Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Probabilistic b-metric spaces and nonlinear contractions

Abderrahim Mbarki[1*] and Rachid Oubrahim[2]

*Correspondence:
dr.mbarki@gmail.com
[1]ANO Laboratory, National School of Applied Sciences, Oujda University, P.O. Box 669, Oujda, Morocco
Full list of author information is available at the end of the article

Abstract

This work is for giving the probabilistic aspect to the known b-metric spaces (Czerwik in Atti Semin. Mat. Fis. Univ. Modena 46(2):263-276, 1998), which leads to studying the fixed point property for nonlinear contractions in this new class of spaces.

MSC: Primary 54E70; 54H25; secondary 47S50; 34B15

Keywords: probabilistic (fuzzy) b-metric spaces; fixed points; nonlinear contractions

1 Introduction

Fixed point theory plays a basic role in applications of many branches of mathematics. Finding a fixed point of contractive mappings has become the center of strong research activity. There are many works about the fixed point of contractive maps (see, for example, [2, 3]). In [3], Polish mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle, in 1922.

After that, based on this finding, a large number of fixed point results have appeared in recent years. Generally speaking, there usually are two generalizations on them. One is from mappings. The other is from spaces.

Concretely, for one thing, from mappings, for example, the concept of a φ-contraction mapping was introduced in 1968 by Browder [4].

For another thing, from spaces, there are too many generalizations of metric spaces. For instance, recently, Bakhtin [5], introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. Starting with the paper of Bakhtin, many fixed point results have been established in those interesting spaces (see [1, 6–8]).

Let us recall the notion of a b-metric space.

Definition 1.1 ([1]) Let M be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : M \times M \to \mathbb{R}^+$ is a b-metric iff, for all $x, y, z \in M$, the following conditions hold:

(1) $d(x,y) = 0$ iff $x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (M,d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than the class of metric spaces since a b-metric is a metric when $s = 1$.

This paper is organized as follows. In Section 2, we present some basic concepts and relevant lemmas on probabilistic metric spaces (pms). In Section 3, we generalize the concept of pms by defining a probabilistic (fuzzy) b-metric space and discuss some topological proprieties of these new structures. In Section 4, we prove the main theorem in this paper, i.e., a new fixed point theorem for probabilistic (fuzzy) φ-contraction in probabilistic (fuzzy) b-metric spaces. Subsequently, as an application of our results, in Sections 5, we provide an example and prove a fixed point theorem in b-metric spaces. Our results generalize some well-known results in the literature.

2 Preliminaries

We begin by briefly recalling some definitions and notions from probabilistic metric spaces theory that we will use in the sequel. For more details, we refer the reader to [9].

A nonnegative real function f defined on $\mathbb{R}^+ \cup \{\infty\}$ is called a distance distribution function (briefly, a d.d.f.) if it is nondecreasing, left-continuous on $(0,\infty)$, with $f(0) = 0$ and $f(\infty) = 1$. The set of all d.d.f's will be denoted by Δ^+; and the set of all $f \in \Delta^+$ for which $\lim_{s\to\infty} f(s) = 1$ by D^+.

A simple example of distribution function is a Heavyside function in D^+

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 2.1 Consider f and g being in Δ^+, $h \in (0,1]$, and let $(f,g;h)$ denote the condition

$$0 \leq g(x) \leq f(x+h) + h$$

for all x in $(0, \frac{1}{h})$.

The modified Lévy distance is the function d_L defined on $\Delta^+ \times \Delta^+$ by

$$d_L(f,g) = \inf\bigl\{h : \text{both conditions } (f,g;h) \text{ and } (g,f;h) \text{ hold}\bigr\}.$$

Note that, for any f and g in Δ^+, both $(f,g;1)$ and $(g,f;1)$ hold, hence d_L is well defined and $d_L(f,g) \leq 1$.

Lemma 2.1 ([9]) *The function d_L is a metric on Δ^+.*

Lemma 2.2 ([9]) *The metric spaces (Δ^+, d_L) are compact, and hence complete.*

Lemma 2.3 ([9]) *For any F in Δ^+ and $t > 0$,*

$$F(t) > 1 - t \quad \textit{iff} \quad d_L(F,H) < t.$$

Lemma 2.4 ([9]) *If F and G are in Δ^+ and $F \leq G$, then $d_L(G,H) \leq d_L(F,H)$.*

τ on Δ^+ is a triangle function if it assigns a d.d.f. in Δ^+ to every pair of d.d.f.'s in $\Delta^+ \times \Delta^+$ and satisfies the following conditions:

$$\tau(F,G) = \tau(G,F),$$

$$\tau(F,G) \leq \tau(K,R) \quad \text{whenever } F \leq K, G \leq R,$$

$$\tau(F,H) = F,$$

$$\tau\bigl(\tau(F,G),R\bigr) = \tau\bigl(F,\tau(G,R)\bigr).$$

A commutative, associative and nondecreasing mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if and only if

(i) $T(a,1) = a \quad$ for all $a \in [0,1]$,

(ii) $T(0,0) = 0$.

As examples we mention the tree typical examples of continuous t-norms as follows: $T_p(a,b) = ab$, $T_M(a,b) = \mathrm{Min}(a,b)$ and $T_L(a,b) = \mathrm{Max}\{a+b-1,0\}$.

Moreover, if T is left-continuous, then the operation $\tau_T : \Delta^+ \times \Delta^+ \to \Delta^+$ defined by

$$\tau_T(F,G)(x) = \sup\bigl\{T\bigl(F(u),G(v)\bigr) : u+v = x\bigr\}$$

is a triangle function.

We say (O Hadzić [10]) that a t-norm T is of H-type if the family $\{T^n(t)\}$ is equicontinuous at $t = 1$, that is,

$$\forall \epsilon \in (0,1)\ \exists \lambda \in (0,1): \quad t > 1-\lambda \Rightarrow T^n(t) > 1-\epsilon \quad \text{for all } n \geq 1,$$

where $T^1(x) = T(x,x)$, $T^n(x) = T(x, T^{n-1}(x))$ for every $n \geq 2$.

The t-norm T_M is a trivial example of t-norm of H-type (see [10]).

Finally, we also have the following.

Lemma 2.5 ([9]) *If T is continuous, then τ_T is continuous.*

3 Probabilistic *b*-metric space

Having introduced the necessary terms, we now turn to our main topic. Developing a theory of probabilistic b-metric spaces, we start with the following definition.

Definition 3.1 A probabilistic b-metric space (briefly a pbms) is a quadruple (M,F,τ,s) where M is a nonempty set, F is a function from $M \times M$ into Δ^+, τ is a triangle function, $s \geq 1$ is a real number, and the following conditions are satisfied: for all $p,r;q \in M$ and $y > 0$,

(i) $F_{pp} = H$,
(ii) $F_{pq} = H \Rightarrow p = q$,
(iii) $F_{pq} = F_{qp}$,
(iv) $F_{pq}(sy) \geq \tau(F_{pr},F_{rq})(y)$.

If $\tau = \tau_T$ for some t-norm T, then (M,F,τ_T,s) is called a b-Menger space.

It should be noted that if T is a continuous t-norm, then (M,F) satisfies (iv) under τ_T if and only if it satisfies

$$\text{(v)} \quad F_{pq}\big(s(y+x)\big) \geq T\big(F_{pr}(y), F_{rq}(x)\big)$$

for all $p,r,q \in M$ and for all $x,y > 0$, under T.

Recall that a probabilistic metric space is a triple (M,F,τ) satisfying (i)-(iii) and the following inequality:

$$\text{(vi)} \quad F_{pq} \geq \tau(F_{pr}, F_{rq}) \quad \text{(triangle inequality)}$$

for all $p,r,q \in M$.

By setting F_{xy} by $F_{xy}(0) = 0$ and $F_{xy}(t) = M(x;y;t)$ for $t > 0$, the fuzzy b-metric space is defined in the following manner.

Definition 3.2 The quadruple $(X;M;*,s)$ is said to be a fuzzy b-metric space if X is an arbitrary set, $*$ is a continuous t-norm, $s \geq 1$ is a real number and M is a fuzzy set on $X^2 \times [0;\infty)$ satisfying the following conditions:

$M(x,y,0) = 0$,

$M(x,y,q) = 1$ for all $q > 0$ iff $x = y$,

$M(x,y,q) = M(y,x,q)$,

$M(x,z,t) * M(z;y;q) \leq M(x;y;s(t+q))$,

$M(x,y,\cdot) : [0,\infty[\to [0,1]$ is left-continuous and nondecreasing

for all $x,y,z \in X$ and $q,t > 0$.

From [11, Lemma 2.6], $M(x,y,\cdot)$ is a nondecreasing mapping for $x;y \in X$. Hence, every fuzzy metric space (in the sense of Kramosil and Michalek [12]) is a fuzzy b-metric space with the constant $s = 1$.

It is clear that every probabilistic (fuzzy) metric space (PM space) is a probabilistic (fuzzy) b-metric space with $s = 1$. But the converse is not true. We confirm this by the following examples.

Example 3.1 Let $M = \{1,2,3,4\}$. Define $F : M \times M \to \Delta^+$ as follows:

$$F_{pq}(t) = \begin{cases} H(t) & \text{if } p = q, \\ H(t-3) & \text{if } p = 2 \text{ and } q = 3 \text{ or } p = 3 \text{ and } q = 2, \\ H(t-1) & \text{otherwise.} \end{cases}$$

It is easy to check that $(M,F,\tau_{T_M},3)$ is a pbms, but (M,F,τ_{T_M}) is not a standard probabilistic metric space because it lacks the triangle inequality:

$$F_{32}\left(\frac{5}{2}\right) = 0 < 1 = H\left(\frac{1}{4}\right) = \operatorname{Min}\left(F_{31}\left(\frac{5}{4}\right), F_{12}\left(\frac{5}{4}\right)\right).$$

Example 3.2 Let $M = [0,\infty)$. Define $F : M \times M \to \Delta^+$ as follows:

$$F_{pq}(t) = H\big(t - |p-q|^2\big).$$

It is easy to check that $(M, F, \tau_{T_M}, 2)$ is a pbms, but (M, F, τ_{T_M}) is not a standard probabilistic metric space because it lacks the triangle inequality:

$$F_{32}\left(\frac{2}{3}\right) = 0 < 1 = H\left(\frac{1}{12}\right) = \operatorname{Min}\left(F_{3\frac{5}{2}}\left(\frac{1}{3}\right), F_{\frac{5}{2}2}\left(\frac{1}{3}\right)\right).$$

Definition 3.3 Let (M, F) be a probabilistic semimetric space (i.e., (i), (ii) and (iii) of Definition 3.1 are satisfied). For p in M and $t > 0$, the strong t-neighborhood of p is the set

$$N_p(t) = \{q \in M : F_{pq}(t) > 1 - t\}.$$

The strong neighborhood system at p is the collection $\wp_p = \{N_p(t) : t > 0\}$, and the strong neighborhood system for M is the union $\wp = \bigcup_{p\in M} \wp_p$.

An immediate consequence of Lemma 2.3 is

$$N_p(t) = \{q \in M : d_L(F_{pq}, H) < t\}.$$

Definition 3.4 Let $\{x_n\}$ be a sequence in a probabilistic semimetric space (M, F).

(1) A sequence $\{x_n\}$ in M is said to be convergent to x in M if, for every $\epsilon > 0$ and $\delta \in (0, 1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{x_n x}(\epsilon) > 1 - \delta$, whenever $n \geq N(\epsilon, \delta)$.

(2) A sequence $\{x_n\}$ in M is called Cauchy sequence if, for every $\epsilon > 0$ and $\delta \in (0, 1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{x_n x_m}(\epsilon) > 1 - \delta$, whenever $n, m \geq N(\epsilon, \delta)$.

(3) (M, F) is said to be complete if every Cauchy sequence has a limit.

Every b-metric space is a probabilistic b-metric space. Moreover, we have the following.

Lemma 3.1 *Let (M, d) be a b-metric space (bms) with the constant s. Define $F : M \times M \to \Delta^+$ by*

$$F_{pq}(t) = H(t - d(p, q)).$$

Then

(a) *(M, F, τ_{T_M}, s) is a pbms.*

(b) *(M, F, τ_{T_M}, s) is complete if and only if (M, d) is complete.*

Proof (a) It is easy to check the conditions (i)-(iii) of Definition 3.1. So, for condition (v), let p, r, q in M, let t_1, t_2 in $[0, \infty)$.

If

$$\operatorname{Min}(F_{pr}(t_1), F_{rq}(t_2)) = 0,$$

then

$$F_{pq}(s(t_1 + t_2)) \geq \operatorname{Min}(F_{pr}(t_1), F_{rq}(t_2)).$$

Else if

$$\mathrm{Min}\big(F_{pr}(t_1), F_{rq}(t_2)\big) = 1,$$

then $t_1 > d(p,r)$ and $t_2 > d(r,q)$. Since (M,d) is a b-metric space with the constant s, we have

$$\begin{aligned} d(p,q) &\leq s\big(d(p,r) + d(r,q)\big) \\ &< s(t_1 + t_2). \end{aligned}$$

Then we get

$$F_{pq}\big(s(t_1 + t_2)\big) = 1.$$

Thus

$$F_{pq}\big(s(t_1 + t_2)\big) \geq \mathrm{Min}\big(F_{pr}(t_1), F_{rq}(t_2)\big).$$

Hence condition (v) holds. So (M, F, τ_{T_M}, s) is a probabilistic b-metric space.
(b) By Definition 3.2 we get, for every $t > 0$,

$$N_p(t) = \big\{q \in M : d(p,q) < t\big\}.$$

So (M, F, T_M, s) is a complete pbms if and only if (M,d) is a complete bms. □

By using the above lemma, we present some typical examples of a probabilistic b-metric space.

Example 3.3 Let (M,d) be a metric space and $d'(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. We show that d' is a b-metric with $s = 2^{p-1}$.

Obviously, conditions (1) and (2) of Definition 1.1 are satisfied. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p (x > 0)$ implies

$$\begin{aligned} d'(x,y) = \big(d(x,y)\big)^p &\leq \big(d(x,z) + d(z,y)\big)^p \\ &\leq 2^{p-1}\big(d(x,z)^p + d(z,y)^p\big) \\ &= 2^{p-1}\big(d'(x,z) + d'(z,y)\big) \end{aligned}$$

for each $x, y \in M$. So, condition (3) of Definition 1.1 is also satisfied and $(M, F, \tau_{T_M}, 2^{p-1})$ is a pbms with $F_{pq}(t) = H(t - d'(p,q))$.

Scheizer and Sklar [9] proved that if (M, F, τ) is a PM space with τ being continuous, then the family $\Im$ consisting of $\emptyset$ and all unions of elements of this strong neighborhood system for M determines a Hausdorff topology for M. Consequently, there exists a topology Λ on M such that the strong neighborhood system $\wp$ is a basis for Λ.

But in a probabilistic b-metric space in general the last assertion is false as shown in the following example.

Example 3.4 Let $a > 0$, $M_a = [0, a] \cup \{4a\}$. Define $F^a : M_a \times M_a \to \Delta^+$ as follows:

$$F^a_{pq}(t) = \begin{cases} H(t-2a) & \text{if } p \text{ and } q \text{ are in } \{a, 4a\}\ p \neq q, \\ H(t-|p-q|) & \text{otherwise.} \end{cases}$$

It is easy to show that $(M_a, F^a, \tau_{T_M}, 2)$ is a pbms with τ_{T_M} being continuous, but (M_a, F^a, τ_{T_M}) is not a standard probabilistic metric space because it lacks the triangle inequality:

$$F_{4a\frac{a}{2}}\left(\frac{7a}{2}\right) = 0 < 1 = \operatorname{Min}\left(F_{4aa}\left(\frac{5a}{2}\right), F_{a\frac{a}{2}}(a)\right),$$

in which $N_{4a}(3a) = \{a, 4a\}$ and there does not exist $t > 0$ such that $N_a(t) \subseteq N_{4a}(3a)$. Hence the strong neighborhood system $\wp$ is not a basis for any topology on M_a.

It is well known that in a probabilistic metric space (M, F, τ) with τ being continuous M is endowed with the topology $\Im$ and $M \times M$ with the corresponding product topology. Then the probabilistic metric F is a continuous mapping from $M \times M$ into Δ^+ [9].

However, in a probabilistic b-metric space (M, F, τ) the probabilistic b-metric F is not continuous in general even though τ is continuous. The following example illustrates this fact.

Example 3.5 Let $M = \mathbb{N} \cup \{\infty\}$, $0 < a \leq 1$. Define $F^a : M \times M \to \Delta^+$ as follows:

$$F^a_{pq}(t) = \begin{cases} H(t) & \text{if } p = q, \\ H(t-7) & \text{if } p \text{ and } q \text{ are odd and } p \neq q, \\ H(t - |\frac{a}{p} - \frac{a}{q}|) & \text{if } p \text{ and } q \text{ are even or } pq = \infty, \\ H(t-3) & \text{otherwise.} \end{cases}$$

It is easy to show that $(M, F^a, \tau_{T_M}, 4)$ is a pbms with τ_{T_M} being continuous. In the sequel, we take $a = 1$. Consider the sequence $x_n = 2n$, $n \in \mathbb{N}$. Then $F_{2n\infty}(t) = H(t - \frac{1}{2n})$. Therefore $x_n \to \infty$, but $F_{2n1}(t) = H(t-3) \neq H(t-1) = F_{\infty 1}(t)$. Hence F is not continuous at ∞.

In the following result we show that a pbms is a Hausdorff space.

Lemma 3.2 *Let (M, F, τ, s) be a pbms if τ is continuous, then the strong neighborhood system $\wp$ satisfies: if $p \neq q$, then there are $t, t' > 0$ such that $N_p(t) \cap N_q(t') = \emptyset$.*

Proof Note that since $p \neq q$, $F'_{pq} \neq H$ with $F'_{pq}(t) = F_{pq}(st)$, whence $\varrho = d_L(F'_{pq}, H) > 0$. By the uniform continuity of τ, there exists $t > 0$ such that

$$d_L(\tau(G, G'), H) < \varrho, \tag{3.1}$$

whenever $d_L(G, H) < t$ and $d_L(G', H) < t$. Suppose that $N_p(t) \cap N_q(t) \neq \emptyset$. So, let $r \in N_p(t) \cap N_q(t)$. Then $d_L(F_{pr}, H) < t$ and $d_L(F_{rq}, H) < t$, whence Lemma 2.4 and (3.1) yield

$$d_L(F'_{pq}, H) \leq d_L(\tau(F_{pr}, F_{rq}), H) < \varrho = d_L(F'_{pq}, H),$$

an impossibility. Hence $N_p(t) \cap N_q(t)$ is empty and the proof is complete. $\square$

4 φ-Probabilistic contraction in a probabilistic b-metric space

Over this section, the letter Ψ denotes the set of all functions $\varphi : [0,\infty) \to [0,\infty)$ such that

$$0 < \varphi(t) < t \quad \text{and} \quad \lim_{n\to\infty} \varphi^n(t) = 0 \quad \text{for each } t > 0.$$

Example 4.1 Let $c \in [1,\infty)$, and let $\varphi_c : [0,\infty) \to [0,\infty)$ be defined by

$$\varphi_c(t) = \frac{t}{c+t}.$$

Then

(i) φ_c is a strictly increasing and continuous function,

(ii) $\varphi_c \in \Psi$.

(i) trivially holds. For (ii) it is very easy to check by induction that

$$\varphi_c^n(t) = \frac{t}{c^n + t\sum_{i=0}^{n-1} c^i} \quad \text{for all } t > 0, n \in \mathbb{N}^*.$$

Consequently,

$$\varphi_c^n(t) \le \frac{t}{1 + t\sum_{i=0}^{n-1} c^i},$$

which yields $\varphi_c^n(t) \to 0$, and since $0 < \varphi_c(t) < t$ for each $t > 0$, we conclude that (ii) holds.

Before stating the main fixed point theorems, we introduce the following concept.

Definition 4.1 Let $\varphi : [0,\infty) \to [0,\infty)$ be a function such that $\varphi(t) < t$ for $t > 0$, and f be a selfmap of a probabilistic b-metric space (M, F, τ, s). We say that f is a φ-probabilistic contraction if

$$F_{fpfq}\big(\varphi(t)\big) \ge F_{pq}(st) \tag{4.1}$$

for all $p, q \in M$ and $t > 0$.

It should be noted, when $s = 1$, the above definition coincides with the concept of φ-probabilistic contraction according to the definition in [13] and [14].

The following definition can be considered as a fuzzy version of Definition 4.1.

Definition 4.2 Let $\varphi : [0,\infty) \to [0,\infty)$ be a function such that $\varphi(t) < t$ for $t > 0$, and f be a selfmap of a fuzzy b-metric space $(X; M; *, s)$. We say that f is a φ-fuzzy contraction if

$$M\big(fp, fq, \varphi(t)\big) \ge M(p, q, st) \tag{4.2}$$

for all $p, q \in X$ and $t > 0$.

In the proof of our first theorem, we use the following lemma.

Lemma 4.1 *Let (M,F,τ_T,s) be a pbms with a t-norm T of H-type and* $\operatorname{Ran} F \subset D^+$. *Let $\{x_n\}$ be a sequence in M. If there exists a function $\varphi \in \Psi$ such that*

$$F_{x_{m+1}x_{n+1}}\bigl(\varphi(t)\bigr) \ge F_{x_m x_n}(st) \quad (n,m \ge 0, t > 0), \tag{4.3}$$

then $\{x_n\}$ is a Cauchy sequence.

Proof Let $\{x_n\} \subset M$ be a sequence satisfying (4.3). Firstly, we prove that

$$F_{x_n x_{n+1}}(t) \to 1 \quad \text{as } n \to \infty \text{ for all } t > 0. \tag{4.4}$$

Let $t > 0$, $n \ge 1$. From (4.3) we obtain

$$\begin{aligned} F_{x_n x_{n+1}}\bigl(\varphi^n(t)\bigr) &\ge F_{x_{n-1}x_n}\bigl(s\varphi^{n-1}(t)\bigr) \\ &\ge F_{x_{n-1}x_n}\bigl(\varphi^{n-1}(t)\bigr) \\ &\ge F_{x_{n-2}x_{n-1}}\bigl(s\varphi^{n-2}(t)\bigr) \\ &\ge F_{x_{n-2}x_{n-1}}\bigl(\varphi^{n-2}(t)\bigr) \\ &\vdots \\ &\ge F_{x_0 x_1}(st) \\ &\ge F_{x_0 x_1}(t). \end{aligned}$$

On the other hand, let $t > 0$ and $\delta \in (0,1)$ be given. Since $\operatorname{Ran} F \subset D^+$, there exists $t_0 > 0$ such that $F_{x_0 x_1}(t_0) > 1 - \delta$, and since $\varphi^n(t_0) \to 0$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^n(t_0) < \epsilon$ whenever $n \ge n_0$. Using the fact that F is increasing, we get

$$F_{x_n x_{n+1}}(t) \ge F_{x_n x_{n+1}}\bigl(\varphi^n(t_0)\bigr) \quad \text{whenever } n \ge n_0,$$

which gives that

$$F_{x_n x_{n+1}}(t) \ge F_{x_0 x_1}(t_0) > 1 - \delta \quad \text{whenever } n \ge n_0.$$

Thus, (4.4) is proved.

Next, let $t > 0$ and $n \in \mathbb{N}$. We shall apply induction to show that, for any $m \ge n$,

$$F_{x_n x_m}(st) \ge T^{m-n}\bigl(F_{x_n x_{n+1}}\bigl(t - \varphi(t)\bigr)\bigr). \tag{4.5}$$

This is obvious for $m = n$ since $F_{x_n x_n}(st) = 1$. Next, suppose that (4.5) is true for some positive integer $m \ge n$. Hence, by (4.3) and the monotonicity of T, we have

$$\begin{aligned} F_{x_n x_{m+1}}(st) &= F_{x_n x_{m+1}}\bigl(s\bigl(t - \varphi(t)\bigr) + s\varphi(t)\bigr) \\ &\ge T\bigl(F_{x_n x_{n+1}}\bigl(t - \varphi(t)\bigr), F_{x_{n+1}x_{m+1}}\bigl(\varphi(t)\bigr)\bigr) \\ &\ge T\bigl(F_{x_n x_{n+1}}\bigl(t - \varphi(t)\bigr), F_{x_n x_m}(st)\bigr). \end{aligned}$$

So, by the induction hypothesis,

$$\begin{aligned} F_{x_n x_{m+1}}(st) &\geq T\big(F_{x_n x_{n+1}}\big(t-\varphi(t)\big), F_{x_n x_m}(st)\big) \\ &\geq T\big(F_{x_n x_{n+1}}\big(t-\varphi(t)\big), T^{m-n}\big(F_{x_n x_{n+1}}\big(t-\varphi(t)\big)\big)\big) \\ &\geq T^{m+1-n}\big(F_{x_n x_{n+1}}\big(t-\varphi(t)\big)\big), \end{aligned}$$

which completes the induction. Now let $\epsilon > 0$ and $\delta \in (0,1)$ be given. Since T is a t-norm of H-type, there exists $\lambda \in (0,1)$ such that $T^n(t) > 1-\delta$ for all $n \geq 1$ when $t > 1-\lambda$.

On the other hand, from (4.4) we have

$$F_{x_n x_{n+1}}\left(\frac{\epsilon}{s} - \varphi\left(\frac{\epsilon}{s}\right)\right) \to 1 \quad \text{as } n \to \infty.$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$F_{x_n x_{n+1}}\left(\frac{\epsilon}{s} - \varphi\left(\frac{\epsilon}{s}\right)\right) > 1-\lambda \quad \text{for all } n \geq n_0.$$

Therefore, from (4.5) it follows

$$F_{x_n x_m}(\epsilon) > 1-\delta \quad \text{whenever } m > n \geq n_0.$$

So we conclude that $\{x_n\}$ is a Cauchy sequence in M. □

Now, we can state and prove the first main fixed point theorem of this paper.

Theorem 4.1 *Let (M, F, τ_T, s) be a complete pbms under a continuous t-norm T of H-type such that* $\operatorname{Ran} F \subset D^+$. *Let $f : M \to M$ be a φ-probabilistic contraction where $\varphi \in \Psi$. Then f has a unique fixed point u and, for any $x \in M$,* $\lim_{n\to\infty} f^n(x) = \overline{x}$.

Proof Let $x_0 \in M$ be arbitrary, and we consider the sequence $\{x_n\}$ defined by

$$x_n = f(x_{n-1}) = f^n(x_0) \quad \text{for each } n \geq 1.$$

By (4.1), we have

$$F_{x_{m+1} x_{n+1}}\big(\varphi(t)\big) \geq F_{x_m x_n}(st).$$

Now, by Lemma 4.1, $\{x_n\}$ is a Cauchy sequence. Since M is complete, there is some $u \in M$ such that

$$x_n \to u \quad \text{as } n \to \infty. \tag{4.6}$$

Now we will show that u is a fixed point of f. Let $\epsilon > 0$ and $\delta \in (0,1)$, since $\varphi(\epsilon) < \epsilon$, by the monotonicity of F and (4.1), we get

$$\begin{aligned} F_{x_{n+1} fu}(\epsilon) &\geq F_{x_{n+1} fu}\big(\varphi(\epsilon)\big) \\ &= F_{f x_n fu}\big(\varphi(\epsilon)\big) \\ &\geq F_{x_n u}(s\epsilon). \end{aligned}$$

Since $\{x_n\}$ converges to u, there exists $n_0 \in \mathbb{N}$ such that

$$F_{x_n u}(s\epsilon) > 1 - \delta \quad \text{for each } n \geq n_0.$$

So,

$$F_{x_{n+1} fu}(\epsilon) > 1 - \delta \quad \text{for each } n \geq n_0.$$

Then

$$\lim_{n\to\infty} x_{n+1} = fu. \tag{4.7}$$

By the inequality (v), we obtain

$$F_{fuu}(t) \geq T\left(F_{fux_{n+1}}\left(\frac{t}{2s}\right), F_{x_{n+1}u}\left(\frac{t}{2s}\right)\right) \tag{4.8}$$

for all $t > 0$, $n \geq 1$.

Letting $n \to \infty$ in (4.8) and using (4.6), (4.7), we get that

$$F_{fuu}(t) \geq 1 \quad \text{for all } t > 0,$$

which holds unless $F_{fuu} = H$, so u is a fixed point of f.

To prove uniqueness, suppose that there exists another fixed point v in M of f. Then, let $t > 0$, from (4.1), by using the monotonicity of F and the fact that $\varphi(t) < t$, we get

$$\begin{aligned} F_{uv}\big(\varphi(t)\big) &= F_{fufv}\big(\varphi(t)\big) \\ &\geq F_{uv}(st) \\ &\geq F_{uv}(t) \\ &\geq F_{uv}\big(\varphi(t)\big). \end{aligned}$$

Hence

$$F_{uv}\big(\varphi(t)\big) = F_{uv}(t).$$

Inductively, we obtain

$$F_{uv}\big(\varphi^n(t)\big) = F_{uv}(t).$$

Now we shall show that

$$F_{uv}(t) = 1 \quad \text{for all } t > 0.$$

Suppose, to the contrary, that there exists $t_0 > 0$ such that $F_{uv}(t_0) < 1$. Since $F_{uv} \in D^+$, then $F_{uv}(t) \to 1$ as $t \to \infty$, so there exists $t_1 > t_0$ such that

$$F_{uv}(t_1) > F_{uv}(t_0).$$

Since $\lim_{n\to\infty}\varphi^n(t_1)\to 0$, there exists a positive integer $n > 1$ such that $\varphi^n(t_1) < t_0$. Then, by the monotonicity of F_{uv}, we have

$$F_{uv}\big(\varphi^n(t_1)\big)\le F_{uv}(t_0).$$

Thus

$$F_{uv}(t_1) = F_{uv}\big(\varphi^n(t_1)\big)\le F_{uv}(t_0),$$

a contradiction. Therefore $F_{uv}(t) = 1$ for all $t > 0$, since $F_{uv}\in D^+$. Hence $F_{uv} = H$.

Then, in view of (ii) of Definition 3.1, we conclude that $u = v$. This completes the proof. □

Since in the proof of Theorem 4.1 the condition $F_{xy}(\infty) = 1$ plays no role, this leads to the following.

Theorem 4.2 *Let $(X; M; *; s)$ be a complete fuzzy b-metric space with the t-norm $*$ of H-type such that $M(x, y, t)\to 1$ as $t\to\infty$ for all $x, y\in X$. Let $f : X\to X$ be a φ-fuzzy contraction where $\varphi\in\Psi$. Then f has a unique fixed point u and, for any $x\in M$, $\lim_{n\to\infty} f^n(x) = \overline{x}$.*

By taking $s = 1$ in Theorem 4.1, we obtain the following result.

Corollary 4.1 ([13]) *Let (M, F, τ_T) be a complete pms under a continuous t-norm T of H-type such that* $\operatorname{Ran} F\subset D^+$. *Let $f : M\to M$ be a φ-probabilistic contraction where $\varphi\in\Psi$. Then f has a unique fixed point u and, for any $x\in M$, $\lim_{n\to\infty} f^n(x) = \overline{x}$.*

5 Applications

As consequences of the above results, we can obtain the following fixed point theorems in usual b-metric spaces.

Proposition 5.1 *Let (M, d) be a complete b-metric space. Let f be a mapping of (M, d) into itself satisfying*

$$d(fx, fy)\le\alpha\left(\frac{d(x,y)}{s}\right),\quad x, y\in M,$$

where the function $\alpha : [0,\infty)\to[0,\infty)$ satisfies the following conditions:

$$\alpha(0) = 0,\qquad \alpha(t) < t,\quad \textit{and}\quad \limsup_{r\to t}\alpha(r) < t\quad \textit{for each } t > 0.$$

Then f has a unique fixed point u, and $f^n(x)\to u$ for all $x\in M$.

Proof From Lemma 3.1, (M, F, τ_{T_M}, s) is a complete probabilistic b-metric space, where $F_{xy}(t) = H(t - d(x, y))$ for all $x, y\in M$. Let f be a mapping such that there exists α satisfying the conditions of Proposition 5.1.

By [15, Lemma 1], there exists a strictly increasing and continuous function $\varphi : [0,\infty)\to [0,\infty)$ such that

$$\alpha(t) < \varphi(t) < t$$

for all $t > 0$.

It is clear that $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$. Then

$$d(fx,fy) \leq \alpha\left(\frac{d(x,y)}{s}\right) \leq \varphi\left(\frac{d(x,y)}{s}\right) \tag{5.1}$$

for all $x, y \in M$. Now, we prove that f is a φ-probabilistic contraction in (M, F, τ_{T_M}, s). Indeed, let $t > 0$ and $x, y \in M$, from (5.1) and the monotonicity of H, we have

$$\begin{aligned} F_{fxfy}\bigl(\varphi(t)\bigr) &= H\bigl(\varphi(t) - d(fx,fy)\bigr) \\ &\geq H\bigl(t - \varphi^{-1}\bigl(d(fx,fy)\bigr)\bigr) \\ &\geq H\left(t - \frac{d(x,y)}{s}\right) \\ &\geq H\bigl(st - d(x,y)\bigr) \\ &\geq F_{xy}(st). \end{aligned}$$

Hence f is a φ-probabilistic contraction in (M, F, τ_{T_M}, s). The existence and uniqueness of the fixed point follow immediately by Theorem 4.1. □

If in Proposition 5.1 we take the function $\alpha(t) = skt$, then we have the following corollary.

Corollary 5.1 ([8]) *Let (M,d) be a complete b-metric space. Let f be a mapping of (M,d) into itself satisfying*

$$d(fx,fy) \leq kd(x,y), \quad x, y \in M,$$

with the restrictions $k \in [0,1)$ and $ks < 1$. Then f has a unique fixed point z, and $f^n(x) \to z$ for all $x \in M$.

Example 5.1 Let $M = [0,1]$, $n \in \mathbb{N}^* - \{1\}$ and F be defined by $F_{xy}(t) = H(t - |x - y|^n)$. Then (M, F, τ_{T_M}) is a complete probabilistic b-metric space with $s = 2^{n-1}$. But in general it is not a probabilistic metric space.

Now we define the mapping $f : [0,1] \to [0,1]$ by

$$f(x) = \frac{x}{s+x}.$$

For all $x; y \in M$ with $x \geq y$, we have

$$\begin{aligned} |fx - fy|^n &= \left|\frac{x}{s+x} - \frac{y}{s+y}\right|^n \\ &\leq \frac{s^n(x-y)^n}{[(s+x)(s+y)]^n} \\ &\leq \frac{s^n(x-y)^n}{[s^2 + s(x+y) + xy]^n} \\ &\leq \frac{s^n(x-y)^n}{[s^2 + s(x-y)]^n} \\ &\leq \frac{s^n(x-y)^n}{s^{2n} + s^n(x-y)^n}. \end{aligned}$$

Similarly, for $x \le y$, we also conclude that

$$|fx - fy|^n \le \frac{s^n |x-y|^n}{s^{2n} + s^n |x-y|^n} = \varphi_{s^{n-1}}\left(\frac{|x-y|^n}{s}\right). \tag{5.2}$$

Now, suppose that $F_{xy}(st) = H(st - |x-y|^n) > 0$, this implies that $st > |x-y|^n$. Then from Example 4.1 we have

$$\varphi_{s^{n-1}}(t) > \varphi_{s^{n-1}}\left(\frac{|x-y|^n}{s}\right). \tag{5.3}$$

It follows from (5.2) and (5.3) that

$$\varphi_{s^{n-1}}(t) > |fx - fy|^n.$$

From the previous inequality, we get

$$F_{xy}\left(\varphi_{s^{n-1}}(t)\right) = H\left(\varphi_{s^{n-1}}(t) - |fx - fy|^n\right) > 0.$$

Hence

$$F_{fxfy}\left(\varphi_{s^{n-1}}(t)\right) \ge F_{xy}(st).$$

Thus, f satisfies the $\varphi_{s^{n-1}}$-probabilistic contraction of Theorem 4.1 and 0 is the unique fixed point of f.

6 Conclusion

The paper deals with the achievement of introducing the notion of probabilistic b-metric space as a generalization of probabilistic metric space and b-metric space and studying some of its topological properties. Also, here we define φ-contraction maps for such spaces. Moreover, we investigate some fixed points for mappings satisfying such conditions in the new framework. Our main theorems extend and unify the existing results in the recent literature. An example is constructed to support our result.

Acknowledgements

The authors are thankful to the referees and the editor for their constructive comments and suggestions which have been useful for the improvement of the paper.

Funding

No sources of funding may be declared.

Competing interests

The authors declare that they do not have any competing interests.

Authors' contributions

Each author equally contributed to this paper, read and approved the final manuscript.

Author details

[1] ANO Laboratory, National School of Applied Sciences, Oujda University, P.O. Box 669, Oujda, Morocco. [2] ANO Laboratory, Faculty of Sciences, Oujda University, Oujda, Morocco.

Received: 13 June 2017 Accepted: 7 December 2017 Published online: 22 December 2017

References

1. Czerwik, S: Nonlinear set-valued contraction mappings in b-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46**(2), 263-276 (1998)
2. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)
3. Banach, S: Sur les operations dans les ensembles abstrait et leur application aux equations, integrals. Fundam. Math. **3**, 133-181 (1922)
4. Browder, F: On the convergence of successive approximations for nonlinear functional equations. Indag. Math. **30**, 27-35 (1968)
5. Bakhtin, IA: The contraction principle in quasimetric spaces. In: Functional Analysis, vol. 30, pp. 26-37. Ul'yanovsk. Gos. Ped. Inst., Ul'yanovsk (1989) (in Russian)
6. Aydi, H, Bota, M-F, Karapınar, E, Mitrović, S: A fixed point theorem for set valued quasicontractions in b-metric spaces. Fixed Point Theory Appl. **2012**, 88 (2012)
7. Hussain, N, Dorić, D, Kadelburg, Z, Radenović, S: Suzuki-type fixed point results in metric type spaces. Fixed Point Theory Appl. **2012**, 126 (2012)
8. Kir, M, Kiziltunc, H: On some well known fixed point theorems in b-metric spaces. Turk. J. Anal. Number Theory **1**(1), 13-16 (2013)
9. Schweizer, B, Sklar, A: Probabilistic Metric Spaces. North-Holland Series in Probability and Applied Mathimatics, vol. 5 (1983)
10. Hadzić, O: A fixed point theorem in Menger spaces. Publ. Inst. Math. (Belgr.) **20**, 107-112 (1979)
11. Gregori, V, Sapena, A: On fixed-point theorems in fuzzy metric spaces. Fuzzy Sets Syst. **125**, 245-252 (2002)
12. Kramosil, O, Michalek, J: Fuzzy metric and statistical metric spaces. Kybernetika **11**, 326-334 (1975)
13. Jachymski, J: On probabilistic φ-contractions on Menger spaces. Nonlinear Anal. **73**, 2199-2203 (2010)
14. Mbarki, A, Naciri, R: Probabilistic generalized metric spaces and nonlinear contractions. Demonstr. Math. **49**(4), 437-452 (2016)
15. Jachymski, JR: Equivalence of some contractivity properties over metrical structures. Proc. Am. Math. Soc. **125**(8), 2327-2335 (1997)