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Axiom of Infinite Choice, transversal ordered spring spaces and fixed points

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Abstract

This paper continues the study of the Axiom of Infinite Choice on transversal ordered spring spaces in terms of fixed point and increasing inductive sets. These principles unify a number of diverse results (about three thousand papers) in fixed point theory, especially recently published. Applications in partially ordered spaces and fixed point theory are also considered.

MSC: Primary 47H10; 05A15; 06A10; 04A25; secondary 05A05; 54H25

Keywords: Fixed points; Axiom of Infinite Choice; Transversal ordered spring spaces; Spaces with the non-numerical transversals; Partially ordered spaces; Increasing mappings; Transversal edges (upper, lower, and middle) spaces; Lower and upper (distribution) functions; Noncomplete spaces; Lemma of Infinite Maximality; Increasing inductiveness; Partially ordered metric spaces

1 Introduction and history

Call a poset (:= partially ordered set) P *increasing inductive* (*increasing chain complete*) when every increasing sequence has an upper bound (the least upper bound, i.e., *supremum*) in P.

Lemma 1 (Increasing inductiveness, Tasković [1]) *If P is an increasing inductive partially ordered set, then P has at least countable or finite maximal elements.*

A brief proof of a special case of this fact may be found in Tasković [1]. This statement is *de facto* an equivalent of Lemma of Infinite Maximality by Tasković [2].

Theorem 1 (Axiom of Infinite Choice, Tasković [2]) *Let $P := (P, \preccurlyeq)$ be a partially ordered set. Then the following statements are equivalent:*

(a) (*Lemma of Infinite Maximality, Tasković* [2].) *Let P be an inductive partially ordered set. Then P has at least countable or finite maximal elements.*

(b) (*Local form, Tasković* [1].) *Let P be an increasing inductive poset and f be an increasing mapping from P into P such that*

$$a \preccurlyeq f(a) \quad \textit{for some } a \in P, \tag{T}$$

then f has at least countable or finite fixed points in P.

(c) (*Tasković* [3].) *Let P be an increasing inductive poset and f be a mapping from P into P such that*

$$x \preccurlyeq f(x) \quad \text{for all } x \in P, \tag{M}$$

then f has at least countable or finite fixed points in P.

We note that a brief proof of this statement based on the Axiom of Infinite Choice may be found in Tasković [2].

Based on Lemma 1 and Theorem 1 above, we are now in a position to formulate the following new statements and facts as *direct* consequences of the Axiom of Infinite Choice.

Corollary 1 (Tasković [4, p. 244]) *Let $P := (P, \preccurlyeq)$ be a partially ordered set and f be an increasing mapping from P into P. If the following set $P(\preccurlyeq f) := \{x \in P : x \preccurlyeq f(x)\}$ is nonempty such that there exists the supremum $s := \sup P(\preccurlyeq f)$, then f has at least countable or finite fixed points in P.*

We notice that this result is a *direct* consequence of an application of the Axiom of Infinite Choice on the set $P(\preccurlyeq f)$. In 1980 Tasković proved that f has at least one fixed point for the preceding case of Corollary 1.

Corollary 2 *Let $P := (P, \preccurlyeq)$ be a partially ordered set and f be an increasing mapping from P into P such that*

$$a \preccurlyeq f(a) \quad \text{for some } a \in P, \tag{M}$$

where every increasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in P is bounded, then f has at least countable or finite fixed points in P.

Proof Let us consider the subset $P(\preccurlyeq f)$ of P given by $P(\preccurlyeq f) := \{x \in P : x \preccurlyeq f(x)\}$. By the hypothesis, we see that $P(\preccurlyeq f)$ is a nonempty poset. Since $x \preccurlyeq f(x)$ implies $f(x) \preccurlyeq f(f(x))$, we see that f maps $P(\preccurlyeq f)$ into $P(\preccurlyeq f)$.

Because $a \preccurlyeq f(a)$ and f is isotone, we find $f(a) \preccurlyeq f^2(a)$ and inductively that $f^n(a) \preccurlyeq f^{n+1}(a)$ for each $n \in \mathbb{N} \cup \{0\}$ and some $x \in P$. Thus, $\{f^n(a) : n \in \mathbb{N} \cup \{0\}\}$ is an increasing sequence of iterates which is bounded, and so the least upper bound of $\{f^n(a) : n \in \mathbb{N} \cup \{0\}\}$ exists. The system of chain C for which

$$x \in C \quad \text{implies} \quad f(x) \in C \quad \text{and} \quad x \preccurlyeq f(x) \tag{SC}$$

contains the nonempty chain $\{f^n(a) \mid n \in \mathbb{N} \cup \{0\}\}$ and therefore contains maximal chains M_k by Lemma 1 (or by the Lemma of Infinite Maximality). By assumption $\mu_k = \sup M_k \in P$ exists, where $\sup M_k$ is the least upper bound of M_k. Since M_k satisfies (SC), we have $x \preccurlyeq f(x) \preccurlyeq f(\mu_k)$ for all $x \in M_k$, so that $\mu_k \preccurlyeq f(\mu_k)$.

On the other hand, if $\mu_k \notin M_k$, then the chain $M_k \cup \{f^n(\mu_k) \mid n \in \mathbb{N} \cup \{0\}\}$ properly contains M_k and satisfies (SC) in contradiction to the maximality of M_k. Therefore, $\mu_k \in M_k$ and also $f(\mu_k) \in M_k$, hence $f(\mu_k) \preccurlyeq \mu_k$. This makes μ_k a fixed point of f, i.e., $f(\mu_k) = \mu_k$. The proof is complete. □

Annotation From the above it follows that $P(\preccurlyeq f)$ is a nonempty poset with the property that each nonempty increasing chain of $P(\preccurlyeq f)$ has an upper bound, i.e., $P(\preccurlyeq f)$ is an increasing inductive set, and f maps $P(\preccurlyeq f)$ into $P(\preccurlyeq f)$, and thus, by (c) of Theorem 1, we see that f has at least countable or finite fixed points as desired.

2 Preliminaries and main results

Spring upper ordered spaces. Let X be a nonempty set, and in what follows, let $P := (P, \preccurlyeq)$ be a partially ordered set such that $a, b \in P$ and $a \prec b$. The set (*interval*) $[a, b)$ is further defined by

$$[a, b) := \{t \in P : a \preccurlyeq t \prec b\}.$$

The function $A : X \times X \to [a, b) \subset P$ for $a \prec b$ is called an *upper spring ordered transverse* (or *upper spring ordered transversal*) on a nonempty set X iff $A(x, y) = a$ if and only if $x = y$ for all $x, y \in X$.

An *upper spring ordered transversal space* $X := (X, \preccurlyeq, A)$ is a nonempty partially ordered set X (with ordering $\preccurlyeq$) together with a given upper spring ordered transverse A on X, where every decreasing sequence $\{u_n\}_{n\in\mathbb{N}}$ of elements in $[a, b)$ has a unique element $u \in [a, b)$ as limit (in notation $u_n \to u$ $(n \to \infty)$). The element $a \in [a, b) \subset P$ is called *spring of space* X (cf. [5]).

In 1986 we investigated the concept of upper spring ordered TCS-convergence in a space X, i.e., an upper spring ordered transversal space $X := (X, A)$ satisfies the condition of *upper spring ordered TCS-convergence* iff $x \in X$ and if $A(T^n(x), T^{n+1}(x)) \to a$ $(n \to \infty)$ implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence in X, see Tasković [6].

In connection with the above, we shall introduce the concept of upper MCS-convergence, i.e., an upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ satisfies the condition of *upper MCS-convergence* (i.e., *upper MSC-completeness*) iff $x \in X$ and if $A(f^n(x), f^{n+1}(x)) \to a$ $(n \to \infty)$ implies that $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded in X.

We notice that the sequence $\{x_n\}_{n\in\mathbb{N}}$ in the upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ is *convergent* (or *upper convergent*) in notation $x_n \to x$ $(n \to \infty)$ iff $A(x_n, x) \to a$ as $n \to \infty$; or equivalently, for a decreasing sequence $\{a_n\}_{n\in\mathbb{N}} \in [a, b)$ which converges to a, the following inequality holds:

$$A(x_n, x) \prec a_n \quad \text{for every } n \in \mathbb{N},$$

or for n large enough.

On the other hand, in connection with this, the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X will be called *upper fundamental* (or *upper spring fundamental*) if the following inequality holds:

$$A(x_n, x_m) \prec a_n \quad \text{for all } n, m \in \mathbb{N}\ (n < m),$$

or for n and m large enough, where the decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ in $[a, b)$ converges to a (cf. [5]).

An upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ is called *upper complete* (or *upper spring complete*) if any upper fundamental sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is upper convergent (to a point of X, of course).

On the other hand, in this paper we shall introduce the concept of upper MBV-convergence, i.e., an upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ satisfies the condition of *upper MBV-convergence* (i.e., *upper MBV-completeness*) if every increasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded in X.

An immediate *direct* consequence of the preceding Corollary 2 is the following new result for upper spring ordered transversal spaces.

Corollary 3 *Let f be an increasing mapping of an upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where X satisfies the condition of upper MBV-convergence. If*

$$a \preccurlyeq f(a) \quad \textit{for some } a \in X, \tag{M}$$

then f has at least countable or at least finite fixed points in X.

The proof of this statement is an elementary fact because the condition of upper MBV-completeness implies that every increasing sequence of iterates is bounded in Corollary 2.

As an immediate application of Corollary 3 *directly*, we obtain the following new consequence on upper spring ordered transversal spaces.

Corollary 4 *Let f be an increasing mapping of an upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where every increasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is upper fundamental. If*

$$a \preccurlyeq f(a) \quad \textit{for some } a \in X, \tag{M}$$

then f has at least countable or at least finite fixed points in X.

Proof Because $a \preccurlyeq f(a)$ and f is isotone, we find $\{f^n(a)\}_{n\in\mathbb{N}\cup\{0\}}$ is an increasing sequence of iterates which is upper fundamental, i.e., bounded in X. It is easy to see that X satisfies all the required hypotheses in Corollary 3. Applying Corollary 3 to this case, we obtain this statement. The proof is complete. □

Corollary 5 (Partially ordered metric spaces) *Let f be an increasing mapping of an ordered metric space $X := (X, \preccurlyeq, \rho)$ into itself, where every increasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is a Cauchy sequence. If*

$$a \preccurlyeq f(a) \quad \textit{for some } a \in X, \tag{M}$$

then f has at least countable or at least finite fixed points in X.

Proof Since an ordered metric space is an example of an upper spring ordered transversal space, thus this statement follows *directly* from Corollary 4. The proof is complete. □

Corollary 6 (Upper spring ordered transversal spaces) *Let f be an increasing mapping of an upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where every increasing and decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is bounded. If*

$$a \preccurlyeq f(a) \quad \textit{for some } a \in X, \tag{M}$$

or

$$f(x) \prec x \quad \textit{for all } x \in X, \tag{N}$$

then f has at least countable or at least finite fixed points in X.

The proof of this statement is a total analogy with the preceding proofs of Corollaries 2 and 3. Thus we omit it.

Corollary 7 (Partially ordered metric spaces) *Let f be an increasing mapping of an ordered metric space* $X := (X, \preccurlyeq, \rho)$ *into itself, where every increasing and decreasing sequence of iterates* $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ *in X is a Cauchy sequence. If* (M) *or* (N) *holds, then f has at least countable or finite fixed points in X*.

This result is an example for Corollary 6. Thus we omit the proof.

3 Applications on partially ordered metric and other spaces

This section is mainly devoted to some applications on partially ordered metric and other spaces. Some sufficient conditions for the upper MBV-completeness are given, which has a direct consequence on the Cauchyness of sequences in partially ordered metric spaces. On the other hand, every increasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is also a Cauchy sequence in $X := (X, \preccurlyeq, A)$. This fact (in this sense) includes (unifies) at least three thousand papers published recently. We note that a partially ordered metric space *directly* is an example of upper spring ordered transversal spaces.

Diametral φ-contraction on metric spaces. In 1980 I proved the following result of a fixed point on metric space, which is one of the most known sufficient conditions (linear and nonlinear) for the existence of a unique fixed point, cf. Tasković [7, 8], and [9]. This result generalizes a great number of known results.

Theorem 2 (Tasković [7, p. 250, Theorem 1]) *Let T be a mapping of a metric space* (X, ρ) *into itself, and let X be T-orbital complete. Suppose that there exists a function* $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+ := [0, +\infty)$ *satisfying*

$$\bigl(\forall t \in \mathbb{R}_+ := (0, +\infty)\bigr) \quad \Bigl(\varphi(t) < t \textit{ and } \limsup_{z\to t+0} \varphi(z) < t\Bigr) \tag{Iφ}$$

such that the following inequality holds:

$$\rho\bigl(T(x), T(y)\bigr) \le \varphi\bigl(\operatorname{diam}\bigl\{x, y, T(x), T(y), T^2(x), T^2(y), \dots\bigr\}\bigr) \tag{J}$$

for all $x, y \in X$. *If* $\operatorname{diam}\mathcal{O}(x) \in \mathbb{R}^0_+$ *for every* $x \in X$, *then T has a unique fixed point* $\xi \in X$ *and* $\{T^n(b)\}_{n\in\mathbb{N}}$ *converges to* ξ *for arbitrary* $b \in X$.

In connection with this result, we notice that this statement is well known as "*the finest theorem of nonlinear functional analysis*" for metric spaces.

In the context of Theorem 2, applying Corollary 5 to this case *directly*, we obtain the following result for *diametral φ-contractions*, i.e., for mappings with properties (J) and (Iφ).

Corollary 8 *Let T be an increasing mapping of an ordered metric space* $X := (X, \preccurlyeq, \rho)$ *into itself. Suppose that there exists a function* $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+$ *satisfying* (Iφ) *such that* (J). *If*

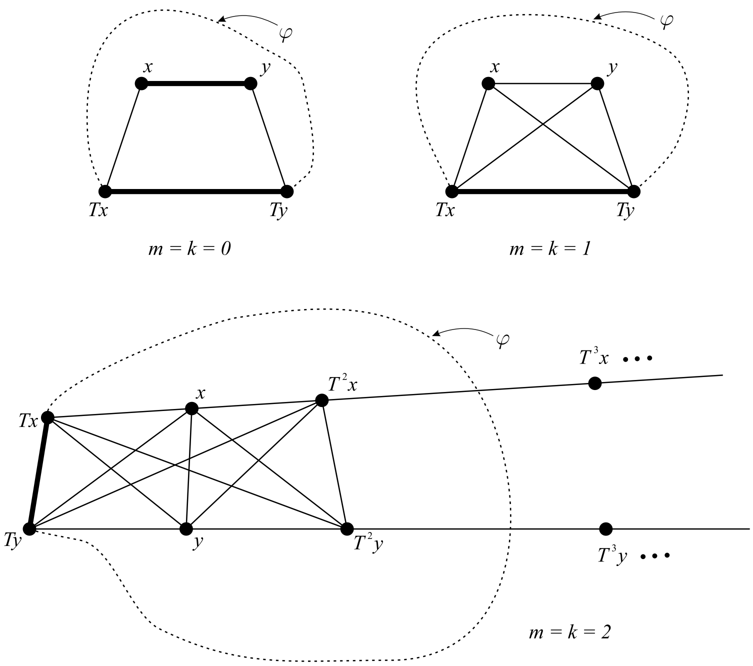

Figure 1 Geometric interpretation of φ

$a \preccurlyeq T(a)$ *for some* $a \in X$ *and if* $\operatorname{diam} \mathcal{O}(x) \in \mathbb{R}_+^0$ *for every* $x \in X$*, then* T *has a unique fixed point* $\xi \in X$ *and* $\{T^n(b)\}_{n \in \mathbb{N}\cup\{0\}}$ *converges to* ξ *for arbitrary* $b \in X$.

Annotation In connection with the preceding inequality (J), we notice that Tasković [10] considered a special case of this condition. Also see Kurepa [11], Ohta–Nikaido [12], and Tasković [13]. It is interesting that in 1976 Kurepa proved a geometric interpretation of condition (J). See Fig. 1.

Transversal upper edges spaces. Let X be a nonempty set. The function $\rho : X \times X \to [a,b]$ (or $\rho : X \times X \to [a,b)$) for $a < b$ ($a, b \in \mathbb{R}_+^0 := [0,+\infty)$) is called an *upper edges transverse* on X (or *upper edges transversal*) iff $\rho(x,y) = \rho(y,x)$, $\rho(x,y) = a$ if and only if $x = y$, and if there is a function $\psi : [a,b]^2 \to [a,b]$ such that

$$\rho(x,y) \le \max\bigl\{\rho(x,z), \rho(z,y), \psi\bigl(\rho(x,z), \rho(z,y)\bigr)\bigr\} \tag{A}$$

for all $x, y, z \in X$.

An *upper edges transversal space* (or *upper edges space*) is a set X together with a given upper edges transverse on X. The function $\psi : [a,b]^2 \to [a,b]$ in (A) is called an *upper bisection function*.

From (A) it follows by induction that there is a function $\mathfrak{G} : [a,b]^n \to [a,b]$ for $a < b$ such that the following inequality holds:

$$\rho(x_0, x_n) \le \max\bigl\{\rho(x_0,x_1), \dots, \rho(x_{n-1},x_n), \mathfrak{G}\bigl(\rho(x_0,x_1), \dots, \rho(x_{n-1},x_n)\bigr)\bigr\}$$

for all $x_0, x_1, \dots, x_n \in X$ and for an arbitrary fixed integer $n \ge 2$.

Example 1 (Metric spaces) A fundamental first example of upper edges space is a metric space. Indeed, if (X,d) is a metric space, then for the upper bisection function $\psi(r,t)=r+t$, we have the following upper edges transverse $\rho : X \times X \to [a,b] \subset \mathbb{R}^0_+$ for $a<b$ defined by

$$\rho(x,y)=\frac{(b-a)d(x,y)}{1+d(x,y)}+a$$

for all $x,y \in X$. Thus (X,ρ) is an example of an upper edges transversal space. In general, every metric space is an example of an upper edges transversal space.

Example 2 (The extended real line $\overline{\mathbb{R}}$) The function f defined in $\mathbb{R}$ by $f(x)=x/(1+|x|)$ is a bijection on $\mathbb{R}$ on the open interval $(-1,1)\subset\mathbb{R}$, and the inverse mapping g being defined by $g(x)=x/(1-|x|)$ for $|x|<1$. Let $\overline{\mathbb{R}}$ be the set which is the union of $\mathbb{R}$ and two new elements written $+\infty$ and $-\infty$ (points at infinity); then we extend f to a bijection of $\overline{\mathbb{R}}$ onto $[-1,1]$ by putting $f(+\infty)=1, f(-\infty)=-1$, and write again g for the inverse mapping.

We can apply this process described to define $\overline{\mathbb{R}}$ as an upper edges transversal space by putting for the upper edges transverse $\rho : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to [0,2]$, that is,

$$\rho(x,y)=\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right|$$

for all $x,y \in \overline{\mathbb{R}}$. (We notice that for $x \geq 0$ is $\rho(+\infty,x)=1/(1+|x|)$, and for $x \leq 0$ that is $\rho(-\infty,x)=1/(1+|x|)$).

Example 3 (Transversal upper edges r-spaces) A fundamental third example of a transversal upper edges space is a transversal upper edges r-space for $r \geq 1$. Indeed, in 1998 this space was known as a transversal upper space. For $\psi(s,t) := rs+rt$ $(r \geq 1)$, we obtain *directly* in this case the transversal upper edges r-space. This space is well known for $r \geq 1$ as an *r-metric space* (or as *almost metric space*) introduced in 1989 by Bakhtin [14]. Also see Czerwik [15].

On the other hand, a fundamental fourth example of a transversal upper edges space is a *transversal upper edges r-max space* (for $r \geq 1$), known as *r-max edges space*. Indeed, in 1998 this space was known as a transversal r-max edges space. For $\psi(s,t)=r\max\{s,t\}$ for $r \geq 1$, we obtain *directly* in this case the transversal upper edges r-space.

Annotation We notice that the set (class) of all well-known b-metric spaces $(b \geq 1)$ can be a proper subset (subclass) of the set (class) all transversal r-max edges space for $r \geq 1$. See Tasković [16].

For any nonempty set Y in the upper edges transversal space X, the *diameter* of Y is defined as $\operatorname{diam}(Y) := \sup\{\rho(x,y) : x,y \in Y\}$; it is a real number in $[a,b]$, $A \subset B$ implies $\operatorname{diam}(A) \leq \operatorname{diam}(B)$. The relation $\operatorname{diam}(Y)=a$ holds if and only if Y is a one-point set.

Elements of an upper edges transversal space will usually be called points. Given an upper edges transversal space (X,ρ) with the bisection function $g : [a,b]^2 \to [a,b]$ and a point $z \in X$, the *open ball* of center z and radius $r>0$ is the set

$$g\big(B(z,r)\big)=\big\{x \in X : \rho(z,x) < a+r\big\}.$$

The *convergence* $x_n \to x$ as $n \to \infty$ in the upper edges transversal space (X, ρ) means that $\rho(x_n, x) \to a$ as $n \to \infty$, or equivalently, for every $\varepsilon > 0$ there exists an integer n_0 such that the relation $n \geq n_0$ implies $\rho(x_n, x) < a + \varepsilon$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ in the upper edges transversal space (X, ρ) is called a *transversal sequence* (or *upper Cauchy sequence*) iff for every $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ such that

$$\rho(x_n, x_m) < a + \varepsilon \quad \text{for all } n, m \geq n_0.$$

Let (X, ρ) be an upper edges transversal space and $T : X \to X$. We notice, from Tasković [17], that a sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ in X is said to be a transversal sequence if and only if

$$\lim_{n\to\infty} \left(\operatorname{diam}\left\{T^k(x) : k \geq n\right\}\right) = a.$$

In this sense, an upper edges transversal space is called *upper complete* iff every transversal sequence converges. Also, a space (X, ρ) is said to be *upper orbitally complete* (or *upper T-orbitally complete*) iff every transversal sequence which is contained in $\mathcal{O}(x) := \{x, T(x), T^2(x), \ldots\}$ for some $x \in X$ converges in X.

For further facts on upper edges transversal spaces, see Tasković [6]. A function T mapping X into the reals is *T-orbitally lower semicontinuous* at $p \in X$ if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{O}(x)$ and $x_n \to p$ $(n \to \infty)$ implies that $T(p) \leq \lim.\inf T(x_n)$.

Let (X, ρ) be an upper edges transversal space. A mapping $T : X \to X$ is said to be *upper edges contraction* if there exists $0 \leq \lambda < 1$ such that

$$\rho\bigl(T(x), T(y)\bigr) \leq \lambda\rho(x, y) + a(1 - \lambda)$$

for all points $x, y \in X$. In addition, let (X, ρ_X) and (Y, ρ_Y) be two upper edges transversal spaces and $T : X \to Y$. We notice, from Tasković [6], that T is *upper edges continuous* at $x_0 \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that the relation

$$\rho_X(x_0, x) < a + \delta \quad \text{implies} \quad \rho_Y\bigl(T(x_0), T(x)\bigr) < a + \varepsilon.$$

A typical first example of an upper edges continuous mapping is the upper edges contraction on the upper edges transversal space (X, ρ). For further facts on the upper edges continuous mappings, see Tasković [6].

Proposition 1 *Let f be an increasing mapping of a partially ordered transversal upper edges space $X := (X, \preccurlyeq, \rho)$ into itself, where X satisfies the condition of upper MCS-convergence. Suppose that there exists a number* $0 \leq \lambda < 1$ *such that*

$$\rho\bigl(f(x), f^2(x)\bigr) \leq \lambda\rho\bigl(x, f(x)\bigr) + a(1 - \lambda) \tag{Cu}$$

for every $x \in X$. If $r \preccurlyeq f(r)$ for some $r \in X$, then f has at least countable or finite fixed points in X.

Proof We see that a partially ordered transversal upper edges space $X := (X, \preccurlyeq, \rho)$ is an example of an upper spring ordered transversal space. From (Cu) it follows $\rho(f^n(x),$

$f^{n+1}(x)) \to a$ $(n \to \infty)$, hence by upper MCS-completeness we obtain that every increasing sequence of iterates in the form $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded in X, i.e., X satisfies the condition of upper MBV-completeness. It is easy to see that f and X satisfy all the required hypotheses in Corollary 3, thus f has at least countable or finite fixed points in X. The proof is complete. □

Proposition 2 *Let f be an increasing mapping of a partially ordered transversal upper edges space $X := (X, \preccurlyeq, \rho)$ into itself. Suppose that there exists $0 \le \lambda < 1$ such that*

$$\rho\bigl(f(x), f(y)\bigr) \le \lambda\rho(x,y) + a(1-\lambda) \tag{Lc}$$

for all $x, y \in X$. If $r \preccurlyeq f(r)$ for some $r \in X$, then f has a unique fixed point in X. (The space X in this statement is not obligatory complete.)

Proof Let $y = f(x)$ in (Lc), then it is easy to see that f and X satisfy all the required hypotheses in Proposition 1. Uniqueness follows immediately from condition (Lc). The proof is complete. □

We are now in a position to formulate the following statement, which is a roof for a great number of known results on metric spaces in the fixed point theory.

Proposition 3 *Let f be an increasing mapping of a partially ordered transversal upper edges space $X := (X, \preccurlyeq, \rho)$ into itself. Suppose that there exists a function $\varphi : [a,b] \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ satisfying*

$$\bigl(\forall (t \in (a,b]\bigr) \quad \Bigl(\varphi(t) < t \textit{ and } \limsup_{z\to t+0} \varphi(z) < t\Bigr) \tag{Iu}$$

such that

$$\rho\bigl(f(x), f(y)\bigr) \le \varphi\bigl(\operatorname{diam}\bigl\{x, y, f(x), f(y), f^2(x), f^2(y), \dots\bigr\}\bigr) \tag{B}$$

for all $x, y \in X$. If $r \preccurlyeq f(r)$ for some $r \in X$, then f has a unique fixed point in X. (The space X in this statement is not obligatory complete.)

Proof We begin the proof with the following lemma (as a well-known lemma) which is essential in the following context.

Lemma 2 (Tasković [13]) *Let the mapping $\varphi : [a,b] \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ have the property* (Iu). *If the sequence (x_n) of nonnegative real numbers satisfies the condition*

$$x_{n+1} \le \varphi(x_n), \quad n \in \mathbb{N},$$

then the sequence (x_n) tends to a. The velocity of this convergence is not necessarily geometrical.

A brief first proof of this statement may be found in Tasković [6]. We can see other brief proofs for this in Tasković [7, 8], and [13].

Proof of Proposition 3 Let x be an arbitrary point in X. We can show then that the sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is a transversal sequence. It is easy to verify (cf. [7, 9], and [13]) that the sequence $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies the following inequality:

$$\operatorname{diam}\mathcal{O}\bigl(f^{n+1}(x)\bigr) \leq \varphi(\operatorname{diam}\mathcal{O}\bigl(f^n(x)\bigr)$$

for $n \in \mathbb{N}$, and hence applying Lemma 2 to the sequence $(\operatorname{diam}\mathcal{O}(f^n(x)))$, we obtain $\lim_{n\to\infty}\operatorname{diam}\mathcal{O}(f^n(x)) = a$. This implies that $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is a transversal sequence in X. Hence, $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded.

Thus f and X satisfy all the required hypotheses in Corollary 4. Uniqueness follows immediately from (B). The proof is complete. □

As an immediate consequence of the preceding Proposition 3, we obtain directly the following interesting cases of (B):

(1) There exists a nondecreasing function $\varphi : [a,b] \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ satisfying $\limsup_{z\to t+0}\varphi(z) < t$ for every $t \in (a,b]$ such that

$$\rho\bigl(f(x),f(y)\bigr) \leq \varphi\bigl(\operatorname{diam}\bigl\{x,y,f(x),f(y)\bigr\}\bigr)$$

for all $x,y \in X$.

(2) (*Special affine case of condition* (B) *for* $\varphi(t) = \alpha t + a(1-\alpha)$.) There exists a constant $\alpha \in [0,1)$ such that, for all $x,y \in X$, the following inequality holds:

$$\rho\bigl(f(x),f(y)\bigr) \leq \alpha\operatorname{diam}\bigl\{x,y,f(x),f(y)\bigr\} + a(1-\alpha),$$

i.e., equivalent to

$$\rho\bigl(f(x),f(y)\bigr) \leq \alpha\max\bigl\{\rho(x,y),\rho\bigl(x,f(x)\bigr),\rho\bigl(y,f(y)\bigr),\rho\bigl(x,f(y)\bigr),\rho\bigl(y,f(x)\bigr)\bigr\} + a(1-\alpha).$$

(3) (*The condition of* $(m+k)$*-polygon*.) There exists a constant $\lambda \in [0,1)$ such that, for all $x,y \in X$, the following inequality holds:

$$\rho\bigl(f(x),f(y)\bigr) \leq \lambda\operatorname{diam}\bigl\{x,y,f(x),f(y),\ldots,f^m(x),f^k(y)\bigr\} + a(1-\lambda)$$

for arbitrary fixed integers $m,k \geq 0$. (This is a linear condition for the diameter of a finite number of points).

(4) There exists a nondecreasing function $\varphi : [a,b] \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ satisfying $\limsup_{z\to t+0}\varphi(z) < t$ for every $t \in (a,b]$ such that

$$\rho\bigl(f(x),f(y)\bigr) \leq \varphi\bigl(\operatorname{diam}\bigl\{x,y,f(x),f(y),\ldots,f^k(x),f^m(y)\bigr\}\bigr)$$

for arbitrary fixed integers $m,k \geq 0$ and for all $x,y \in X$.

(5) There exists an increasing mapping, i.e., $x_i \leq y_i$ $(i = 1,\ldots,5)$ implies $\varphi(x_1,\ldots,x_5) \leq \varphi(y_1,\ldots,y_5)$, $\varphi : [a,b]^5 \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ satisfying $\limsup_{z\to t+0}\varphi(z,z,z,z,z) < t$ for every $t \in (a,b]$ such that

$$\rho\bigl(f(x),f(y)\bigr) \leq \varphi\bigl(\rho(x,y),\rho\bigl(x,f(x)\bigr),\rho\bigl(y,f(y)\bigr),\rho\bigl(x,f(y)\bigr),\rho\bigl(y,f(x)\bigr)\bigr)$$

for all $x,y \in X$.

In connection with the preceding facts, we are now in a position to formulate a localization of Proposition 3 in the following form.

Proposition 4 *Let f be an increasing mapping of a partially ordered transversal upper edges space $X := (X, \preccurlyeq, \rho)$ into itself. Suppose that there exists a function $\varphi : [a, b] \to [a, b] \subset \mathbb{R}^0_+$ for $a < b$ satisfying* (Iu) *such that*

$$\operatorname{diam}\bigl\{f(x), f^2(x), \ldots\bigr\} \leq \varphi\bigl(\operatorname{diam}\bigl\{x, f(x), f^2(x), \ldots\bigr\}\bigr)$$

for every $x \in X$. If $r \preccurlyeq f(r)$ for some $r \in X$, then f has at least countable or finite fixed points in X. (*The space X in this statement is not obligatory complete.*)

The proof of this statement is a total analogy with the preceding proof of Proposition 3. Thus we omit it.

Corollary 9 (Non-complete partially ordered metric spaces) *Let f be an increasing mapping of a partially ordered metric space $X := (X, \preccurlyeq, \rho)$ into itself. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$\rho\bigl(f(x), f^2(x)\bigr) \leq \lambda\rho\bigl(x, f(x)\bigr) \tag{1}$$

for every $x \in X$. If $r \preccurlyeq f(r)$ for some $r \in X$, then f has at least countable or finite fixed points in X.

Proof From (1) it follows that the sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is a sequence of bounded variation, i.e., it is a Cauchy sequence. Applying Corollary 5 to this case, we obtain this statement. The proof is complete. □

Corollary 10 (Non-complete partially ordered metric spaces) *Let f be an increasing mapping of a partially ordered metric space $X := (X, \preccurlyeq, \rho)$ into itself. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$\rho\bigl(f(x), f(y)\bigr) \leq \lambda\rho(x, y)) \tag{2}$$

for all $x, y \in X$. If $r \preccurlyeq f(r)$ for some $r \in X$, then f has a unique fixed point in X.

Proof Let $y = f(x)$ in (2), then it is easy to see that f and X satisfy all the required hypotheses in Corollary 9. Uniqueness follows immediately from condition (2). The proof is complete. □

On the other hand, in this paper we shall introduce the concept of spring sup MCS-convergence, i.e., an upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ satisfies the condition of *spring sup MCS-convergence* iff $x \in X$ and if $\sup_{i,j\geq n} A(T^i(x), T^j(x))$ or $\sup_{i,j\geq 2n} A(T^i(x), T^j(x))$ or $\sup_{i,j\geq 2n+1} A(T^i(x), T^j(x))$ converges to $u, v, c \in [a, b)$ respectively implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n}(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n+1}(x)\}_{n\in\mathbb{N}}$ is a bounded sequence in X, respectively.

Also, if $T : X \to X$, then a function $x \mapsto A(x, T(x))$ is *ordered T-orbitally lower semicontinuous* at $\xi \in X$ if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\sigma(x, y, \infty) := \{x, y, T(x), T(y), T^2(x), T^2(y), \ldots\}$ and $x_n \to \xi\, (n \to \infty)$ implies that $A(\xi, T(\xi)) \preccurlyeq \lim_{n\to\infty} A(T^n(x), T^{n+1}(x))$.

Theorem 3 (Monotone principle of fixed point) *Let T be an increasing mapping of an upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where X satisfies the condition of spring sup MCS-convergence. Suppose that there exists a controlling function $B : X \times X \to [a, b)$ for $a \prec b$ such that*

$$A\bigl(T(x), T(y)\bigr) \preccurlyeq B(x, y) \prec \sup_{z,r\in\sigma(x,y,\infty)} A(z, r) \prec b \tag{3}$$

for all $x, y \in X$ or

$$A\bigl(T(x), T(y)\bigr) \prec B(x, y) \preccurlyeq \sup_{z,r\in\sigma(x,y,\infty)} A(z, r) \prec b \tag{3$'$}$$

for all $x, y \in X$. If $r \preccurlyeq T(r)$ for some $r \in X$, then T has at least countable or finite fixed points in X. If additionally $A(t, t) \preccurlyeq \sup\{A(s, t), A(t, s)\}$ for all $s, t \in X$, then T has a unique fixed point in X.

Let $X := (X, \preccurlyeq, M)$ be a partially ordered (with ordering $\preccurlyeq$) topological space and $T : X \to X$, where $M : X \to [a, b) \subset P$ for $a \prec b$. In this part (by Tasković [5]), we shall introduce the concept of local sup MCS-convergence in a space X, i.e., a topological space X satisfies the condition of *local spring sup MCS-convergence* iff $x \in X$ and $\sup_{i\geq n} M(T^i(x))$ or $\sup_{i\geq 2n} M(T^i(x))$ or $\sup_{i\geq 2n+1} M(T^i(x))$ converges to $u, v, c \succcurlyeq a$ respectively implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n}(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n+1}(x)\}_{n\in\mathbb{N}}$ is a bounded sequence in X, respectively.

We are now in a position to formulate the following theorem on partially ordered (with ordering $\preccurlyeq$) topological spaces with non-numerical transverses.

Theorem 4 (Localization monotone principle) *Let T be an increasing mapping of a partially ordered (with ordering $\preccurlyeq$) topological space $X := (X, \preccurlyeq, M)$ into itself, where X satisfies the condition of local spring sup MCS-convergence. Suppose that there exists a controlling function $N : X \to [a, b) \subset P$ for $a \prec b$ such that*

$$M\bigl(T(x)\bigr) \preccurlyeq N(x) \preccurlyeq \sup_{z\in\sigma(x,\infty)} M(z) \prec b \quad \textit{for every } x \in X, \tag{D}$$

and $r \preccurlyeq T(r)$ for some $r \in X$. Then T has at least countable or finite fixed points in X.

An immediate consequence of the preceding statement is the following result.

Corollary 11 *Let T be an increasing mapping of a partially ordered with ordering $\preccurlyeq$ topological space $X := (X, \preccurlyeq, M)$ into itself, where X satisfies the condition of local spring sup MCS-convergence. Suppose that there exists a controlling function $N : X \to [a, b) \subset P$ for $a \prec b$ such that*

$$M\bigl(T(x)\bigr) \preccurlyeq N(x) \preccurlyeq M(x) \quad \textit{for every } x \in X, \tag{D$'$}$$

and $r \preccurlyeq T(r)$ for some $r \in X$. Then T has at least countable or finite fixed points in X.

The proof of this statement is an elementary fact because condition (D′) implies condition (D).

Proof of Theorem 4 Let $x \in X$ be an arbitrary point and $n \in \mathbb{N} \cup \{0\}$ be any nonnegative integers. From (D) for $T^i(x)$ we have $M(T^{i+1}(x)) \preccurlyeq N(T^i(x)) \preccurlyeq \sup_{z\in\sigma(T^i(x),\infty)} M(z)$, and hence

$$\sup_{i\geq n+1} M\big(T^i(x)\big) \preccurlyeq \sup_{i\geq n} N\big(T^i(x)\big) \preccurlyeq \sup_{i\geq n} M\big(T^i(x)\big), \tag{4}$$

i.e., we obtain that $\{\sup_{i\geq n} M(T^i(x))\}_{n\in\mathbb{N}}$ is a decreasing convergent sequence in $[a,b) \subset P$. This implies (from local spring sup MCS-convergence) that its sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ is a bounded sequence in X, i.e., X satisfies the condition of upper MBV-completeness.

In the cases of other two sequences, in local spring sup MCS-convergence, the proof is a total analogy. Hence we omit the proof in these cases. Also, from [5] we obtain that every partially ordered topological space is a spring ordered transversal space. It is easy to see that T and X satisfy all the required hypotheses in Corollary 3, thus T has at least countable or finite fixed points in X. The proof is complete. □

Proof of Theorem 3 Let $M(x) := A(x, T(x))$ and $N(x) := B(x, T(x))$, then it is easy to see that A, B, and X satisfy all the required hypotheses in Theorem 4. Uniqueness follows immediately from conditions (3) and (3′). The proof is complete. □

In 1976 Tasković proved a localization theorem on a Cartesian product of metric spaces as a solution of Kuratowski's problem of 1932, see Brown [18], Reny George and Brian Fisher [19], Tasković [5], etc. In this context the following result holds.

Let X be an arbitrary topological space. By Tasković [10], for a mapping $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number), we will construct the *iteration sequence* $\{T^n(u)\}_{n\in\mathbb{N}}$ for an arbitrary point $u := (u_1, \dots, u_k) \in X^k$ in the following sense. Let T^0 :=Identical mapping and

$$T^n := T\psi^{n-1} \quad (n = 1, 2, \dots), \tag{Is}$$

where $\psi : X^k \to X^k$ defined by $\psi(u_1, \dots, u_k) = (u_2, \dots, u_{k+1})$ for the element of the form $u_{k+1} = T(u_1, \dots, u_k)$ and ψ^0 = Identical mapping.

We are now in a position to formulate the following statement for mappings of Cartesian product topological spaces.

Let $\mathfrak{O}(x, T(x)) := \{x_k, T(x), T^2(x), \dots\}$ for $x := (x_1, \dots, x_k)$ and $T : X^k \to X$ ($k \in \mathbb{N}$ is a fixed number). Also, $\mathfrak{O}(t, T(t, \dots, t)) := \{t, T(t, \dots, t), T^2(t, \dots, t), \dots\}$. A function $t \mapsto A(t, T(t, \dots, t))$ is *T-orbital lower semicontinuous* at $p \in X$ iff $T^n(x) \to p$ ($n \to \infty$) implies that $A(p, T(p, \dots, p)) \leq \liminf_{n\to\infty} A(T^n(x), T(T^n(x), \dots, T^{n+k-1}(x)))$.

In the following, we consider an *increasing function* $T : X^k \to X$ (for fixed $k \in \mathbb{N}$) on the partially ordered set X with ordering $\preccurlyeq$, i.e., if $x_i \preccurlyeq y_i$ ($i = 1, \dots, k$) implies that $T(x_1, \dots, x_k) \preccurlyeq T(y_1, \dots, y_k)$.

Corollary 12 *Let T be an increasing mapping of a Cartesian product of topological spaces X^k ($k \in \mathbb{N}$ is a fixed number) into X which is partially ordered with ordering $\preccurlyeq$, where*

$X := (X, \preccurlyeq, A)$ satisfies the condition of spring sup MCS-convergence. Suppose that there exists a controlling function $B : X \times X \to [a, b) \subset P$ for $a \prec b$ such that

$$A\big(T(u_1, \ldots, u_k), T(u_2, \ldots, u_{k+1})\big) \preccurlyeq B(u_k, u_{k+1}) \prec \sup_{z,r \in \mathfrak{O}(x,Tx)} A(z, r) \prec b \tag{5}$$

for all $u_1, \ldots, u_k, u_{k+1} \in X$ or

$$A\big(T(u_1, \ldots, u_k), T(u_2, \ldots, u_{k+1})\big) \prec B(u_k, u_{k+1}) \preccurlyeq \sup_{z,r \in \mathfrak{O}(x,Tx)} A(z, r) \prec b \tag{5$'$}$$

for all $u_1, \ldots, u_k, u_{k+1} \in X$, where $x := (u_1, \ldots, u_k)$. If $r \preccurlyeq T(r, \ldots, r)$ for some $r \in X$, then the equation $x = T(x, \ldots, x)$ has at least countable or finite solutions on X. If additionally $A(t, t) \preccurlyeq \sup\{A(s, t), A(t, s)\}$ for all $s, t \in X$, then there is unique $\zeta \in X$ such that $T(\zeta, \ldots, \zeta) = \zeta$.

The proof of this statement is a total analogy with the former proof of Corollary 3. Thus we omit it.

Further, a mapping $M : \mathbb{R} \to [a, b] \subset \mathbb{R}^0_+$ for $a < b$ is called an *upper (distribution) function* if it is nonincreasing, left-continuous with $\inf M = a$, and $\sup M = b$. We will denote by $\mathfrak{D}$ the set of all upper (distribution) functions.

The next two spaces are very interesting examples of transversal upper spaces. *First*, an *upper statistical space* is a pair $(X, \mathcal{R})$, where X is an abstract set and $\mathcal{R}$ is a mapping of $X \times X$ into the set of all upper (distribution) functions $\mathfrak{D}$. We shall denote the upper (distribution) function $\mathcal{R}(u, v)$ by $M_{u,v}(x)$ or $M_{u,v}$, whence the symbol $M_{u,v}(x)$ will denote the value of $M_{u,v}$ at $x \in \mathbb{R}$. The functions $M_{u,v}$ are assumed to satisfy the following conditions: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathbb{R}$, and

$$M_{u,v}(x) = a \quad \text{for } x > c \quad \text{if and only if} \quad u = v, \tag{Eq}$$

and if $M_{u,r}(x) = a$ and $M_{r,v}(y) = a$ imply $M_{u,v}(x + y) = a$ for all $u, v, r \in X$ and for all $x, y \in \mathbb{R}$.

In view of the condition $M_{u,v}(c) = b$, which evidently implies that $M_{u,v}(x) = b$ for every $x \leq c$, condition (Eq) is equivalent to the statement $u = v$ if and only if $M_{u,v}(x) = A(x)$, where $A(x) = b$ if $x \leq c$ and $A(x) = a$ if $x > c$. See Fig. 2.

Obviously, every metric space may be regarded as an upper statistical space of a special kind. One has only to set $M_{u,v}(x) = A(x - d(u, v))$ for every pair of points (u, v) in the metric

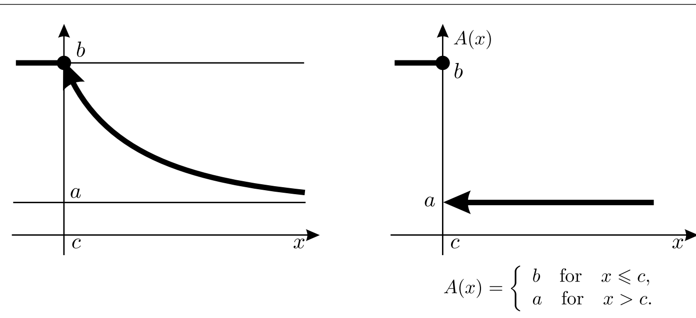

Figure 2 Geometric interpretation of the upper distribution function

space (X, d). Also, $M_{u,v}(x)$ may be interpreted as the "measure" that the distance between u and v is less than x.

A very characteristic example, *for further work*, of the transversal upper edges spaces is the following space in the following form.

A *transversal upper edges T-space* is a pair (X, ρ), where X is a transversal upper edges space and where the upper (edges) transverse $\rho[u, v] = M_{u,v}(x)$ satisfies $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathbb{R}$ and (Eq).

Next, the concept of a neighborhood can be introduced and defined with the aid of the upper edges transverse. In fact, neighborhoods in transversal upper edges spaces may be defined in several nonequivalent ways. Here, we shall consider only one of these.

If $p \in X$, $\mu > c$ for some $c \in \mathbb{R}$ and r is a positive real, then a *(μ, r)-neighborhood* of p, denoted by $U_p(\mu, r)$, is defined by

$$U_p(\mu, r) = \{q \in X : \rho(p, q) = M_{p,q}(\mu) < a + r\}.$$

Corollary 13 *Let T be an increasing mapping of a transversal upper edges T-space, partially ordered with ordering $\preccurlyeq$, $X := (X, \preccurlyeq, M_{u,v}(t))$ into itself, where X satisfies the condition of spring sup MCS-convergence. Suppose that there exists an upper function $K_{x,y}(t)$, as a controlling function, such that*

$$M_{Tx,Ty}(t) \leq K_{x,y}\bigl(\varphi(t)\bigr) < \sup_{z,r \in \sigma(x,y,\infty)} M_{z,r}\bigl(\varphi(t)\bigr) \tag{6}$$

for all $x, y \in X$ or

$$M_{Tx,Ty}(t) < K_{x,y}\bigl(\varphi(t)\bigr) \leq \sup_{z,r \in \sigma(x,y,\infty)} M_{z,r}\bigl(\varphi(t)\bigr) \tag{6'}$$

for all $x, y \in X$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing function satisfying $\varphi^n(t) \to +\infty$ $(n \to \infty)$. If $r \preccurlyeq T(r)$ for some $r \in X$, then T has a unique fixed point in X.

The proof of this statement is a total analogy with the preceding proofs as consequences of the main statements. Thus we omit it.

In connection with the above, applying our *general principle of transpose* for nonnumerical transverses (see Tasković [5, p. 89]) to Corollary 13, we get an extended and generalized version of this result in the following sense.

Indeed, let $\mathfrak{S} := (\mathfrak{S}, \preccurlyeq)$ be a totally ordered set. A mapping $M : \mathfrak{S} \to [a, b] \subset P$ for $a \prec b$, where $P := (P, \preccurlyeq)$ is a partially ordered set is called an *upper ordered (distribution) function* if it is nonincreasing with $\inf M = a$ and $\sup M = b$. We will denote by $\mathfrak{D}$ the set of all upper ordered (distribution) functions.

The next two spaces are very interesting examples of transversal ordered upper spaces. *First*, an *upper ordered statistical space* is a pair $(X, \mathcal{R})$, where X is an abstract set and $\mathcal{R}$ is a mapping of $X \times X$ into the set of all upper ordered (distribution) functions $\mathfrak{D}$. We shall denote the upper ordered (distribution) function $\mathcal{R}(u, v)$ by $M_{u,v}(x)$ or $M_{u,v}$, hence the symbol $M_{u,v}(x)$ will denote the value of $M_{u,v}$ at $x \in \mathfrak{S}$. The functions $M_{u,v}$ are assumed to satisfy the following conditions: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathfrak{S}$, and

$$M_{u,v}(x) = a \quad \text{for } x \succ c \quad \text{if and only if} \quad u = v, \tag{Eq}$$

and if $M_{u,r}(x) = a$ and $M_{r,v}(y) = a$ imply $M_{u,v}(x + y) = a$ for all $u, v, r \in X$ and for all $x, y \in \mathfrak{G}$.

In view of the condition $M_{u,v}(c) = b$, which evidently implies that $M_{u,v}(x) = b$ for every $x \preccurlyeq c$, condition (Eq) is equivalent to the statement $u = v$ if and only if $M_{u,v}(x) = A(x)$, where $A(x) = b$ if $x \preccurlyeq c$ and $A(x) = a$ if $x \succ c$. See Fig. 2.

Also, $M_{u,v}(x)$ may be interpreted as the "measure" that the distance between u and v is less than $x \in \mathfrak{G}$.

A very characteristic example, *for further work*, of the transversal ordered upper edges spaces is the following space in the following form.

A *transversal upper ordered edges T-space* is a pair (X, ρ), where X is a transversal upper ordered edges space and where the upper ordered (edges) transverse $\rho(u, v) = M_{u,v}(x)$ satisfies $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in \mathfrak{G}$, and (Eq).

If $p \in X$, $\mu \succ c$ for some $c \in \mathfrak{G}$ and $r \succ a$, then a (μ, r)-*neighborhood* of p, denoted by $U_p(\mu, r)$, is defined by

$$U_p(\mu, r) = \{q \in X : \rho(p, q) = M_{p,q}(\mu) \prec r\}.$$

Applying our general principle of transpose for partially ordered sets to Corollary 13 and to upper spring ordered transversal space $X := (X, \preccurlyeq, A)$ for $A(u, v) := M_{u,v}(x)$ *directly*, we get an extended and generalized version of Corollary 13 in the following form.

Corollary 14 *Let T be an increasing mapping of an upper transversal spring ordered space $X := (X, \preccurlyeq, A)$ for $A := M_{u,v}(t)$ with the property* (Eq) *into itself, where X satisfies the condition of spring sup MCS-convergence. Suppose that there exists an increasing function $\varphi : \mathfrak{G} \to \mathfrak{G}$ ($\mathfrak{G}$ is a totally ordered set) such that $t \prec \varphi(t)$ for every $t \in \mathfrak{G}$ and for some $c \in \mathfrak{G}$ the following equality holds*:

$$\lim_{n\to\infty} M_{u,v}(\varphi^n(t)) = a \quad \textit{for every } t \succ c,$$

and there exists an upper function $K_{x,y}(t)$, as a controlling function, such that

$$M_{Tx,Ty}(t) \preccurlyeq K_{x,y}(\varphi(t)) \prec \sup_{z,r\in\sigma(x,y,\infty)} M_{z,r}(\varphi(t)) \tag{7}$$

for all $x, y \in X$ or

$$M_{Tx,Ty}(t) \prec K_{x,y}(\varphi(t)) \preccurlyeq \sup_{z,r\in\sigma(x,y,\infty)} M_{z,r}(\varphi(t)) \tag{7'}$$

for all $x, y \in X$. If $r \preccurlyeq T(r)$ for some $r \in X$, then T has a unique fixed point in X.

The proof of this statement is a total analogy with the preceding proofs as consequences of the main statements. Thus we omit it.

Corollary 15 (Tasković [6, p. 549]) *Let T be an increasing mapping of a transversal upper edges T-space (partially ordered with ordering $\preccurlyeq$), $X := (X, \preccurlyeq, M_{u,v}(t))$ into itself, where*

X satisfies the condition of spring sup MCS-convergence. Suppose that there exists an increasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some fixed $c \in \mathbb{R}$ satisfying

$$\lim_{n\to\infty} \varphi^n(t) = +\infty \quad \textit{for every } t > c \tag{As}$$

such that ($\mathfrak{G}$):

$$M_{Tu,Tv}(x) \le \max\{M_{u,v}(\varphi(x)), M_{u,Tu}(\varphi(x)), M_{v,Tv}(\varphi(x)), M_{u,Tv}(\varphi(x)), M_{v,T(u)}(\varphi(x))\}$$

for all $u, v \in X$ and for every $x > c$. If $r \precsim T(r)$ for some $r \in X$, then T has a unique fixed point in X.

We notice that in 2001 Tasković proved a special case of this statement on a transversal upper edges T-space $X := (X, M_{u,v}(t))$. Also, in 1998 Tasković introduced the concept of upper, lower, and middle transversal edges spaces.

In the context of this statement the following conditions are special cases of condition ($\mathfrak{G}$):

(1) There exists a constant $0 < k < 1$ such that, for all $p, q \in X$, for some fixed $c \in \mathbb{R}$, and for every $x > c$, the following inequality holds:

$$M_{Tp,Tq}(x) \le M_{p,q}\left(\frac{x}{k}\right).$$

(2) There exists a nondecreasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some fixed $c \in \mathbb{R}$ satisfying (As) such that, for some fixed $0 < k < 1$, for all $u, v \in X$, and for every $x > c$, the following inequality holds:

$$M_{Tu,Tv}(\varphi(kx)) \le \max\{M_{u,v}(\varphi(x)), M_{u,Tu}(\varphi(x)), M_{v,Tv}(\varphi(x)), M_{u,Tv}(\varphi(x)), M_{v,Tu}(\varphi(x))\}.$$

(3) There exists a constant $0 < k < 1$ such that, for all $u, v \in X$, for some fixed $x \in \mathbb{R}$, and for every $x > c$, the following inequality holds:

$$M_{Tu,Tv}(kx) \le \max\{M_{u,v}(x), M_{u,Tu}(x), M_{v,Tv}(x), M_{u,Tv}(x), M_{v,Tu}(x)\}$$

or in an equivalently form as

$$M_{Tu,Tv}(x) \le \max\left\{M_{u,v}\left(\frac{x}{k}\right), M_{u,Tu}\left(\frac{x}{k}\right), M_{v,Tv}\left(\frac{x}{k}\right), M_{u,Tv}\left(\frac{x}{k}\right), M_{v,Tu}\left(\frac{x}{k}\right)\right\}.$$

Corollary 16 (Tasković [6, p. 558]) *Let T be an increasing mapping of an upper spring ordered transversal space $X := (X, \precsim, A)$ for $A(u, v) := M_{u,v}(t)$ with the property* (Eq) *into itself, where X satisfies the condition of upper MCS-convergence. Suppose that there exists an increasing function $\varphi : \mathfrak{G} \to \mathfrak{G}$ ($\mathfrak{G}$ is a totally ordered set) such that $t \prec \varphi(t)$ for every $t \in \mathfrak{G}$ and for some fixed $c \in \mathfrak{G}$ the following equality holds*:

$$\lim_{n\to\infty} M_{u,v}(\varphi^n(t)) = a \quad \textit{for every } t \succ c,$$

and

$$M_{Tu,Tv}(x) \preccurlyeq \sup\{M_{u,v}(\varphi(x)), M_{u,Tu}(\varphi(x)), M_{v,Tv}(\varphi(x)), M_{u,Tv}(\varphi(x)), M_{v,T(u)}(\varphi(x))\}$$

for all $u, v \in X$ and for every $x \succ c$. If $r \circledcurlyeq T(r)$ for some $r \in X$, then T has a unique fixed point in X.

Proof Applying the general principle of transpose for posets to Corollary 15, we get a *directly* extended and generalized version of a result given in Corollary 15. The proof is complete. □

Annotations The results of Corollaries 15 and 16 are two immediate consequences of Corollary 3. Uniqueness follows immediately from the conditions of given inequalities.

Spring lower ordered spaces. Let X be a nonempty set, and let $P := (P, \preccurlyeq)$ be a partially ordered set such that $a, b \in P$ and $a \prec b$. The set (*interval*) $(a, b]$ is defined by

$$(a, b] := \{t \in P : a \prec t \preccurlyeq b\}.$$

The function $A : X \times X \to (a, b] \subset P$ for $a \prec b$ is called a *lower spring ordered transverse* (or *lower spring ordered transversal*) on a nonempty set X iff $A(x, y) = b$ if and only if $x = y$ for all $x, y \in X$.

A *lower spring ordered transversal space* $X := (X, \circledcurlyeq, A)$ is a nonempty partially ordered set X (with ordering $\circledcurlyeq$) together with a given lower spring ordered transverse A on X, where every increasing sequence $\{u_n\}_{n\in\mathbb{N}}$ of elements in $(a, b]$ has a unique element u in $(a, b]$ as limit (in notation $u_n \to u$ $(n \to \infty)$). The element $b \in (a, b] \subset P$ is called *spring of space* X (cf. [5]).

In 1986 we investigated the concept of lower spring ordered TCS-convergence in a space X, i.e., a lower spring ordered transversal space $X := (X, A)$ satisfies the condition of *lower spring ordered TCS-convergence* iff $x \in X$ and if $A(T^n(x), T^{n+1}(x)) \to b$ $(n \to \infty)$ implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ has a convergent subsequence in X, see Tasković [5].

In connection with the above, we shall introduce the concept of lower MCS-convergence, i.e., a lower spring ordered transversal space $X := (X, \circledcurlyeq, A)$ satisfies the condition of *lower MCS-convergence* (i.e., *lower MSC-completeness*) iff $x \in X$ and if $A(f^n(x), f^{n+1}(x)) \to b$ $(n \to \infty)$ implies that $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded X.

We notice that the sequence $\{x_n\}_{n\in\mathbb{N}}$ in the lower spring ordered transversal space $X := (X, A)$ is *convergent* (or *lower convergent*) in notation $x_n \to x$ $(n \to \infty)$ iff $A(x_n, x) \to b$ as $n \to \infty$; or equivalently, for an increasing sequence $\{b_n\}_{n\in\mathbb{N}} \in (a, b]$ which converges to b, the following inequality holds:

$$A(x_n, x) \succ b_n \quad \text{for every } n \in \mathbb{N},$$

or for n large enough.

On the other hand, in connection with this, the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X will be called *lower fundamental* (or *lower spring fundamental*) if the following inequality holds:

$$A(x_n, x_m) \succ b_n \quad \text{for all } n, m \in \mathbb{N}\ (n < m),$$

or for n and m large enough, where the increasing sequence $\{b_n\}_{n\in\mathbb{N}} \in (a, b]$ converges to b (cf. [5]).

A lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ is called *lower complete* (or *lower spring complete*) if any lower fundamental sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is lower convergent (to a point of X, of course).

In connection with the preceding facts, based on Lemma 1 and Theorem 1, we are now in a position to formulate the following new statements as *direct* consequences of the Axiom of Infinite Choice.

Corollary 17 (Tasković [4, p. 244]) *Let $P := (P, \preccurlyeq)$ be a partially ordered set and f be an increasing mapping from P into P. If the following set $P(f \preccurlyeq) := \{x \in P : f(x) \preccurlyeq x\}$ is nonempty such that there exists the infimum $I := \inf P(f \preccurlyeq)$, then f has at least countable or finite fixed points.*

For the first time, in 1980 Tasković proved that f has at least one fixed point for the preceding case of Corollary 17. Further application of the Axiom of Infinite Choice follows Corollary 17, which is a dual form of Corollary 1. The proof is a total analogy with the proof of Corollary 2.

Corollary 18 *Let $P := (P, \preccurlyeq)$ be a partially ordered set and f be an increasing mapping from P into P such that*

$$f(a) \preccurlyeq a \quad \text{for some } a \in P, \tag{R}$$

where every decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in P is bounded, then f has at least countable or finite fixed points in P.

This statement is a dual form of Corollary 2. Thus we omit the proof of Corollary 18.

Lower spring ordered transversal spaces. In this paper we shall introduce the concept of lower MBV-convergence, i.e., a lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ satisfies the condition of *lower MBV-convergence* (i.e., *lower MBV-completeness*) if every decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is bounded.

As an immediate *direct* consequence of the preceding Corollary 18, we give the following new result for lower spring ordered transversal spaces.

Corollary 19 *Let f be an increasing mapping of a lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where X satisfies the condition of lower MBV-convergence. If*

$$f(a) \preccurlyeq a \quad \text{for some } a \in X, \tag{R}$$

then f has at least countable or at least finite fixed points in X.

The proof of this statement is an elementary fact because the condition of lower MBV-completeness implies that every decreasing sequence of iterates is bounded in Corollary 18.

Corollary 20 *Let f be an increasing mapping of a lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where every decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is lower fundamental. If*

$$f(a) \preccurlyeq a \quad \textit{for some } a \in X, \tag{R}$$

then f has at least countable or at least finite fixed points in X.

Proof Since $f(a) \preccurlyeq a$ and f is isotone, we find $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ is a decreasing sequence of iterates which is lower fundamental, i.e., bounded in X. It is easy to see that X satisfies all the required hypotheses in Corollary 19. Applying Corollary 19 to this case, we obtain this statement. The proof is complete. □

Corollary 21 (Partially ordered metric spaces) *Let f be an increasing mapping of an ordered metric space $X := (X, \preccurlyeq, A)$ into itself, where every decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is a Cauchy sequence. If*

$$f(a) \preccurlyeq a \quad \textit{for some } a \in X, \tag{R}$$

then f has at least countable or at least finite fixed points in X.

Proof Since an ordered metric space is an example of a lower spring ordered transversal space, thus this statement follows *directly* from Corollary 20. The proof is complete. □

Corollary 22 (Lower spring ordered transversal spaces) *Let f be an increasing mapping of a lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where every increasing and decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is bounded. If*

$$f(a) \preccurlyeq a \quad \textit{for some } a \in X, \tag{R}$$

or

$$x \prec f(x) \quad \textit{for every } x \in X, \tag{E}$$

then f has at least countable or at least finite fixed points in X.

The proof of this statement is a total analogy with the preceding proofs of Corollaries 2 and 3 (because Corollary 22 is a dual form of Corollary 6). Thus we omit it.

Corollary 23 (Partially ordered metric spaces) *Let f be an increasing mapping of an ordered metric space $X := (X, \preccurlyeq, \rho)$ into itself, where every increasing and decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is a Cauchy sequence. If* (R) *or* (E) *holds, then f has at least countable or finite fixed points in X.*

This result is an example for Corollary 22. Also, Corollary 23 is a dual form of Corollary 7. Thus we omit the proof.

Transversal lower edges spaces. Let X be a nonempty set. The function $\rho : X \times X \to [a, b] \subset \mathbb{R}^0_+$ (or $\rho : X \times X \to (a, b] \subset \mathbb{R}^0_+$) for $a < b$ is called a *lower edges transverse* on X (or *lower edges transversal*) iff $\rho(x,y) = \rho(y,x)$, $\rho(x,y) = b$ if and only if $x = y$, and if there is a function $d : [a,b]^2 \to [a,b]$ such that

$$\rho(x,y) \geq \min\bigl\{\rho(x,z), \rho(z,y), d\bigl(\rho(x,z), \rho(z,y)\bigr)\bigr\} \tag{Ca}$$

for all $x,y,z \in X$.

A *lower edges transversal space* (or *lower edges space*) is a set X together with a given lower edges transverse on X. The function $d : [a,b]^2 \to [a,b]$ in (Ca) is called a *lower bisection function*.

From (Ca) it follows by induction that there is a function $\mathcal{D} : [a,b]^n \to [a,b]$ for $a < b$ such that the following inequality holds:

$$\rho(x_0,x_n) \geq \min\bigl\{\rho(x_0,x_1), \dots, \rho(x_{n-1},x_n), \mathcal{D}\bigl(\rho(x_0,x_1), \dots, \rho(x_{n-1},x_n)\bigr)\bigr\}$$

for all $x_0, x_1, \dots, x_n \in X$ and for an arbitrary fixed integer $n \geq 2$.

Example 4 (Metric spaces) A fundamental first example of lower edges transversal space is a metric space. Indeed, if (X,q) is a metric space, then for the lower bisection function $d(r,t) = r + t$, we have the following lower edges transverse $\rho : X \times X \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ defined by

$$\rho(x,y) = \frac{(a-b)q(x,y)}{1 + q(x,y)} + b$$

for all $x,y \in X$. Thus (X,ρ) is an example of a lower edges transversal space. In general, every metric space is an example of a lower edges transversal space.

Example 5 (Lower probabilistic spaces) A mapping $F : \mathbb{R} \to \mathbb{R}^0_+$ is called a *distribution function* if it is nondecreasing, left-continuous with $\inf F = 0$ and $\sup F = 1$. We will denote by $\mathcal{L}$ the set of all distribution functions. We shall denote the distribution function $\mathcal{F}(p,q)$ by $F_{p,q}(x)$, whence $F_{p,q}(x)$ will denote the value of $F_{p,q}$ at $x \in \mathbb{R}$.

An example of lower edges transversal space is a *lower probabilistic space* which is a nonempty set X together with the functions $F_{p,q}(x)$ with the following properties: $F_{p,q}(x) = F_{q,p}(x)$, $F_{p,q}(0) = 0$,

$$F_{p,q}(x) = 1 \quad \text{for } x > 0 \quad \text{if and only if} \quad p = q,$$

and if there is a nondecreasing function $\tau : [0,1]^2 \to [0,1]$ with the property $\tau(t,t) \geq t$ for all $t \in [0,1]$ such that

$$F_{p,q}(x+y) \geq \tau\bigl(F_{p,r}(x), F_{r,q}(y)\bigr) \tag{Nm}$$

for all $p,q,r \in X$ and for all $x,y \geq 0$. If we choose a lower bisection function $d : [0,1]^2 \to [0,1]$ such that $d = \tau$ (from (Nm)), then we immediately obtain that every lower probabilistic space, for $\rho[p,q] = F_{p,q}(x) : X \times X \to [0,1]$, is a lower edges transversal space.

Example 6 (Transversal lower edges r-spaces) A fundamental example of a transversal lower edges space is a *transversal lower edges r-space* (for $0 < r \leq 1$). In 1998 this space was known as a transversal lower space. For $d(s,t) := r\min\{s,t\}$ and $0 < r \leq 1$, we obtain *directly* in this case the transversal lower edges r-space. This space is known for $0 < r \leq 1$ as a *transversal lower edges r-min space* or only as an *r-min edges space*. For this, see Tasković [16].

For any nonempty set S in the lower edges transversal space X, the *diameter* of S is defined as $\operatorname{diam}(S) := \inf\{\rho(x,y) : x,y \in S\}$; it is a real number in $[a,b]$, $A \subset B$ implies $\operatorname{diam}(B) \leq \operatorname{diam}(A)$. The relation $\operatorname{diam}(S) = b$ holds if and only if S is a one-point set.

Elements of a lower edges transversal space will usually be called points. Given a lower edges transversal space (X,ρ), with the bisection function $d : [a,b]^2 \to [a,b]$ and a point $z \in X$, the *open ball* of *center* z and radius $r > 0$ is the set

$$d\bigl(B(z,r)\bigr) = \bigl\{x \in X : \rho(z,x) > b - r\bigr\}.$$

The *convergence* $x_n \to x$ as $n \to \infty$ in the lower edges transversal space (X,ρ) means that $\rho(x_n,x) \to b$ as $n \to \infty$; or equivalently, for every $\varepsilon > 0$, there exists an integer n_0 such that the relation $n \geq n_0$ implies $\rho(x_n,x) > b - \varepsilon$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ in the lower edges transversal space (X,ρ) is called a *transversal sequence* (or a *lower Cauchy sequence*) iff for every $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ such that

$$\rho(x_n,x_m) > b - \varepsilon \quad \text{for all } n,m \geq n_0.$$

Let (X,ρ) be a lower edges transversal space and $T : X \to X$. We notice, from Tasković [16], that a sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ in X is said to be a transversal sequence if and only if

$$\lim_{n\to\infty}\bigl(\operatorname{diam}\bigl\{T^k(x) : k \geq n\bigr\}\bigr) = b.$$

In this sense, a lower edges transversal space is called *lower complete* iff every transversal sequence converges.

Also, a space (X,ρ) is said to be *lower orbitally complete* (or *lower T-orbitally complete*) iff every transversal sequence which is contained in $\mathcal{O}(x)$ for some $x \in X$ converges in X.

A function T mapping X into the reals is *T-orbitally upper semicontinuous* at $p \in X$ iff $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{O}(x)$ and $x_n \to p$ $(n \to \infty)$ implies that $T(p) \geq \lim.\sup T(x_n)$.

Let (X,ρ) be a lower edges transversal space. A mapping $T : X \to X$ is said to be *lower edges contraction* if there exists $0 \leq \lambda < 1$ such that

$$\rho\bigl(T(x),T(y)\bigr) \geq \lambda\rho(x,y) + b(1-\lambda) \tag{Le}$$

for all points $x,y \in X$. For further facts on the lower edges contraction, see Tasković [6].

Let (X,ρ_X) and (Y,ρ_Y) be two lower edges transversal spaces, and let $T : X \to Y$.

We notice from Tasković [6] that T is *lower edges continuous* at $x_0 \in X$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that the relation

$$\rho_X(x_0,x) > b - \delta \quad \text{implies} \quad \rho_Y\bigl(T(x_0),T(x)\bigr) > b - \varepsilon.$$

A typical first example of a lower edges continuous mapping is the lower edges contraction on the lower edges transversal space (X,ρ). For further facts on the lower edges continuous mappings, see *Tasković* [6].

In connection with the preceding facts, we are now in a position to formulate a localization for lower edges contractions in the following form.

Proposition 5 *Let f be an increasing mapping of a partially ordered transversal lower edges space $X := (X,\preccurlyeq,\rho)$ into itself, where X satisfies the condition of the lower MCS-convergence. Suppose that there exists $0 \leq \lambda < 1$ such that*

$$\rho\bigl(f(x),f^2(x)\bigr) \geq \lambda\rho\bigl(x,f(x)\bigr) + b(1-\lambda) \tag{Cl}$$

for every $x \in X$. If $f(r) \preccurlyeq r$ for some $r \in X$, then f has at least countable or finite fixed points in X.

Proof We notice that a partially ordered transversal lower edges space $X := (X,\preccurlyeq,\rho)$ is an example of a lower spring ordered transversal space. From (Cl) it follows $\rho(f^n(x), f^{n+1}(x)) \to b$ $(n \to \infty)$, hence by lower MCS-completeness, we obtain that every decreasing sequence of iterates $\{f^n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ in X is bounded, i.e., X satisfies the condition of lower MBV-completeness. It is easy to see that f and X satisfy all the required hypotheses in Corollary 19, thus f has at least countable or finite fixed points in X. The proof is complete. □

Proposition 6 *Let f be an increasing mapping of a partially ordered transversal lower edges space $X := (X,\preccurlyeq,\rho)$ into itself. Suppose that there exists $0 \leq \lambda < 1$ such that*

$$\rho\bigl(f(x),f(y)\bigr) \geq \lambda\rho(x,y) + b(1-\lambda) \tag{Ld}$$

for all $x,y \in X$. If $f(r) \preccurlyeq r$ for some $r \in X$, then f has a unique fixed point in X. (The space X in this statement is not obligatory complete).

Proof Let $y = f(x)$ in (Ld), then it is easy to see that f and X satisfy all the required hypotheses in Proposition 5. Uniqueness follows immediately from condition (Ld). The proof is complete. □

Proposition 7 *Let f be an increasing mapping of a partially ordered transversal lower edges space $X := (X,\preccurlyeq,\rho)$ into itself. Suppose that there exists a function $\varphi : [a,b] \to [a,b]$ satisfying*

$$\bigl(\forall t \in [a,b)\bigr) \quad \Bigl(\varphi(t) > t \text{ and } \liminf_{z\to t-0}\varphi(z) > t\Bigr) \tag{Id}$$

such that

$$\rho\bigl(f(x),f(y)\bigr) \geq \varphi\bigl(\operatorname{diam}\bigl\{x,y,f(x),f(y),f^2(x),f^2(y),\dots\bigr\}\bigr) \tag{D}$$

for all $x,y \in X$. If $f(r) \preccurlyeq r$ for some $r \in X$, then f has a unique fixed point in X. (The space X in this statement is not obligatory complete.)

The proof of this statement is a total analogy with the preceding proofs of Propositions 5 and 6. Thus we omit it.

A brief proof of this statement may be found in Tasković [6] with application of the following context.

Lemma 3 (Tasković [6]) *Let the mapping $\varphi : [a,b] \to [a,b] \subset \mathbb{R}_+^0$ for $a < b$ have the property* (Id). *If the sequence (x_n) of nonnegative real numbers satisfies the condition*

$$x_{n+1} \geq \varphi(x_n), \quad n \in \mathbb{N},$$

then the sequence (x_n) tends to b. The velocity of this convergence is not necessarily geometrical.

A brief first proof of this statement may be found in Tasković [6]. We notice that Lemma 3 is a dual form of Lemma 2.

As an immediate consequence of the preceding Proposition 7, we obtain directly the following interesting cases of (D):

(6) There exists a nondecreasing function $\varphi : [a,b] \to [a,b] \subset \mathbb{R}_+^0$ for $a < b$ satisfying $\liminf_{z \to t-0} \varphi(z) > t$ for every $t \in [a,b)$ such that

$$\rho\bigl(f(x),f(y)\bigr) \geq \varphi\bigl(\operatorname{diam}\bigl\{x,y,f(x),f(y)\bigr\}\bigr)$$

for all $x,y \in X$.

(7) (*Special affine case of condition* (D) *for* $\varphi(t) = \alpha t + b(1-\alpha)$.) There exists a constant $\alpha \in [0,1)$ such that, for all $x,y \in X$, the following inequality holds:

$$\rho\bigl(f(x),f(y)\bigr) \geq \alpha \operatorname{diam}\bigl\{x,y,f(x),f(y)\bigr\} + b(1-\alpha),$$

i.e., equivalent to

$$\rho\bigl(f(x),f(y)\bigr) \geq \alpha \min\bigl\{\rho(x,y), \rho\bigl(x,f(x)\bigr), \rho\bigl(y,f(y)\bigr), \rho\bigl(x,f(y)\bigr), \rho\bigl(y,f(x)\bigr)\bigr\} + b(1-\alpha).$$

(8) (*The condition of* $(m+k)$*-polygon*.) There exists a constant $\lambda \in [0,1)$ such that, for all $x,y \in X$, the following inequality holds:

$$\rho\bigl(f(x),f(y)\bigr) \geq \lambda \operatorname{diam}\bigl\{x,y,f(x),f(y),\ldots,f^m(x),f^k(y)\bigr\} + b(1-\lambda)$$

for arbitrary fixed integers $m,k \geq 0$. (This is a linear condition for the diameter of a finite number of points.)

(9) There exists a nondecreasing function $\varphi : [a,b] \to [a,b] \subset \mathbb{R}_+^0$ for $a < b$ satisfying $\liminf_{z \to t-0} \varphi(z) > t$ for every $t \in [a,b)$ such that

$$\rho\bigl(f(x),f(y)\bigr) \geq \varphi\bigl(\operatorname{diam}\bigl\{x,y,f(x),f(y),\ldots,f^m(x),f^k(y)\bigr\}\bigr)$$

for arbitrary fixed integers $m,k \geq 0$ and for all $x,y \in X$.

(10) There exists an increasing mapping $\psi : [a,b]^5 \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ satisfying $\liminf_{z\to t-0} \psi(z,z,z,z,z) > t$ for every $t \in [a,b)$ such that

$$\rho\big(f(x),f(y)\big) \geq \psi\big(\rho(x,y),\rho\big(x,f(x)\big),\rho\big(y,f(y)\big),\rho\big(x,f(y)\big),\rho\big(y,f(x)\big)\big)$$

for all $x,y \in X$.

In connection with the preceding facts, we are now in a position to formulate a localization of Proposition 7 in the following form.

Proposition 8 *Let f be an increasing mapping of a partially ordered transversal lower edges space $X := (X, \preccurlyeq, \rho)$ into itself. Suppose that there exists a function $\varphi : [a,b] \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ satisfying* (Id) *such that*

$$\operatorname{diam}\big\{f(x),f^2(x),\ldots\big\} \geq \varphi\big(\operatorname{diam}\big\{x,f(x),f^2(x),\ldots\big\}\big)$$

for every $x \in X$. If $f(r) \preccurlyeq r$ for some $r \in X$, then f has at least countable or finite fixed points in X. (*The space X in this statement is not obligatory complete.*)

The proof of this statement is a total analogy with the preceding proof of Proposition 7. Thus we omit it.

A lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ is called *lower complete* (or *lower spring complete*) if any lower fundamental sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is lower convergent (to a point of X, of course).

On the other hand, in this paper we shall introduce the concept of spring inf MCS-convergence, i.e., a lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ satisfies the condition of *spring inf MCS-convergence* iff $x \in X$ and if $\inf_{i,j\geq n} A(T^i(x), T^j(x))$ or $\inf_{i,j\geq 2n} A(T^i(x), T^j(x))$ or $\inf_{i,j\geq 2n+1} A(T^i(x), T^j(x))$ converges to $u, v, c \in (a,b]$ respectively implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n}(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n+1}(x)\}_{n\in\mathbb{N}}$ is a bounded sequence in X, respectively.

Theorem 5 (Monotone principle of fixed point) *Let T be an increasing mapping of a lower spring ordered transversal space $X := (X, \preccurlyeq, A)$ into itself, where X satisfies the condition of spring inf MCS-convergence. Suppose that there exists a controlling function $B : X \times X \to (a,b]$ for $a \prec b$ such that*

$$A\big(T(x), T(y)\big) \succcurlyeq B(x,y) \succ \inf_{z,r\in\sigma(x,y,\infty)} A(z,r) \succ a \tag{8}$$

for all $x, y \in X$ or

$$A\big(T(x), T(y)\big) \succ B(x,y) \succcurlyeq \inf_{z,r\in\sigma(x,y,\infty)} A(z,r) \succ a \tag{8'}$$

for all $x, y \in X$. If $T(r) \preccurlyeq r$ for some $r \in X$, then T has at least countable or finite fixed points in X.

If additionally $A(t,t) \succcurlyeq \inf\{A(s,t), A(t,s)\}$ for all $s, t \in X$, then T has a unique fixed point in X.

Let $X := (X, \textcircled{\preccurlyeq}, M)$ be a partially ordered with ordering $\textcircled{\preccurlyeq}$ topological space and $T : X \to X$, where $M : X \to (a, b] \subset P$ for $a \prec b$. In this part, we shall introduce the concept of local spring inf MCS-convergence in a space X, i.e., a topological space X satisfies the condition of *local spring inf MCS-convergence* iff $x \in X$ and $\inf_{i\geq n} M(T^i(x))$ or $\inf_{i\geq 2n} M(T^i(x))$ or $\inf_{i\geq 2n+1} M(T^i(x))$ converges to $u, v, c \preccurlyeq b$ respectively implies that $\{T^n(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n}(x)\}_{n\in\mathbb{N}}$ or $\{T^{2n+1}(x)\}_{n\in\mathbb{N}}$ is a bounded sequence in X, respectively.

We are now in a position to formulate the following theorem on partially ordered (with ordering $\textcircled{\preccurlyeq}$) topological spaces with non-numerical transverses.

Theorem 6 (Localization monotone principle) *Let T be an increasing mapping of a partially ordered with ordering $\textcircled{\preccurlyeq}$ topological space $X := (X, \textcircled{\preccurlyeq}, M)$ into itself, where X satisfies the condition of local spring inf MCS-convergence. Suppose that there exists a controlling function $N : X \to (a, b] \subset P$ for $a \prec b$ such that*

$$M\big(T(x)\big) \succcurlyeq N(x) \succcurlyeq \inf_{z\in\sigma(x,\infty)} M(z) \succ a \quad \textit{for every } x \in X \tag{D}$$

and $T(r) \textcircled{\preccurlyeq} r$ for some $r \in X$. Then T has at least countable or finite fixed points in X.

An immediate consequence of the preceding statement is the following result.

Corollary 24 *Let T be an increasing mapping of a partially ordered with ordering $\textcircled{\preccurlyeq}$ topological space $X := (X, \textcircled{\preccurlyeq}, M)$ into itself, where X satisfies the condition of local spring inf MCS-convergence. Suppose that there exists a controlling function $N : X \to (a, b] \subset P$ for $a \prec b$ such that*

$$M\big(T(x)\big) \succcurlyeq N(x) \succcurlyeq M(x) \quad \textit{for every } x \in X, \tag{D$'$}$$

and $T(r) \textcircled{\preccurlyeq} r$ for some $r \in X$. Then T has at least countable or finite fixed points in X.

The proof of this statement is an elementary fact because condition (D$'$) implies condition (D).

Proof of Theorem 6 Let $x \in X$ be an arbitrary point and $n \in \mathbb{N} \cup \{0\}$ be any nonnegative integers. From (D) for $T^i(x)$, we have $M(T^{i+1}(x)) \succcurlyeq N(T^i(x)) \succcurlyeq \inf_{z\in\sigma(T^i(x),\infty)} M(z)$, and hence

$$\inf_{i\geq n+1} M\big(T^i(x)\big) \succcurlyeq \inf_{i\geq n} N\big(T^i(x)\big) \succcurlyeq \inf_{i\geq n} M\big(T^i(x)\big), \tag{9}$$

i.e., we obtain that $\{\inf_{i\geq n} M(T^i(x))\}_{n\in\mathbb{N}}$ is an increasing convergent sequence in $(a, b] \subset P$. This implies (from local spring inf MCS-convergence) that its sequence of iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ is a bounded sequence in X, i.e., X satisfies the condition of lower MBV-completeness.

In the cases of other two sequences, in local spring inf MCS-convergence, the proof is a total analogy. Hence we omit the proof in these cases. It is easy to see that T and X satisfy all the required hypotheses in Corollary 19, thus T has at least countable or finite fixed points in X. The proof is complete. □

Proof of Theorem 5 Let $M(x) := A(x, T(x))$ and $N(x) := B(x, T(x))$, then it is easy to see that A, B, and X satisfy all the required hypotheses in Theorem 6. Uniqueness follows immediately from conditions (8) and (8′). The proof is complete. □

Corollary 25 *Let T be an increasing mapping of a Cartesian product of topological spaces X^k ($k \in \mathbb{N}$ is a fixed number) into X which is partially ordered with ordering $\preccurlyeq$, where $X := (X, \preccurlyeq, A)$ satisfies the condition of spring inf MCS-convergence. Suppose that there exists a controlling function $B : X \times X \to (a, b] \subset P$ for $a \prec b$ such that*

$$A\big(T(u_1,\dots,u_k), T(u_2,\dots,u_{k+1})\big) \succcurlyeq B(u_k,u_{k+1}) \succ \inf_{z,r\in\mathfrak{O}(x,Tx)} A(z,r) \succ a \tag{10}$$

for all $u_1,\dots,u_k,u_{k+1} \in X$ or

$$A\big(T(u_1,\dots,u_k), T(u_2,\dots,u_{k+1})\big) \succ B(u_k,u_{k+1}) \succcurlyeq \inf_{z,r\in\mathfrak{O}(x,Tx)} A(z,r) \succ a \tag{10′}$$

for all $u_1,\dots,u_k,u_{k+1} \in X$, where $x := (u_1,\dots,u_k)$. If $T(r,\dots,r) \preccurlyeq r$ for some $r \in X$, then the equation $x = T(x,\dots,x)$ has at least countable or finite solutions on X. If additionally $A(t,t) \succcurlyeq \inf\{A(s,t), A(t,s)\}$ for all $s,t \in X$, then there is a unique $\zeta \in X$ such that $T(\zeta,\dots,\zeta) = \zeta$.

Otherwise, a *transversal edges space* (or a *middle transversal edges space*) is an upper and a lower transversal edges space simultaneously.

As an important example of transversal lower edges spaces, we have Menger's (probabilistic) space. Karl Menger introduced the notion of probabilistic metric space in 1942.

In this sense, a mapping $N : \mathbb{R} \to [a,b] \subset \mathbb{R}^0_+$ for $a < b$ is called a *lower (distribution) function* if it is nondecreasing, left-continuous with $\inf N = a$ and $\sup N = b$. We will denote by $\mathcal{L}$ the set of all lower (distribution) functions.

A *lower statistical space* is a pair $(X, \mathcal{D})$, where X is an abstract set and $\mathcal{D}$ is a mapping of $X \times X$ into the set of all lower (distribution) functions $\mathcal{L}$. We shall denote the lower ordered (distribution) function $\mathcal{D}(p,q)$ by $N_{p,q}(x)$ or $N_{p,q}$, whence the symbol $N_{p,q}(x)$ will denote the value of $N_{p,q}$ at $x \in \mathbb{R}$. The functions $N_{p,q}$ are assumed to satisfy the following conditions: $N_{p,q} = N_{q,p}, N_{p,q}(c) = a$ for some $c \in \mathbb{R}$, and

$$N_{p,q}(x) = b \quad \text{for } x > c \quad \text{if and only if} \quad p = q, \tag{Em}$$

and if $N_{p,q}(x) = b$ and $N_{q,r}(y) = b$ imply $N_{p,r}(x + y) = b$ for all $p,q,r \in X$ and for all $x,y \in \mathbb{R}$.

In view of the condition $N_{p,q}(c) = a$ for some $c \in \mathbb{R}$, which evidently implies that $N_{p,q}(x) = a$ for all $x \le c$, condition (Em) is equivalent to the statement: $p = q$ if and only if $N_{p,q}(x) = H(x)$, where $H(x) = a$ if $x \le c$ and $H(x) = b$ if $x > c$. See Fig. 3.

Every metric space may be regarded as a statistical lower space of a special kind. One has only to set $N_{p,q}(x) = H(x - \rho(p,q))$ for every pair of points (p,q) in the metric space (X, ρ).

In connection with the above, a *transversal lower edges T-space* is a pair (X, ρ), where X is a transversal lower edges space and where the lower (edges) transverse $\rho[u,v] = N_{u,v}(x)$ satisfies $N_{u,v} = N_{v,u}$, $N_{u,v}(c) = a$ for some $c \in \mathbb{R}$ and (Em). This space is a very characteristic

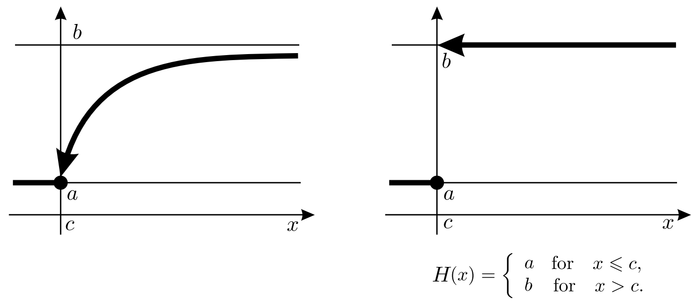

Figure 3 Geometric interpretation of the lower distribution function

example of transversal lower edges spaces for further work. Every Menger's space is also a lower edges space.

The concept of a neighborhood in a lower transversal edges space X for the lower edges transverse $\rho(p,q) = N_{p,q}(x)$ in $[a,b] \subset \mathbb{R}^0_+$ for $a < b$ is the following. If $p \in X$, $\mu > c$ for some $c \in \mathbb{R}$ and σ is a positive real, then a (μ,σ)-*neighborhood* of p, denoted by $\mathcal{O}_p(\mu,\sigma)$, is defined by

$$\mathcal{O}_p(\mu,\sigma) = \{q \in X : \rho(p,q) = N_{p,q}(\mu) > b - \sigma\}.$$

Corollary 26 *Let T be an increasing mapping of a transversal lower edges T-space (partially ordered with ordering $\preccurlyeq$) $X := (X, \preccurlyeq, N_{u,v}(t))$ into itself, where X satisfies the condition of spring inf MCS-convergence. Suppose that there exists a lower function $K_{x,y}(t)$, as a controlling function, such that*

$$N_{Tx,Ty}(t) \geq K_{x,y}\bigl(\varphi(t)\bigr) > \inf_{z,r\in\sigma(x,y,\infty)} N_{z,r}\bigl(\varphi(t)\bigr) \tag{11}$$

for all $x,y \in X$ or

$$N_{Tx,Ty}(t) > K_{x,y}\bigl(\varphi(t)\bigr) \geq \inf_{z,r\in\sigma(x,y,\infty)} N_{z,r}\bigl(\varphi(t)\bigr) \tag{11'}$$

for all $x,y \in X$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing function satisfying $\varphi^n(t) \to +\infty$ $(n \to \infty)$. If $T(r) \preccurlyeq r$ for some $r \in X$, then T has a unique fixed point in X.

The proof of this statement is a total analogy with the preceding proofs as consequences of the main statements. Thus we omit it.

In connection with the above, applying our *general principle of transpose* for nonnumerical transverses (see Tasković [5, p. 89]) to Corollary 26, we get an extended and generalized version of this result in the following sense.

Indeed, let $\mathfrak{G} := (\mathfrak{G}, \preccurlyeq)$ be a totally ordered set. A mapping $N : \mathfrak{G} \to [a,b] \subset P$ for $a \prec b$, where $P := (P, \preccurlyeq)$ is a partially ordered set, is called a *lower ordered (distribution) function* if it is nonincreasing with $\inf N = a$ and $\sup N = b$. We will denote by $\mathfrak{L}$ the set of all lower ordered (distribution) functions.

The next two spaces are very interesting examples of transversal ordered lower spaces. *First*, a *lower statistical space* is a pair $(X, \mathcal{R})$, where X is an abstract set and $\mathcal{R}$ is a mapping of $X \times X$ into the set of all lower ordered (distribution) functions $\mathfrak{L}$. We shall denote the lower ordered (distribution) function $\mathcal{R}(u,v)$ by $N_{u,v}(x)$ or $N_{u,v}$, whence the symbol $N_{u,v}(x)$ will denote the value of $N_{u,v}$ at $x \in \mathfrak{G}$. The functions $N_{u,v}$ are assumed to satisfy the following conditions: $N_{u,v} = N_{v,u}$, $N_{u,v}(c) = a$ for some $c \in \mathfrak{G}$, and

$$N_{u,v}(x) = b \quad \text{for } x \succ c \quad \text{if and only if} \quad u = v, \tag{Em}$$

and if $N_{u,r}(x) = b$ and $N_{r,v}(y) = b$ imply $N_{u,v}(x+y) = b$ for all $u, v, r \in X$ and for all $x, y \in \mathfrak{G}$.

In view of the condition $N_{u,v}(c) = a$, for some $c \in \mathfrak{G}$, which evidently implies that $N_{u,v}(x) = a$ for every $x \preccurlyeq c$, condition (Em) is equivalent to the statement: $u = v$ if and only if $N_{u,v}(x) = H(x)$, where $H(x) = a$ if $x \preccurlyeq c$ and $H(x) = b$ if $x \succ c$. See Fig. 3.

Also, $N_{u,v}(x)$ may be interpreted as the "measure" that the distance between u and v is less than $x \in \mathfrak{G}$.

A very characteristic example, *for further work*, of the transversal lower ordered edges spaces is the following space in the following form.

A *transversal lower ordered edges T-space* is a pair (X, ρ), where X is a transversal lower ordered edges space and where the lower ordered (edges) transverse $\rho(u,v) = N_{u,v}(x)$ satisfies $N_{u,v} = N_{v,u}$, $N_{u,v}(c) = a$ for some $c \in \mathfrak{G}$ and (Em).

Furthermore, the concept of a neighborhood can be introduced and defined with the aid of the lower ordered edges transverse. In fact, neighborhoods in transversal lower ordered edges spaces may be defined in several nonequivalent ways. Here, we shall consider only one of these.

If $p \in X$, $\mu \succ c$ for some $c \in \mathfrak{G}$ and $r \prec b$, then a (μ, r)-*neighborhood* of p, denoted by $U_p(\mu, r)$, is defined by

$$U_p(\mu, r) = \bigl\{q \in X : \rho(p,q) = N_{p,q}(\mu) \succ r\bigr\}.$$

Applying our general principle of transpose for partially ordered sets to Corollary 26 and to lower spring ordered transversal space $X := (X, \circledcirc\!\!\preccurlyeq, A)$ for $A(u,v) := N_{u,v}(x)$ *directly*, we get an extended and generalized version of Corollary 26 in the following form.

Corollary 27 *Let T be an increasing mapping of a lower spring ordered transversal space $X := (X, \circledcirc\!\!\preccurlyeq, A)$ for $A := N_{u,v}(t)$ with the property* (Em) *into itself, where X satisfies the condition of spring inf MCS-convergence. Suppose that there exists an increasing function $\varphi : \mathfrak{G} \to \mathfrak{G}$ ($\mathfrak{G}$ is a totally ordered set) such that $t \prec \varphi(t)$ for every $t \in \mathfrak{G}$ and for some $c \in \mathfrak{G}$ the following equality holds*:

$$\lim_{n\to\infty} N_{u,v}\bigl(\varphi^n(t)\bigr) = b \quad \textit{for every } t \succ c,$$

and there exists a lower function $K_{x,y}(t)$, as a controlling function, such that

$$N_{Tx,Ty}(t) \succcurlyeq K_{x,y}\bigl(\varphi(t)\bigr) \succ \inf_{z,r \in \sigma(x,y,\infty)} N_{z,r}\bigl(\varphi(t)\bigr) \tag{12}$$

for all $x, y \in X$ or

$$N_{Tx,Ty}(t) \succ K_{x,y}\bigl(\varphi(t)\bigr) \succcurlyeq \inf_{z,r \in \sigma(x,y,\infty)} N_{z,r}\bigl(\varphi(t)\bigr) \tag{12'}$$

for all $x, y \in X$. If $T(r) \circledcirc\!\!\preccurlyeq r$ for some $r \in X$, then T has a unique fixed point in X.

The proof of this statement is a total analogy with the preceding proofs as consequences of the main statements. Thus we omit it.

Corollary 28 (Tasković [6, p. 607]) *Let T be an increasing mapping of a transversal lower edges T-space (partially ordered with ordering $\circledcirc$), $X := (X, \circledcirc, N_{u,v}(t))$ into itself, where X satisfies the condition of spring inf MCS-convergence. Suppose that there exists an increasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some fixed $c \in \mathbb{R}$ satisfying*

$$\lim_{n\to\infty} \varphi^n(t) = +\infty \quad \text{for every } t > c \tag{As}$$

such that (F):

$$N_{Tu,Tv}(x) \geq \min\{N_{u,v}(\varphi(x)), N_{u,Tu}(\varphi(x)), N_{v,Tv}(\varphi(x)), N_{u,Tv}(\varphi(x)), N_{v,T(u)}(\varphi(x))\}$$

for all $u, v \in X$ and for every $x > c$. If $T(r) \circledcirc r$ for some $r \in X$, then T has a unique fixed point in X.

We notice that in 2001 Tasković proved a special case of this statement on a transversal lower edges T-space $X := (X, M_{u,v}(t))$. Also, in 1998 Tasković introduced the concept of upper, lower, and middle transversal edges spaces. Also see: [20–22] and [23].

In context of this statement, the following conditions are special cases of condition (F):

(1) There exists a constant $0 < k < 1$ such that, for all $p, q \in X$, for some fixed $c \in \mathbb{R}$, and for every $x > c$, the following inequality holds:

$$N_{Tp,Tq}(x) \geq N_{p,q}\left(\frac{x}{k}\right).$$

(2) There exists a nondecreasing function $\varphi : [c, +\infty) \to [c, +\infty)$ for some fixed $c \in \mathbb{R}$ satisfying (As) such that, for some fixed $0 < k < 1$, for all $u, v \in X$, and for every $x > c$, the following inequality holds:

$$N_{Tu,Tv}(\varphi(kx)) \geq \min\{N_{u,v}(\varphi(x)), N_{u,Tu}(\varphi(x)), N_{v,Tv}(\varphi(x)), N_{u,Tv}(\varphi(x)), N_{v,Tu}(\varphi(x))\}.$$

(3) There exists a constant $0 < k < 1$ such that, for all $u, v \in X$, for some fixed $x \in \mathbb{R}$, and for every $x > c$, the following inequality holds:

$$N_{Tu,Tv}(kx) \geq \min\{N_{u,v}(x), N_{u,Tu}(x), N_{v,Tv}(x), N_{u,Tv}(x), N_{v,Tu}(x)\}$$

or in an equivalently form as

$$N_{Tu,Tv}(x) \geq \min\left\{N_{u,v}\left(\frac{x}{k}\right), N_{u,Tu}\left(\frac{x}{k}\right), N_{v,Tv}\left(\frac{x}{k}\right), N_{u,Tv}\left(\frac{x}{k}\right), N_{v,Tu}\left(\frac{x}{k}\right)\right\}.$$

Corollary 29 (Tasković [6, p. 616]) *Let T be an increasing mapping of a lower spring ordered transversal space $X := (X, \circledcirc, A)$ for $A(u, v) := N_{u,v}(t)$ with the property* (Em) *into itself, where X satisfies the condition of lower MCS-convergence. Suppose that there exists* *an increasing function* $\varphi : \mathfrak{G} \to \mathfrak{G}$ ($\mathfrak{G}$ *is a totally ordered set*) *such that* $t \prec \varphi(t)$ *for every* $t \in \mathfrak{G}$ *and for some fixed* $c \in \mathfrak{G}$ *the following equality holds*:

$$\lim_{n\to\infty} N_{u,v}\bigl(\varphi^n(t)\bigr) = b \quad \textit{for every } t \succ c,$$

and

$$N_{Tu,Tv}(x) \succcurlyeq \inf\bigl\{N_{u,v}\bigl(\varphi(x)\bigr), N_{u,Tu}\bigl(\varphi(x)\bigr), N_{v,Tv}\bigl(\varphi(x)\bigr), N_{u,Tv}\bigl(\varphi(x)\bigr), N_{v,T(u)}\bigl(\varphi(x)\bigr)\bigr\}$$

for all $u, v \in X$ *and for every* $x \succ c$. *If* $T(r) \mathrel{\textcircled{$\preccurlyeq$}} r$ *for some* $r \in X$, *then* T *has a unique fixed point in* X.

Proof Applying the general principle of transpose for posets to Corollary 28, we get *directly* an extended and generalized version of the result given in Corollary 28. The proof is complete. □

Annotations The results of Corollaries 28 and 29 are two immediate consequences of Corollary 19. Uniqueness follows immediately from the conditions of given inequalities.

4 Conclusions

This paper presents new consequences of the Axiom of Infinite Choice in terms of ordered spring spaces and increasing mappings. Applications in nonlinear functional analysis and fixed point theory are also considered.

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