned}
&\frac{1}{2} \|\overline{\upsilon}_n - \hat{\upsilon}\|^2_{\mathbf{L}^2(\Omega)}(T) \\
&\qquad + \int_0^T \int_\Omega \big(\mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0 \xi, D\big(\overline{\upsilon}_n + \upsilon^0 \xi\big)\big) \\
&\qquad - \mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0 \xi, D\big(\hat{\upsilon} + \upsilon^0 \xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\, dx\, dt \\
&\quad \leq \int_0^T \int_\Omega \big(\mathcal{F}\big(\theta, \overline{\upsilon} + \upsilon^0 \xi, D\big(\hat{\upsilon} + \upsilon^0 \xi\big)\big) \\
&\qquad - \mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0 \xi, D\big(\hat{\upsilon} + \upsilon^0 \xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\, dx\, dt.
\end{aligned} \tag{4.27}$$

We perform the same kind of computations as in Theorem 3.5. More precisely, we split the second term of the left-hand side as follows:

$$\begin{aligned}
&\int_0^T \int_\Omega \big(\mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_n + \upsilon^0\xi\big)\big) \\
&\quad - \mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D\big(\hat{\upsilon} + \upsilon^0\xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt \\
&= \int_0^T \int_\Omega \big(\mathcal{F}_1\big(D\big(\overline{\upsilon}_n + \upsilon^0\xi\big)\big) - \mathcal{F}_1\big(D\big(\hat{\upsilon} + \upsilon^0\xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \big(\mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_n + \upsilon^0\xi\big)\big) \\
&\quad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D\big(\hat{\upsilon} + \upsilon^0\xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt,
\end{aligned}$$

where we recall that

$$\mathcal{F}_1(\lambda_2) = \mu_0 \|\lambda_2\|^{p-2}\lambda_2 \quad \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3\times 3}}, \qquad \mathcal{F}_1(\lambda_2) = 0_{\mathbb{R}^{3\times 3}} \quad \text{otherwise},$$

$$\begin{cases} \mathcal{F}_2(\lambda_0, \lambda_1, \lambda_2) = 2\overline{\mu}(\lambda_0, \lambda_1, \|\lambda_2\|)\|\lambda_2\|^{p-2}\lambda_2 & \text{if } \lambda_2 \neq 0_{\mathbb{R}^{3\times 3}}, \\ \mathcal{F}_2(\lambda_0, \lambda_1, \lambda_2) = 0_{\mathbb{R}^{3\times 3}} & \text{otherwise} \end{cases}$$

and $\overline{\mu} = \mu - \frac{\mu_0}{2}$. Since $\lambda_2 \mapsto \mathcal{F}_2(\cdot, \cdot, \lambda_2)$ is monotone in $\mathbb{R}^{3\times 3}$ (see Lemma 1 in [9]), we have

$$\begin{aligned}
&\int_0^T \int_\Omega \big(\mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D\big(\overline{\upsilon}_n + \upsilon^0\xi\big)\big) \\
&\quad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D\big(\hat{\upsilon} + \upsilon^0\xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt \geq 0,
\end{aligned}$$

and (4.27) yields

$$\begin{aligned}
&\frac{1}{2}\|\overline{\upsilon}_n - \hat{\upsilon}\|^2_{\mathbf{L}^2(\Omega)}(T) \\
&\quad + \int_0^T \int_\Omega \big(\mathcal{F}_1\big(D\big(\overline{\upsilon}_n + \upsilon^0\xi\big)\big) - \mathcal{F}_1\big(D\big(\hat{\upsilon} + \upsilon^0\xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt \\
&\leq \int_0^T \int_\Omega \big(\mathcal{F}_2\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D\big(\hat{\upsilon} + \upsilon^0\xi\big)\big) \\
&\quad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D\big(\hat{\upsilon} + \upsilon^0\xi\big)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt.
\end{aligned} \tag{4.28}$$

Then we distinguish two cases.

Case 1: $p \in [6/5, 2)$

Recalling that

$$\big(\|\lambda\|^p + \|\lambda'\|^p\big)^{\frac{2-p}{2}} \big(\big(\mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda')\big) : (\lambda - \lambda')\big)^{\frac{p}{2}} \geq \frac{(\mu_0(p-1))^{\frac{p}{2}}}{2^{\frac{(p-1)(2-p)}{2}}} \|\lambda - \lambda'\|^p$$

for all $(\lambda, \lambda') \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$, we obtain

$$\begin{aligned}
&\frac{(\mu_0(p-1))^{\frac{p}{2}}}{2^{\frac{(p-1)(2-p)}{2}}} \int_0^T \int_\Omega \|D(\overline{\upsilon}_n - \hat{\upsilon})\|^p \,dx\,dt \\
&\leq \left(\int_0^T \int_\Omega \left(\mathcal{F}_1\big(D(\overline{\upsilon}_n + \upsilon^0\xi)\big) - \mathcal{F}_1\big(D(\hat{\upsilon} + \upsilon^0\xi)\big)\right) : D(\overline{\upsilon}_n - \hat{\upsilon}) \,dx\,dt\right)^{\frac{p}{2}} \\
&\quad \times \left(\int_0^T \int_\Omega \left(\|D(\overline{\upsilon}_n + \upsilon^0\xi)\|^p + \|D(\hat{\upsilon} + \upsilon^0\xi)\|^p\right) dx\,dt\right)^{\frac{2-p}{2}} \\
&\leq \left(\int_0^T \int_\Omega \left(\mathcal{F}_1\big(D(\overline{\upsilon}_n + \upsilon^0\xi)\big) - \mathcal{F}_1\big(D(\hat{\upsilon} + \upsilon^0\xi)\big)\right) : D(\overline{\upsilon}_n - \hat{\upsilon}) \,dx\,dt\right)^{\frac{p}{2}} \\
&\quad \times \left(\|\overline{\upsilon}_n + \upsilon^0\xi\|^p_{L^p(0,T;V^p_{\Gamma_1})} + \|\hat{\upsilon} + \upsilon^0\xi\|^p_{L^p(0,T;V^p_{\Gamma_1})}\right)^{\frac{2-p}{2}}.
\end{aligned}$$

Thus with (3.11) and (4.6), we get

$$\begin{aligned}
&\frac{(\mu_0(p-1))^{\frac{p}{2}}}{2^{\frac{(p-1)(2-p)}{2}}} \|D(\overline{\upsilon}_n - \hat{\upsilon})\|^p_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\leq \left(\int_0^T \int_\Omega \left(\mathcal{F}_1\big(D(\overline{\upsilon}_n + \upsilon^0\xi)\big) - \mathcal{F}_1\big(D(\hat{\upsilon} + \upsilon^0\xi)\big)\right) : D(\overline{\upsilon}_n - \hat{\upsilon}) \,dx\,dt\right)^{\frac{p}{2}} \\
&\quad \times \left(\left(\widetilde{C}^{\text{flow}} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\right)^p + \left(C^{\text{flow}} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})}\right)^p\right)^{\frac{2-p}{2}}.
\end{aligned}$$

By observing that $\widetilde{C}^{\text{flow}} \geq C^{\text{flow}}$, we have finally

$$\begin{aligned}
&\frac{\mu_0(p-1)}{2^{(2-p)}(\widetilde{C}^{\text{flow}} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})})^{(2-p)}} \|D(\overline{\upsilon}_n - \hat{\upsilon})\|^2_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\leq \int_0^T \int_\Omega \left(\mathcal{F}_1\big(D(\overline{\upsilon}_n + \upsilon^0\xi)\big) - \mathcal{F}_1\big(D(\hat{\upsilon} + \upsilon^0\xi)\big)\right) : D(\overline{\upsilon}_n - \hat{\upsilon}) \,dx\,dt \\
&\leq \int_0^T \int_\Omega \Big(\mathcal{F}_2\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \\
&\quad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big)\Big) : D(\overline{\upsilon}_n - \hat{\upsilon}) \,dx\,dt \\
&\leq \Big\|\mathcal{F}_2\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \\
&\quad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big)\Big\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})} \\
&\quad \times \|D(\overline{\upsilon}_n - \hat{\upsilon})\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})}
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{\mu_0(p-1)}{2^{(2-p)}(\widetilde{C}^{\text{flow}} + \|\upsilon^0\xi\|_{L^p(0,T;V^p_{\Gamma_1})})^{(2-p)}} \|D(\overline{\upsilon}_n - \hat{\upsilon})\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\leq \Big\|\mathcal{F}_2\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \\
&\quad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big)\Big\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})}.
\end{aligned} \tag{4.29}$$

Case 2: $p \in [2,+\infty)$
Recalling that

$$\big(\mathcal{F}_1(\lambda) - \mathcal{F}_1(\lambda')\big) : (\lambda - \lambda') \geq \frac{\mu_0}{2^{p-1}} \|\lambda - \lambda'\|^p$$

for all $(\lambda, \lambda') \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$, we obtain

$$\begin{aligned}
&\frac{\mu_0}{2^{p-1}} \|D(\overline{\upsilon}_n - \hat{\upsilon})\|^p_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\quad \leq \int_0^T \int_\Omega \big(\mathcal{F}_1\big(D(\overline{\upsilon}_n + \upsilon^0\xi)\big) - \mathcal{F}_1\big(D(\hat{\upsilon} + \upsilon^0\xi)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt \\
&\quad \leq \int_0^T \int_\Omega \big(\mathcal{F}_2\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \\
&\qquad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big)\big) : D(\overline{\upsilon}_n - \hat{\upsilon})\,dx\,dt \\
&\quad \leq \big\|\mathcal{F}_2\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \\
&\qquad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big)\big\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})} \\
&\qquad \times \|D(\overline{\upsilon}_n - \hat{\upsilon})\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})}
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{\mu_0}{2^{p-1}} \|D(\overline{\upsilon}_n - \hat{\upsilon})\|^{p-1}_{L^p(0,T;(L^p(\Omega))^{3\times 3})} \\
&\quad \leq \big\|\mathcal{F}_2\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \\
&\qquad - \mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big)\big\|_{L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})}.
\end{aligned} \tag{4.30}$$

By using the convergence properties (4.24) for the sequences $(\theta_n)_{n\geq 1}$ and $(\overline{\upsilon}_n)_{n\geq 1}$ and the continuity and boundedness assumptions (2.8) and (2.10) for the mapping μ, we obtain

$$\begin{aligned}
&\mathcal{F}_2\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \longrightarrow \mathcal{F}_2\big(\theta, \hat{\upsilon} + \upsilon^0\xi, D(\hat{\upsilon} + \upsilon^0\xi)\big) \\
&\quad \text{strongly in } L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3}).
\end{aligned}$$

Then with (4.29) if $p \in [6/5, 2)$ and (4.30) if $p \in [2,+\infty)$, we obtain

$$\lim_{n\to+\infty} \|D(\overline{\upsilon}_n - \hat{\upsilon})\|_{L^p(0,T;(L^p(\Omega))^{3\times 3})} = 0.$$

Finally, Korn's inequality implies that the sequence $(\overline{\upsilon}_n)_{n\geq 1}$ converges strongly to $\hat{\upsilon}$ in $L^p(0,T;V^p_{0,\mathrm{div}})$, which yields $\hat{\upsilon} = \overline{\upsilon}$. □

By possibly extracting another subsequence, still denoted $(\overline{\upsilon}_n, \pi_n, \theta_n)_{n\geq 1}$, we obtain that

$$D(\overline{\upsilon}_n) \longrightarrow D(\overline{\upsilon}) \quad \text{a.e. in } (0,T) \times \Omega.$$

By using (2.8) and (2.10), we infer from Lebesgue's theorem that

$$\begin{aligned}
&\mathcal{F}\big(\theta_n, \overline{\upsilon}_n + \upsilon^0\xi, D(\overline{\upsilon}_n + \upsilon^0\xi)\big) \longrightarrow \mathcal{F}\big(\theta, \overline{\upsilon} + \upsilon^0\xi, D(\overline{\upsilon} + \upsilon^0\xi)\big) \\
&\quad \text{strongly in } L^{p'}(0,T;(L^{p'}(\Omega))^{3\times 3})
\end{aligned}$$

and

$$g_{\delta_n}(\theta_n,\overline{\upsilon}_n) \longrightarrow 2\mu\big(\theta,\overline{\upsilon}+\upsilon^0\xi,\big\|D\big(\overline{\upsilon}+\upsilon^0\xi\big)\big\|\big)\big\|D(\overline{\upsilon}+\upsilon_0\xi)\big\|^p$$
$$= \mathcal{F}\big(\theta,\overline{\upsilon}+\upsilon^0\xi,D\big(\overline{\upsilon}+\upsilon^0\xi\big)\big):D\big(\overline{\upsilon}+\upsilon^0\xi\big)$$
$$\text{strongly in } L^1(0,T;L^1(\Omega)).$$

Moreover,

$$\overline{\upsilon}_n \longrightarrow \hat{\upsilon}=\overline{\upsilon} \quad \text{strongly in } L^p(0,T;V^p_{0.\,\mathrm{div}}).$$

Reminding that the mapping J is continuous on $L^p(0,T;V^p_0)$, we get

$$J(\overline{\upsilon}_n+\widetilde{\vartheta}\zeta) \longrightarrow J(\overline{\upsilon}+\widetilde{\vartheta}\zeta) \quad \forall\widetilde{\vartheta}\in V^p_0,\forall\zeta\in\mathcal{D}(0,T).$$

Finally, by using the continuity and boundedness assumptions (2.6)–(2.7) for the mapping r, we also have

$$r(\theta_n) \longrightarrow r(\theta) \quad \text{strongly in } L^1(0,T;L^1(\Omega)).$$

It follows that we can pass to the limit in (4.1) and (4.5), which allows us to conclude that the triplet $(\overline{\upsilon},\pi,\theta)$ is a solution to the coupled fluid flow/heat transfer problem (P).

Remark 4.5 We may observe that $\pi=\hat{\pi}$. Indeed,

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}(\overline{\upsilon},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)},\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad+\int_0^T\int_\Omega \mathcal{F}\big(\theta,\overline{\upsilon}+\upsilon^0\xi,D\big(\overline{\upsilon}+\upsilon^0\xi\big)\big):D(\tilde{\vartheta})\zeta\,dx\,dt \\
&\quad-\left\langle\int_\Omega \pi\,\mathrm{div}(\tilde{\vartheta})\,dx,\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}+J(\overline{\upsilon}+\tilde{\vartheta}\zeta)-J(\overline{\upsilon}) \\
&\ge\int_0^T\left(f+\upsilon^0\frac{\partial\xi}{\partial t},\tilde{\vartheta}\right)_{\mathbf{L}^2(\Omega)}\zeta\,dt \quad \forall\tilde{\vartheta}\in V^p_0,\forall\zeta\in\mathcal{D}(0,T)
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}(\hat{\upsilon},\tilde{\vartheta})_{\mathbf{L}^2(\Omega)},\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad+\int_0^T\int_\Omega \mathcal{F}\big(\theta,\overline{\upsilon}+\upsilon^0\xi,D\big(\hat{\upsilon}+\upsilon^0\xi\big)\big):D(\tilde{\vartheta})\zeta\,dx\,dt \\
&\quad-\left\langle\int_\Omega \hat{\pi}\,\mathrm{div}(\tilde{\vartheta})\,dx,\zeta\right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}+J(\hat{\upsilon}+\tilde{\vartheta}\zeta)-J(\hat{\upsilon}) \\
&\ge\int_0^T\left(f+\upsilon^0\frac{\partial\xi}{\partial t},\tilde{\vartheta}\right)_{\mathbf{L}^2(\Omega)}\zeta\,dt \quad \forall\tilde{\vartheta}\in V^p_0,\forall\zeta\in\mathcal{D}(0,T).
\end{aligned}$$

Owing to that $\overline{\upsilon} = \hat{\upsilon}$, we obtain that

$$\left\langle \int_\Omega (\pi - \hat{\pi}) \operatorname{div}(\tilde{\vartheta})\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = 0 \quad \forall \tilde{\vartheta} \in \mathbf{W}_0^{1,p}(\Omega), \forall \zeta \in \mathcal{D}(0,T).$$

Hence, for any $\varpi \in L_0^p(\Omega)$, we may choose $\widetilde{\vartheta} = P_p(\varpi)$, where P_p is the operator introduced in Proposition 4.1, and we get

$$\left\langle \int_\Omega (\pi - \hat{\pi}) \varpi \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = 0.$$

Furthermore, for any $\varpi^* \in L^p(\Omega)$, we may define $\varpi \in L_0^p(\Omega)$ by

$$\varpi = \varpi^* - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega \varpi^* \, dx,$$

and since $\pi - \hat{\pi} \in H^{-1}(0,T; L_0^p(\Omega))$, we have

$$\begin{aligned}
&\left\langle \int_\Omega (\pi - \hat{\pi}) \varpi \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad = \left\langle \int_\Omega (\pi - \hat{\pi}) \varpi^* \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\qquad - \frac{1}{\operatorname{meas}(\Omega)} \left(\int_\Omega \varpi^* \, dx \right) \left\langle \int_\Omega (\pi - \hat{\pi})\, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} \\
&\quad = \left\langle \int_\Omega (\pi - \hat{\pi}) \varpi^* \, dx, \zeta \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)} = 0 \quad \forall \varpi^* \in L^p(\Omega), \forall \zeta \in \mathcal{D}(0,T),
\end{aligned}$$

which implies $\pi = \hat{\pi}$.

Remark 4.6 The strong convergence of $(g_{\delta_n}(\theta_n, \overline{\upsilon}_n))_{n \ge 1}$ and $(r(\theta_n))_{n \ge 1}$ in the space $L^1(0,T; L^1(\Omega))$ implies that $\theta \in C^0([0,T]; L^1(\Omega))$ and $\theta(0,\cdot) = \theta^0$. Indeed, starting from (4.11), we have also

$$\begin{aligned}
&\int_0^T \int_\Omega c \frac{\partial(\theta_n - \theta_{n'})}{\partial t} \widetilde{w} \, dx\, dt + \int_0^T \int_\Omega \big(K \nabla(\theta_n - \theta_{n'}) \big) \cdot \nabla \widetilde{w} \, dx\, dt \\
&\quad = \int_0^T \int_\Omega \big(g_{\delta_n}(\theta_n, \overline{\upsilon}_n) - g_{\delta_{n'}}(\theta_{n'}, \overline{\upsilon}_{n'}) \big) \widetilde{w} \, dx\, dt + \int_0^T \int_\Omega \big(r(\theta_n) - r(\theta_{n'}) \big) \widetilde{w} \, dx\, dt \\
&\qquad + \int_0^T \int_\omega \big(\theta_{\delta_n}^b - \theta_{\delta_{n'}}^b \big) \widetilde{w} \, dx'\, dt
\end{aligned}$$

for all $\widetilde{w} \in L^2(0,T; W^{1,2}_{\Gamma_1 \cup \Gamma_L}(\Omega))$, for all $n \ge 1$, and $n' \ge 1$. By choosing $\widetilde{w} = \psi_0^m(\theta_n - \theta_{n'}) \mathbf{1}_{[0,t]}$, with $t \in (0,T]$ and by passing to the limit, as m tends to $+\infty$, we get

$$\begin{aligned}
&\int_\Omega c(x) \Phi_0 \big(\theta_n(t,x) - \theta_{n'}(t,x) \big)\, dx \\
&\qquad + \int_0^t \int_\Omega \Psi_0 \big(\theta_n(s,x) - \theta_{n'}(s,x) \big)
\end{aligned}$$

$$
\begin{aligned}
&\times \big(K(x)\nabla\big(\theta_n(s,x)-\theta_{n'}(s,x)\big)\big)\cdot\nabla\big(\theta_n(s,x)-\theta_{n'}(s,x)\big)\,dx\,ds\\
&= \int_0^t\int_\Omega \big(g_{\delta_n}(\theta_n,\overline{v}_n)-g_{\delta_{n'}}(\theta_{n'},\overline{v}_{n'})\big)\psi_0(\theta_n-\theta_{n'})\,dx\,ds\\
&\quad+\int_0^t\int_\Omega \big(r(\theta_n)-r(\theta_{n'})\big)\psi_0(\theta_n-\theta_{n'})\,dx\,ds\\
&\quad+\int_0^t\int_\omega \big(\theta^b_{\delta_n}-\theta^b_{\delta_{n'}}\big)\psi_0(\theta_n-\theta_{n'})\,dx'\,ds+\int_\Omega c(x)\Phi_0\big(\theta^0_{\delta_n}-\theta^0_{\delta_{n'}}\big)\,dx,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&c_0\int_\Omega \Phi_0\big(\theta_n(t,x)-\theta_{n'}\big)\,dx\\
&\le \big\|g_{\delta_n}(\theta_n,\overline{v}_n)-g_{\delta_{n'}}(\theta_{n'},\overline{v}_{n'})\big\|_{L^1(0,T;L^1(\Omega))}\\
&\quad+\big\|r(\theta_n)-r(\theta_{n'})\big\|_{L^1(0,T;L^1(\Omega))}+\big\|\theta^b_{\delta_n}-\theta^b_{\delta_{n'}}\big\|_{L^1((0,T)\times\omega)}+\big\|c\big(\theta^0_{\delta_n}-\theta^0_{\delta_{n'}}\big)\big\|_{L^1(\Omega)}.
\end{aligned}
$$

Next, we observe that

$$
\Phi_0(s)=\begin{cases}\frac{s^2}{2} & \text{if } |s|\le 1,\\ s-\frac12 & \text{if } s>1,\\ -s-\frac12 & \text{if } s<-1\end{cases}
$$

which implies $\Phi_0(s)\ge\frac{|s|}{2}$ if $|s|\ge 1$. It follows that

$$
\begin{aligned}
&c_0\int_\Omega \big|\theta_n(t,x)-\theta_{n'}(t,x)\,dx\big|\\
&\le c_0\bigg\{\bigg(\int_\Omega \Phi_0\big(\theta_n(t,x)-\theta_{n'}(t,x)\big)\bigg)^{1/2}\operatorname{meas}(\Omega)^{1/2}+2\int_\Omega \Phi_0\big(\theta_n(t,x)-\theta_{n'}(t,x)\big)\bigg\},
\end{aligned}
$$

which allows us to conclude that $(\theta_n)_{n\ge1}$ is a Cauchy sequence in $C^0([0,T];L^1(\Omega))$. As a consequence, we may relax the regularity of the test-function $\widetilde{\zeta}$ from $C^\infty([0,T])$ in (2.15).

Moreover, following [28], we can prove that θ is an entropy solution of (1.3)–(1.9). For more details on the definition of entropy (or renormalized) solutions, the reader is referred to [5, 12, 28] and the references therein.

Finally, let us emphasize that the strong convergence of $(\theta_n)_{n\ge1}$ to θ in the space $C^0([0,T];L^1(\Omega))$, which is the key point to show that θ is an entropy solution of (1.3)–(1.9), will not be true anymore if convective effects are taken into account in the heat equation.

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Authors' contributions
I declare that I am the sole author of this work. The author read and approved the final manuscript.

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