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Some convergence theorems involving proximal point and common fixed points for asymptotically nonexpansive mappings in CAT(0) spaces

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Abstract

In this paper, a new modified proximal point algorithm involving fixed point iterates of asymptotically nonexpansive mappings in CAT(0) spaces is proposed and the existence of a sequence generated by our iterative process converging to a minimizer of a convex function and a common fixed point of asymptotically nonexpansive mappings is proved.

Keywords: convex minimization problem; resolvent identity; CAT(0) space; proximal point algorithm; asymptotically nonexpansive mapping

1 Introduction

Recently, many convergence results by *the proximal point algorithm* (shortly, the PPA) which was initiated by Martinet [1] in 1970 for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces, and Banach spaces to the setting of manifolds (see [1–9]).

For example, in 2013, Bačák [6] introduced the PPA in a CAT(0) space (X, d) as follows: $x_1 \in X$ and

$$x_{n+1} = \arg\min_{y\in X}\left(f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)\right), \quad \forall n \geq 1, \tag{1.1}$$

where $\lambda_n > 0$, $\forall n \geq 1$. It was shown that if f has a minimizer and $\Sigma_{n=1}^{\infty}\lambda_n = \infty$, then the sequence $\{x_n\}$ $\triangle$-converges to its minimizer (see [7]).

Also in 2015, Cholamjiak-Abdou-Cho [10] established the strong convergence of the sequence to minimizers of a convex function and to fixed points of nonexpansive mappings in CAT(0) spaces.

Motivated and inspired by the research going on in this direction, it is naturally to put forward the following.

Open question Can we establish the strong convergence of the sequence to minimizers of a convex function and to a common fixed point of asymptotically nonexpansive mappings in CAT(0) spaces?

The purpose of this paper is to propose the modified proximal point algorithm using the S-type iteration process for four asymptotically nonexpansive mappings in CAT(0) spaces and to prove some $\triangle$- and strong convergence theorems of the proposed processes under suitable conditions.

Our results not only give an affirmative answer to the above open question but also generalize the corresponding results of Bačák [6], Ariza-Ruiz *et al.* [7], Cholamjiak-Abdou-Cho [10], Agarwal *et al.* [11], Dhompongsa-Panyanak [12], Khan-Abbas [13], and many others.

2 Preliminaries

Recall that a metric space (X,d) is called a CAT(0) space, if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. A subset K of a CAT(0) space X is convex if, for any $x,y\in K$, we have $[x,y]\subset K$, where $[x,y]:=\{\lambda x\oplus(1-\lambda)y:0\le\lambda\le1\}$ is the unique geodesic joining x and y.

It is well known that a geodesic space (X,d) is a CAT(0) space, if and only if the inequality

$$d^2\bigl((1-t)x\oplus ty,z\bigr)\le(1-t)d^2(x,z)+td^2(y,z)-t(1-t)d^2(x,y) \tag{2.1}$$

is satisfied for all $x,y,z\in X$ and $t\in[0,1]$. In particular, if x, y, z are points in a CAT(0) space (X,d) and $t\in[0,1]$, then

$$d\bigl((1-t)x\oplus ty,z\bigr)\le(1-t)d(x,z)+td(y,z). \tag{2.2}$$

In order to save space, we will not repeat the geometric properties, some conclusions, and the $\triangle$-convergence of CAT(0) space here. The interested reader may refer to (for example) [12, 14–16].

In the sequel, we denote by $F(T)$ the fixed point set of a mapping T.

Recall that a mapping $T:C\to C$ is said to be *asymptotically nonexpansive*, if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $k_n\to1$ such that

$$d\bigl(T^nx,T^ny\bigr)\le k_nd(x,y),\quad\forall x,y\in C,n\ge1. \tag{2.3}$$

Recall that a function $f:C\to(-\infty,\infty]$ defined on a convex subset C of a CAT(0) space is convex if, for any geodesic $[x,y]:=\{\gamma_{x,y}(\lambda):0\le\lambda\le1\}:=\{\lambda x\oplus(1-\lambda)y:0\le\lambda\le1\}$ joining $x,y\in C$, the function $f\circ\gamma$ is convex, *i.e.*,

$$f\bigl(\gamma_{x,y}(\lambda)\bigr):=f\bigl(\lambda x\oplus(1-\lambda)y\bigr)\le\lambda f(x)+(1-\lambda)f(y).$$

Examples of convex functions in CAT(0) *space X*:

Example 1 The function $y\mapsto d(x,y):X\to[0,\infty)$ is convex.

Example 2 For a nonempty, closed, and convex subset $C\subset X$, the indicator function defined by

$$\delta_C:X\to R,\qquad \delta_C(x)=\begin{cases}0, & \text{if } x\in C,\\ +\infty, & \text{otherwise,}\end{cases} \tag{2.4}$$

is a proper, convex, and lower semi-continuous function.

Example 3 The function $y \mapsto d^2(z,y) : X \to [0,\infty)$ is convex.

Indeed, for each two points $x, y \in X$, there is a unique geodesic $\gamma_{x,y}(\lambda)$ joining x and y such that

$$\begin{aligned} d^2\bigl(z, \gamma_{x,y}(\lambda)\bigr) &\le (1-\lambda)d^2(z,x) + \lambda d^2(z,y) - \lambda(1-\lambda)d^2(x,y) \\ &\le (1-\lambda)d^2(z,x) + \lambda d^2(z,y). \end{aligned} \tag{2.5}$$

This implies that the function $y \mapsto d^2(z,y) : X \to [0,\infty)$ is convex.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in CAT(0) space X as

$$J_\lambda(x) = \arg\min_{y \in X}\left[f(y) + \frac{1}{2\lambda}d^2(y,x)\right], \quad \forall x \in X. \tag{2.6}$$

Let $f : X \to (-\infty,\infty]$ be a proper convex and lower semi-continuous function. It was shown in [11] that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $\arg\min y_X f(y)$ of minimizers of f. Also for any $\lambda > 0$, the resolvent J_λ of f is nonexpansive [17].

Lemma 2.1 (Sub-differential inequality [18]) *Let (X,d) be a complete* CAT(0) *space and $f : X \to (-\infty,\infty]$ be proper convex and lower semi-continuous. Then, for all $x, y \in X$ and $\lambda > 0$, the following inequality holds*:

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x,y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \le f(y). \tag{2.7}$$

Lemma 2.2 (Demi-closed principle [16]) *Assume C is a closed convex subset of a complete* CAT(0) *space X and $T : C \to C$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\triangle$-$\lim x_n = p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tp = p$.*

Lemma 2.3 [19] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following conditions*:

$$a_{n+1} \le (1 + b_n)a_n, \quad \forall n \ge \infty,$$

where $b_n \ge 0$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 2.4 [20, 21] *Let X be a* CAT(0) *space, C be a nonempty, closed, and convex subset of X. Let $\{x_i\}_{i=1}^n$ be any finite subset of C, and $\alpha_i \in (0,1)$, $i = 1, 2, \ldots, n$ such that $\sum_{n=1}^n \alpha_i = 1$. Then the following inequalities hold*:

$$d\left(\bigoplus_{i=1}^n \alpha_i x_i, z\right) \le \sum_{i=1}^n \alpha_i d(x_i, z), \quad \forall z \in C, \tag{2.8}$$

$$d\left(\bigoplus_{i=1}^n \alpha_i x_i, z\right)^2 \le \sum_{i=1}^n \alpha_i d(x_i, z)^2 - \sum_{i,j=1, i\ne j}^n \alpha_i \alpha_j d(x_i, x_j)^2, \quad \forall z \in C. \tag{2.9}$$

Lemma 2.5 (The resolvent identity [17]) *Let* (X,d) *be a complete* CAT(0) *space and* $f : X \to (-\infty,\infty]$ *be a proper convex and lower semi-continuous function. Then the following identity holds*:

$$J_\lambda x = J_\mu\left(\frac{\lambda-\mu}{\lambda}J_\lambda x \oplus \frac{\mu}{\lambda}x\right), \quad \forall x \in X \text{ and } \lambda > \mu > 0. \tag{2.10}$$

3 Some △-convergence theorems involving proximal point and common fixed points for asymptotically nonexpansive mappings in CAT(0) spaces

We are now in a position to give the main results of the paper.

Theorem 3.1 *Assume that*

(1) (X,d) *is a complete* CAT(0) *space, and* C *is a nonempty, closed, and convex subset of* X;

(2) $f : C \to (-\infty,\infty]$ *is a proper convex and lower continuous function*;

(3) $T_i : C \to C$ *and* $S_i : C \to C$, $i = 1,2$ *all are* $\{k_n\}$*-asymptotically nonexpansive mappings with* $k_n \in [1,\infty)$, $k_n \to 1$ *and* $\sum_{i=1}^\infty (k_n - 1) < \infty$ *such that*

$$\Omega := F(T_1) \cap F(T_2) \cap F(S_1) \cap F(S_2) \cap \arg\min_{y\in C} f(y) \neq \emptyset; \tag{3.1}$$

(4) $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\eta_n\}$, $\{\xi_n\}$ *are sequences in* $[0,1]$ *with*

$$\begin{aligned} &\alpha_n + \beta_n + \gamma_n = 1, \\ &\delta_n + \eta_n + \xi_n = 1, \quad 0 < a \le \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n, \xi_n < 1, \forall n \ge 1, \end{aligned} \tag{3.2}$$

where a *is a positive constant in* $(0,1)$;

(5) $\{\lambda_n\}$ *is a sequence such that* $\lambda_n \ge \lambda > 0$ *for all* $n \ge 1$ *and some* λ.

Let $\{x_n\}$ *be the sequence generated in the following manner*:

$$\begin{cases} z_n = \arg\min y_{y\in C}[f(y) + \frac{1}{2\lambda_n}d^2(y,x_n)], \\ y_n = \alpha_n x_n \oplus \beta_n T_1^n x_n \oplus \gamma_n T_2^n z_n, \qquad \forall n \ge 1. \\ x_{n+1} = \delta_n T_2^n x_n \oplus \eta_n S_1^n x_n \oplus \xi_n S_2^n y_n, \end{cases} \tag{3.3}$$

Then $\{x_n\}$ △*-converges to a point* $x^* \in \Omega$ *which is a minimizer of* f *in* C *as well as a common fixed point of* T_i, S_i, $i = 1,2$.

Proof Let $q \in \Omega$. Then $q = T_1 q = T_2 q = S_1 q = S_2 q$ and $f(q) \le f(y)$, $\forall y \in C$. Therefore we have

$$f(q) + \frac{1}{2\lambda_n}d^2(q,q) \le f(y) + \frac{1}{2\lambda_n}d^2(y,q), \quad \forall y \in C,$$

and hence $q = J_{\lambda_n} q$, $\forall n \ge 1$.

(I) *First we prove that the limit* $\lim_{n\to\infty} d(x_n,q)$ *exists.*

Indeed, $z_n = J_{\lambda_n} x_n$, and J_{λ_n} is nonexpansive [17]. Hence we have

$$d(z_n,q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \le d(x_n,q). \tag{3.4}$$

Also, by (3.3), (3.4), and (2.8), we have

$$\begin{aligned} d(y_n, q) &= d\big(\alpha_n x_n \oplus \beta_n T_1^n x_n \oplus \gamma_n T_2^n z_n, q\big) \\ &\le \alpha_n d(x_n, q) + \beta_n d\big(T_1^n x_n, q\big) + \gamma_n d\big(T_2^n z_n, q\big) \\ &\le \alpha_n d(x_n, q) + \beta_n k_n d(x_n, q) + \gamma_n k_n d(z_n, q) \\ &\le k_n d(x_n, q). \end{aligned} \tag{3.5}$$

Similarly, by (3.3) and (3.5), we obtain

$$\begin{aligned} d(x_{n+1}, q) &= d\big(\delta_n T_2^n x_n \oplus \eta_n S_1^n x_n \oplus \xi_n S_2^n y_n, q\big) \\ &\le \delta_n d\big(T_2^n x_n, q\big) + \eta_n d\big(S_1^n x_n, q\big) + \xi_n d\big(S_2^n y_n, q\big) \\ &\le k_n\big[\delta_n d(x_n, q) + \eta_n d(x_n, q)\big] + \xi_n k_n d(y_n, q) \\ &\le k_n^2 d(x_n, q) = \big(1 + \big(k_n^2 - 1\big)\big) d(x_n, q) \\ &= \big(1 + (k_n - 1)(k_n + 1)\big) d(x_n, q) \\ &\le \big(1 + (k_n - 1)L\big) d(x_n, q), \quad \forall n \ge 1, \end{aligned} \tag{3.6}$$

where $L = 1 + \sup_{n\ge1} k_n$. By Lemma 2.3, the limit $\lim_{n\to\infty} d(x_n, q)$ exists. Without loss of generality, we can assume that

$$\lim_{n\to\infty} d(x_n, q) = c \ge 0. \tag{3.7}$$

Therefore $\{x_n\}$ is bounded, and so are $\{z_n\}$, $\{y_n\}$, $\{T_i^n x_n\}$, $i = 1, 2$, $\{S_1^n x_n\}$, $\{T_2^n z_n\}$, $\{S_2^n y_n\}$.

(II) *Now we prove that* $\lim_{n\to\infty} d(x_n, z_n) = 0$.

Indeed, by the sub-differential inequality (2.7) we have

$$\frac{1}{2\lambda_n}\big\{d^2(z_n, q) - d^2(x_n, q) + d^2(x_n, z_n)\big\} \le f(q) - f(z_n).$$

Since $f(q) \le f(z_n)$, $\forall n \ge 1$, it follows that

$$d^2(x_n, z_n) \le d^2(x_n, q) - d^2(z_n, q). \tag{3.8}$$

Furthermore, it follows from (3.6) that

$$\begin{aligned} d(x_{n+1}, q) &\le k_n\big[\delta_n d(x_n, q) + \eta_n d(x_n, q)\big] + \xi_n k_n d(y_n, q) \\ &\le k_n\big[(1 - \xi_n) d(x_n, q) + \xi_n d(y_n, q)\big]. \end{aligned}$$

Simplifying we have

$$\begin{aligned} d(x_n, q) &\le \frac{1}{\xi_n k_n}\big[k_n d(x_n, q) - d(x_{n+1}, q)\big] + d(y_n, q) \\ &\le \frac{1}{a k_n}\big[k_n d(x_n, q) - d(x_{n+1}, q)\big] + d(y_n, q). \end{aligned}$$

This together with (3.7) shows that

$$c = \liminf_{n\to\infty} d(x_n, q) \le \liminf_{n\to\infty} d(y_n, q). \tag{3.9}$$

On the other hand it follows from (3.5) that

$$\limsup_{n\to\infty} d(y_n, q) \le \limsup_{n\to\infty} \bigl(k_n d(x_n, q)\bigr) = c.$$

This together with (3.9) implies that

$$\lim_{n\to\infty} d(y_n, q) = c. \tag{3.10}$$

Also, by (3.5) we have

$$\begin{aligned} d(y_n, q) &\le \alpha_n d(x_n, q) + \beta_n k_n d(x_n, q) + \gamma_n k_n d(z_n, q) \\ &= k_n\bigl[(1-\gamma_n) d(x_n, q) + \gamma_n d(z_n, q)\bigr], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} d(x_n, q) &\le \frac{1}{\gamma_n k_n}\bigl[k_n d(x_n, q) - d(y_n, q)\bigr] + d(z_n, q) \\ &\le \frac{1}{a k_n}\bigl[k_n d(x_n, q) - d(y_n, q)\bigr] + d(z_n, q). \end{aligned}$$

This together with (3.10) shows that

$$c = \liminf_{n\to\infty} d(x_n, q) \le \liminf_{n\to\infty} d(z_n, q). \tag{3.11}$$

From (3.4), it follows that

$$\limsup_{n\to\infty} d(z_n, q) \le \limsup_{n\to\infty} d(x_n, q) = c.$$

This shows that $\lim_{n\to\infty} d(z_n, q) = c$. Therefore it follows from (3.8) that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{3.12}$$

(III) *Now we prove that*

$$\lim_{n\to\infty} d\bigl(x_n, T_i^n x_n\bigr) = 0, \quad i = 1, 2 \quad \textit{and} \quad \lim_{n\to\infty} d(y_n, z_n) = 0.$$

Indeed, it follows from (2.9) that

$$\begin{aligned} d^2(y_n, q) &= d^2\bigl(\alpha_n x_n \oplus \beta_n T_1^n x_n \oplus \gamma_n T_2^n z_n, q\bigr) \\ &\le \alpha_n d^2(x_n, q) + \beta_n d^2\bigl(T_1^n x_n, q\bigr) + \gamma_n d^2\bigl(T_2^n z_n, q\bigr) \\ &\quad - \alpha_n\beta_n d^2\bigl(x_n, T_1^n x_n\bigr) - \alpha_n\gamma_n d^2\bigl(x_n, T_2^n z_n\bigr) - \beta_n\gamma_n d^2\bigl(T_1^n x_n, T_2^n z_n\bigr) \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n d^2(x_n,q) + \beta_n k_n^2 d^2(x_n,q) + \gamma_n k_n^2 d^2(z_n,q) \\
&\quad - \alpha_n\beta_n d^2\big(x_n, T_1^n x_n\big) - \alpha_n\gamma_n d^2\big(x_n, T_2^n z_n\big) - \beta_n\gamma_n d^2\big(T_1^n x_n, T_2^n z_n\big) \\
&\le k_n^2 d^2(x_n,q) - \alpha_n\beta_n d^2\big(x_n, T_1^n x_n\big) \\
&\quad - \alpha_n\gamma_n d^2\big(x_n, T_2^n z_n\big) - \beta_n\gamma_n d^2\big(T_1^n x_n, T_2^n z_n\big). \qquad (3.13)
\end{aligned}$$

By virtue of (3.7) and (3.10) we have

$$\begin{aligned}
&\alpha_n\beta_n d^2\big(x_n, T_1^n x_n\big) + \alpha_n\gamma_n d^2\big(x_n, T_2^n z_n\big) + \beta_n\gamma_n d^2\big(T_1^n x_n, T_2^n z_n\big) \\
&\quad \le k_n^2 d^2(x_n,q) - d^2(y_n,q) \to 0 \quad (\text{as } n\to\infty).
\end{aligned}$$

By condition (4) we have

$$\begin{aligned}
&d\big(x_n, T_1^n x_n\big) \to 0, \qquad d\big(x_n, T_2^n z_n\big) \to 0 \quad \text{and} \\
&d\big(T_1^n x_n, T_2^n z_n\big) \to 0 \quad (\text{as } n\to\infty).
\end{aligned} \qquad (3.14)$$

Since

$$\begin{aligned}
d\big(x_n, T_2^n x_n\big) &\le d\big(x_n, T_1^n x_n\big) + d\big(T_1^n x_n, T_2^n z_n\big) + d\big(T_2^n z_n, T_2^n x_n\big) \\
&\le d\big(x_n, T_1^n x_n\big) + d\big(T_1^n x_n, T_2^n z_n\big) + k_n d(z_n, x_n),
\end{aligned}$$

this together with (3.14) and (3.12) shows that

$$\lim_{n\to\infty} d\big(x_n, T_2^n x_n\big) = 0. \qquad (3.15)$$

Also from (3.12), (3.14), (3.15), and (2.9) we have

$$\begin{aligned}
d(y_n, z_n) &\le \alpha_n d(x_n, z_n) + \beta_n d\big(T_1^n x_n, z_n\big) + \gamma_n d\big(T_2^n z_n, z_n\big) \\
&\le \alpha_n d(x_n, z_n) + \beta_n\big\{d\big(T_1^n x_n, x_n\big) + d(x_n, z_n)\big\} \\
&\quad + \gamma_n\big\{d\big(T_2^n z_n, T_2^n x_n\big) + d\big(T_2^n x_n, x_n\big) + d(x_n, z_n)\big\} \\
&\le \alpha_n d(x_n, z_n) + \beta_n\big\{d\big(T_1^n x_n, x_n\big) + d(x_n, z_n)\big\} \\
&\quad + \gamma_n\big\{k_n d(z_n, x_n) + d\big(T_2^n x_n, x_n\big) + d(x_n, z_n)\big\} \to 0 \quad (\text{as } n\to\infty). \qquad (3.16)
\end{aligned}$$

(IV) *Now we prove that*

$$\lim_{n\to\infty} d\big(x_n, S_i^n x_n\big) = 0, \quad i = 1,2 \quad \textit{and} \quad \lim_{n\to\infty} d(y_n, z_n) = 0.$$

In fact, it follows from (3.3) and (2.9) that

$$\begin{aligned}
d^2(x_{n+1}, q) &\le \delta_n d^2\big(T_2^n x_n, q\big) + \eta_n d^2\big(S_1^n x_n, q\big) + \xi_n d^2\big(S_2^n y_n, q\big) \\
&\quad - \delta_n\eta_n d^2\big(T_2^n x_n, S_1^n x_n\big) - \delta_n\xi_n d^2\big(T_2^n x_n, S_2^n y_n\big) - \eta_n\xi_n d^2\big(S_1^n x_n, S_2^n y_n\big) \\
&\le \delta_n k_n^2 d^2(x_n, q) + \eta_n k_n^2 d^2(x_n, q) + \xi_n k_n^2 d^2(y_n, q)
\end{aligned}$$

$$\begin{aligned}
&- \delta_n\eta_n d^2\big(T_2^n x_n, S_1^n x_n\big) - \delta_n\xi_n d^2\big(T_2^n x_n, S_2^n y_n\big) - \eta_n\xi_n d^2\big(S_1^n x_n, S_2^n y_n\big) \\
&\le k_n^4 d^2(x_n, q) - \delta_n\eta_n d^2\big(T_2^n x_n, S_1^n x_n\big) - \delta_n\xi_n d^2\big(T_2^n x_n, S_2^n y_n\big) \\
&\quad - \eta_n\xi_n d^2\big(S_1^n x_n, S_2^n y_n\big),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\delta_n\eta_n d^2\big(T_2^n x_n, S_1^n x_n\big) + \delta_n\xi_n d^2\big(T_2^n x_n, S_2^n y_n\big) + \eta_n\xi_n d^2\big(S_1^n x_n, S_2^n y_n\big) \\
&\quad \le k_n^4 d^2(x_n, q) - d^2(x_{n+1}, q) \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

This implies that

$$\begin{aligned}
&d^2\big(T_2^n x_n, S_1^n x_n\big) \to 0, \qquad d^2\big(T_2^n x_n, S_2^n y_n\big) \to 0 \quad \text{and} \\
&d^2\big(S_1^n x_n, S_2^n y_n\big) \to 0 \quad (\text{as } n \to \infty).
\end{aligned} \tag{3.17}$$

This together with $\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0$, $\lim_{n\to\infty} d(x_n, z_n) = 0$, $\lim_{n\to\infty} d(y_n, z_n) = 0$ shows that

$$\lim_{n\to\infty} d\big(x_n, S_i^n x_n\big) = 0, \quad i = 1,2, \qquad \lim_{n\to\infty} d\big(x_n, S_2^n y_n\big) = 0. \tag{3.18}$$

By the way, it follows from (3.18) that

$$d(x_{n+1}, x_n) \le \delta_n d\big(T_2^n x_n, x_n\big) + \eta_n d\big(S_1^n x_n, x_n\big) + \xi_n d\big(S_2^n y_n, x_n\big) \to 0 \quad (\text{as } n \to \infty). \tag{3.19}$$

(V) *Next we prove that*

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \qquad \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i = 1,2. \tag{3.20}$$

In fact, it follows from (3.14), (3.15), and (3.19) that, for each $i = 1,2$,

$$\begin{aligned}
d(x_n, T_i x_n) &\le d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i^{n+1} x_{n+1}\big) + d\big(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n\big) + d\big(T_i^{n+1} x_n, T_i x_n\big) \\
&\le d(x_n, x_{n+1}) + d\big(x_{n+1}, T_i^{n+1} x_{n+1}\big) + k_{n+1} d(x_{n+1}, x_n) \\
&\quad + k_1 d\big(T_i^n x_n, x_n\big) \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Similarly we can also prove that $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$, $i = 1,2$.

(VI) *Next we prove that*

$$\lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0, \quad \textit{where } \lambda_n \ge \lambda > 0. \tag{3.21}$$

In fact, it follows from (3.12) and Lemma 2.5 that

$$\begin{aligned}
d(J_\lambda x_n, x_n) &\le d(J_\lambda x_n, z_n) + d(z_n, x_n) = d(J_\lambda x_n, J_{\lambda_n} x_n) + d(z_n, x_n) \\
&= d\left(J_\lambda x_n, J_\lambda\left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right)\right) + d(z_n, x_n)
\end{aligned}$$

$$\begin{aligned} &\leq d\left(x_n, \left(1-\frac{\lambda}{\lambda_n}\right)J_{\lambda_n}x_n \oplus \frac{\lambda}{\lambda_n}x_n\right) + d(z_n, x_n) \\ &= \left(1-\frac{\lambda}{\lambda_n}\right)d(x_n, z_n) + d(z_n, x_n) \to 0. \end{aligned} \tag{3.22}$$

(VII) *Next we prove that*

$$w_\triangle(x_n) := \bigcup_{\{u_n\}\subset\{x_n\}} \bigl\{A\bigl(\{u_n\}\bigr)\bigr\} \subset \Omega, \tag{3.23}$$

where $A(\{u_n\})$ *is the asymptotic center of* $\{u_n\}$ (*for the definition of the asymptotic center see, for example*, [15, 16]).

Let $u \in w_\triangle(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Therefore there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{n\to\infty} v_n = v$ for some $v \in C$. In view of (3.14), (3.18), and (3.21)

$$\lim_{n\to\infty} d(v_n, T_i v_n) = 0, \qquad \lim_{n\to\infty} d(v_n, S_i v_n) = 0, \quad i = 1,2 \quad \text{and} \quad \lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0.$$

By Lemma 2.2, $v \in \Omega$. So, by (3.7), the limit $\lim_{n\to\infty} d(x_n, v)$ exists and $u = v$ [12]. This shows that $w_\triangle(x_n) \subset \Omega$.

Finally, we show that the sequence $\{x_n\}$ $\triangle$-converges to a point in Ω. To this end, it suffices to show that $w_\triangle(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in w_\triangle(x_n) \subset \Omega$ and $\{d(x_n, u)\}$ converges, we have $x = u$ [12]. Hence $w_\triangle(x_n) = \{x\}$.

This completes the proof of Theorem 3.1. □

Remark 3.2

1. Theorem 3.1 generalizes the main results in Agarwal *et al.* [11] and Khan-Abbas [13] from one nonexpansive mapping to four asymptotically nonexpansive mappings involving the convex and lower semi-continuous function in CAT(0) spaces.
2. Theorem 3.1 extends the main result in Bačák [6], and the corresponding results in Ariza-Ruiz *et al.* [7] and Cholamjiak *et al.* [10]. In fact, we present a new modified proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of asymptotically nonexpansive mappings in CAT(0) spaces.

Since every real Hilbert space H is a complete CAT(0) space, the following result can be obtained from Theorem 3.1 immediately.

Corollary 3.3 *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let T_1, T_2, S_1, S_2, $\{k_n\}$, f, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\eta_n\}$, $\{\xi_n\}$, $\{\lambda_n\}$, λ, and Ω satisfy the conditions* (1)-(5) *in Theorem* 3.1. *Let $\{x_n\}$ be the sequence generated in the following manner*:

$$\begin{cases} z_n = \arg\min y_{y\in C}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ y_n = \alpha_n x_n + \beta_n T_1^n x_n + \gamma_n T_2^n z_n, \\ x_{n+1} = \delta_n T_2^n x_n + \eta_n S_1^n x_n + \xi_n S_2^n y_n, \end{cases} \qquad \forall n \geq 1. \tag{3.24}$$

Then the sequence $\{x_n\}$ converges weakly to an element in Ω.

Remark 3.4 Corollary 3.3 is an improvement and generalization of the main result in Agarwal *et al.* [11], Rockafellar [2], and Güler [3].

4 Some strong convergence theorems involving proximal point and common fixed points for asymptotically nonexpansive mappings in CAT(0) spaces

Let (X, d) be a CAT(0) space, and C be a nonempty, closed, and convex subset of X.

Recall that a mapping $T : C \to C$ is said to be *demi-compact*, if for any bounded sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ (as $n \to \infty$), there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly (*i.e.*, in metric topology) to some point $p \in C$.

Theorem 4.1 *Under the assumptions of Theorem* 3.1, *if, in addition, one of* S_1, S_2, T_1, *and* T_2 *is demi-compact, then the sequence* $\{x_n\}$ *defined by* (3.3) *converges strongly* (*i.e., in metric topology*) *to a point* $x^* \in \Omega$.

Proof In fact, it follows from (3.20) and (3.21) that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \qquad \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i = 1, 2, \tag{4.1}$$

and

$$\lim_{n\to\infty} d\big(x_n, J_\lambda(x_n)\big) = 0. \tag{4.2}$$

Again by the assumption that one of S_1, S_2, T_1, and T_2 is demi-compact, without loss of generality, we can assume T_1 is demi-compact, and it follows from (4.1) that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $p \in C$. Since J_λ is nonexpansive, it is demi-closed at 0. Again since S_1, S_2, T_1, and T_2 are asymptotically nonexpansive, by Lemma 2.2, they are also demi-closed at 0. Hence $p \in \Omega$. Again by (3.7) the limit $\lim_{n\to\infty} d(x_n, p)$ exists. Hence we have $\lim_{n\to\infty} d(x_n, p) = 0$.

This completes the proof of Theorem 4.1. □

Theorem 4.2 *Under the assumptions of Theorem* 3.1, *assume, in addition, there exists a nondecreasing function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$, $g(r) > 0$, $\forall r > 0$, *such that*

$$g\big(d(x, \Omega)\big) \le d(x, J_\lambda x) + d(x, S_1 x) + d(x, S_2 x) + d(x, T_1 x) + d(x, T_2 x), \quad \forall x \in C. \tag{4.3}$$

Then the sequence $\{x_n\}$ *defined by* (3.3) *converges strongly* (*i.e., in metric topology*) *to a point* $p^* \in \Omega$.

Proof It follows from (3.20) and (3.21) that for each $i = 1, 2$ and each λ, $0 < \lambda \le \lambda_n$ we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \qquad \lim_{n\to\infty} d(x_n, S_i x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(x_n, J_\lambda(x_n)\big) = 0. \tag{4.4}$$

Therefore we have $\lim_{n\to\infty} g(d(x_n, \Omega)) = 0$. Since g is nondecreasing with $g(0) = 0$ and $g(r) > 0$, $r > 0$, we have

$$\lim_{n\to\infty} d(x_n, \Omega) = 0. \tag{4.5}$$

By virtue of the definition of $\{x_n\}$ and (4.5), it is easy to prove that $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset in a complete CAT(0) space X, it is complete. Without loss of generality, we can assume that $\{x_n\}$ converges strongly to some point p^*. It is easy to see that $F(J_\lambda)$, $F(T_i)$, and $F(S_i)$, $i = 1, 2$, all are closed subsets in C, so is Ω. Since $\lim_{n\to\infty} d(x_n, \Omega) = 0$, $p^* \in \Omega$. This completes the proof of Theorem 4.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of the present article. They also read and approved the final paper.

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