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Common coupled fixed point results for operators without mixed monotone type properties and application to nonlinear integral equations

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Abstract

We establish some new common coupled fixed point theorems for a pair of operators not assumed to satisfy mixed monotone type properties in the ordered Banach space setting. For that purpose, the notions of weakly inflationary and weakly deflationary operators are introduced in the two-dimensional setting, and existence and uniqueness common coupled fixed point theorems for such operators are established under certain condensing and contractive conditions involving the two-dimensional setting. As an application, we study the existence and uniqueness of nonnegative solutions for nonlinear integral equations.

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Keywords: ordered Banach space; coupled fixed point; mixed monotone property; inflationary mapping

1 Introduction and preliminaries

The notion of a coupled fixed point was introduced and studied by Opoitsev [1] and investigated later by Guo-Lakshmikantham [2]. Among investigations of coupled fixed point results for nonlinear operators in the ordered Banach space setting, there are more results on the existence of coupled fixed points than on the existence of common coupled fixed points of a pair of operators, and most of these results were established for mixed monotone operators; see for instance [2–8] and the references therein. The notion of a mixed monotone operator was introduced in [2] as follows.

Definition 1.1 Let $(\Omega, \leq)$ be a partially ordered set. An operator $A : \Omega \times \Omega \to \Omega$ is said to be mixed monotone if $A(x,y)$ is nondecreasing in x and nonincreasing in y, in the sense that

$$(\forall x_1, x_2, y \in \Omega) \quad x_1 \leq x_2 \quad \Rightarrow \quad A(x_1,y) \leq A(x_2,y)$$

and

$$(\forall x, y_1, y_2 \in \Omega) \quad y_1 \leq y_2 \quad \Rightarrow \quad A(x,y_1) \geq A(x,y_2).$$

In some recent work, the authors have followed a line of research consisting in replacing the mixed monotonicity of operators by other properties, since the mixed monotonicity is not often easy to check; see [9] for a property based on the comparability of elements in ordered metric spaces, and [3, 10] for the following alternative mixed monotone property.

Definition 1.2 Let $(\Omega, \leq)$ be a partially ordered set, and let $A : \Omega \times \Omega \to \Omega$, $g : \Omega \to \Omega$ be two operators. A is said to be mixed g-monotone if $A(x,y)$ is g-nondecreasing in x and g-nonincreasing in y, in the sense that

$$(\forall x_1, x_2, y \in \Omega) \quad g(x_1) \leq g(x_2) \quad \Rightarrow \quad A(x_1, y) \leq A(x_2, y)$$

and

$$(\forall x, y_1, y_2 \in \Omega) \quad g(y_1) \leq g(y_2) \quad \Rightarrow \quad A(x, y_1) \geq A(x, y_2).$$

Hence, the mixed g-monotone property of an operator $A : \Omega \times \Omega \to \Omega$ extends its mixed monotone property (which is the mixed Id-monotone property where Id is the identity mapping on Ω).

In this paper, we continue on this path by investigating the existence of common coupled fixed points of a pair of operators $A, B : \Omega \times \Omega \to \Omega$, where mixed monotone type properties of the operators are not assumed, while the pair of operators is assumed to satisfy a new two-dimensional order type property, extending a well-known one-dimensional equivalent property (Definition 2.3 and the remarks following), and guaranteeing the use of monotone iterative technique. To prove the existence of a common coupled fixed point for such pair of operators, it is assumed for operators to satisfy a useful condensing and contractive conditions (conditions (C_1) and (C_2) hereafter) involving the two-dimensional setting. The consideration of the latter condensing and contractive conditions is motivated by the fact that these conditions are satisfied in particular when the operators satisfy the standard condensing and contractive conditions as defined in the literature; see Definition 1.5 and the remarks following Lemma 2.1.

The first main result extends the well-known results in the literature on a fixed point theorem for monotone operators and coupled fixed point theorem for mixed monotone operators; see [4], Theorem 2.1.1 and Theorem 2.1.7. We look, also, at the equivalent of our main result when the Banach space is endowed with its weak topology. Finally, we illustrate the applicability of our results by studying the existence and uniqueness of nonnegative solution for a two-dimensional nonlinear integral equations.

Throughout this paper X will be a real Banach space, B_r will denote the closed ball in X centered at 0 with radius $r > 0$. In particular, we use the notation $B_X := B_1$. For a subset $\Omega \subset X$, $\pi_\Omega : \Omega \times \Omega \to \Omega$ will denote the first projection mapping, *i.e.* $\pi_\Omega(x,y) = x$ $(x, y \in \Omega)$. We will mean in the sequel by the term 'operator' between two Banach spaces a mapping which is (nonlinear in general) continuous and bounded (*i.e.* takes bounded sets to bounded sets).

A cone K in X is a subset of X with $K + K \subset K$, $\alpha K \subset K$ for all $\alpha \geq 0$, and $K \cap (-K) = \{0\}$. As usual X will be ordered by the (partial) order relation

$$x \leq y \quad \Leftrightarrow \quad y - x \in K$$

and the cone $K = \{x \in X : x \geq 0\}$ will be denoted by X^+. $(X, \leq)$ is said to be an ordered Banach space, if the positive cone X^+ is closed. For two vectors $x, y \in X$, $[x)$, $(y]$, and the order interval $[x, y]$ are the sets defined by $[x) = \{z \in X : x \leq z\}$, $(y] = \{z \in X : z \leq y\}$, and $[x, y] = \{z \in X : x \leq z \leq y\}$. Note that if $x \nleq y$ then $[x, y] = \phi$. A cone X^+ of an ordered Banach space X is said to be normal whenever there is a constant $N > 0$ (called the normal constant of X^+ when N is the smallest constant) such that for every $x, y \in X$

$$0 \leq x \leq y \quad \Rightarrow \quad \|x\| \leq N\|y\|.$$

The following two lemmas will be useful in the proofs of our results.

Lemma 1.3 ([11], Lemmas 2.3 and 2.22) *Let X be an ordered Banach space. Then the following assertions hold:*

(1) *X is Hausdorff and the order intervals of X are closed;*
(2) *if the cone X^+ is normal, then every order interval is bounded.*

Lemma 1.4 *Let X be an ordered Banach space with a normal cone X^+. Then a monotone sequence $(u_n) \subset X$ is convergent if and only if it has a weakly convergent subsequence.*

Proof The 'only if' part is obvious. For the 'if' part, assume that (u_n) is nondecreasing and let $(u_{n_k}) \subset (u_n)$ be a subsequence such that $u_{n_k} \to u$ weakly for some $u \in X$. Let $m \in \mathbb{N}$ be fixed. For each $k \geq m$, we see that

$$u_m \leq u_k \leq u_{n_k}. \tag{1.1}$$

Since the cone X^+ is convex and closed, it is weakly closed. Thus, since $u_{n_k} \to u$ weakly, we see from (1.1) that $u_m \leq u$ for each $m \in \mathbb{N}$. Thus, it follows from [11], Lemma 2.28 that $\lim u_{n_k} = u$. Now, if $m \geq n_k$ then $0 \leq u - u_m \leq u - u_{n_k}$ and hence $\|u - u_m\| \leq N\|u - u_{n_k}\|$, where N denotes the normal constant of X^+. Letting $k \to \infty$, we see that $\lim u_m = u$ as required. The desired conclusion is proved similarly when (u_n) is nonincreasing. □

For a subset $D \subset X$, denote by $\mathcal{B}(D)$, $\mathcal{W}(D)$, $\mathcal{W}_r(D)$ the family of all bounded subsets of D, weakly compact subsets of D and relatively weakly compact subsets of D. A function $\phi : \mathcal{B}(X) \to \mathbb{R}^+$ is said to be a Sadovskij functional [12] if it satisfies the following requirements ($U, V \in \mathcal{B}(X)$, $\lambda \in \mathbb{R}$):

(1) $\phi(U \cup V) = \max(\phi(U), \phi(V))$ (the set additivity);
(2) $\phi(U + V) \leq \phi(U) + \phi(V)$ (the algebraic subadditivity);
(3) $\phi(\lambda U) = |\lambda|\phi(U)$ (the homogeneity);
(4) $\phi(U) \leq \phi(V)$ if $U \subset V$ (the monotonicity);
(5) $\phi([0,1] \cdot U) = \phi(U)$ (the absorption invariance);
(6) $\phi(\overline{\text{co}}U) = \phi(U)$ (the convex closure invariance).

ϕ is said to be a measure of noncompactness if it is a regular Sadovskij functional in the sense that

$$\phi(U) = 0 \quad \text{if and only if} \quad U \text{ is relatively compact.}$$

An important example of a measure of noncompactness (see [12] for other examples) is the so called Kuratowski measure of noncompactness defined for $U \in \mathcal{B}(X)$ by

$$\alpha(U) = \inf\{d > 0 : U \text{ is covered by a finite number of sets with diameter} \leq d\},$$

where the diameter of a bounded subset V of X is defined by

$$\operatorname{dia}(V) = \sup\bigl\{\|x - y\| : x, y \in V\bigr\}.$$

Another type of Sadovskij functional is the so called measure of weak noncompactness introduced by De Blasi in [13], and defined for $U \in \mathcal{B}(X)$ by

$$\omega(U) = \inf\bigl\{r > 0 : \text{there exists } W \in \mathcal{W}(X) \text{ such that } U \subseteq W + B_r\bigr\}.$$

Also, ω satisfies the following regular property (see for more properties [14]):

$$\omega(U) = 0 \quad \text{if and only if} \quad U \text{ is relatively weakly compact.}$$

In L_1-spaces, ω enjoys the following useful formula, which was established by Appell and De Pascale [14]:

$$\omega(U) = \lim_{\varepsilon \to 0} \sup_{f \in U} \left\{ \sup \left\{ \int_M \bigl|f(t)\bigr| \, dt : M \subset S, \lambda(M) \leq \varepsilon \right\} \right\} \tag{1.2}$$

for every $M \in \mathcal{B}(L_1(S))$, where S is a measure space and λ is the Lebesgue measure.

Definition 1.5 ([15], p.21) Let X and Y be two real Banach spaces, let ϕ and ψ be two measures of (weak) noncompactness in X and Y, respectively, and let $\Omega \subset X$ be a subset. An operator $T : \Omega \to Y$ is called (ϕ, ψ)-condensing if

$$\psi\bigl(T(U)\bigr) < \phi(U)$$

for all $U \in \mathcal{B}(\Omega)$ with $\phi(U) > 0$. The operator T is said to be a k-(ϕ, ψ)-contraction ($k \geq 0$) if

$$\psi\bigl(T(U)\bigr) \leq k\phi(U)$$

for all $U \in \mathcal{B}(\Omega)$. A k-(ϕ, ψ)-contraction is called a strict (ϕ, ψ)-contraction if $k < 1$.

When $X = Y$ and $\phi = \psi$, we shall simply say 'ϕ-condensing' and 'k-ϕ-contraction'. The following implications are now evident:

$$\begin{aligned} T \text{ is a strict } (\phi, \psi)\text{-contraction} \quad &\Rightarrow \quad T \text{ is } (\phi, \psi)\text{-condensing} \\ &\Rightarrow \quad T \text{ is } 1 - (\phi, \psi)\text{-contraction.} \end{aligned}$$

Furthermore, $T : \Omega \to Y$ is called a φ-nonlinear contraction, if there exists a continuous and nondecreasing real function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(r) < r$ for $r > 0$, such that

$$\|Tu - Tv\|_Y \leq \varphi\bigl(\|u - v\|_X\bigr)$$

for all $u, v \in \Omega$ (in particular, if $\varphi(r) = kr$; $0 \leq k < 1$, then T is called a contraction operator with contraction constant k). Also, $T : \Omega \to Y$ is called a nonlinear (ϕ, ψ)-set-contraction, if

$$\psi\bigl(T(U)\bigr) \leq \varphi\bigl(\phi(U)\bigr)$$

for all $U \in \mathcal{B}(\Omega)$.

Remark 1.6 Clearly every nonlinear (ϕ, ψ)-set-contraction $T : \Omega \to Y$ is (ϕ, ψ)-condensing. Also, it is easy to see that every φ-nonlinear contraction $T : \Omega \to Y$ is a nonlinear (α_X, α_Y)-set-contraction, where α_X, α_Y denote, respectively, the Kuratowski measures of noncompactness in X, Y.

When $Y = X$, an operator $T : \Omega \to \Omega$ is said to be nondecreasing (resp. nonincreasing) if for all $x, y \in \Omega$, $x \leq y$ implies $Tx \leq Ty$ (resp. $Tx \geq Ty$).

A point $x_* \in \Omega$ is called a fixed point (resp. lower fixed point, resp. upper fixed point) if $x_* = Tx_*$ (resp. $x_* \leq Tx_*$, resp. $x_* \geq Tx_*$). A lower (resp. upper) fixed points are also called a post-fixed points (resp. pre-fixed points); see [16], p.264. Also, for an operator $A : \Omega \times \Omega \to \Omega$, let us recall from [4] the following two-dimensional fixed point notions:

- A point $(x_*, y_*) \in \Omega \times \Omega$ is said to be a coupled fixed point of A if

  $$A(x_*, y_*) = x_* \quad \text{and} \quad A(y_*, x_*) = y_*.$$

  $x_* \in \Omega$ is called a fixed point of A if $A(x_*, x_*) = x_*$. Evidently, if (x_*, y_*) is a coupled fixed point of A, then (y_*, x_*) is also a coupled fixed point of A. Also, (x_*, x_*) is a coupled fixed point of A whenever x_* is a fixed point of A.
- A point $(x, y) \in \Omega \times \Omega$ is said to be a lower (resp. upper) coupled fixed point of A if

  $$x \leq A(x, y) \quad \bigl(\text{resp. } x \geq A(x, y)\bigr)$$

  and

  $$A(y, x) \leq y \quad \bigl(\text{resp. } A(y, x) \geq y\bigr).$$

  Clearly, if (x, y) is a lower (resp. upper) coupled fixed point of A then (y, x) is an upper (resp. lower) coupled fixed point of A.

2 Main results

In the sequel, we consider the product space $X \times X$ equipped with $\|(x, y)\|_\infty = \max\{\|x\|, \|y\|\}$ and the Kuratowski measure of noncompactness and the De Blasi measure of weak noncompactness $\alpha^\times$ and $\omega^\times$, respectively, and $\gamma = \alpha$ or ω. The facts in the following lemma are obtained in a simple way and therefore their proofs are omitted.

Lemma 2.1 *Let X be a Banach space. For each $D_1, D_2 \in \mathcal{B}(X)$ the following assertions hold*:

(1) $\operatorname{dia}(D_1 \times D_2) = \max\{\operatorname{dia}(D_1), \operatorname{dia}(D_2)\}$.

(2) $\gamma^{\times}(D_1 \times D_2) = \max\{\gamma(D_1), \gamma(D_2)\}$.

For a subset $\Omega \subset X$, it follows from Lemma 2.1 that if an operator $A : \Omega \times \Omega \to \Omega$ is $(\gamma^{\times}, \gamma)$-condensing then A satisfies the following γ-condensing condition:

$$(C_1) \qquad \gamma\big(A(U \times V)\big) < \max\big(\gamma(U), \gamma(V)\big)$$

for all $U, V \in \mathcal{B}(\Omega)$ with $\gamma(U) > 0$ or $\gamma(V) > 0$. Also, if A is a k-$(\gamma^{\times}, \gamma)$-contraction then A satisfies the following k-γ-contraction condition:

$$(C_2) \qquad \gamma\big(A(U \times V)\big) \leq k \max\big(\gamma(U), \gamma(V)\big)$$

for all $U, V \in \mathcal{B}(\Omega)$.

Example 2.2

(1) Let $0 < k < 1$. Define the (continuous and bounded) operator $A : B_X \times B_X \to B_X$ by

$$A(x, y) = \frac{k}{2}\big(\|x\| \cdot x + \|y\| \cdot y\big).$$

Then A satisfies the α-condensing condition (C_1). Indeed, let $G : B_X \times B_X \to B_X$ and $T : B_X \to B_X$ be such that

$$G(x, y) = k \cdot \|y\| \cdot x,$$
$$T(x) = G(x, x).$$

Since $G(\cdot, y)$ is contractive with contraction constant k for every $y \in B_X$, and $G(x, \cdot)$ is compact for every $x \in B_X$ (because the range of $G(x, \cdot)$ lies in a finite dimensional subspace of X, for every $x \in B_X$), T is α-condensing (see [12], p.198). Now, let $U, V \subset B_X$ be such that $\alpha(U) > 0$ or $\alpha(V) > 0$. Then, from $A(U \times V) = \frac{1}{2}(T(U) + T(V))$, we see that

$$\begin{aligned} \alpha\big(A(U \times V)\big) &\leq \frac{1}{2}\big(\alpha\big(T(U)\big) + \alpha\big(T(V)\big)\big) \\ &< \frac{1}{2}\big(\alpha(U) + \alpha(V)\big) \\ &\leq \max\big(\alpha(U), \alpha(V)\big), \end{aligned}$$

which proves that A satisfies the α-condensing condition (C_1).

(2) More generally, if $T, S : \Omega \to \Omega$ are two operators such that both T and S are γ-condensing, then the operator $A : \Omega \times \Omega \to \Omega$ defined by

$$A(x, y) = \frac{1}{2}\big(T(x) + S(y)\big)$$

satisfies the γ-condensing condition (C_1).

(3) In (1), if $B = k' \cdot A$ ($k' \geq 0$), then it follows that the operator B satisfies the k'-α-contraction condition (C_2).

Recall from [16], p.263 that an operator $T : \Omega \to \Omega$ on a partially ordered set is said to be inflationary (or progressing, see [17], p.34) if $Tx \geq x$ for every $x \in \Omega$. An example of such operator is the operator that associates to every element of a vector lattice its positive part. We introduce the following similar two-dimensional concepts.

Definition 2.3 Let $(\Omega, \leq)$ be a partially ordered set. For an operator $A : \Omega \times \Omega \to \Omega$, let $T_A : \Omega \times \Omega \to \Omega \times \Omega$ be the operator defined by

$$T_A(x,y) = \big(A(x,y), A(y,x)\big).$$

(1) An operator $A : \Omega \times \Omega \to \Omega$ is said to be inflationary, if A is inflationary with respect to its first argument, that is, $A(x,y) \geq x$ for every $x, y \in \Omega$;
(2) a pair of operators $A, B : \Omega \times \Omega \to \Omega$ is said to be weakly inflationary if A is inflationary on $T_B(\Omega \times \Omega)$ and B is inflationary on $T_A(\Omega \times \Omega)$, that is,

$$A\big(B(x,y), B(y,x)\big) \geq B(x,y)$$

and

$$B\big(A(x,y), A(y,x)\big) \geq A(x,y)$$

for all $x, y \in \Omega$. If the preceding inequalities are satisfied only on $T_B(D)$ and $T_A(D)$ respectively, where D is a subset of $\Omega \times \Omega$, then the pair A, B is said to be weakly inflationary on D;
(3) an operator $A : \Omega \times \Omega \to \Omega$ is said to be weakly inflationary, if the pair A, A is weakly inflationary, that is,

$$A\big(A(x,y), A(y,x)\big) \geq A(x,y)$$

for all $x, y \in \Omega$.

If in the preceding definition the order is reversed then we will use the term 'deflationary' instead of inflationary. On the other hand, if we take $A = T \circ \pi_\Omega$ and $B = S \circ \pi_\Omega$, where $T, S : \Omega \to \Omega$ are two operators, then we obtain the equivalent one-dimensional notion of [18], that is, T and S are weakly isotone increasing (resp. weakly isotone decreasing if the order is reversed) in the sense that

$$T(Sx) \geq Sx \quad \text{and} \quad S(Tx) \geq Tx$$

for all $x \in \Omega$; we opted for the terms (weakly) inflationary-(weakly) deflationary instead of (weakly) isotone increasing-(weakly) isotone decreasing, as the latter are used also to mean that the operators are increasing-decreasing (resp. strictly increasing-strictly decreasing); see for instance [19], p.4, [16], p.263.

Clearly, if A and B are both inflationary (resp. deflationary), then the pair A, B is weakly inflationary (resp. weakly deflationary). The converse does not hold in general.

Example 2.4 Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}, f(x,y)=e^{-(x+y)}+1$, $\Delta=\{(x,x):x\in[0,1]\}$, $\Delta'=\{(y,y):y\in f(\Delta)\}$, $g(x)=f(x,x)$, $x\in\mathbb{R}$, and $h(x)=g(x)-x$, $x\in\mathbb{R}$.

(1) Since $h'(x)=-2e^{-2x}-1<0$ for all $x\in\mathbb{R}$, it follows that

$$h(x)\geq h(1)=e^{-2}>0$$

for all $x\in[0,1]$. This shows that $f(x,x)\geq x$ for all $x\in[0,1]$, *i.e.*, f is inflationary on Δ. Since g is decreasing,

$$f\bigl(f(x,x),f(x,x)\bigr)\leq f(x,x) \tag{2.1}$$

for all $x\in[0,1]$. This shows that f is weakly deflationary on Δ. Therefore, the restriction of f on Δ is a weakly deflationary mapping which is not deflationary.

(2) On the other hand, it follows from (2.1) that f is deflationary on Δ', and since g is decreasing, then

$$f\bigl(f(y,y),f(y,y)\bigr)\geq f(y,y)$$

for all $y\in f(\Delta)$. This shows that the restriction of f on Δ' is a weakly inflationary mapping which is not inflationary.

(3) The function f does not satisfy the mixed monotone property since f is decreasing in x.

(4) From (1) and (2), it follows that the pair f, g (*i.e.* f, $g\circ\pi_{\mathbb{R}}$) is weakly deflationary on Δ (resp. weakly inflationary on Δ'), but f is not mixed g-monotone. Indeed, if $x,y_1,y_2\in\mathbb{R}$ such that $g(y_1)\leq g(y_2)$, since g is decreasing and one to one from $\mathbb{R}$ to $(1,+\infty)$, then $y_1\geq y_2$ and hence $f(x,y_1)\leq f(x,y_2)$ since f is decreasing in y. This shows that f is not g-nonincreasing in y.

Theorem 2.5 *Let X be an ordered Banach space with a normal cone X^+ and let Ω be a nonempty closed and bounded subset of X. Consider two operators $A,B:\Omega\times\Omega\to\Omega$ such that*:

(1) *A satisfies the α-condensing condition* (C_1);

(2) *B satisfies the $1-\alpha$-contraction condition* (C_2);

(3) *the pair A, B is weakly inflationary* (*resp. weakly deflationary*).

Then A and B have at least one common coupled fixed point (u_*,v_*) in $\Omega\times\Omega$; *moreover, we have*

$$u_*=\lim u_n \quad \textit{and} \quad v_*=\lim v_n,$$

where the common coupled fixed point iteration is given by

$$u_{2n+1}=A(u_{2n},v_{2n}),\qquad u_{2n+2}=B(u_{2n+1},v_{2n+1}) \tag{2.2}$$

and

$$v_{2n+1}=A(v_{2n},u_{2n}),\qquad v_{2n+2}=B(v_{2n+1},u_{2n+1}) \tag{2.3}$$

for each $n=0,1,2,\dots$, and (u_0,v_0) is any element in $\Omega\times\Omega$.

From the above theorem, we derive the following two corollaries, which extend the well-known results in the literature; see Remark 2.8 hereafter.

Corollary 2.6 *Let X be an ordered Banach space with a normal cone X^+, and let Ω be a nonempty closed and bounded subset of X. Let $T, S : \Omega \to \Omega$ be two nondecreasing operators such that*

(1) *T and S commute, that is, $TSx = STx$ for each $x \in \Omega$;*
(2) *T and S have at least one common lower (resp. upper) fixed point $u_0 \in \Omega$ (resp. $v_0 \in \Omega$);*
(3) *T is α-condensing;*
(4) *S is a $1 - \alpha$-contraction.*

Then T and S have a minimal common fixed point u_* in $[u_0) \cap \Omega$ (resp. maximal common fixed point u^* in $(v_0] \cap \Omega$); moreover, we have

$$u_* = \lim u_n \quad \textit{and} \quad u^* = \lim v_n$$

where the common fixed point iteration is given by

$$u_{2n+1} = Tu_{2n}, \qquad u_{2n+2} = Su_{2n+1}, \tag{2.4}$$

and

$$v_{2n+1} = Tv_{2n}, \qquad v_{2n+2} = Sv_{2n+1} \tag{2.5}$$

for each $n = 0, 1, 2, \ldots$.

Proof Consider the bounded set $\Gamma = \{u \in [u_0) \cap \Omega : u \leq Tu \text{ and } u \leq Su\}$. By assumption (2) Γ is nonempty ($u_0 \in \Gamma$). Since T and S are continuous and the cone X^+ is closed, Γ is closed. Now, consider the operators $A, B : \Gamma \times \Gamma \to \Gamma$ defined by $A = T \circ \pi_\Gamma$ and $B = S \circ \pi_\Gamma$. Since T and S are nondecreasing and commute, it is easy to verify that $A(\Gamma \times \Gamma) \subset \Gamma$, $B(\Gamma \times \Gamma) \subset \Gamma$, and the pair A, B is weakly inflationary. Since T and S are continuous and bounded, A and B are so. The condition (2) of Theorem 2.5 follows from the $1 - \alpha$-contraction property of S. For the condition (1) of Theorem 2.5, let $U, V \subset \Omega$ with $\alpha(U) > 0$ or $\alpha(V) > 0$. If $\alpha(U) > 0$, then

$$\alpha\bigl(A(U \times V)\bigr) = \alpha\bigl(T(U)\bigr) < \alpha(U) \leq \max\bigl\{\alpha(U), \alpha(V)\bigr\}.$$

If $\alpha(U) = 0$, then U is relatively compact and hence so is $T(U)$. Therefore, since $\alpha(V) > 0$, we have

$$\alpha\bigl(A(U \times V)\bigr) = \alpha\bigl(T(U)\bigr) = 0 < \alpha(V) \leq \max\bigl\{\alpha(U), \alpha(V)\bigr\}.$$

Now, applying Theorem 2.5, there exists a common coupled fixed point $(u_*, v_*) \in \Gamma \times \Gamma$ of A and B, that is, u_* is a common fixed point of T and S. From (2.2) we see that $u_* = \lim u_n$ and (2.4) holds true. Finally, to prove the minimality of u_*, let $u \in [u_0) \cap \Omega$ be such that $Tu = Su = u$. Since T is nondecreasing, it follows from $u_0 \leq u$ that $Tu_0 \leq Tu$, that is, $u_1 \leq u$.

Again, since S is nondecreasing, $Su_1 \leq Su$, that is, $u_2 \leq u$. Proceeding inductively, we get $u_n \leq u$ for each $n = 0, 1, 2, \dots$. Now, taking the limit $n \to \infty$, we obtain $u_* \leq u$ as desired.

For the existence of a maximal common fixed point u^* of T and S, consider the subset $\Gamma' = \{v \in (v_0] \cap \Omega : Tv \leq v \text{ and } Sv \leq v\}$, and by the same preceding arguments, such common fixed point exists with $u^* = \lim v_n$ and (2.5) holds true. □

Now, consider in the product space $(X \times X, \|\cdot\|_\infty)$ the following partial order:

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{if } x_1 \leq x_2 \text{ and } y_1 \geq y_2. \tag{2.6}$$

It is easy to see that if X^+ is a normal cone in X then $(X \times X)^+$ is also a normal cone in $X \times X$.

Corollary 2.7 *Let X be an ordered Banach space with a normal cone X^+, and let Ω be a nonempty closed subset of X. Let $A, B : \Omega \times \Omega \to \Omega$ be two mixed monotone operators such that*

(1) *A and B satisfy the following commutation property*:

$$A\big(B(x,y), B(y,x)\big) = B\big(A(x,y), A(y,x)\big)$$

for all $x, y \in \Omega$;

(2) *A and B have at least one common lower* (*resp. upper*) *coupled fixed point* (u_0, v_0) *with $u_0 \leq v_0$ (resp. $u_0 \geq v_0$)*;

(3) *A is $(\alpha^\times, \alpha)$-condensing*;

(4) *B is a $1 - (\alpha^\times, \alpha)$-contraction.*

Then A and B have a minimal common coupled fixed point $(u_*, v_*) \in \Omega_0 \times \Omega_0$ such that (v_*, u_*) is a maximal common coupled fixed point of A and B, that is,

$$(u_*, v_*) \leq (\widetilde{u}, \widetilde{v}) \leq (v_*, u_*)$$

for each common coupled fixed point $(\widetilde{u}, \widetilde{v}) \in \Omega_0 \times \Omega_0$ of A and B, where $\Omega_0 = [u_0, v_0] \cap \Omega$; moreover, we have

$$u_* = \lim u_n \quad \textit{and} \quad v_* = \lim v_n$$

where the common coupled fixed point iteration is given by (2.2) *and* (2.3).

Proof Consider the (continuous and bounded) operators $T_A, T_B : \Omega_0 \times \Omega_0 \to \Omega_0 \times \Omega_0$ (where Ω_0 is closed and bounded since X^+ is normal) defined by

$$T_A(u,v) = \big(A(u,v), A(v,u)\big) \quad \text{and} \quad T_B(u,v) = \big(B(u,v), B(v,u)\big).$$

Clearly $(u,v) \in X \times X$ is a common coupled fixed point of A and B if and only if (u,v) is a common fixed point of T_A and T_B. From assumption (2) and (2.6) we see that (u_0, v_0) is a common lower fixed point of T_A and T_B. Also, it can be shown easily from the mixed monotone property of A and B that T_A and T_B are nondecreasing. Furthermore, from assumption (1) we see that T_A and T_B commute. Let us show that T_A is $\alpha^\times$-condensing.

To this end, let $D \subset \Omega_0 \times \Omega_0$ with $\alpha^\times(D) > 0$, and set $D^* = \{(v,u) : (u,v) \in D\}$, and note by a simple computation that $\alpha^\times(D^*) = \alpha^\times(D)$. Then, since A is $(\alpha^\times, \alpha)$-condensing and taking into account Lemma 2.1, we see from $T_A(D) \subset A(D) \times A(D^*)$ that

$$\begin{aligned}\alpha^\times\bigl(T_A(D)\bigr) &\leq \alpha^\times\bigl(A(D) \times A\bigl(D^*\bigr)\bigr) \\ &= \max\bigl\{\alpha\bigl(A(D)\bigr), \alpha\bigl(A\bigl(D^*\bigr)\bigr)\bigr\} \\ &< \max\bigl\{\alpha^\times(D), \alpha^\times\bigl(D^*\bigr)\bigr\} \\ &= \alpha^\times(D).\end{aligned}$$

Therefore T_A is $\alpha^\times$-condensing. Similarly, it can be shown that T_B is a $1 - \alpha^\times$-contraction.

Now, applying Corollary 2.6, the operators T_A and T_B have a minimal common fixed point $(u_*, v_*) \in \Omega_0 \times \Omega_0$, that is, (u_*, v_*) is a minimal common coupled fixed point of A and B, and (2.2) and (2.3) follow from (2.4) and (2.5) applied for T_A and T_B.

Now, since (u_*, v_*) is a common coupled fixed point of A and B, then so is (v_*, u_*). Finally, to prove the maximality of (v_*, u_*), let $(u,v) \in \Omega_0 \times \Omega_0$ be any common coupled fixed point of A and B. Since A is mixed monotone, it follows from $u_0 \leq u \leq v_0$ and $u_0 \leq v \leq v_0$ that

$$u_1 = A(u_0, v_0) \leq A(u_0, v) \leq A(u, v) = u \leq A(v_0, v) \leq A(v_0, u_0) = v_1$$

and

$$u_1 = A(u_0, v_0) \leq A(u_0, u) \leq A(v, u) = v \leq A(v_0, u) \leq A(v_0, u_0) = v_1.$$

Again, since B is mixed monotone, it follows by similar arguments from $u_1 \leq u \leq v_1$ and $u_1 \leq v \leq v_1$ that

$$u_2 \leq B(u,v) = u \leq v_2 \quad \text{and} \quad u_2 \leq B(v,u) = v \leq v_2.$$

Proceeding inductively, we get

$$u_n \leq u \leq v_n \quad \text{and} \quad u_n \leq v \leq v_n$$

for each $n = 0, 1, 2, \ldots$. Now, taking the limit $n \to \infty$ in the preceding inequalities, we obtain $u_* \leq u \leq v_*$ and $u_* \leq v \leq v_*$, that is, $(u_*, v_*) \leq (u, v) \leq (v_*, u_*)$.

Note that in the case (u_0, v_0) is a common upper coupled fixed point of A and B, (v_0, u_0) is a common lower coupled fixed point of A and B, and the required conclusions follow from the preceding case. □

Remark 2.8

(1) If we take $S = T$ in Corollary 2.6 and $A = B$ in Corollary 2.7 then we obtain the well-known results [4], Theorem 2.1.1 and Theorem 2.1.7.

(2) In Corollary 2.7, the operator A is supposed to be only $(\alpha^\times, \alpha)$-condensing, which is an hypothesis weaker than the complete continuity supposed in [4], Theorem 2.1.7.

In order to establish the equivalent of Theorem 2.5 for the De Blasi measure of weak noncompactness, we need for an operator $A : \Omega \times \Omega \to \Omega$ the following two conditions:

(C_3) $A(U \times V)$ is relatively weakly compact for every $U, V \in \mathcal{W}_r(\Omega)$,

(C_4) $A(U \times V)$ is relatively weakly compact for every $U, V \in \mathcal{B}(\Omega)$.

Remark 2.9

(1) Weakly continuous operators $A : \Omega \times \Omega \to \Omega$ satisfy the condition (C_3). However, the converse is false in general. Indeed, if $\mathcal{N}_\varphi : L_1[0,1] \to L_1[0,1]$ is the Nemytskii operator generated by a Caratheodory function $\varphi : [0,1] \times \mathbb{R} \to \mathbb{R}$, then $\mathcal{N}_\varphi \circ \pi_{L_1[0,1]}$ satisfies the condition (C_3) (see Lemma 4.2 hereafter for $n = 1$), but $\mathcal{N}_\varphi$ is weakly continuous if and only if φ is linear (see [20], Theorem 2.6). Furthermore, it is clear that the condition (C_4) is in particular satisfied by weakly compact operators (*i.e.*, operators that map bounded sets to a relatively weakly compact ones);

(2) clearly, if A satisfies the condition (C_4), then A satisfies the condition (C_3) and the ω-condensing condition (C_1).

Theorem 2.10 *Let X be an ordered Banach space with a normal cone X^+ and let Ω be a nonempty closed and bounded subset of X. Consider two operators $A, B : \Omega \times \Omega \to \Omega$ such that*:

(1) *A satisfies the condition* (C_3);

(2) *A satisfies the ω-condensing condition* (C_1);

(3) *B satisfies the $1 - \omega$-contraction condition* (C_2);

(4) *the pair A, B is weakly inflationary* (*resp. weakly deflationary*).

Then A and B have at least one common coupled fixed point (u_*, v_*) in $\Omega \times \Omega$; moreover, we have

$$u_* = \lim u_n \quad \textit{and} \quad v_* = \lim v_n$$

where (u_n) and (v_n) are defined, respectively, as in (2.2) *and* (2.3).

Corollary 2.11 *Let X be an ordered Banach space with a normal cone X^+ and let Ω be a nonempty closed and bounded subset of X. Consider two operators $A, B : \Omega \times \Omega \to \Omega$ such that*:

(1) *A satisfies the condition* (C_4);

(2) *B satisfies the $1 - \omega$-contraction condition* (C_2);

(3) *the pair A, B is weakly inflationary* (*resp. weakly deflationary*).

Then A and B have at least one common coupled fixed point (u_*, v_*) in $\Omega \times \Omega$; moreover, we have

$$u_* = \lim u_n \quad \textit{and} \quad v_* = \lim v_n$$

where (u_n) and (v_n) are defined as in (2.2) *and* (2.3).

Note that, as shown in Remark 1.6, if $A : \Omega \times \Omega \to \Omega$ is a φ-nonlinear contraction, that is,

$$\|Au - Av\| \leq \phi\bigl(\|u - v\|_\infty\bigr)$$

for all $u, v \in \Omega \times \Omega$, then A is a nonlinear $(\alpha^{\times}, \alpha)$-set-contraction and hence satisfies the α-condensing condition (C_1). For the De Blasi measure of weak noncompactness, we have the following lemma.

Lemma 2.12 *For a subset $\Omega \subset X$, every φ-nonlinear contraction $A : \Omega \times \Omega \to \Omega$ satisfying the condition (C_3) satisfies the following nonlinear ω-set-contraction condition*:

$$\omega\bigl(A(U \times V)\bigr) \le \varphi\bigl(\max\bigl(\omega(U), \omega(V)\bigr)\bigr)$$

for all $U, V \in \mathcal{B}(\Omega)$. In particular, A satisfies the ω-condensing condition (C_1).

Proof Let $r > \max(\omega(U), \omega(V))$. Then there exist $W, W' \in \mathcal{W}(\Omega)$ such that $U \subset W + B_r$ and $V \subset W' + B_r$. Since A is a φ-nonlinear contraction, it follows easily that

$$A(U \times V) \subset A\bigl(W \times W'\bigr) + B_{\varphi(r)} \subset \overline{A\bigl(W \times W'\bigr)}^{\sigma(X,X')} + B_{\varphi(r)}.$$

Therefore, from the condition (C_3), we see that $\omega(A(U \times V)) \le \varphi(r)$. Now, since φ is continuous, letting $r \to \max(\omega(U), \omega(V))$, we get the required conclusion. □

Corollary 2.13 *Let assumptions of Theorem* 2.5 (*resp. Theorem* 2.10) *be satisfied. Then the following assertions hold true*:

(1) *The operators A and B have at least one common fixed point $u_* \in \Omega$, and the common fixed point iteration is given by*

$$u_{2n+1} = A(u_{2n}, u_{2n}) \quad \textit{and} \quad u_{2n+2} = B(u_{2n+1}, u_{2n+1}) \quad (n = 0, 1, 2, \ldots).$$

(2) *If for the operator A, we replace the condensing condition with "A is a φ-nonlinear contraction", then A and B have a unique common fixed point u_* and (u_*, u_*) is their unique common coupled fixed point.*

(3) *If A satisfies in addition the following weak φ-nonlinear contraction condition*:

$$\bigl\|A(x, y) - A(y, x)\bigr\| \le \varphi\bigl(\|x - y\|\bigr)$$

for all $x, y \in \Omega$, then all coupled fixed points of A (and hence all common coupled fixed points of A and B) are in the form (u_*, u_*), u_* is a fixed point of A (common fixed point of A and B).

3 Proofs of the main theorems

Proof of Theorem 2.5 Assume that the pair A, B is weakly inflationary. Set $U_1 = \{u_{2n} : n = 0, 1, 2, \ldots\}$, $U_2 = \{u_{2n+1} : n = 0, 1, 2, \ldots\}$ and $U = U_1 \cup U_2$, where the sequence $(u_n) \subset \Omega$ is defined as in (2.2). Similarly, the sets V_1, V_2, and V are defined by the sequence $(v_n) \subset \Omega$ instead of (u_n). Since A, B are weakly inflationary, it follows that

$$\begin{aligned} u_{2n+1} &= A(u_{2n}, v_{2n}) \le B\bigl(A(u_{2n}, v_{2n}), A(v_{2n}, u_{2n})\bigr) \\ &= B(u_{2n+1}, v_{2n+1}) = u_{2n+2} \end{aligned}$$

and

$$u_{2n+2} = B(u_{2n+1}, v_{2n+1}) \le A\big(B(u_{2n+1}, v_{2n+1}), B(v_{2n+1}, u_{2n+1})\big)$$
$$= A(u_{2n+2}, v_{2n+2}) = u_{2n+3}$$

hold for each $n = 0, 1, 2, \ldots$, which proves that the sequence (u_n) $(n \ge 1)$ is nondecreasing. Similarly, (v_n) $(n \ge 1)$ is also nondecreasing.

Now, set $\Delta = \{(u_{2n}, v_{2n}) : n = 0, 1, 2, \ldots\}$, $\Delta^* = \{(v_{2n}, u_{2n}) : n = 0, 1, 2, \ldots\}$, $\Sigma = \{(u_{2n+1}, v_{2n+1}) : n = 0, 1, 2, \ldots\}$, and $\Sigma^* = \{(v_{2n+1}, u_{2n+1}) : n = 0, 1, 2, \ldots\}$. We shall see that the sets U_1 and V_1 are relatively compact. Assume by way of contradiction that $\alpha(U_1) > 0$ or $\alpha(V_1) > 0$. It follows from $U_2 = A(\Delta) \subset A(U_1 \times V_1)$, $U_1 = B(\Sigma) \cup \{u_0\} \subset B(U_2 \times V_2) \cup \{u_0\}$, $V_1 = B(\Sigma^*) \cup \{v_0\} \subset B(V_2 \times U_2) \cup \{v_0\}$, and assumptions (1) and (2) that

$$\begin{aligned}\alpha(U_2) &\le \alpha\big(A(U_1 \times V_1)\big)\\ &< \max\big\{\alpha(U_1), \alpha(V_1)\big\}\\ &\le \max\big\{\alpha\big(B(U_2 \times V_2)\big), \alpha\big(B(V_2 \times U_2)\big)\big\}\\ &\le \max\big\{\alpha(U_2), \alpha(V_2)\big\}.\end{aligned}$$

Similarly, we have $\alpha(V_2) < \max\{\alpha(V_2), \alpha(U_2)\}$ which is a contradiction. Therefore, U_1 and V_1 are a relatively compact sets. Since A is continuous, $A(U_1 \times V_1)$ and $A(V_1 \times U_1)$ are relatively compact, and hence so are $U_2 \subset A(U_1 \times V_1)$ and $V_2 = A(\Delta^*) \subset A(V_1 \times U_1)$. Thus, we conclude that U and V are a relatively compact sets. It follows from Lemma 1.4 that $\lim u_n = u_*$ and $\lim v_n = v_*$ for some $(u_*, v_*) \in \Omega \times \Omega$. Now, since A and B are continuous then by (2.2) and (2.3) we get $A(u_*, v_*) = B(u_*, v_*) = u_*$ and $A(v_*, u_*) = B(v_*, u_*) = v_*$, that is, (u_*, v_*) is a common coupled fixed point of A and B.

If A, B is weakly deflationary, then in this case the sequences of common coupled fixed point iteration are nonincreasing, and the desired conclusions are obtained by similar arguments. □

Proof of Theorem 2.10 Let U and V be the sets defined as in the proof of Theorem 2.5. Using the condition (C_3), it can be shown by a similar arguments of the proof of Theorem 2.5 that U and V are relatively weakly compact. Since the sequences (u_n) and (v_n) are nondecreasing, it follows from Lemma 1.4 that $\lim u_n = u_*$ and $\lim v_n = v_*$ for some $(u_*, v_*) \in \Omega \times \Omega$. Now, since A and B are continuous, by (2.2) and (2.3) we get $A(u_*, v_*) = B(u_*, v_*) = u_*$ and $A(v_*, u_*) = B(v_*, u_*) = v_*$, that is, (u_*, v_*) is a common coupled fixed point of A and B. □

4 Application to nonlinear integral equations

We give an application of Corollary 2.11 to the following nonlinear integral equation:

$$g\big(t, u(t)\big) = \int_0^1 K(t,x) f\big(x, \big(u(x), v(x)\big)\big)\,dx, \quad t \in I = [0,1], u, v \in X = L_1(I), \tag{4.1}$$

where the kernel mapping $K : I \times I \to \mathbb{R}$ is measurable on $I \times I$, and $g : I \times \mathbb{R} \to \mathbb{R}$, $f : I \times \mathbb{R}^2 \to \mathbb{R}$ are two given mappings.

A pair $u, v \in X$ is said to be a coupled solution of (4.1) if both (u, v) and (v, u) are solutions of (4.1), *i.e.*,

$$g\big(t, u(t)\big) = \int_0^1 K(t,x) f\big(x, \big(u(x), v(x)\big)\big)\,dx, \quad t \in I,$$

and

$$g\big(t, v(t)\big) = \int_0^1 K(t,x) f\big(x, \big(v(x), u(x)\big)\big)\,dx, \quad t \in I.$$

An element $u \in X$ is called a solution of (4.1) if (u, u) is a solution of (4.1).

Now, recall that a function $\varphi : I \times \mathbb{R}^n \to \mathbb{R}$ is said to be a Caratheodory function if it satisfies the Caratheodory condition in the following sense:

- the function $t \to \varphi(t, u)$ is measurable on I for any $u \in \mathbb{R}^n$;
- the function $u \to \varphi(t, u)$ is continuous on $\mathbb{R}^n$ for almost all $t \in I$.

A function $\varphi : I \times \mathbb{R}^n \to \mathbb{R}$ is said to have the separated domination property if there exist a constant $k > 0$ and a positive function $u_0 \in X$ such that

$$\big|\varphi(t, x)\big| \leq u_0(t) + k|x| \tag{4.2}$$

for almost all $t \in I$ and for all $x \in \mathbb{R}^n$, where $|x| := \sum_{i=1}^n |x_i|$, $x = (x_1, x_2, \ldots, x_n)$. If the function φ is a Caratheodory function and satisfies the separated domination property, then clearly φ defines a mapping $\mathcal{N}_\varphi : X^n \to X$ by $\mathcal{N}_\varphi(u_1, \ldots, u_n)(t) = \varphi(t, (u_1(t), \ldots, u_n(t)))$, where X^n is the Cartesian product of n copies of X. In the case $n = 1$, this mapping is called the Nemytskii operator generated by φ. The following lemmas will be useful in the proof of our results.

Lemma 4.1 ([21], p.154) *If a Caratheodory function $\varphi : I \times \mathbb{R}^n \to \mathbb{R}$ satisfies the separated domination property, then the operator $\mathcal{N}_\varphi : X^n \to X$ is continuous and bounded.*

Lemma 4.2 *Let $\varphi : I \times \mathbb{R}^n \to \mathbb{R}$ be a Caratheodory function satisfying the separated domination property* (4.2). *Then*

$$\omega\big(\mathcal{N}_\varphi(W_1 \times \cdots \times W_n)\big) \leq k\big(\omega(W_1) + \cdots + \omega(W_n)\big)$$

for every $W_1, \ldots, W_n \in \mathcal{B}(X)$. In particular, $\mathcal{N}_\varphi(W_1 \times \cdots \times W_n)$ is relatively weakly compact for every relatively weakly compact sets $W_1, \ldots, W_n \subset X$.

Proof Let $\varepsilon > 0$ and $(u_1, \ldots, u_n) \in W_1 \times \cdots \times W_n$. Let $M \subset I$ be a measurable set with $\lambda(M) \leq \varepsilon$. From (4.2) it follows that

$$\int_M \big|\mathcal{N}_\varphi(u_1, \ldots, u_n)(t)\big|\,dt \leq \int_M u_0(t)\,dt + k\bigg(\int_M \big|u_1(t)\big|\,dt + \cdots + \int_M \big|u_n(t)\big|\,dt\bigg).$$

So, since $\omega(\{u_0\}) = 0$, under the supremum on all subsets M and all $(u_1, \ldots, u_n) \in W_1 \times \cdots \times W_n$ respectively, and letting $\varepsilon \to 0$, the desired conclusion follows from equation (1.2). □

Now, we consider the following assumptions:

(A_1) f is a Caratheodory function satisfying the following positivity property:

$$f\big(t,(x,y)\big) > 0$$

for almost all $t \in I$ and for all $x, y \in \mathbb{R}^+$.

(A_2) f satisfies the separated domination property, that is, there exist a constant $k > 0$ and a positive function $w_0 \in X$ such that

$$\big|f(t,u)\big| \leq w_0(t) + k|u| \tag{4.3}$$

for almost all $t \in I$ and for all $u \in \mathbb{R}^2$.

(A_3) The kernel mapping K is positive, *i.e.* $K(t,x) > 0$ for all $t, x \in I$, and satisfies the following upper estimate:

$$K(t,x) \leq \psi(t)\varphi(x)$$

for almost all $t, x \in I$, and for some positive functions $\psi, \varphi \in L_\infty(I)$ such that

$$\delta = \|\psi\|_\infty \cdot \|\varphi\|_\infty \leq \frac{\rho}{1 + 2\rho k}, \tag{4.4}$$

where ρ is some (arbitrary) positive number.

(A_4) g is a Caratheodory function such that there exists a nondecreasing mapping $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(r) \leq r$, $r > 0$, such that, for almost all $t \in I$ and for all $x \geq 0$, we have

$$\frac{\mu_0}{\|\psi\|_\infty}\psi(t) \leq g(t,x) \leq \phi(x), \tag{4.5}$$

where $\mu_0 = \rho\|w_0\|$.

(A_5) g is nonexpansive with respect to the second variable, that is, for almost all $t \in I$ and every $x, y \in \mathbb{R}$, we have

$$\big|g(t,x) - g(t,y)\big| \leq |x - y|.$$

Theorem 4.3 *Under the assumptions* (A_1)-(A_5), *the integral equation* (4.1) *has in the interval* $[\theta, \boldsymbol{\mu}_0] \subset X$ *at least one coupled solution* u_*, v_*, *where* $\theta, \boldsymbol{\mu}_0 \in X$ *are respectively the almost everywhere null function and the constant function equal to* μ_0. *Moreover, the sequences* (u_n) *and* (v_n) *converge in* X *monotonically to* u_* *and* v_*, *respectively, where*

$$u_{2n+1}(t) = \int_0^1 K(t,x)f\big(x, \big(u_{2n}(x), v_{2n}(x)\big)\big)\,dx, \qquad u_{2n+2}(t) = g\big(t, u_{2n+1}(t)\big), \tag{4.6}$$

and

$$v_{2n+1}(t) = \int_0^1 K(t,x)f\big(x, \big(v_{2n}(x), u_{2n}(x)\big)\big)\,dx, \qquad v_{2n+2}(t) = g\big(t, v_{2n+1}(t)\big), \tag{4.7}$$

for each $n = 0, 1, 2, \ldots$ *and each* $t \in I$, *and* u_0, v_0 *are any two elements in* X *such that*

$$0 \leq u_0(t), v_0(t) \leq c$$

for each $t \in I$, *where* $c = \frac{\delta\|w_0\|}{1-2k\delta}$.

Proof We know that X is an ordered Banach space under the standard λ-almost everywhere pointwise order and under the L_1-norm (denoted here $\|\cdot\|$), and that $X^+ := \{u \in X : u \geq 0 \text{ a.e.}\}$ is a normal cone in X (since $0 \leq u \leq v$ implies $\|u\| \leq \|v\|$). Set $\Omega = [\theta, \mathbf{c}]$, where $\mathbf{c}$ is the constant function in X equal to c, and note from (4.4) that $0 < c \leq \mu_0$. Define the operators $A, B : \Omega \times \Omega \to \Omega$ by

$$A(u,v)(t) = L \circ \mathcal{N}_f(u,v)(t) \quad \text{and} \quad B(u,v)(t) = \mathcal{N}_g \circ \pi_X(u,v)(t), \quad t \in I,$$

where $L : X \to X$ is the linear integral operator defined by

$$L(u)(t) = \int_0^1 K(t,x)u(x)\,dx, \quad t \in I.$$

Let $u, v \in \Omega$. Then from the separated domination property of f, we have

$$\begin{aligned} 0 &\leq A(u,v)(t) \leq \delta\bigl(\|w_0\| + k\bigl(\|u\| + \|v\|\bigr)\bigr) \\ &\leq \delta\bigl(\|w_0\| + 2kc\bigr) = c \end{aligned}$$

for almost all $t \in I$. Therefore, A is well defined and bounded (since Ω is bounded). From assumption (A_3) it follows that L is a bounded linear operator, and from assumption (A_2) and Lemma 4.1 $\mathcal{N}_f$ is continuous. Therefore, the operator A is continuous. Now, if $u \in \Omega$, it follows from assumption (A_4) that

$$0 \leq g\bigl(t, u(t)\bigr) \leq \phi\bigl(u(t)\bigr) \leq \phi(c) < c$$

for almost all $t \in I$, and hence the operator B is also well defined and bounded. Since g is nonexpansive with respect to the second variable,

$$\bigl|g(t,x)\bigr| \leq \bigl|g(t,0)\bigr| + \bigl|g(t,x) - g(t,0)\bigr| \leq \bigl|g(t,0)\bigr| + |x| = \bigl|\mathcal{N}_g(\theta)(t)\bigr| + |x| \tag{4.8}$$

for almost all $t \in I$ and all $x \in \mathbb{R}$. Therefore, by Lemma 4.1 again, the operator B is continuous.

Now, clearly a pair u, v is a coupled solution of (4.1) if and only if (u,v) is a common coupled fixed point of A and B. In the following we prove

(a) B satisfies the $1 - \omega$-contraction condition (C_2);
(b) A satisfies the condition (C_4);
(c) the pair A, B is weakly deflationary.

By (4.8) and Lemma 4.2 (applied for $\varphi = g$ and $n = 1$) we see that

$$\omega\bigl(B(U \times V)\bigr) = \omega\bigl(\mathcal{N}_g(U)\bigr) \leq \omega(U) \leq \max\bigl(\omega(U), \omega(V)\bigr)$$

for all $U, V \subset \Omega$. This shows that the operator B satisfies the $1 - \omega$-contraction condition (C_2), and (a) is proved.

Now, to prove (b), let $U, V \subset \Omega$. Let $\varepsilon > 0$ and $(u,v) \in U \times V$, and let $M \subset I$ be a measurable set with $\lambda(M) \leq \varepsilon$. Since X^+ is normal, $\|u\|$, $\|v\| \leq c$ and hence, for almost all $t \in I$, we have

$$\begin{aligned}\bigl|A(u,v)(t)\bigr| &\leq \int_0^1 K(t,x)\bigl|\mathcal{N}_f(u,v)(x)\bigr|\,dx\\ &\leq \psi(t)\|\varphi\|_\infty\bigl(\|w_0\| + k\bigl(\|u\| + \|v\|\bigr)\bigr)\\ &\leq \psi(t)\|\varphi\|_\infty\bigl(\|w_0\| + 2kc\bigr).\end{aligned}$$

Hence, integrating again we get

$$\int_M \bigl|A(u,v)(t)\bigr|\,dt \leq \|\varphi\|_\infty\bigl(\|w_0\| + 2kc\bigr)\int_M \psi(t)\,dt.$$

So, under the supremum on all subsets M and all $(u,v)\in U\times V$, respectively, and letting $\varepsilon\to 0$, we get from equation (1.2)

$$\omega\bigl(A(U\times V)\bigr) \leq \|\varphi\|_\infty\bigl(\|w_0\| + 2kc\bigr)\omega\bigl(\{\psi\}\bigr) = 0,$$

since the set $\{\psi\}$ is weakly compact in X. Therefore, $A(U\times V)$ is a relatively weakly compact set in X, and (b) is proved.

For the proof of (c), let $u,v\in\Omega$. From assumption (A_1) and the positivity of the Kernel mapping K we see that $A(u,v)(t) > 0$ for all $t\in I$, and hence by assumption (A_4), for almost all $t\in I$ we have

$$\begin{aligned}B\bigl(A(u,v),A(v,u)\bigr)(t) &= \mathcal{N}_g\bigl(A(u,v)\bigr)(t)\\ &= g\bigl(t,A(u,v)(t)\bigr)\\ &\leq \phi\bigl(A(u,v)(t)\bigr) < A(u,v)(t).\end{aligned}$$

Therefore, $B(A(u,v),A(v,u)) \leq A(u,v)$. Now, from assumptions (A_2), (A_3), for almost all $t\in I$ we have

$$\begin{aligned}A\bigl(B(u,v),B(v,u)\bigr)(t) &= \int_0^1 K(t,x)\mathcal{N}_f\bigl(\mathcal{N}_g(u),B(v,u)\bigr)(x)\,dx\\ &\leq \psi(t)\|\varphi\|_\infty\bigl(\|w_0\| + k\bigl\|\mathcal{N}_g(u)\bigr\| + k\bigl\|B(v,u)\bigr\|\bigr)\\ &\leq \psi(t)\|\varphi\|_\infty\bigl(\|w_0\| + kc + k\bigl\|\mathcal{N}_g(u)\bigr\|\bigr)\\ &= \frac{\psi(t)}{\|\psi\|_\infty}\delta\left(\frac{c}{\delta} - kc + k\bigl\|\mathcal{N}_g(u)\bigr\|\right)\\ &= \frac{\psi(t)}{\|\psi\|_\infty}\bigl((1-k\delta)c + k\delta\bigl\|\mathcal{N}_g(u)\bigr\|\bigr).\end{aligned}$$

Since $\mathcal{N}_g(u)\in\Omega$ and X^+ is normal, it follows that $\|\mathcal{N}_g(u)\| \leq c$. Therefore, by assumption (A_4) the preceding inequality becomes

$$\begin{aligned}A\bigl(B(u,v),B(v,u)\bigr)(t) &\leq \frac{\psi(t)}{\|\psi\|_\infty}c\\ &\leq \frac{\psi(t)}{\|\psi\|_\infty}\mu_0\\ &\leq g\bigl(t,u(t)\bigr) = B(u,v)(t)\end{aligned}$$

for almost all $t\in I$. Thus, $A(B(u,v),B(v,u)) \leq B(u,v)$, and (c) is proved.

Finally, applying Corollary 2.11, we obtain all conclusions of Theorem 4.3. □

Corollary 4.4 *Under the assumptions* (A_1), (A_3)-(A_5) *and* f *is nonexpansive with respect to the second variable, in the sense that*

$$\left|f(t,x)-f(t,y)\right| \leq |x-y|$$

for almost all $t \in I$ *and for all* $x, y \in \mathbb{R}^2$, *the integral equation* (4.1) *has a unique solution* u *in* $[\theta, \boldsymbol{\mu}_0]$ *and* (u,u) *is its unique coupled solution.*

Proof Since f is nonexpansive with respect to the second variable,

$$\left|f(t,x)\right| \leq f\bigl(t,(0,0)\bigr) + |x| \tag{4.9}$$

for almost all $t \in I$ and for all $x \in \mathbb{R}^2$. Therefore f satisfies assumption (A_2) with $w_0 = \mathcal{N}_f(\theta,\theta)$ and $k = 1$. Now, by assumption (A_2) again and the nonexpansive property of f, it follows by an easy computation that

$$\begin{aligned}\left\|A(u,v)-A\left(u',v'\right)\right\| &\leq \delta\left(\left\|u-u'\right\| + \left\|v-v'\right\|\right)\\ &\leq 2\delta \max\left(\left\|u-u'\right\|, \left\|v-v'\right\|\right)\\ &= 2\delta\left\|(u,v)-\left(u',v'\right)\right\|_\infty\end{aligned}$$

for every $u, v, u', v' \in \Omega$. From (4.4) we have $2\delta \leq \frac{2\rho}{1+2\rho} < 1$. Now, the desired conclusions follow from Theorem 4.3 and Corollary 2.13. □

Remark 4.5

(1) If we assume g to be nondecreasing with respect to the second variable, *i.e.*,

$$x \leq y \quad \Rightarrow \quad g(t,x) \leq g(t,y)$$

for almost all $t \in I$ and for all $x, y \in \mathbb{R}$, then assumption (A_4) may be reduced to

$$\frac{\mu_0}{\|\psi\|_\infty}\psi(t) \leq g(t,0) \tag{4.10}$$

and

$$0 \leq g(t,x) \leq x$$

for almost all $t \in I$ and for all $x > 0$.

(2) If the function $\mathcal{N}_g(\theta) \in L_\infty(I)$, then it is easy to see that the upper estimate of K in assumption (A_3) and assumption (4.10) are, respectively, guaranteed by the following inequalities:

$$K(t,x) \leq g(t,0) \leq \frac{\rho}{1+2\rho k}$$

and

$$w_0(t) \leq \frac{1}{\rho} g(t,0)$$

for almost all $t, x \in I$ (here $\psi = \mathcal{N}_g(\theta)$ and φ is the constant function equal to one).

(3) If f is nonexpansive with respect to the second variable, then from (4.9) we have $w_0 = \mathcal{N}_f(\theta, \theta)$ and $k = 1$, and hence the preceding inequalities become

$$K(t,x) \leq g(t,0) \leq \frac{\rho}{1+2\rho}$$

and

$$f\bigl(t,(0,0)\bigr) \leq \frac{1}{\rho} g(t,0)$$

for almost all $t, x \in I$.

Competing interests
The author declares that he has no competing interests.

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