Karahan *Fixed Point Theory and Applications* (2016) 2016:101
DOI 10.1186/s13663-016-0592-6

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Strong and weak convergence theorems for split equality generalized mixed equilibrium problem

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Abstract

In this paper, we consider split equality generalized mixed equilibrium problem, which is more general than many problems such as split feasibility problem, split equality problem, split equilibrium problem, and so on. We propose a new modified algorithm to obtain strong and weak convergence theorems for split equality generalized mixed equilibrium problem for nonexpansive mappings in Hilbert spaces. Also, we give some applications to other problems. Our results extend some results in the literature.

MSC: 47H09; 47J25

Keywords: split equality generalized mixed equilibrium problem; nonexpansive mappings; fixed point; demicompactness

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H, and $F: C\times C\to\mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of all real numbers. The scalar-valued equilibrium problem is finding a point $x\in C$ such that

$$F(x,y)\geq 0 \tag{1.1}$$

for all $y\in C$. The equilibrium problem has been extensively studied, beginning with Blum and Oettli [1]. In 2014, Ahmad and Rahaman [2] introduced the generalized vector equilibrium problem to find a point $x\in C$ such that

$$F\bigl(\lambda x+(1-\lambda)z,y\bigr)\nsubseteq -C\backslash\{0\}$$

for all $y,z\in C$ and $\lambda\in(0,1]$, where $F: C\times C\to 2^H$ is a set-valued mapping such that $F(\lambda x+(1-\lambda)z,x)\supseteq\{0\}$. In the scalar case, the generalized equilibrium problem takes the form of finding $x\in C$ such that

$$F\bigl(\lambda x+(1-\lambda)z,y\bigr)\geq 0 \tag{1.2}$$

for all $y,z\in C$ and $\lambda\in(0,1]$ under the condition $F(\lambda x+(1-\lambda)z,x)=0$. If $\lambda=1$, then problem (1.2) is reduced to problem (1.1).

Let $T : C \to C$ be a continuous mapping, and $\phi : C \to \mathbb{R}$ be a mapping. Very recently, Rahaman *et al.* [3] considered the generalized mixed equilibrium problem of finding a point $x \in C$ such that

$$F\bigl(\lambda x + (1-\lambda)z, y\bigr) + \langle Tx, y - x\rangle + \phi(y) - \phi(x) \geq 0 \tag{1.3}$$

for all $y, z \in C$ and $\lambda \in (0,1]$. The set of solutions of problem (1.3) is denoted by $\mathrm{GMEP}(F, T, \phi)$.

Let H_1, H_2, and H_3 be real Hilbert spaces, and C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions, $T : C \to C$ and $S : Q \to Q$ be nonlinear mappings, $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex mappings such that $C \cap \operatorname{dom}\phi \neq \emptyset$ and $Q \cap \operatorname{dom}\varphi \neq \emptyset$, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear mappings. In 2015, Rahaman *et al.* [3] introduced the following split equality generalized mixed equilibrium problem (SEGMEP): find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} &F\bigl(\lambda_1 x^* + (1-\lambda_1)b, x\bigr) + \bigl\langle Tx^*, x - x^*\bigr\rangle + \phi(x) - \phi\bigl(x^*\bigr) \geq 0, \\ &G\bigl(\lambda_2 y^* + (1-\lambda_2)c, y\bigr) + \bigl\langle Sy^*, y - y^*\bigr\rangle + \varphi(y) - \varphi\bigl(y^*\bigr) \geq 0, \quad \text{and} \\ &Ax^* = By^* \end{aligned} \tag{1.4}$$

for all $x, b \in C$, $y, c \in Q$, and $\lambda_1, \lambda_2 \in (0,1]$. The solution set of problem (1.4) is denoted by $\mathrm{SEGMEP}(F, G, T, S, \phi, \varphi)$. This problem is a generalization of all the following problems.

1. If $T = S = 0$ and $\lambda_1 = \lambda_2 = 1$, then the split equality generalized mixed equilibrium problem (SEGMEP) is reduced to the split equality mixed equilibrium problem (SEMEP) introduced by Ma *et al.* [4]: find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} &F\bigl(x^*, x\bigr) + \phi(x) - \phi\bigl(x^*\bigr) \geq 0, \quad \forall x \in C, \\ &G\bigl(y^*, y\bigr) + \varphi(y) - \varphi\bigl(y^*\bigr) \geq 0, \quad \forall y \in Q, \quad \text{and} \\ &Ax^* = By^*. \end{aligned} \tag{1.5}$$

2. If $T = S = \phi = \varphi = 0$, $B = I$, $H_2 = H_3$, and $\lambda_1 = \lambda_2 = 1$, then problem (1.4) is reduced to the split equilibrium problem ($\mathrm{SE_qP}$) introduced by He [5]: find $x^* \in C$ such that

$$\begin{aligned} &F\bigl(x^*, x\bigr) \geq 0, \quad \forall x \in C, \quad \text{and} \\ &Ax^* = y^* \in Q \quad \text{solves} \quad G\bigl(y^*, y\bigr) \geq 0, \quad \forall y \in Q. \end{aligned} \tag{1.6}$$

3. If $T = S = \phi = \varphi = 0$ and $\lambda_1 = \lambda_2 = 1$, then problem (1.4) is reduced to the split equality equilibrium problem (SEEP) of finding $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} &F\bigl(x^*, x\bigr) \geq 0, \quad \forall x \in C, \\ &G\bigl(y^*, y\bigr) \geq 0, \quad \forall y \in Q, \quad \text{and} \\ &Ax^* = By^*. \end{aligned} \tag{1.7}$$

4. If $F = G = T = S = 0$, then problem (1.4) is reduced to the split equality convex minimization problem (SECMP) of finding $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} &\phi(x) \geq \phi(x^*), \quad \forall x \in C, \qquad \varphi(y) \geq \varphi(y^*), \quad \forall y \in Q, \quad \text{and} \\ &Ax^* = By^*. \end{aligned} \tag{1.8}$$

5. If $F = G = T = S = 0$, $B = I$, and $H_2 = H_3$, then problem (1.4) is reduced to the split convex minimization problem (SCMP) of finding $x^* \in C$ such that

$$\begin{aligned} &\phi(x) \geq \phi(x^*), \quad \forall x \in C, \qquad \varphi(y) \geq \varphi(y^*), \quad \forall y \in Q, \quad \text{and} \\ &Ax^* = y^* \in Q. \end{aligned} \tag{1.9}$$

6. If $F = G = \phi = \varphi = T = S = 0$, then problem (1.4) is reduced to the split equality problem (SEP) of finding $x^* \in C$ and $y^* \in Q$ such that

$$Ax^* = By^*. \tag{1.10}$$

7. If $F = G = \phi = \varphi = T = S = 0$, $B = I$, and $H_2 = H_3$, then problem (1.4) is reduced to the split feasibility problem (SFP) of finding $x^* \in C$ such that

$$Ax^* \in Q. \tag{1.11}$$

This problem was introduced by Censor and Elfving [6].

Since these kinds of problems are related implicitly or explicitly to many areas, such as engineering, science optimization, economics, transportation, network and structural analysis, Nash equilibrium problems in noncooperative games, computer tomograph, radiation therapy treatment planing, physics, inverse problems that arise from phase retrievals and in medical image reconstruction, and so on, it is very important in mathematics. So, many authors have proposed some algorithms to solve such problems; see, for instance, [7–14]. We further give some of them.

In 2015, Ma *et al.* [4] introduced the following simultaneous iterative algorithm to obtain weak and strong convergence theorems for (SEMEP):

$$\begin{cases} F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ G(v_n, v) + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)T(u_n - \gamma_n A^*(Au_n - Bv_n)), \\ y_{n+1} = \alpha_n u_n + (1 - \alpha_n)S(v_n + \gamma_n B^*(Au_n - Bv_n)) \end{cases} \tag{1.12}$$

for all $u \in C$, $v \in Q$, where $n \geq 1$, and $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are nonexpansive mappings. In the same year, Rahaman *et al.* [3] gave the following method as a generalization of algorithm (1.12):

$$\begin{cases} F(\lambda_1 u_n + (1 - \lambda_1)b, u) + \phi(u) - \phi(u_n) \\ \quad + \langle Tu_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ G(\lambda_2 v_n + (1 - \lambda_2)c, v) + \varphi(v) - \varphi(v_n) \\ \quad + \langle Sv_n, v - v_n\rangle + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1 - \alpha_n)(u_n - \delta_n A^*(Au_n - Bv_n)) + \alpha_n P(u_n - \delta_n A^*(Au_n - Bv_n)), \\ y_{n+1} = (1 - \alpha_n)(v_n + \delta_n B^*(Au_n - Bv_n)) + \alpha_n Q(v_n + \delta_n B^*(Au_n - Bv_n)) \end{cases} \tag{1.13}$$

for all $u, b \in C$, $v, c \in Q$, where $n \geq 1$, and $P : H_1 \to H_1$ and $Q : H_2 \to H_2$ are two demicontractive mappings. They also proved that the sequence $\{(x_n, y_n)\}$ generated by algorithm (1.13) converges weakly and strongly to the solution of the split equality generalized mixed equilibrium problem (1.4) under some suitable conditions.

In this paper, inspired by algorithm (1.13), we introduce a modified algorithm to obtain weak and strong convergence results for the split equality generalized mixed equilibrium problem. Also, we give some corollaries and applications for the split equality problem, the split feasibility problem, the split equality mixed convex differentiable optimization problem, the split equality convex minimization problem, and the split equality mixed equilibrium problem. Our results extend some correspoing results of many authors.

2 Preliminaries

Throughout this paper, we use the symbols $\to$ and $\rightharpoonup$ for the strong and weak convergence, respectively. Now, we recall some definitions, lemmas, and properties, which we need in the proof of our main theorem.

Let T be a mapping on a Hilbert space H. The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in H : Tx = x\}$. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ it is said to be

(i) a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(ii) a firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in C.$$

Lemma 1 ([15]) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Lemma 2 ([16]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and T be a nonexpansive self-mapping on C. If $F(T) \neq \emptyset$, then $I - T$ is demiclosed, that is, if $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, then $(I - T)x = y$. Here, I is the identity operator of H.*

Recall that T is said to be demicompact if every bounded sequence $\{x_n\}$ in C such that $\{(I - T)x_n\}$ converges strongly contains a strongly convergent subsequence.

To solve a generalized mixed equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$ and mappings $T : C \to C$ and $\phi : C \to \mathbb{R}$, let us assume that the following conditions are satisfied:

A1. $F(\lambda x + (1 - \lambda)b, x) = 0$, $\forall x \in C$;

A2. F is monotone, that is, for all $x, y \in C$,

$$F\bigl(\lambda x + (1 - \lambda)b, y\bigr) + F\bigl(\lambda y + (1 - \lambda)b, x\bigr) \leq 0;$$

A3. T is monotone, that is, for all $x, y \in C$,

$$\langle Tx - Ty, x - y\rangle \geq 0;$$

A4. $\forall x \in C$, $y \mapsto F(\lambda x + (1-\lambda)b, y)$ is convex and lower semicontinuous;
A5. F is hemicontinuous in the first argument;
A6. T is weakly upper semicontinuous;
A7. For all $x \in C$, $\lambda \in (0,1]$, and $r > 0$, there exist a bounded subset $D \subseteq C$ and $a \in C$ such that, for all $z \in C \setminus D$ and $b \in C$,

$$-F\big(\lambda a + (1-\lambda)b, z\big) + \langle Tz, a - z\rangle + \phi(a) - \phi(z) + \frac{1}{r}\langle a - z, z - x\rangle < 0.$$

Lemma 3 ([3]) *Let C be a nonempty closed convex subset of a Hilbert space H_1. Suppose that the bifunction $F : C \times C \to \mathbb{R}$ and the mapping $T : C \to C$ satisfy conditions* (A1)-(A7). *Let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex mapping such that $C \cap \operatorname{dom}\phi = \emptyset$. For $r > 0$, $\lambda_1 \in (0,1]$, and $x \in H$, let $J_r^{F,T} : H_1 \to C$ be the resolvent operator of F and T defined by*

$$J_r^{F,T}(x) = \Big\{ z \in C : F\big(\lambda_1 z + (1-\lambda_1)b, y\big) + \langle Tz, y - z\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y, b \in C \Big\}. \tag{2.1}$$

Then:
(i) *For each $x \in H_1$, $J_r^{F,T}(x) \neq \emptyset$*;
(ii) *$J_r^{F,T}$ is single valued*;
(iii) *$J_r^{F,T}$ is firmly nonexpansive, that is,*

$$\big\| J_r^{F,T}(x) - J_r^{F,T}(y)\big\|^2 \leq \big\langle J_r^{F,T}(x) - J_r^{F,T}(y), x - y\big\rangle, \quad \forall x, y \in H_1;$$

(iv) *$F(J_r^{F,T}) = \mathrm{GMEP}(F, T, \phi)$, and it is closed and convex.*

Let the bifunction $G : Q \times Q \to \mathbb{R}$ and the mapping $S : Q \to Q$ satisfy conditions (A1)-(A7). Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex mapping such that $Q \cap \operatorname{dom}\varphi = \emptyset$. For $s > 0$, $\lambda_2 \in (0,1]$, and $u \in H_2$, let $J_s^{G,S} : H_2 \to Q$ be the resolvent operator of G and S defined by

$$J_s^{G,S}(u) = \Big\{ v \in Q : G\big(\lambda_2 v + (1-\lambda_2)c, w\big) + \langle Sv, w - v\rangle + \varphi(w) - \varphi(v) + \frac{1}{s}\langle w - v, v - u\rangle \geq 0, \forall w, c \in Q \Big\}. \tag{2.2}$$

Then, clearly, $J_s^{G,S}$ satisfies (i)-(iv) of Lemma 3, and $F(J_s^{G,S}) = \mathrm{GMEP}(G, S, \varphi)$.

Lemma 4 (Opial's lemma [17]) *Let H be a real Hilbert space, and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying the following conditions*:
(i) *for every $\mu \in W$, $\lim_{n\to\infty} \|\mu_n - \mu\|$ exists*;
(ii) *any weak cluster point of the sequence $\{\mu_n\}$ belongs to W.*
Then there exists $w^* \in W$ such that $\{\mu_n\}$ converges weakly to w^*.

Lemma 5 ([18]) *Let H be a real Hilbert space. Then, we have*

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$$

and

$$\big\|\lambda x + (1-\lambda)y\big\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

3 Main results

Now, we give a new modified iterative algorithm to solve the split equality generalized mixed equilibrium problem. Moreover, we prove strong and weak convergence theorems for nonexpansive mappings in Hilbert spaces. Throughout this section, we always assume that:

B1. H_1, H_2, and H_3 are real Hilbert spaces, and $C \subseteq H_1$ and $Q \subseteq H_2$ are nonempty closed convex subsets;

B2. $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions satisfying conditions (A1), (A2), (A4), (A5), and (A7);

B3. $T : C \to C$ and $S : Q \times Q \to \mathbb{R}$ are mappings satisfying conditions (A3), (A6), and (A7);

B4. $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex mappings such that $C \cap \operatorname{dom}\phi \neq \emptyset$ and $Q \cap \operatorname{dom}\varphi \neq \emptyset$;

B5. $P_1, P_2 : H_1 \to H_1$ and $P_3, P_4 : H_2 \to H_2$ are nonexpansive mapping;

B6. $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear mappings.

For an arbitrary initial value $(x_1, y_1) \in C \times Q$, define the sequence $\{(x_n, y_n)\}$ in $C \times Q$ generated by

$$\begin{cases} F(\lambda_1 u_n + (1-\lambda_1)b, u) + \phi(u) - \phi(u_n) \\ \quad + \langle Tu_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ G(\lambda_2 v_n + (1-\lambda_2)c, v) + \varphi(v) - \varphi(v_n) \\ \quad + \langle Sv_n, v - v_n\rangle + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1-\alpha_n)P_1(u_n - \delta_n A^*(Au_n - Bv_n)) + \alpha_n P_2(u_n - \delta_n A^*(Au_n - Bv_n)), \\ y_{n+1} = (1-\alpha_n)P_3(v_n + \delta_n B^*(Au_n - Bv_n)) + \alpha_n P_4(v_n + \delta_n B^*(Au_n - Bv_n)) \end{cases} \tag{3.1}$$

for all $u, b \in C$ and $v, c \in Q$, where $n \geq 1$, $\lambda_1, \lambda_2 \in (0,1]$, and the sequences $\{\delta_n\}$, $\{\alpha_n\}$, and $\{r_n\}$ satisfy the following conditions:

C1. $\{\delta_n\}$ is a positive real sequence such that $\delta_n \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon)$ for sufficiently small ε, where λ_A and λ_B are the spectral radii of A^*A and B^*B, respectively;

C2. $\{\alpha_n\}$ is a sequence in $(0,1)$ such that, for some $\alpha, \beta \in (0,1)$, $0 < \alpha \leq \alpha_n \leq \beta < 1$;

C3. $\{r_n\} \subset (0,\infty)$ is such that $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Theorem 1 *Let $H_1, H_2, H_3, F, G, T, S, P_1, P_2, P_3, P_4, \phi, \varphi, A$, and B satisfy conditions* (B1)-(B6). *Let $\{(x_n, y_n)\}$ be a sequence generated by* (3.1). *If $\mathcal{F} := \bigcap_{i=1}^4 F(P_i) \cap \mathrm{SEGMEP}(F, G, P_i, \phi, \varphi) \neq \emptyset$, then*:

(i) *the sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution of problem* (1.4);

(ii) *if* P_i, $i = 1, 2, 3, 4$, *are demicompact, then the sequence* $\{(x_n, y_n)\}$ *converges strongly to a solution of problem* (1.4).

Proof (i) Let $(x, y) \in \mathcal{F}$. So, $x \in F(P_1) \cap F(P_2)$ and $y \in F(P_3) \cap F(P_4)$. It is easy to see from Lemma 3 that

$$\|u_n - x\| = \left\|J_{r_n}^{F,T}(x_n) - J_{r_n}^{F,T}(x)\right\| \leq \|x_n - x\| \tag{3.2}$$

and

$$\|v_n - y\| = \left\|J_{r_n}^{F,T}(y_n) - J_{r_n}^{F,T}(y)\right\| \leq \|y_n - y\|. \tag{3.3}$$

Since P_i, $i = 1, 2, 3, 4$, are nonexpansive mappings and

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle y, x\rangle$$

for all $x, y \in H$, we get from Lemma 5 that

$$\begin{aligned}
\|x_{n+1} - x\|^2 &= \big\|(1-\alpha_n)P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad + \alpha_n P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) - x\big\|^2 \\
&= \big\|(1-\alpha_n)\big(P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) - x\big) \\
&\quad + \alpha_n\big(P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) - x\big)\big\|^2 \\
&\leq (1-\alpha_n)\big\|u_n - \delta_n A^*(Au_n - Bv_n) - x\big\|^2 \\
&\quad + \alpha_n\big\|u_n - \delta_n A^*(Au_n - Bv_n) - x\big\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad - P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big)\big\|^2 \\
&= \big\|u_n - \delta_n A^*(Au_n - Bv_n) - x\big\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad - P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big)\big\|^2 \\
&= \|u_n - x\|^2 + \delta_n^2\big\|A^*(Au_n - Bv_n)\big\|^2 \\
&\quad - 2\delta_n\big\langle A^*(Au_n - Bv_n), u_n - x\big\rangle \\
&\quad - \alpha_n(1-\alpha_n)\big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad - P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big)\big\|^2 \\
&\leq \|x_n - x\|^2 + \delta_n^2\big\|A^*(Au_n - Bv_n)\big\|^2 \\
&\quad - 2\delta_n\big\langle A^*(Au_n - Bv_n), u_n - x\big\rangle \\
&\quad - \alpha_n(1-\alpha_n)\big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad - P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big)\big\|^2.
\end{aligned} \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}\delta_n^2\big\|A^*(Au_n - Bv_n)\big\|^2 &= \delta_n^2\big\langle Au_n - Bv_n, AA^*(Au_n - Bv_n)\big\rangle \\ &\leq \lambda_A\delta_n^2\|Au_n - Bv_n\|^2. \end{aligned}\tag{3.5}$$

So, it follows from (3.4) and (3.5) that

$$\begin{aligned}\|x_{n+1} - x\|^2 \leq{}& \|x_n - x\|^2 + \lambda_A\delta_n^2\|Au_n - Bv_n\|^2 \\ &- 2\delta_n\langle Au_n - Bv_n, Au_n - Ax\rangle \\ &- \alpha_n(1-\alpha_n)\big\|P_1\big(u_n - \delta_nA^*(Au_n - Bv_n)\big) \\ &- P_2\big(u_n - \delta_nA^*(Au_n - Bv_n)\big)\big\|^2. \end{aligned}\tag{3.6}$$

In a similar way, we get

$$\begin{aligned}\|y_{n+1} - y\|^2 \leq{}& \|y_n - y\|^2 + \lambda_B\delta_n^2\|Au_n - Bv_n\|^2 \\ &+ 2\delta_n\langle Au_n - Bv_n, Bv_n - By\rangle \\ &- \alpha_n(1-\alpha_n)\big\|P_3\big(v_n + \delta_nB^*(Au_n - Bv_n)\big) \\ &- P_4\big(v_n + \delta_nB^*(Au_n - Bv_n)\big)\big\|^2. \end{aligned}\tag{3.7}$$

By adding inequalities (3.6) and (3.7) side by side and using $Ax = By$, we obtain

$$\begin{aligned}&\|x_{n+1} - x\|^2 + \|y_{n+1} - y\|^2 \\ &\quad\leq \|x_n - x\|^2 + \|y_n - y\|^2 + \delta_n^2(\lambda_A + \lambda_B)\|Au_n - Bv_n\|^2 \\ &\qquad - 2\delta_n\|Au_n - Bv_n\|^2 - \alpha_n(1-\alpha_n)\big\{\big\|P_1\big(u_n - \delta_nA^*(Au_n - Bv_n)\big) \\ &\qquad - P_2\big(u_n - \delta_nA^*(Au_n - Bv_n)\big)\big\|^2 + \big\|P_3\big(v_n + \delta_nB^*(Au_n - Bv_n)\big) \\ &\qquad - P_4\big(v_n + \delta_nB^*(Au_n - Bv_n)\big)\big\|^2\big\} \\ &\quad= \|x_n - x\|^2 + \|y_n - y\|^2 - \delta_n\big(2 - \delta_n(\lambda_A + \lambda_B)\big)\|Au_n - Bv_n\|^2 \\ &\qquad - \alpha_n(1-\alpha_n)\big\{\big\|P_1\big(u_n - \delta_nA^*(Au_n - Bv_n)\big) \\ &\qquad - P_2\big(u_n - \delta_nA^*(Au_n - Bv_n)\big)\big\|^2 + \big\|P_3\big(v_n + \delta_nB^*(Au_n - Bv_n)\big) \\ &\qquad - P_4\big(v_n + \delta_nB^*(Au_n - Bv_n)\big)\big\|^2\big\}. \end{aligned}\tag{3.8}$$

Let $\xi_n(x,y) = \|x_n - x\|^2 + \|y_n - y\|^2$. Thus, we have from (3.8) that

$$\begin{aligned}\xi_{n+1}(x,y) \leq{}& \xi_n(x,y) - \delta_n\big(2 - \delta_n(\lambda_A + \lambda_B)\big)\|Au_n - Bv_n\|^2 \\ &- \alpha_n(1-\alpha_n)\big\{\big\|P_1\big(u_n - \delta_nA^*(Au_n - Bv_n)\big) \\ &- P_2\big(u_n - \delta_nA^*(Au_n - Bv_n)\big)\big\|^2 + \big\|P_3\big(v_n + \delta_nB^*(Au_n - Bv_n)\big) \\ &- P_4\big(v_n + \delta_nB^*(Au_n - Bv_n)\big)\big\|^2\big\}. \end{aligned}\tag{3.9}$$

Since $\alpha_n \in (0,1)$ and $\delta_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$, we get $2 - \delta_n(\lambda_A + \lambda_B) > 0$. So, from (3.9) we obtain

$$\xi_{n+1}(x,y) \leq \xi_n(x,y).$$

Therefore, the sequence $\{\xi_n(x,y)\}$ is nonincreasing and lower bounded by 0. Hence, $\lim_{n\to\infty} \xi_n(x,y)$ exists. Let $\lim_{n\to\infty} \xi_n(x,y) = \sigma(x,y)$. So condition (i) of Lemma 4 is satisfied with $\mu_n = (x_n, y_n)$, $\mu^* = (x,y)$, and $W = \mathcal{F}$. Since the sequence $\{\xi_n(x,y)\}$ converges to a finite limit, we have from inequality (3.9) that

$$\lim_{n\to\infty} \|Au_n - Bv_n\| = 0, \tag{3.10}$$

$$\lim_{n\to\infty} \bigl\|P_1\bigl(u_n - \delta_n A^*(Au_n - Bv_n)\bigr) - P_2\bigl(u_n - \delta_n A^*(Au_n - Bv_n)\bigr)\bigr\| = 0, \tag{3.11}$$

and

$$\lim_{n\to\infty} \bigl\|P_3\bigl(v_n + \delta_n B^*(Au_n - Bv_n)\bigr) - P_4\bigl(v_n + \delta_n B^*(Au_n - Bv_n)\bigr)\bigr\| = 0. \tag{3.12}$$

Moreover, since $\|x_n - x\|^2 \leq \xi_n(x,y)$ and $\|y_n - y\|^2 \leq \xi_n(x,y)$, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded, and $\limsup_{n\to\infty} \|x_n - x\|$ and $\limsup_{n\to\infty} \|y_n - y\|$ exist. Also, it follows from (3.2) and (3.3) that $\limsup_{n\to\infty} \|u_n - x\|$ and $\limsup_{n\to\infty} \|v_n - y\|$ exist. Let us assume that the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to points x^* and y^*, respectively. So, by (3.10), the sequence $\{u_n - \delta_n A^*(Au_n - Bv_n)\}$ converges weakly to x^*, and $\{v_n + \delta_n B^*(Au_n - Bv_n)\}$ converges weakly to y^*. By Lemma 5 we get

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x - x_n + x\|^2 \\
&= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_{n+1} - x_n, x_n - x\rangle \\
&= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 \\
&\quad - 2\bigl\langle x_{n+1} - x^*, x_n - x\bigr\rangle + 2\bigl\langle x_n - x^*, x_n - x\bigr\rangle.
\end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{3.13}$$

and, similarly,

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.14}$$

By Lemma 3, since $u_n = J_{r_n}^{F,T}(x_n)$ and $u_{n+1} = J_{r_{n+1}}^{F,T}(x_{n+1})$, we have that, for all $u \in C$,

$$\begin{aligned}
&F\bigl(\lambda_1 u_n + (1-\lambda_1)b, u\bigr) + \langle Tu_n, u - u_n\rangle \\
&\quad + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&F\bigl(\lambda_1 u_{n+1} + (1-\lambda_1)b, u\bigr) + \langle Tu_{n+1}, u - u_{n+1}\rangle \\
&\quad + \phi(u) - \phi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.
\end{aligned} \tag{3.16}$$

Taking $u = u_n$ in (3.16) and $u = u_{n+1}$ in (3.15) and adding the resulting inequalities side by side, we obtain

$$\begin{aligned}
0 \leq{} & F\big(\lambda_1 u_n + (1-\lambda_1)b, u_{n+1}\big) + F\big(\lambda_1 u_{n+1} + (1-\lambda_1)b, u_n\big) \\
& + \langle Tu_n, u_{n+1} - u_n\rangle + \langle Tu_{n+1}, u_n - u_{n+1}\rangle \\
& + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle.
\end{aligned}$$

Using conditions (A2)-(A3), we have

$$\begin{aligned}
0 &\leq \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \\
&\leq \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \\
&= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \\
&= \langle u_{n+1} - u_n, u_n - u_{n+1}\rangle \\
&\quad + \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\
&= -\|u_{n+1} - u_n\|^2 \\
&\quad + \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle,
\end{aligned}$$

which implies that

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\|\left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right).$$

Thus, we get

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|. \tag{3.17}$$

Using (3.13) and (C3), from (3.17) we get

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.18}$$

In a similar way, we get

$$\lim_{n\to\infty}\|v_{n+1} - v_n\| = 0. \tag{3.19}$$

On the other hand, from (3.6) and (3.7) we get

$$\begin{aligned}
\|x_{n+1} - x\|^2 \leq{} & \|u_n - x\|^2 + \delta_n^2\lambda_A\|Au_n - Bv_n\|^2 \\
& - 2\delta_n\langle Au_n - Bv_n, Au_n - Ax\rangle \\
& - \alpha_n(1-\alpha_n)\big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
& - P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big)\big\|^2
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\|y_{n+1}-y\|^2 \le{}& \|v_n-y\|^2+\delta_n^2\lambda_B\|Au_n-Bv_n\|^2\\&+2\delta_n\langle Au_n-Bv_n,Bv_n-By\rangle\\&-\alpha_n(1-\alpha_n)\big\|P_2\big(v_n+\delta_nB^*(Au_n-Bv_n)\big)\\&-P_4\big(v_n+\delta_nB^*(Au_n-Bv_n)\big)\big\|^2.\end{aligned}\tag{3.21}$$

Using $Ax = By$ and adding inequalities (3.20) and (3.21) side by side, we have

$$\begin{aligned}&\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2\\&\quad\le\|u_n-x\|^2+\|v_n-y\|^2\\&\qquad-\delta_n\big(2-\delta_n(\lambda_A+\lambda_B)\big)\|Au_n-Bv_n\|^2\\&\qquad-\alpha_n(1-\alpha_n)\big\{\big\|P_1\big(u_n-\delta_nA^*(Au_n-Bv_n)\big)\\&\qquad-P_2\big(u_n-\delta_nA^*(Au_n-Bv_n)\big)\big\|^2\\&\qquad+\big\|P_3\big(v_n+\delta_nB^*(Au_n-Bv_n)\big)\\&\qquad-P_4\big(v_n+\delta_nB^*(Au_n-Bv_n)\big)\big\|^2\big\},\end{aligned}\tag{3.22}$$

where

$$\begin{aligned}\|u_n-x\|^2&=\big\|J_{r_n}^{F,T}(x_n)-J_{r_n}^{F,T}(x)\big\|^2\le\langle x_n-x,u_n-x\rangle\\&=\frac{1}{2}\big\{\|x_n-x\|^2+\|u_n-x\|^2-\|x_n-u_n\|^2\big\}\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}\|v_n-y\|^2&=\big\|J_{r_n}^{G,S}(y_n)-J_{r_n}^{F,T}(y)\big\|^2\le\langle y_n-y,v_n-y\rangle\\&=\frac{1}{2}\big\{\|y_n-y\|^2+\|v_n-y\|^2-\|y_n-v_n\|^2\big\}.\end{aligned}\tag{3.24}$$

From (3.22)-(3.24) we conclude that

$$\begin{aligned}&\|x_n-u_n\|^2+\|y_n-v_n\|^2\\&\quad\le\|x_n-x\|^2-\|x_{n+1}-x\|^2+\|y_n-y\|^2-\|y_{n+1}-y\|^2\\&\qquad-\delta_n\big(2-\delta_n(\lambda_A+\lambda_B)\big)\|Au_n-Bv_n\|^2\\&\qquad-\alpha_n(1-\alpha_n)\big\{\big\|P_1\big(u_n-\delta_nA^*(Au_n-Bv_n)\big)\\&\qquad-P_2\big(u_n-\delta_nA^*(Au_n-Bv_n)\big)\big\|^2\\&\qquad+\big\|P_3\big(v_n+\delta_nB^*(Au_n-Bv_n)\big)\\&\qquad-P_4\big(v_n+\delta_nB^*(Au_n-Bv_n)\big)\big\|^2\big\}.\end{aligned}\tag{3.25}$$

Using (3.10)-(3.14), we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0\tag{3.26}$$

and

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \tag{3.27}$$

Hence, $u_n \rightharpoonup x^*$ and $v_n \rightharpoonup y^*$.

Since P_i, $i = 1, 2, 3, 4$, are nonexpansive mappings, we obtain

$$\begin{aligned}
\|u_n - P_1u_n\| &= \|u_n - x_{n+1} + x_{n+1} - P_1u_n\| \\
&\le \|u_n - x_{n+1}\| + \|x_{n+1} - P_1u_n\| \\
&= \|u_n - u_{n+1} + u_{n+1} - x_{n+1}\| \\
&\quad + \big\|(1-\alpha_n)P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad + \alpha_n P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) - P_1u_n\big\| \\
&\le \|u_n - u_{n+1}\| + \|u_{n+1} - x_{n+1}\| \\
&\quad + \big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) - P_1u_n\big\| \\
&\quad + \alpha_n\big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad - P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big)\big\| \\
&\le \|u_n - u_{n+1}\| + \|u_{n+1} - x_{n+1}\| \\
&\quad + |\delta_n|\big\|A^*\big\|\|Au_n - Bv_n\| + \alpha_n\big\|P_1\big(u_n - \delta_n A^*(Au_n - Bv_n)\big) \\
&\quad - P_2\big(u_n - \delta_n A^*(Au_n - Bv_n)\big)\big\|.
\end{aligned}$$

Using (3.10), (3.11), (3.18), and (3.26), we have

$$\lim_{n\to\infty} \|u_n - P_1u_n\| = 0. \tag{3.28}$$

Similarly, using the same steps as before for P_2, P_3, and P_4, we get

$$\lim_{n\to\infty} \|u_n - P_2u_n\| = 0, \qquad \lim_{n\to\infty} \|v_n - P_3v_n\| = 0, \quad \text{and} \quad \lim_{n\to\infty} \|v_n - P_4v_n\| = 0. \tag{3.29}$$

Since

$$\begin{aligned}
\|x_n - P_1x_n\| &= \|x_n - u_n + u_n - P_1u_n + P_1u_n - P_1x_n\| \\
&\le \|x_n - u_n\| + \|u_n - P_1u_n\| + \|u_n - x_n\| \\
&= 2\|x_n - u_n\| + \|u_n - P_1u_n\|,
\end{aligned}$$

we have from (3.26) and (3.28) that

$$\lim_{n\to\infty} \|x_n - P_1x_n\| = 0. \tag{3.30}$$

Similarly, we have

$$\lim_{n\to\infty} \|x_n - P_2x_n\| = 0, \qquad \lim_{n\to\infty} \|y_n - P_3y_n\| = 0, \quad \text{and} \quad \lim_{n\to\infty} \|y_n - P_4y_n\| = 0. \tag{3.31}$$

Since the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to x^* and y^*, respectively, and $(I - P_i)$, $i = 1, 2, 3, 4$, are demiclosed at zero, it follows from (3.30) and (3.31) that $x^* \in F(P_1) \cap F(P_2)$ and $y^* \in F(P_3) \cap F(P_4)$. On the other hand, it is well known that every Hilbert space satisfies Opial's condition. So, we have that the weakly subsequential limit of $\{(x_n, y_n)\}$ is unique.

Now, we show that $x^* \in \mathrm{GMEP}(F, T, \phi)$ and $y^* \in \mathrm{GMEP}(G, S, \varphi)$. Since $u_n = J_{r_n}^{F,T}(x_n)$, we have

$$F\big(\lambda_1 u_n + (1 - \lambda_1)b, u\big) + \langle Tu_n, u - u_n\rangle + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0$$

for all $b, u \in C$ and $\lambda \in (0, 1]$. From conditions (A2) and (A3) we obtain

$$\begin{aligned}\phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle &\geq -F\big(\lambda_1 u_n + (1 - \lambda_1)b, u\big) - \langle Tu_n, u - u_n\rangle \\ &\geq F\big(\lambda_1 u + (1 - \lambda_1)b, u_n\big) + \langle Tu, u_n - u\rangle,\end{aligned}$$

and hence

$$\phi(u) - \phi(u_{n_k}) + \frac{1}{r_{n_k}}\langle u - u_{n_k}, u_{n_k} - x_{n_k}\rangle \geq F\big(\lambda_1 u + (1 - \lambda_1)b, u_{n_k}\big) + \langle Tu, u_{n_k} - u\rangle.$$

From (3.26) it is easy to see that $u_{n_k} \rightharpoonup x^*$. So, we can write $\lim_{k\to\infty} \frac{\|u_{n_k} - x_{n_k}\|}{r_{n_k}} = 0$, and from the lower semicontinuity of ϕ we get

$$F\big(\lambda_1 u + (1 - \lambda_1)b, x^*\big) + \big\langle Tu, x^* - u\big\rangle + \phi\big(x^*\big) - \phi(u) \leq 0 \tag{3.32}$$

for all $b, u \in C$. Set $u_t = tu + (1 - t)x^*$ for $t \in (0, 1]$ and $u \in C$. Since C is a convex set, we have $u_t \in C$. Hence, from (3.32) we have

$$F\big(\lambda_1 u_t + (1 - \lambda_1)b, x^*\big) + \big\langle Tu_t, x^* - u_t\big\rangle + \phi\big(x^*\big) - \phi(u_t) \leq 0. \tag{3.33}$$

Using inequality (3.33), the convexity of ϕ, and conditions (A1)-(A4), we get

$$\begin{aligned}0 &= F\big(\lambda_1 u_t + (1 - \lambda_1)b, u_t\big) + (1 - t)\langle Tu_t, u_t - u_t\rangle + \phi(u_t) - \phi(u_t) \\ &\leq tF\big(\lambda_1 u_t + (1 - \lambda_1)b, u\big) + (1 - t)F\big(\lambda_1 u_t + (1 - \lambda_1)b, x^*\big) \\ &\quad + t\phi(u) + (1 - t)\phi\big(x^*\big) - \phi(u_t) + (1 - t)\big\langle Tu_t, u_t - x^*\big\rangle \\ &\quad + (1 - t)\big\langle Tu_t, x^* - u_t\big\rangle \\ &= t\big\{F\big(\lambda_1 u_t + (1 - \lambda_1)b, u\big) + (1 - t)\big\langle Tu_t, u - x^*\big\rangle + \phi(u) - \phi(u_t)\big\} \\ &\quad + (1 - t)\big\{F\big(\lambda_1 u_t + (1 - \lambda_1)b, x^*\big) + \big\langle Tu_t, x^* - u_t\big\rangle + \phi\big(x^*\big) - \phi(u_t)\big\} \\ &\leq t\big\{F\big(\lambda_1 u_t + (1 - \lambda_1)b, u\big) + (1 - t)\big\langle Tu_t, u - x^*\big\rangle + \phi(u) - \phi(u_t)\big\},\end{aligned}$$

which implies that

$$F\big(\lambda_1 u_t + (1 - \lambda_1)b, u\big) + (1 - t)\big\langle Tu_t, u - x^*\big\rangle + \phi(u) - \phi(u_t) \geq 0$$

for all $b, u \in C$. From the definition of u_t it is clear that $u_t \to x^*$ as $t \to 0$. Using conditions (A5) and (A6) and the proper lower semicontinuity of ϕ, we obtain

$$F\big(\lambda_1 x^* + (1-\lambda_1)b, u\big) + (1-t)\big\langle Tx^*, u - x^*\big\rangle + \phi(u) - \phi\big(x^*\big) \geq 0$$

for all $b, u \in C$, which shows that $x^* \in \mathrm{GMEP}(F, T, \phi)$. By using similar steps we have that $y^* \in \mathrm{GMEP}(G, S, \varphi)$.

Since $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear mappings and $\{u_n\}$ and $\{v_n\}$ converge weakly to x^* and y^*, respectively, for arbitrary $f \in H_3^*$, we have

$$f(Au_n) \to f\big(Ax^*\big).$$

Similarly,

$$f(Bv_n) \to f\big(By^*\big).$$

Hence, we get

$$Au_n - Bv_n \rightharpoonup Ax^* - By^*,$$

which implies that

$$\big\|Ax^* - By^*\big\| \leq \liminf_{n\to\infty} \|Au_n - Bv_n\| = 0,$$

so that $Ax^* = By^*$. So, it follows that $(x^*, y^*) \in \mathrm{SEGMEP}(F, G, T, S, \phi, \varphi)$. Therefore, $(x^*, y^*) \in \mathcal{F}$.

Finally, we conclude that, for each $(x^*, y^*) \in \mathcal{F}$, $\lim_{n\to\infty}(\|x_n - x^*\| + \|y_n - y^*\|)$ exists and each weak cluster point of the sequence $\|(x^*, y^*)\|$ belongs to $\mathcal{F}$. Let $H = H_1 \times H_2$ with norm $\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$, $W = \mathcal{F}$, $\mu_n = (x_n, y_n)$, and $\mu = (x^*, y^*)$. From Lemma 4 we see that there exists $(\overline{x}, \overline{y}) \in \mathcal{F}$ such that $x_n \rightharpoonup \overline{x}$ and $y_n \rightharpoonup \overline{y}$. Therefore, the sequence $\{(x_n, y_n)\}$ generated by the iterative algorithm (3.1) converges weakly to a solution of problem (1.4) in $\mathcal{F}$. This completes the proof.

(ii) Now, we prove the strong convergence of the sequence $\{(x_n, y_n)\}$ generated by the iterative algorithm (3.1) under the demicompact condition.

Since P_i, $i = 1, 2, 3, 4$, are demicompact, $\{x_n\}$ and $\{y_n\}$ are bounded sequences, and $\lim_{n\to\infty} \|x_n - P_1 x_n\| = 0$, $\lim_{n\to\infty} \|x_n - P_2 x_n\| = 0$, $\lim_{n\to\infty} \|y_n - P_3 y_n\| = 0$, and $\lim_{n\to\infty} \|y_n - P_4 y_n\| = 0$, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converge strongly to some points u^* and v^*, respectively. The weak convergence of $\{x_{n_k}\}$ and $\{y_{n_k}\}$ to x^* and y^*, respectively, implies that $x^* = u^*$ and $y^* = v^*$. It follows from the demiclosedness of P_i that $x^* \in F(P_1) \cap F(P_2)$ and $y^* \in F(P_3) \cap F(P_4)$. Using similar steps to the previous ones, we get that $x^* \in \mathrm{GMEP}(F, T, \phi)$ and $y^* \in \mathrm{GMEP}(G, S, \varphi)$. Thus, we have

$$\big\|Ax^* - By^*\big\| = \lim_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0,$$

which implies that $Ax^* = By^*$. Hence, $(x^*, y^*) \in \mathcal{F}$. On the other hand, since $\xi_n(x, y) = \|x_n - x\|^2 + \|y_n - y\|^2$ for $(x, y) \in \mathcal{F}$, we know that $\lim_{k\to\infty} \xi_{n_k}(x^*, y^*) = 0$. From conjecture

(i) we see that $\lim_{n\to\infty}\xi_n(x^*,y^*)$ exists; therefore, $\lim_{n\to\infty}\xi_n(x^*,y^*)=0$. So, the iterative scheme (3.1) converges strongly to a solution of problem (1.4). This completes the proof of conjecture (ii). □

Taking $F=G=T=S=\phi=\varphi=0$ in Theorem 1, we get the following convergence theorem for the split equality problem (1.10).

Corollary 1 *Let $H_1, H_2, H_3, P_1, P_2, P_3, P_4, A$, and B satisfy conditions* (B1), (B5), *and* (B6). *For an arbitrary initial value $(x_1,y_1)\in C\times Q$, define the sequence $\{(x_n,y_n)\}$ in $C\times Q$ generated by*

$$\begin{cases} x_{n+1}=(1-\alpha_n)P_1(x_n-\delta_nA^*(Ax_n-By_n))+\alpha_nP_2(x_n-\delta_nA^*(Ax_n-By_n)),\\ y_{n+1}=(1-\alpha_n)P_3(y_n+\delta_nB^*(Ax_n-By_n))+\alpha_nP_4(y_n+\delta_nB^*(Ax_n-By_n)),\end{cases}$$

where $n\ge 1$, and the sequences $\{\delta_n\}$ and $\{\alpha_n\}$ satisfy conditions (C1) *and* (C2), *respectively. If $\mathcal{F}:=\bigcap_{i=1}^4F(P_i)\cap\mathrm{SEP}\neq\emptyset$, then*:

(i) *The sequence $\{(x_n,y_n)\}$ converges weakly to a solution of problem* (1.10);

(ii) *If P_i, $i=1,2,3,4$, are demicompact, then the sequence $\{(x_n,y_n)\}$ converges strongly to a solution of problem* (1.10).

Taking $B=I$ and $H_2=H_3$ in Corollary 1, we obtain the following convergence theorem for the split feasibility problem (1.11).

Corollary 2 *Let $H_1, H_2, P_1, P_2, P_3, P_4$, and A satisfy conditions* (B1), (B5), *and* (B6) *with $A:H_1\to H_2$. For an arbitrary initial value $(x_1,y_1)\in C\times Q$, define the sequence $\{(x_n,y_n)\}$ in $C\times Q$ generated by*

$$\begin{cases} x_{n+1}=(1-\alpha_n)P_1(x_n-\delta_nA^*(Ax_n-y_n))+\alpha_nP_2(x_n-\delta_nA^*(Ax_n-y_n)),\\ y_{n+1}=(1-\alpha_n)P_3(y_n+\delta_n(Ax_n-y_n))+\alpha_nP_4(y_n+\delta_n(Ax_n-y_n)),\quad n\ge 1,\end{cases}$$

where $\{\delta_n\}$ is a positive real sequence such that $\delta_n\in(\varepsilon,\frac{1}{\lambda_A}-\varepsilon)$ for sufficiently small ε, where λ_A denotes the spectral radius of A^*A, and $\{\alpha_n\}$ satisfy condition (C2). *If $\mathcal{F}:=\bigcap_{i=1}^4F(P_i)\cap$ SFP $\neq\emptyset$, then*:

(i) *The sequence $\{(x_n,y_n)\}$ converges weakly to a solution of problem* (1.11);

(ii) *If P_i, $i=1,2,3,4$, are demicompact, then the sequence $\{(x_n,y_n)\}$ converges strongly to a solution of problem* (1.11).

4 Applications

Let C be a nonempty closed convex subset of a real Hilbert space H, and $\psi:C\to C$ be a convex and differentiable mapping. It is known that the convex differentiable minimization problem is to find $x^*\in C$ such that

$$\min_{x\in C}\psi(x)=\psi\left(x^*\right). \tag{4.1}$$

Also, it is well known that a point x^* is a solution of problem (4.1) if and only if

$$\left\langle\nabla\psi\left(x^*\right),y-x^*\right\rangle\ge 0 \tag{4.2}$$

for all $y \in C$. Problem (4.2) is called the classical variational inequality problem. If we get $F(x^*, y) = \langle \nabla\psi(x^*), y - x^*\rangle$, then the equilibrium problem (1.1) and the variational inequality problem (4.2) have the same solution.

In 2015, Rahaman *et al.* [3] introduced the split equality mixed convex differentiable optimization problem of finding $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} &\langle \nabla\psi(x^*), x - x^*\rangle + \langle T(x^*), x - x^*\rangle + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in C, \\ &\langle \nabla\sigma(y^*), y - y^*\rangle + \langle S(y^*), y - y^*\rangle + \varphi(y) - \varphi(y^*) \geq 0, \quad \forall y \in Q, \quad \text{and} \\ &Ax^* = By^*, \end{aligned} \tag{4.3}$$

where $\psi : C \to H_1$ and $\sigma : Q \to H_2$ are convex differentiable mappings. The set of solutions of the split equality mixed convex differentiable optimization problem (4.3) is denoted by $\mathrm{SEMCDOP}(\psi, \sigma, T, S, \phi, \varphi)$. If $T = 0$, then this problem is reduced to the split equality mixed variational inequality problem introduced by Ma *et al.* [4] in 2015. Also, if $B = I$ and $H_2 = H_3$, then problem (4.3) is reduced to the split mixed convex differentiable optimization problem of finding $x^* \in C$ such that

$$\langle \nabla\psi(x^*), x - x^*\rangle + \langle T(x^*), x - x^*\rangle + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in C,$$

and such that $Ax^* = y^* \in Q$ solves

$$\langle \nabla\sigma(y^*), y - y^*\rangle + \langle S(y^*), y - y^*\rangle + \varphi(y) - \varphi(y^*) \geq 0, \quad \forall y \in Q. \tag{4.4}$$

The solution set of this problem is denoted by $\mathrm{SMCDOP}(\psi, \sigma, T, S, \phi, \varphi)$.

Since the gradients $\nabla\psi$ and $\nabla\sigma$ are monotone mappings, if $F(x^*, y) = \langle \nabla\psi(x^*), x - x^*\rangle$, $G(x, y^*) = \langle \nabla\sigma(y^*), y - y^*\rangle$, and $\lambda_1 = \lambda_2 = 1$, then F and G satisfy condition (B2). So, we can give the following result.

Theorem 2 *Let H_1, H_2, H_3, T, S, ϕ, φ, P_1, P_2, P_3, P_4, A, and B satisfy conditions* (B1)-(B6) *except* (B2). *Suppose that the mappings $\psi : C \to H_1$ and $\sigma : Q \to H_2$ are convex and differentiable mappings. For an arbitrary initial value $(x_1, y_1) \in C \times Q$, define the sequence $\{(x_n, y_n)\}$ in $C \times Q$ generated by*

$$\begin{cases} \langle \nabla\psi(u_n), u - u_n\rangle + \phi(u) - \phi(u_n) + \langle Tu_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ \langle \nabla\sigma(v_n), v - v_n\rangle + \varphi(v) - \varphi(v_n) + \langle Sv_n, v - v_n\rangle + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1 - \alpha_n)P_1(u_n - \delta_n A^*(Au_n - Bv_n)) + \alpha_n P_2(u_n - \delta_n A^*(Au_n - Bv_n)), \\ y_{n+1} = (1 - \alpha_n)P_3(v_n + \delta_n B^*(Au_n - Bv_n)) + \alpha_n P_4(v_n + \delta_n B^*(Au_n - Bv_n)) \end{cases}$$

for all $u \in C$ and $v \in Q$, where $n \geq 1$, and the sequences $\{\delta_n\}$, $\{\alpha_n\}$, and $\{r_n\}$ satisfy conditions (C1)-(C3), *respectively. If $\mathcal{F} := \bigcap_{i=1}^4 F(P_i) \cap \mathrm{SEMCDOP}(\psi, \sigma, T, S, \phi, \varphi) \neq \emptyset$, then*:

(i) *The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem* (4.3);

(ii) *If P_i, $i = 1, 2, 3, 4$, are demicompact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of problem* (4.3).

In Theorem 2, if we take $B = I$ and $H_2 = H_3$, then we get the following result.

Corollary 3 *Let $H_1, H_2, T, S, \phi, \varphi, P_1, P_2, P_3, P_4$, and A satisfy conditions* (B1)-(B6) *except* (B2) *with $A : H_1 \to H_2$. Suppose that the mappings $\psi : C \to H_1$ and $\sigma : Q \to H_2$ are convex and differentiable mappings. For an arbitrary initial value $(x_1, y_1) \in C \times Q$, define the sequence $\{(x_n, y_n)\}$ in $C \times Q$ generated by*

$$\begin{cases} \langle \nabla\psi(u_n), u - u_n\rangle + \phi(u) - \phi(u_n) + \langle Tu_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ \langle \nabla\sigma(v_n), v - v_n\rangle + \varphi(v) - \varphi(v_n) + \langle Sv_n, v - v_n\rangle + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1 - \alpha_n)P_1(u_n - \delta_n A^*(Au_n - v_n)) + \alpha_n P_2(u_n - \delta_n A^*(Au_n - v_n)), \\ y_{n+1} = (1 - \alpha_n)P_3(v_n + \delta_n(Au_n - v_n)) + \alpha_n P_4(v_n + \delta_n(Au_n - v_n)) \end{cases}$$

for all $u \in C$ and $v \in Q$, where $n \geq 1$, $\{\delta_n\}$ is a positive real sequences such that $\delta_n \in (\varepsilon, \frac{1}{\lambda_A} - \varepsilon)$ for sufficiently small ε, where λ_A denotes the spectral radius of A^*A, and the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy conditions (C2) *and* (C3), *respectively. If $\mathcal{F} := \bigcap_{i=1}^4 F(P_i) \cap$ SMCDOP$(\psi, \sigma, T, S, \phi, \varphi) \neq \emptyset$, then*:

(i) *The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem* (4.4);

(ii) *If P_i, $i = 1, 2, 3, 4$, are demicompact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of problem* (4.4).

In Theorem 1, if we take $F = G = T = S = 0$, then we have the following result for the split equality convex minimization problem (1.8).

Theorem 3 *Let $H_1, H_2, H_3, \phi, \varphi, P_1, P_2, P_3, P_4, A$, and B satisfy conditions* (B1), (B4), (B5), *and* (B6). *For an arbitrary initial value $(x_1, y_1) \in C \times Q$, define the sequence $\{(x_n, y_n)\}$ in $C \times Q$ generated by*

$$\begin{cases} \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1 - \alpha_n)P_1(u_n - \delta_n A^*(Au_n - Bv_n)) + \alpha_n P_2(u_n - \delta_n A^*(Au_n - Bv_n)), \\ y_{n+1} = (1 - \alpha_n)P_3(v_n + \delta_n B^*(Au_n - Bv_n)) + \alpha_n P_4(v_n + \delta_n B^*(Au_n - Bv_n)) \end{cases}$$

for all $u \in C$ and $v \in Q$, where $n \geq 1$, and the sequences $\{\delta_n\}$, $\{\alpha_n\}$, and $\{r_n\}$ satisfy conditions (C1)-(C3), *respectively. If $\mathcal{F} := \bigcap_{i=1}^4 F(P_i) \cap$ SECMP$(\phi, \varphi) \neq \emptyset$, then*:

(i) *The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem* (1.8);

(ii) *If P_i, $i = 1, 2, 3, 4$, are demicompact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of problem* (1.8).

If we take $B = I$ and $H_2 = H_3$ in Theorem 3, then we get the following result for the split convex minimization problem (1.9).

Corollary 4 *Let $H_1, H_2, P_1, P_2, P_3, P_4, \phi, \varphi$, and A satisfy conditions* (B1), (B4), (B5), *and* (B6) *with $A : H_1 \to H_2$. For an arbitrary initial value $(x_1, y_1) \in C \times Q$, define the sequence $\{(x_n, y_n)\}$ in $C \times Q$ generated by*

$$\begin{cases} \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1 - \alpha_n)P_1(u_n - \delta_n A^*(Au_n - v_n)) + \alpha_n P_2(u_n - \delta_n A^*(Au_n - v_n)), \\ y_{n+1} = (1 - \alpha_n)P_3(v_n + \delta_n(Au_n - v_n)) + \alpha_n P_4(v_n + \delta_n(Au_n - v_n)) \end{cases}$$

for all $u \in C$ and $v \in Q$, where $n \geq 1$, $\{\delta_n\}$ is a positive real sequences such that $\delta_n \in (\varepsilon, \frac{1}{\lambda_A} - \varepsilon)$ for sufficiently small ε, where λ_A denotes the spectral radius of A^*A, and the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy conditions (C2) *and* (C3), *respectively. If $\mathcal{F} := \bigcap_{i=1}^{4} F(P_i) \cap \mathrm{SCMP}(\phi, \varphi) \neq \emptyset$, then*:

(i) *The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem* (1.9);

(ii) *If P_i, $i = 1, 2, 3, 4$, are demicompact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of problem* (1.9).

In Theorem 1, if we take $T = S = 0$ and $\lambda_1 = \lambda_2 = 1$, then we have the following convergence result for the split equality mixed equilibrium problem (1.5).

Theorem 4 *Let H_1, H_2, H_3, F, G, ϕ, φ, P_1, P_2, P_3, P_4, A, and B satisfy conditions* (B1)-(B6) *except* (B3). *For an arbitrary initial value $(x_1, y_1) \in C \times Q$, define the sequence $\{(x_n, y_n)\}$ in $C \times Q$ generated by*

$$\begin{cases} F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ G(v_n, v) + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1-\alpha_n)P_1(u_n - \delta_n A^*(Au_n - Bv_n)) + \alpha_n P_2(u_n - \delta_n A^*(Au_n - Bv_n)), \\ y_{n+1} = (1-\alpha_n)P_3(v_n + \delta_n B^*(Au_n - Bv_n)) + \alpha_n P_4(v_n + \delta_n B^*(Au_n - Bv_n)) \end{cases}$$

for all $u \in C$ and $v \in Q$, where $n \geq 1$, and the sequences $\{\delta_n\}$, $\{\alpha_n\}$, and $\{r_n\}$ satisfy conditions (C1)-(C3), *respectively. If $\mathcal{F} := \bigcap_{i=1}^{4} F(P_i) \cap \mathrm{SEMEP}(F, G, \phi, \varphi) \neq \emptyset$, then*:

(i) *The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem* (1.5);

(ii) *If P_i, $i = 1, 2, 3, 4$, are demicompact, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of problem* (1.5).

Competing interests

The author has no competing interests.

Received: 24 February 2016 Accepted: 25 November 2016 Published online: 01 December 2016

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