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Applications and common coupled fixed point results in ordered partial metric spaces

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Abstract
In this paper, we obtain a unique common coupled fixed point theorem by using (ψ, α, β)-contraction in ordered partial metric spaces. We give an application to integral equations as well as homotopy theory. Also we furnish an example which supports our theorem.

Keywords: partial metric; *w*-compatible maps; coupled fixed point; mixed *g*-monotone property; ψ-α-β contraction; homotopy theory

1 Introduction

The notion of a partial metric space (PMS) was introduced by Matthews [1] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that PMSs play an important role in constructing models in the theory of computation and domain theory in computer science (see *e.g.* [2–9]).

Matthews [1, 10], Oltra and Valero [11] and Altun *et al.* [12] proved some fixed point theorems in PMSs for a single map. For more work on fixed, common fixed point theorems in PMSs, we refer to [6, 13–27].

The notion of a coupled fixed point was introduced by Bhaskar and Lakshmikantham [28] and they studied some fixed point theorems in partially ordered metric spaces. Later some authors proved coupled fixed and coupled common fixed point theorems (see [16, 29–35]).

The aim of this paper is to study unique common coupled fixed point theorems of Jungck type maps by using a (ψ, α, β)-contraction condition over partially ordered PMSs.

2 Preliminaries

First we recall some basic definitions and lemmas which play a crucial role in the theory of PMSs.

Definition 2.1 (See [1, 10]) A partial metric on a non-empty set X is a function $p : X \times X \to R^+$ such that, for all $x, y, z \in X$,

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y), p(y,y) \leq p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a PMS.

If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$, given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{1}$$

is a metric on X.

Example 2.2 (See *e.g.* [10, 14, 20]) Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a PMS. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x-y|$.

Example 2.3 (See [19]) Let $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a PMS.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

We now state some basic topological notions (such as convergence, completeness, continuity) on PMSs (see *e.g.* [1, 10, 12, 14, 20, 22]).

Definition 2.4

1. A sequence $\{x_n\}$ in the PMS (X,p) converges to the limit x if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
2. A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
3. A PMS (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
4. A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0,\delta)) \subseteq B_p(Fx_0,\epsilon)$.

The following lemma is one of the basic results as regards PMS [1, 10, 12, 14, 20, 22].

Lemma 2.5

1. *A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).*
2. *A PMS (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \quad \Leftrightarrow \quad p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{2}$$

Next, we give two simple lemmas which will be used in the proofs of our main results. For the proofs we refer [14].

Lemma 2.6 *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

Lemma 2.7 *Let (X,p) be a PMS. Then*

(A) *if $p(x,y)=0$, then $x=y$,*
(B) *if $x\neq y$, then $p(x,y)>0$.*

Remark 2.8 If $x=y$, $p(x,y)$ may not be 0.

Definition 2.9 ([28]) Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\to X$. Then the map F is said to have mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and monotone non-increasing in y; that is, for any $x,y\in X$,

$$x_1\preceq x_2 \quad \text{implies} \quad F(x_1,y)\preceq F(x_2,y) \quad \text{for all } y\in X$$

and

$$y_1\preceq y_2 \quad \text{implies} \quad F(x,y_2)\preceq F(x,y_1) \quad \text{for all } x\in X.$$

Definition 2.10 ([28]) An element $(x,y)\in X\times X$ is called a coupled fixed point of a mapping $F:X\times X\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Definition 2.11 ([30]) An element $(x,y)\in X\times X$ is called

(g_1) a coupled coincident point of mappings $F:X\times X\to X$ and $f:X\to X$ if $fx=F(x,y)$ and $fy=F(y,x)$,
(g_2) a common coupled fixed point of mappings $F:X\times X\to X$ and $f:X\to X$ if $x=fx=F(x,y)$ and $y=fy=F(y,x)$.

Definition 2.12 ([30]) The mappings $F:X\times X\to X$ and $f:X\to X$ are called w-compatible if $f(F(x,y))=F(fx,fy)$ and $f(F(y,x))=F(fy,fx)$ whenever $fx=F(x,y)$ and $fy=F(y,x)$.

Inspired by Definition 2.9, Lakshmikantham and Ćirić in [31] introduced the concept of a g-mixed monotone mapping.

Definition 2.13 ([31]) Let $(X,\preceq)$ be a partially ordered set, $F:X\times X\to X$ and $g:X\to X$ be mappings. Then the map F is said to have a mixed g-monotone property if $F(x,y)$ is monotone g-non-decreasing in x as well as monotone g-non-increasing in y; that is, for any $x,y\in X$,

$$gx_1\preceq gx_2 \quad \text{implies} \quad F(x_1,y)\preceq F(x_2,y) \quad \text{for all } y\in X$$

and

$$gy_1\preceq gy_2 \quad \text{implies} \quad F(x,y_2)\preceq F(x,y_1) \quad \text{for all } x\in X.$$

Now we prove our main results.

3 Results and discussions

Definition 3.1 Let (X, p) be a PMS, let $F : X \times X \to X$ and $g : X \to X$ be mappings. We say that F satisfies a (ψ, α, β)-contraction with respect to g if there exist $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ satisfying the following:

(3.1.1) ψ is continuous and monotonically non-decreasing, α is continuous and β is lower semi continuous,

(3.1.2) $\psi(t) = 0$ if and only if $t = 0, \alpha(0) = \beta(0) = 0$,

(3.1.3) $\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$,

(3.1.4) $\psi(p(F(x,y), F(u,v))) \leq \alpha(M(x,y,u,v)) - \beta(M(x,y,u,v))$, $\forall x,y,u,v \in X, gx \preceq gu, gy \succeq gv$ and

$$M(x,y,u,v) = \max\left\{\begin{array}{c} p(gx,gu), p(gy,gv), p(gx,F(x,y)), p(gy,F(y,x)), p(gu,F(u,v)), p(gv,F(v,u)), \\ \frac{p(gx,F(x,y))p(gy,F(y,x))}{1+p(gx,gu)+p(gy,gv)+p(F(x,y),F(u,v))}, \frac{p(gu,F(u,v))p(gv,F(v,u))}{1+p(gx,gu)+p(gy,gv)+p(F(x,y),F(u,v))} \end{array}\right\}.$$

Theorem 3.2 *Let $(X, \preceq)$ be a partially ordered set and p be a partial metric such that (X, p) is a PMS. Let $F : X \times X \to X$ and $g : X \to X$ be such that*

(3.2.1) *F satisfies a (ψ, α, β)-contraction with respect to g,*

(3.2.2) *$F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X,*

(3.2.3) *F has a mixed g-monotone property,*

(3.2.4) (a) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*
(b) *if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in $X \times X$.

Proof Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we choose $x_1, y_1 \in X$ such that

$$gx_0 \preceq F(x_0, y_0) = gx_1 \quad \text{and} \quad gy_0 \succeq F(y_0, x_0) = gy_1$$

and choose $x_2, y_2 \in X$ such that

$$gx_2 = F(x_1, y_1) \quad \text{and} \quad gy_2 = F(y_1, x_1).$$

Since F has the mixed g-monotone property, we obtain

$$gx_0 \preceq gx_1 \preceq gx_2 \quad \text{and} \quad gy_0 \succeq gy_1 \succeq gy_2.$$

Continuing this process, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n), \quad n = 0, 1, 2, \ldots$$

with

$$\left.\begin{array}{l} gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \quad \text{and} \\ gy_0 \succeq gy_1 \succeq gy_2 \succeq \cdots. \end{array}\right\} \tag{I}$$

Case (a): If $gx_m = gx_{m+1}$ and $gy_m = gy_{m+1}$ for some m, then (x_m, y_m) is a coupled coincidence point in $X \times X$.

Case (b): Assume $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n.

Since $gx_n \preceq gx_{n+1}$ and $gy_n \succeq gy_{n+1}$, from (3.2.1), we obtain

$$\begin{aligned}\psi\big(p(gx_n, gx_{n+1})\big) &= \psi\big(p\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big)\big)\\ &\leq \alpha\big(M(x_{n-1}, y_{n-1}, x_n, y_n)\big) - \beta\big(M(x_{n-1}, y_{n-1}, x_n, y_n)\big),\end{aligned}$$

$$M(x_{n-1}, y_{n-1}, x_n, y_n) = \max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n),\\ p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}),\\ \frac{p(gx_{n-1}, gx_n)p(gy_{n-1}, gy_n)}{1+p(gx_{n-1}, gx_n)+p(gy_{n-1}, gy_n)+p(gx_n, gx_{n+1})},\\ \frac{p(gx_n, gx_{n+1})p(gy_n, gy_{n+1})}{1+p(gx_{n-1}, gx_n)+p(gy_{n-1}, gy_n)+p(gx_n, gx_{n+1})}\end{array}\right\}.$$

But

$$\frac{p(gx_{n-1}, gx_n)p(gy_{n-1}, gy_n)}{1 + p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n) + p(gx_n, gx_{n+1})} \leq \max\big\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\big\}$$

and

$$\frac{p(gx_n, gx_{n+1})p(gy_n, gy_{n+1})}{1 + p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n) + p(gx_n, gx_{n+1})} \leq p(gy_n, gy_{n+1}).$$

Therefore

$$M(x_{n-1}, y_{n-1}, x_n, y_n) = \max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n),\\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\end{array}\right\}.$$

Hence

$$\begin{aligned}\psi\big(p(gx_n, gx_{n+1})\big) \leq{}& \alpha\left(\max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n),\\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\end{array}\right\}\right)\\ &- \beta\left(\max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n),\\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\end{array}\right\}\right).\end{aligned}$$

Similarly

$$\begin{aligned}\psi\big(p(gy_n, gy_{n+1})\big) \leq{}& \alpha\left(\max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n),\\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\end{array}\right\}\right)\\ &- \beta\left(\max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n),\\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\end{array}\right\}\right).\end{aligned}$$

Put $R_n = \max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}$. Let us suppose that

$$R_n \neq 0 \quad \text{for all } n \geq 1. \tag{3}$$

Let, if possible, for some n, $R_{n-1} < R_n$.

Now

$$\begin{aligned}\psi(R_n) &= \psi\big(\max\{p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1})\}\big)\\ &= \max\{\psi\big(p(gx_n,gx_{n+1})\big),\psi\big(p(gy_n,gy_{n+1})\big)\}\\ &\le \alpha\left(\max\left\{\begin{matrix}p(gx_{n-1},gx_n),p(gy_{n-1},gy_n),\\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1})\end{matrix}\right\}\right)\\ &\quad -\beta\left(\max\left\{\begin{matrix}p(gx_{n-1},gx_n),p(gy_{n-1},gy_n),\\ p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1})\end{matrix}\right\}\right)\\ &= \alpha\big(\max\{R_{n-1},R_n\}\big)-\beta\big(\max\{R_{n-1},R_n\}\big)\\ &= \alpha(R_n)-\beta(R_n).\end{aligned}$$

From (3.1.2) and (3.1.3), it follows that $R_n = 0$, a contradiction.

Hence

$$R_n \le R_{n-1}. \tag{4}$$

Thus $\{R_n\}$ is a non-increasing sequence of non-negative real numbers and must converge to a real number $r \ge 0$.

Also

$$\psi(R_n) \le \alpha(R_{n-1}) - \beta(R_{n-1}).$$

Letting $n \to \infty$, we get

$$\psi(r) \le \alpha(r) - \beta(r).$$

From (3.1.2) and (3.1.3), we get $r = 0$. Thus

$$\begin{aligned}&\lim_{n\to\infty}\max\{p(gx_n,gx_{n+1}),p(gy_n,gy_{n+1})\} = 0,\\ &\lim_{n\to\infty}p(gx_n,gx_{n+1}) = 0 = \lim_{n\to\infty}p(gy_n,gy_{n+1}).\end{aligned} \tag{5}$$

Hence from (p_2), we have

$$\lim_{n\to\infty}p(gx_n,gx_n) = 0 = \lim_{n\to\infty}p(gy_n,gy_n). \tag{6}$$

From (5) and (6) and by the definition of d_p, we get

$$\lim_{n\to\infty}d_p(gx_n,gx_{n+1}) = 0 = \lim_{n\to\infty}d_p(gy_n,gy_{n+1}). \tag{7}$$

Now we prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

To the contrary, suppose that $\{gx_n\}$ or $\{gy_n\}$ is not Cauchy.

This implies that $d_p(gx_m,gx_n) \not\to 0$ or $d_p(gy_m,gy_n) \not\to 0$ as $n,m \to \infty$.

Consequently

$$\max\{d_p(gx_m, gx_n), d_p(gy_m, gy_n)\} \nrightarrow 0 \quad \text{as } n, m \to \infty.$$

Then there exist an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$. We have

$$\max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} \geq \epsilon \tag{8}$$

and

$$\max\{d_p(gx_{m_k}, gx_{n_k-1}), d_p(gy_m, gy_{n_k-1})\} < \epsilon. \tag{9}$$

From (8) and (9), we have

$$\begin{aligned}\epsilon &\leq \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\}\\ &\leq \max\{d_p(gx_{m_k}, gx_{n_k-1}), d_p(gy_{m_k}, gy_{n_k-1})\}\\ &\quad + \max\{d_p(gx_{n_k-1}, gx_{n_k}), d_p(gy_{n_k-1}, gy_{n_k})\}\\ &< \epsilon + \max\{d_p(gx_{n_k-1}, gx_{n_k}), d_p(gy_{n_k-1}, gy_{n_k})\}.\end{aligned}$$

Letting $k \to \infty$ and using (7), we get

$$\lim_{k\to\infty} \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\} = \epsilon. \tag{10}$$

By the definition of d_p and using (6) we get

$$\lim_{k\to\infty} \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\} = \frac{\epsilon}{2}. \tag{11}$$

From (8), we have

$$\begin{aligned}\epsilon &\leq \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\}\\ &\leq \max\{d_p(gx_{m_k}, gx_{m_k-1}), d_p(gy_{m_k}, gy_{m_k-1})\}\\ &\quad + \max\{d_p(gx_{m_k-1}, gx_{n_k}), d_p(gy_{m_k-1}, gy_{n_k})\}\\ &\leq 2\max\{d_p(gx_{m_k}, gx_{m_k-1}), d_p(gy_{m_k}, gy_{m_k-1})\}\\ &\quad + \max\{d_p(gx_{m_k}, gx_{n_k}), d_p(gy_{m_k}, gy_{n_k})\}.\end{aligned} \tag{12}$$

Letting $k \to \infty$, using (7), (10) and (12), we get

$$\lim_{k\to\infty} \max\{d_p(gx_{m_k-1}, gx_{n_k}), d_p(gy_{m_k-1}, gy_{n_k})\} = \epsilon. \tag{13}$$

Hence, we get

$$\lim_{k\to\infty} \max\{p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})\} = \frac{\epsilon}{2}. \tag{14}$$

From (9), we have

$$\begin{aligned}
\epsilon &\le \max\{d_p(gx_{m_k},gx_{n_k}),d_p(gy_{m_k},gy_{n_k})\} \\
&\le \max\{d_p(gx_{m_k},gx_{m_k-1}),d_p(gy_{m_k},gy_{m_k-1})\} \\
&\quad + \max\{d_p(gx_{m_k-1},gx_{n_k+1}),d_p(gy_{m_k-1},gy_{n_k+1})\} \\
&\quad + \max\{d_p(gx_{n_k+1},gx_{n_k}),d_p(gy_{n_k+1},gy_{n_k})\} \\
&\le 2\max\{d_p(gx_{m_k},gx_{m_k-1}),d_p(gy_{m_k},gy_{m_k-1})\} \\
&\quad + \max\{d_p(gx_{m_k},gx_{n_k}),d_p(gy_{m_k},gy_{n_k})\} \\
&\quad +2\max\{d_p(gx_{n_k},gx_{n_k+1}),d_p(gy_{n_k},gy_{n_k+1})\}. \qquad (15)
\end{aligned}$$

Letting $k \to \infty$, using (7), (10) and (15), we get

$$\lim_{k\to\infty} \max\{d_p(gx_{m_k-1},gx_{n_k+1}),d_p(gy_{m_k-1},gy_{n_k+1})\} = \epsilon. \qquad (16)$$

Hence, we have

$$\lim_{k\to\infty} \max\{p(gx_{m_k-1},gx_{n_k+1}),p(gy_{m_k-1},gy_{n_k+1})\} = \frac{\epsilon}{2}. \qquad (17)$$

Now from (8), we have

$$\begin{aligned}
\epsilon &\le \max\{d_p(gx_{m_k},gx_{n_k}),d_p(gy_{m_k},gy_{n_k})\} \\
&\le \max\{d_p(gx_{m_k},gx_{n_k+1}),d_p(gy_{m_k},gy_{n_k+1})\} \\
&\quad + \max\{d_p(gx_{n_k+1},gx_{n_k}),d_p(gy_{n_k+1},gy_{2n_k})\}.
\end{aligned}$$

Letting $k \to \infty$ and using (7), we obtain

$$\begin{aligned}
\epsilon &\le \lim_{k\to\infty} \max\{d_p(gx_{m_k},gx_{n_k+1}),d_p(gy_{m_k},gy_{n_k+1})\} + 0 \\
&\le \lim_{k\to\infty} \max\left\{\begin{array}{l} 2p(gx_{m_k},gx_{n_k+1}) - p(gx_{m_k},gx_{m_k}) - p(gx_{n_k+1},gx_{n_k+1}), \\ 2p(gy_{m_k},gy_{n_k+1}) - p(gy_{m_k},gy_{m_k}) - p(gy_{n_k+1},gy_{n_k+1}) \end{array}\right\} \\
&= 2\lim_{k\to\infty} \max\{p(gx_{m_k},gx_{n_k+1}),p(gy_{m_k},gy_{n_k+1})\}, \quad \text{from (6).}
\end{aligned}$$

Thus,

$$\frac{\epsilon}{2} \le \lim_{k\to\infty} \max\{p(gx_{m_k},gx_{n_k+1}),p(gy_{m_k},gy_{n_k+1})\}.$$

By the properties of ψ,

$$\begin{aligned}
\psi\left(\frac{\epsilon}{2}\right) &\le \lim_{k\to\infty} \psi\left(\max\{p(gx_{m_k},gx_{n_k+1}),p(gy_{m_k},gy_{n_k+1})\}\right) \\
&= \lim_{k\to\infty} \max\{\psi\left(p(gx_{m_k},gx_{n_k+1})\right),\psi\left(p(gy_{m_k},gy_{n_k+1})\right)\}. \qquad (18)
\end{aligned}$$

Now

$$\begin{aligned}
\psi\big(p(gx_{m_k}, gx_{n_k+1})\big) &= \psi\big(p\big(F(x_{m_k-1}, y_{m_k-1}), F(x_{n_k}, y_{n_k})\big)\big) \\
&\leq \alpha\big(M(x_{m_k-1}, y_{m_k-1}, x_{n_k}, y_{n_k})\big) - \beta\big(M(x_{m_k-1}, y_{m_k-1}, x_{n_k}, y_{n_k})\big) \\
&= \alpha\left(\max\left\{\begin{array}{c} p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k}), p(gx_{m_k-1}, gx_{m_k}), \\ p(gy_{m_k-1}, gy_{m_k}), p(gx_{n_k}, gx_{n_k+1}), p(gy_{n_k}, gy_{n_k+1}), \\ \frac{p(gx_{m_k-1}, gx_{m_k})p(gy_{m_k-1}, gy_{m_k})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})}, \\ \frac{p(gx_{n_k}, gx_{n_k+1})p(gy_{n_k}, gy_{n_k+1})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})} \end{array}\right\}\right) \\
&\quad - \beta\left(\max\left\{\begin{array}{c} p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k}), p(gx_{m_k-1}, gx_{m_k}), \\ p(gy_{m_k-1}, gy_{m_k}), p(gx_{n_k}, gx_{n_k+1}), p(gy_{n_k}, gy_{n_k+1}), \\ \frac{p(gx_{m_k-1}, gx_{m_k})p(gy_{m_k-1}, gy_{m_k})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})}, \\ \frac{p(gx_{n_k}, gx_{n_k+1})p(gy_{n_k}, gy_{n_k+1})}{1+p(gx_{m_k-1}, gx_{n_k}), p(gy_{m_k-1}, gy_{n_k})+p(gx_{m_k}, gx_{n_k+1})} \end{array}\right\}\right).
\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \psi\big(p(gx_{m_k}, gx_{n_k+1})\big) \leq \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right).$$

Similarly, we obtain

$$\lim_{k\to\infty} \psi\big(p(gy_{m_k}, gy_{n_k+1})\big) \leq \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right).$$

Hence from (18), we have

$$\psi\left(\frac{\epsilon}{2}\right) \leq \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right).$$

From (3.1.2) and (3.1.3), we get $\frac{\epsilon}{2} = 0$, a contradiction.

Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the metric space (X, d_p).

Hence we have $\lim_{n,m\to\infty} d_p(gx_n, gx_m) = 0 = \lim_{n,m\to\infty} d_p(gy_n, gy_m)$.

Now from the definition of d_p and from (6), we have

$$\lim_{n\to\infty} p(gx_n, gx_m) = 0 = \lim_{n\to\infty} p(gy_n, gy_m). \tag{19}$$

Suppose $g(X)$ is a complete subspace of X.

Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in a complete metric space $(g(X), d_p)$. Then $\{gx_n\}$ and $\{gy_n\}$ converges to some u and v in $g(X)$ respectively. Thus

$$\lim_{n\to\infty} d_p(gx_n, u) = 0$$

and

$$\lim_{n\to\infty} d_p(gy_n, v) = 0$$

for some u and v in $g(X)$.

Since $u, v \in g(X)$, there exist $x, y \in X$ such that $u = gx$ and $v = gy$.

Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences, $gx_n \to u$, $gy_n \to v$, $gx_{n+1} \to u$ and $gy_{n+1} \to v$.

From Lemma 2.5(2) and (19), we obtain

$$p(u,u) = \lim_{n\to\infty} p(gx_n, u) = p(v,v) = \lim_{n\to\infty} p(gy_n, v) = 0. \tag{20}$$

Now we prove that $\lim_{n\to\infty} p(F(x,y), gx_n) = p(F(x,y), u)$.

By definition of d_p,

$$d_p\big(F(x,y), gx_n\big) = 2p\big(F(x,y), gx_n\big) - p\big(F(x,y), F(x,y)\big) - p(gx_n, gx_n).$$

Letting $n \to \infty$, we have

$$d_p\big(F(x,y), u\big) = 2\lim_{n\to\infty} p\big(F(x,y), gx_n\big) - p\big(F(x,y), F(x,y)\big) - 0, \text{ from (6)}.$$

By definition of d_p and (19), we have

$$\lim_{n\to\infty} p\big(F(x,y), gx_n\big) = p\big(F(x,y), u\big).$$

Similarly, $\lim_{n\to\infty} p(F(y,x), gy_n) = p(F(y,x), v)$.

From (p_4), we have

$$\begin{aligned} p\big(u, F(x,y)\big) &\leq p(u, gx_{n+1}) + p\big(gx_{n+1}, F(x,y)\big) - p(gx_{n+1}, gx_{n+1}) \\ &= p(u, gx_{n+1}) + p\big(gx_{n+1}, F(x,y)\big). \end{aligned}$$

Letting $n \to \infty$, we have

$$p\big(u, F(x,y)\big) \leq 0 + \lim_{n\to\infty} p\big(F(x_n, y_n), F(x,y)\big).$$

Also from (3.2.4), we get $gx_n \preceq gx$ and $gy_n \succeq gy$. Since ψ is a continuous and non-decreasing function, we get

$$\begin{aligned} \psi\big(p\big(u, F(x,y)\big)\big) &\leq \lim_{n\to\infty} \psi\big(p\big(F(x_n,y_n), F(x,y)\big)\big) \\ &\leq \lim_{n\to\infty} \big[\alpha\big(M(x_n,y_n,x,y)\big) - \beta\big(M(x_n,y_n,x,y)\big)\big], \end{aligned}$$

$$\begin{aligned} M(x_n,y_n,x,y) &= \max\left\{\begin{array}{c} p(gx_n,u), p(gy_n,v), p(gx_n,gx_{n+1}), \\ p(gy_n,gy_{n+1}), p(u,F(x,y)), p(v,F(y,x)), \\ \frac{p(gx_n,gx_{n+1})p(gy_n,gy_{n+1})}{1+p(gx_n,u)+p(gy_n,v)+p(gx_{n+1},F(x,y))}, \\ \frac{p(u,F(x,y))p(v,F(y,x))}{1+p(gx_n,u)+p(gy_n,v)+p(gx_{n+1},F(x,y))} \end{array}\right\} \\ &\to \max\big\{p\big(u,F(x,y)\big), p\big(v,F(y,x)\big)\big\} \quad \text{as } n \to \infty. \end{aligned}$$

Therefore

$$\psi\big(p\big(u,F(x,y)\big)\big) \leq \alpha\left(\max\left\{\begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array}\right\}\right) - \beta\left(\max\left\{\begin{array}{c} p(u,F(x,y)), \\ p(v,F(y,x)) \end{array}\right\}\right).$$

Similarly,

$$\psi\big(p\big(v,F(y,x)\big)\big) \le \alpha\left(\max\left\{\begin{matrix} p(u,F(x,y)), \\ p(v,F(y,x)) \end{matrix}\right\}\right) - \beta\left(\max\left\{\begin{matrix} p(u,F(x,y)), \\ p(v,F(y,x)) \end{matrix}\right\}\right).$$

Hence

$$\begin{aligned} &\psi\big(\max\big\{p\big(u,F(x,y)\big),p\big(v,F(y,x)\big)\big\}\big) \\ &\quad = \max\big\{\psi\big(p\big(u,F(x,y)\big)\big),\psi\big(p\big(v,F(y,x)\big)\big)\big\} \\ &\quad \le \alpha\left(\max\left\{\begin{matrix} p(u,F(x,y)), \\ p(v,F(y,x)) \end{matrix}\right\}\right) - \beta\left(\max\left\{\begin{matrix} p(u,F(x,y)), \\ p(v,F(y,x)) \end{matrix}\right\}\right). \end{aligned}$$

It follows that $\max\{p(u,F(x,y)),p(v,F(y,x))\} = 0$. So $F(x,y) = u$ and $F(y,x) = v$.

Hence $F(x,y) = gx = u$ and $F(y,x) = gy = v$.

Hence F and g have a coincidence point in $X \times X$. □

Theorem 3.3 *In addition to the hypothesis of Theorem* 3.2, *we suppose that for every* $(x,y),(x^1,y^1) \in X \times X$ *there exists* $(u,v) \in X \times X$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(x^1,y^1),F(y^1,x^1))$. *If* (x,y) *and* (x^1,y^1) *are coupled coincidence points of* F *and* g, *then*

$$\begin{aligned} &F(x,y) = gx = gx^1 = F\big(x^1,y^1\big) \quad \textit{and} \\ &T(y,x) = gy = gy^1 = F\big(y^1,x^1\big). \end{aligned}$$

Moreover, if (F,g) *is* w*-compatible, then* F *and* g *have a unique common coupled fixed point in* $X \times X$.

Proof The proof follows from Theorem 3.2 and the definition of comparability. □

Theorem 3.4 *Let* $(X,\preceq)$ *be a partially ordered set and* p *be a partial metric such that* (X,p) *is a complete PMS. Let* $F : X \times X \to X$ *be such that*

(3.4.1) $$\psi\big(p\big(F(x,y),F(u,v)\big)\big) \le \alpha\big(\max\big\{p(x,u),p(y,v)\big\}\big) - \beta\big(\max\big\{p(x,u),p(y,v)\big\}\big),$$

$\forall x,y,u,v \in X$, $x \preceq u$ *and* $y \succeq v$, *where* ψ,α *and* β *are defined in Definition* 3.1 *and*

(3.4.2) (a) *If a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n, *and*

(b) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y \preceq y_n$ *for all* n.

If there exist $x_0,y_0 \in X$ *such that* $x_0 \preceq F(x_0,y_0)$ *and* $y_0 \succeq F(y_0,x_0)$, *then* F *has a unique coupled fixed point in* $X \times X$.

Example 3.5 Let $X = [0,1]$, let $\preceq$ be partially ordered on X by

$$x \preceq y \quad \Leftrightarrow \quad x \ge y.$$

The mapping $F : X \times X \to X$ defined by $F(x,y) = \frac{x^2+y^2}{8(x+y+1)}$ and $p : X \times X \to [0,\infty)$ by $p(x,y) = \max\{x,y\}$ is a complete partial metric on X. Define $\psi,\alpha,\beta : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$,

$\alpha(t) = \frac{t}{2}$ and $\beta(t) = \frac{t}{4}$. We have

$$\begin{aligned} p\bigl(F(x,y),F(u,v)\bigr) &= \max\left\{\frac{x^2+y^2}{8(x+y+1)}, \frac{u^2+v^2}{8(u+v+1)}\right\} \\ &= \frac{1}{4}\left[\max\left\{\frac{x^2}{x+y+1}, \frac{u^2}{u+v+1}\right\} + \max\left\{\frac{y^2}{x+y+1}, \frac{v^2}{u+v+1}\right\}\right] \\ &\leq \frac{1}{8}\left[\max\left\{\frac{x^2}{x+1}, \frac{u^2}{u+1}\right\} + \max\left\{\frac{y^2}{y+1}, \frac{v^2}{v+1}\right\}\right] \\ &\leq \frac{1}{8}\left[\max\left\{\frac{x}{x+1}, \frac{u}{u+1}\right\} + \max\left\{\frac{y}{y+1}, \frac{v}{v+1}\right\}\right] \\ &\leq \frac{1}{8}\bigl[\max\{x,u\} + \max\{y,v\}\bigr] \\ &= \frac{1}{8}\bigl[p(x,u) + p(y,v)\bigr] \\ &\leq \frac{1}{4}\max\bigl\{p(x,u),p(y,v)\bigr\} \\ &= \alpha\bigl(\max\bigl\{p(x,u),p(y,v)\bigr\}\bigr) - \beta\bigl(\max\bigl\{p(x,u),p(y,v)\bigr\}\bigr). \end{aligned}$$

Hence all conditions of Theorem 3.4 hold. From Theorem 3.4, $(0,0)$ is a unique coupled fixed point of F in $X \times X$.

3.1 Application to integral equations

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 3.4.

Consider the initial value problem

$$\begin{aligned} &x^1(t) = f\bigl(t,x(t),x(t)\bigr), \quad t \in I = [0,1], \\ &x(0) = x_0, \end{aligned} \tag{21}$$

where $f : I \times [\frac{x_0}{4},\infty) \times [\frac{x_0}{4},\infty) \to [\frac{x_0}{4},\infty)$ and $x_0 \in \mathbb{R}$.

Theorem 3.6 *Consider the initial value problem* (21) *with* $f \in C(I \times [\frac{x_0}{4},\infty) \times [\frac{x_0}{4},\infty))$ *and*

$$\int_0^t f\bigl(s,x(s),y(s)\bigr)\,ds \leq \max\left\{\begin{array}{l} \frac{1}{4}\int_0^t f(s,x(s),x(s))\,ds - \frac{9x_0}{16}, \\ \frac{1}{4}\int_0^t f(s,y(s),y(s))\,ds - \frac{9x_0}{16} \end{array}\right\}.$$

Then there exists a unique solution in $C(I,[\frac{x_0}{4},\infty))$ *for the initial value problem* (21).

Proof The integral equation corresponding to initial value problem (21) is

$$x(t) = x_0 + \int_0^t f\bigl(s,x(s),x(s)\bigr)\,ds. \tag{22}$$

Let $X = C(I, [\frac{x_0}{4}, \infty))$ and $p(x,y) = \max\{x - \frac{x_0}{4}, y - \frac{x_0}{4}\}$ for $x, y \in X$. Define $\psi, \alpha, \beta : [0,\infty) \to [0,\infty)$ by $\psi(t) = t$, $\alpha(t) = \frac{1}{2}t$ and $\beta(t) = \frac{1}{4}t$. Define $F : X \times X \to X$ by

$$F(x,y)(t) = x_0 + \int_0^t f\bigl(s, x(s), y(s)\bigr)\,ds.$$

Now

$$\begin{aligned}
&p\bigl(F(x,y)(t), F(u,v)(t)\bigr)\\
&\quad= \max\left\{F(x,y) - \frac{x_0}{4}, F(u,v) - \frac{x_0}{4}\right\}\\
&\quad= \max\left\{\frac{3x_0}{4} + \int_0^t f\bigl(s,x(s),y(s)\bigr)\,ds, \frac{3x_0}{4} + \int_0^t f\bigl(s,u(s),v(s)\bigr)\,ds\right\}\\
&\quad\le \max\left\{\begin{array}{l}\frac{3x_0}{4} + \max\left\{\begin{array}{l}\frac{1}{4}\int_0^t f(s,x(s),x(s))\,ds - \frac{9x_0}{16},\\ \frac{1}{4}\int_0^t f(s,y(s),y(s))\,ds - \frac{9x_0}{16}\end{array}\right\},\\ \frac{3x_0}{4} + \max\left\{\begin{array}{l}\frac{1}{4}\int_0^t f(s,u(s),u(s))\,ds - \frac{9x_0}{16},\\ \frac{1}{4}\int_0^t f(s,v(s),v(s))\,ds - \frac{9x_0}{16}\end{array}\right\}\end{array}\right\}\\
&\quad= \max\left\{\begin{array}{l}\max\left\{\frac{x(t)}{4} - \frac{x_0}{16}, \frac{y(t)}{4} - \frac{x_0}{16}\right\},\\ \max\left\{\frac{u(t)}{4} - \frac{x_0}{16}, \frac{v(t)}{4} - \frac{x_0}{16}\right\}\end{array}\right\}\\
&\quad= \frac{1}{4}\max\left\{\max\left\{x(t) - \frac{x_0}{4}, u(t) - \frac{x_0}{4}\right\}, \max\left\{y(t) - \frac{x_0}{4}, v(t) - \frac{x_0}{4}\right\}\right\}\\
&\quad= \frac{1}{4}\max\bigl\{p(x,u), p(y,v)\bigr\}\\
&\quad= \alpha\bigl(\max\bigl\{p(x,u), p(y,v)\bigr\}\bigr) - \beta\bigl(\max\bigl\{p(x,u), p(y,v)\bigr\}\bigr).
\end{aligned}$$

Thus F satisfies the condition (3.4.1) of Theorem 3.4. From Theorem 3.4, we conclude that F has a unique coupled fixed point (x,y) with $x = y$. In particular $x(t)$ is the unique solution of the integral equation (22). □

3.2 Application to homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 3.7 *Let (X,p) be a complete PMS, U be an open subset of X and $\overline{U}$ be a closed subset of X such that $U \subseteq \overline{U}$. Suppose $H : \overline{U} \times \overline{U} \times [0,1] \to X$ is an operator such that the following conditions are satisfied*:

(i) $x \neq H(x,y,\lambda)$ *and* $y \neq H(y,x,\lambda)$ *for each* $x, y \in \partial U$ *and* $\lambda \in [0,1]$ (*here* ∂U *denotes the boundary of* U *in* X),

(ii) $\psi(p(H(x,y,\lambda), H(u,v,\lambda))) \le \alpha(\max\{p(x,y), p(u,v)\}) - \beta(\max\{p(x,y), p(u,v)\})$ $\forall x, y \in \overline{U}$ *and* $\lambda \in [0,1]$, *where* $\psi, \alpha : [0,\infty) \to [0,\infty)$ *is continuous and non-decreasing and* $\beta : [0,\infty) \to [0,\infty)$ *is lower semi continuous with* $\psi(t) - \alpha(t) + \beta(t) > 0$ *for* $t > 0$,

(iii) *there exists* $M \ge 0$ *such that*

$$p\bigl(H(x,y,\lambda), H(x,y,\mu)\bigr) \le M|\lambda - \mu|$$

for every $x \in \overline{U}$ *and* $\lambda, \mu \in [0,1]$.

Then $H(\cdot, 0)$ *has a coupled fixed point if and only if* $H(\cdot, 1)$ *has a coupled fixed point.*

Proof Consider the set

$$A = \{\lambda \in [0,1] : (x,y) = H(x,y,\lambda) \text{ for some } x,y \in U\}.$$

Since $H(\cdot, 0)$ has a coupled fixed point in U, we have $0 \in A$, so that A is a non-empty set.

We will show that A is both open and closed in $[0,1]$ so by the connectedness we have $A = [0,1]$.

As a result, $H(\cdot, 1)$ has a fixed point in U. First we show that A is closed in $[0,1]$.

To see this let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$.

We must show that $\lambda \in A$.

Since $\lambda_n \in A$ for $n = 1, 2, 3, \ldots$, there exist $x_n, y_n \in U$ with $u_n = (x_n, y_n) = H(x_n, y_n \lambda_n)$. Consider

$$\begin{aligned} p(x_n, x_{n+1}) &= p\bigl(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_{n+1})\bigr) \\ &\leq p\bigl(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}\lambda_n)\bigr) \\ &\quad + p\bigl(H(x_{n+1}, y_{n+1}, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_{n+1})\bigr) \\ &\quad - p\bigl(H(x_{n+1}, y_{n+1}, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_n)\bigr) \\ &\leq p\bigl(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_n)\bigr) + M|\lambda_n - \lambda_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} p(x_n, x_{n+1}) \leq \lim_{n\to\infty} p\bigl(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_n)\bigr) + 0.$$

Since ψ is continuous and non-decreasing we obtain

$$\begin{aligned} &\lim_{n\to\infty} \psi\bigl(p(x_n, x_{n+1})\bigr) \\ &\quad \leq \lim_{n\to\infty} \psi\bigl(p\bigl(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_n)\bigr)\bigr) \\ &\quad \leq \lim_{n\to\infty} \bigl[\alpha\bigl(\max\bigl\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\bigr\}\bigr) - \beta\bigl(\max\bigl\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\bigr\}\bigr)\bigr]. \end{aligned}$$

Similarly

$$\begin{aligned} &\lim_{n\to\infty} \psi\bigl(p(y_n, y_{n+1})\bigr) \\ &\quad \leq \lim_{n\to\infty} \bigl[\alpha\bigl(\max\bigl\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\bigr\}\bigr) - \beta\bigl(\max\bigl\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\bigr\}\bigr)\bigr]. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 = \lim_{n\to\infty} p(y_n, y_{n+1}). \tag{23}$$

From (p_2),

$$\lim_{n\to\infty} p(x_n, x_n) = 0 = \lim_{n\to\infty} p(y_n, y_n). \tag{24}$$

By the definition of d_p, we obtain

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0 = \lim_{n\to\infty} d_p(y_n, y_{n+1}). \tag{25}$$

Now we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d_p). Contrary to this hypothesis, suppose that $\{x_n\}$ or $\{s_n\}$ is not Cauchy.

There exists an $\epsilon > 0$ and a monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$\max\{d_p(x_{m_k}, x_{n_k}), d_p(y_{m_k}, y_{n_k})\} \geq \epsilon \tag{26}$$

and

$$\max\{d_p(x_{m_k}, x_{n_k-1}), d_p(y_{m_k}, y_{n_k-1})\} < \epsilon. \tag{27}$$

From (26) and (27), we obtain

$$\begin{aligned}\epsilon &\leq \max\{d_p(x_{m_k}, x_{n_k}), d_p(y_{m_k}, y_{n_k})\}\\ &\leq \max\{d_p(x_{m_k}, x_{n_k-1}), d_p(y_{m_k}, y_{n_k-1})\}\\ &\quad + \max\{d_p(x_{n_k-1}, x_{n_k}), d_p(y_{n_k-1}, y_{n_k})\}\\ &< \epsilon + \max\{d_p(x_{n_k-1}, x_{n_k}), d_p(y_{n_k-1}, y_{n_k})\}.\end{aligned}$$

Letting $k \to \infty$ and then using (25), we get

$$\lim_{k\to\infty} \max\{d_p(x_{m_k}, x_{n_k}), d_p(y_{m_k}, y_{n_k})\} = \epsilon. \tag{28}$$

Hence from the definition of d_p and from (24), we get

$$\lim_{k\to\infty} \max\{p(x_{m_k}, x_{n_k}), p(y_{m_k}, y_{n_k})\} = \frac{\epsilon}{2}. \tag{29}$$

Letting $k \to \infty$ and then using (28) and (25) in

$$\left|d_p(x_{m_k}, x_{n_k+1}) - d_p(x_{m_k}, x_{n_k})\right| \leq d_p(x_{n_k+1}, x_{n_k}),$$

we get

$$\lim_{k\to\infty} d_p(x_{n_k+1}, x_{m_k}) = \epsilon. \tag{30}$$

Hence, we have

$$\lim_{k\to\infty} p(x_{n_k+1}, x_{m_k}) = \frac{\epsilon}{2}. \tag{31}$$

Similarly

$$\lim_{k\to\infty} p(y_{n_k+1}, y_{m_k}) = \frac{\epsilon}{2}. \tag{32}$$

Consider

$$\begin{aligned} p(x_{m_k}, x_{n_k+1}) &= p\big(H(x_{m_k}, y_{m_k}, \lambda_{m_k}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big) \\ &\le p\big(H(x_{m_k}, y_{m_k}, \lambda_{m_k}), H(x_{m_k}, y_{m_k}, \lambda_{n_k+1})\big) \\ &\quad + p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big) \\ &\quad - p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{m_k}, y_{m_k}, \lambda_{n_k+1})\big) \\ &\le M|\lambda_{m_k} - \lambda_{n_k+1}| + p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big). \end{aligned}$$

Since $\{\lambda_n\}$ is Cauchy, letting $k \to \infty$ in the above, we get

$$\frac{\epsilon}{2} \le \lim_{k\to\infty} p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big).$$

Since ψ is continuous and non-decreasing we obtain

$$\begin{aligned} \psi\left(\frac{\epsilon}{2}\right) &\le \lim_{k\to\infty} \psi\left(p\big(H(x_{m_k}, y_{m_k}, \lambda_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \lambda_{n_k+1})\big)\right) \\ &\le \lim_{k\to\infty} \Big[\alpha\left(\max\big\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\big\}\right) \\ &\quad - \beta\big(\max\big\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\big\}\big)\Big] \\ &= \alpha\left(\frac{\epsilon}{2}\right) - \beta\left(\frac{\epsilon}{2}\right). \end{aligned}$$

It follows that $\epsilon \le 0$, which is a contradiction.

Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d_p) and

$$\lim_{n,m\to\infty} d_p(x_n, x_m) = 0 = \lim_{n,m\to\infty} d_p(y_n, y_m).$$

By the definition of d_p and (24), we get $\lim_{n,m\to\infty} p(x_n, x_m) = 0 = \lim_{n,m\to\infty} p(y_n, y_m)$.

From Lemma 2.5, we conclude (a) $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, p).

Since (X, p) is complete, from Lemma 2.5(b), we conclude there exist $u, v \in U$ with

$$p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{n\to\infty} p(x_{n+1}, u) = \lim_{n,m\to\infty} p(x_n, x_m) = 0, \tag{33}$$

$$p(v, v) = \lim_{n\to\infty} p(x_n, v) = \lim_{n\to\infty} p(x_{n+1}, v) = \lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{34}$$

From Lemma 2.6, we get $\lim_{n\to\infty} p(x_n, H(u, v, \lambda)) = p(u, H(u, v, \lambda))$.

Now,

$$\begin{aligned} p\big(x_n, H(u, v, \lambda)\big) &= p\big(H(x_n, y_n, \lambda_n), H(u, v, \lambda)\big) \\ &\le p\big(H(x_n, y_n, \lambda_n), H(x_n, y_n, \lambda)\big) + p\big(H(x_n, y_n, \lambda), H(u, v, \lambda)\big) \end{aligned}$$

$$\begin{aligned}&- p\big(H(x_n, y_n, \lambda), H(x_n, y_n, \lambda)\big)\\ &\leq M|\lambda_n - \lambda| + p\big(H(x_n, y_n, \lambda), H(u, v, \lambda)\big).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$p\big(u, H(u, v, \lambda)\big) \leq \lim_{n\to\infty} p\big(H(x_n, y_n, \lambda), H(u, v, \lambda)\big).$$

Since ψ is continuous and non-decreasing, we obtain

$$\begin{aligned}\psi\big(p\big(u, H(u, v, \lambda)\big)\big) &\leq \lim_{n\to\infty} \psi\big(p\big(H(x_n, y_n, \lambda), H(u, v, \lambda)\big)\big)\\ &\leq \lim_{n\to\infty}\big[\alpha\big(\max\big\{p(x_n, u), p(y_n, v)\big\}\big) - \beta\big(\max\big\{p(x_n, u), p(y_n, v)\big\}\big)\big]\\ &= 0.\end{aligned}$$

It follows that $p(u, H(u, v, \lambda)) = 0$. Thus $u = H(u, v, \lambda)$. Similarly $v = H(v, u, \lambda)$.

Thus $\lambda \in A$. Hence A is closed in $[0, 1]$.

Let $\lambda_0 \in A$. Then there exist $x_0, y_0 \in U$ with $x_0 = H(x_0, y_0, \lambda_0)$.

Since U is open, there exists $r > 0$ such that $B_p(x_0, r) \subseteq U$.

Choose $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that $|\lambda - \lambda_0| \leq \frac{1}{M^n} < \epsilon$.

Then $x \in \overline{B_p(x_0, r)} = \{x \in X / p(x, x_0) \leq r + p(x_0, x_0)\}$. We have

$$\begin{aligned}p\big(H(x, y, \lambda), x_0\big) &= p\big(H(x, y, \lambda), H(x_0, x_0, \lambda_0)\big)\\ &\leq p\big(H(x, y, \lambda), H(x, y, \lambda_0)\big) + p\big(H(x, y, \lambda_0), H(x_0, y_0, \lambda_0)\big)\\ &\quad - p\big(H(x, y, \lambda_0), H(x, y, \lambda_0)\big)\\ &\leq M|\lambda - \lambda_0| + p\big(H(x, y, \lambda_0), H(x_0, y_0, \lambda_0)\big)\\ &\leq \frac{1}{M^{n-1}} + p\big(H(x, y, \lambda_0), H(x_0, y_0, \lambda_0)\big).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$p\big(H(x, y, \lambda), x_0\big) \leq p\big(H(x, y, \lambda_0), H(x_0, y_0, \lambda_0)\big).$$

Since ψ is continuous and non-decreasing, we have

$$\begin{aligned}\psi\big(p\big(H(x, y, \lambda), x_0\big)\big) &\leq \psi\big(p\big(H(x, y, \lambda_0), H(x_0, y_0, \lambda_0)\big)\big)\\ &\leq \alpha\big(\max\big\{p(x, x_0), p(y, y_0)\big\}\big)\\ &\quad - \phi\big(\max\big\{p(x, x_0), p(y, y_0)\big\}\big).\end{aligned}$$

Similarly

$$\begin{aligned}&\psi\big(p\big(H(y, x, \lambda), y_0\big)\big)\\ &\quad\leq \alpha\big(\max\big\{p(x, x_0), p(y, y_0)\big\}\big) - \phi\big(\max\big\{p(x, x_0), p(y, y_0)\big\}\big).\end{aligned}$$

Thus

$$\begin{aligned}&\psi\left(\max\left\{p\left(H(x,y,\lambda),x_0\right),p\left(H(y,x,\lambda),y_0\right)\right\}\right)\\&\quad\le\alpha\left(\max\left\{p(x,x_0),p(y,y_0)\right\}\right)-\phi\left(\max\left\{p(x,x_0),p(y,y_0)\right\}\right)\\&\quad\le\psi\left(\max\left\{p(x,x_0),p(y,y_0)\right\}\right).\end{aligned}$$

Since ψ is non-decreasing, we have

$$\begin{aligned}\max\left\{p\left(H(x,y,\lambda),x_0\right),p\left(H(y,x,\lambda),y_0\right)\right\}&\le\max\left\{p(x,x_0),p(y,y_0)\right\}\\&\le\max\left\{r+p(x_0,x_0),r+p(y_0,y_0)\right\}.\end{aligned}$$

Thus for each fixed $\lambda\in(\lambda_0-\epsilon,\lambda_0+\epsilon)$, $H(\cdot,\lambda):\overline{B_p(x_0,r)}\to\overline{B_p(x_0,r)}$.

Since also (ii) holds and ψ and α are continuous and non-decreasing and β is continuous with $\psi(t)-\alpha(t)+\beta(t)>0$ for $t>0$, all conditions of Theorem 3.4 are satisfied.

Thus we deduce that $H(\cdot,\lambda)$ has a coupled fixed point in $\overline{U}$. But this coupled fixed point must be in U since (i) holds.

Thus $\lambda\in A$ for any $\lambda\in(\lambda_0-\epsilon,\lambda_0+\epsilon)$.

Hence $(\lambda_0-\epsilon,\lambda_0+\epsilon)\subseteq A$ and therefore A is open in $[0,1]$.

For the reverse implication, we use the same strategy. □

Corollary 3.8 *Let (X,p) be a complete PMS, U be an open subset of X and $H:\overline{U}\times\overline{U}\times[0,1]\to X$ with the following properties:*

1. *$x\neq H(x,y,t)$ and $y\neq H(y,x,t)$ for each $x,y\in\partial U$ and each $\lambda\in[0,1]$ (here ∂U denotes the boundary of U in X),*
2. *there exist $x,y\in\overline{U}$ and $\lambda\in[0,1]$, $L\in[0,1)$, such that*

$$p\left(H(x,y,\lambda),H(u,v,\mu)\right)\le L\max\left\{p(x,u),p(y,v)\right\},$$

3. *there exists $M\geq 0$, such that*

$$p\left(H(x,\lambda),H(x,\mu)\right)\le M\cdot|\lambda-\mu|$$

for all $x\in\overline{U}$ and $\lambda,\mu\in[0,1]$.

If $H(\cdot,0)$ has a fixed point in U, then $H(\cdot,1)$ has a fixed point in U.

Proof The proof follows by taking $\psi(x)=x$, $\phi(x)=x-Lx$ with $L\in[0,1)$ in Theorem 3.7. □

4 Conclusions

In this paper we conclude some applications on homotopy theory and integral equations by using coupled fixed point theorems in ordered PMSs.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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