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Existence theorems for single-valued and set-valued mappings with *w*-distances in metric spaces

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Abstract

In this paper, using the concept of *w*-distances, and we prove existence theorems for single-valued mappings and set-valued mappings in a complete metric space which generalize Takahashi, Wong, and Yao's theorems.

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1 Introduction

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm and let $(\ell^\infty)^*$ be the dual space of ℓ^∞. Let μ be an element of $(\ell^\infty)^*$. We denote by $\mu(f)$ the value of μ at $f = \{x_n\} \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = \{1, 1, 1, \dots\}$. Hasegawa *et al.* [1] obtained the following unique fixed point theorem on a complete metric space.

Theorem 1.1 ([1]) *Let (X, d) be a complete metric space and let S be a mapping of X into itself. Let ℓ^∞ be the Banach space of bounded sequences with the supremum norm. Suppose that there exist a real number r with $0 \le r < 1$ and an element $x \in X$ such that $\{S^n x\}$ is bounded and*

$$\mu_n d\bigl(S^n x, Sy\bigr) \le r\mu_n d\bigl(S^n x, y\bigr), \quad \forall y \in X$$

for some mean μ on l^∞. Then the following hold:

(1) *S has a unique fixed point $u \in X$;*
(2) *for every $z \in X$, the sequence $\{S^n z\}$ converges to u.*

By using the idea of Caristi's fixed point theorem [2], Chuang *et al.* [3] proved a unique fixed point theorem for single-valued mappings which generalizes Theorem 1.1. Furthermore, they obtained an existence theorem for set-valued mappings in a complete metric space. Using these results, Chuang *et al.* [3] obtained new and well-known existence theorems in a complete metric space.

On the other hand, in 1996, Kada *et al.* [4] introduced the concept of w-distances on a metric space.

Let (X,d) be a metric space. A function $p: X \times X \to [0,\infty)$ is said to be a *w-distance* [4] on X if the following are satisfied:

(1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x,y,z \in X$;

(2) for any $x \in X$, $p(x,\cdot): X \to [0,\infty)$ is lower semicontinuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

Using the concept of w-distances, they improved important results in complete metric spaces. For example, they improved Caristi's fixed point theorem [2], Ekeland's variational principle [5] and the nonconvex minimization theorem according to Takahashi [6]. Motivated by Chuang *et al.* [3], Takahashi *et al.* [7] improved their unique fixed point theorem for single-valued mappings by using the concept of w-distances. Furthermore, they extended Chuang *et al.*'s existence theorem [3] for set-valued mappings to w-distances. However, Takahashi *et al.* [7] assumed that w-distances are symmetric.

In this paper, without assuming that w-distances are symmetric, we prove Takahashi *et al.*'s unique fixed point theorems for single-valued mappings and their existence theorem for set-valued mappings in a complete metric space. Using these results, we obtained new and well-known existence theorems in a complete metric space. In particular, using this unique fixed point theorem for single-valued mappings, we obtain a unique fixed point theorem of Caristi's type [2] with lower semicontinuous functions and w-distances. It seems that the proofs are technical and useful.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let X be a metric space with metric d. Then we denote by $W(X)$ the set of all w-distances on X. A w-distance p on X is called *symmetric* if $p(x,y) = p(y,x)$ for all $x,y \in X$. We denote by $W_0(X)$ the set of all symmetric w-distances on X. Note that the metric d is an element of $W_0(X)$. We also know that there are many important examples of w-distances on X; see [4, 8].

The following lemma was proved by Kada *et al.* [4]; see also Shioji *et al.* [9].

Lemma 2.1 ([4]) *Let (X,d) be a complete metric space and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{s_n\}$ and $\{t_n\}$ be sequences in $[0,\infty)$ converging to 0, and let $x,y,z \in X$. Then the following hold*:

(1) *If $p(x_n,y) \leq s_n$ and $p(x_n,z) \leq t_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$*;

(2) *if $p(x_n,y_n) \leq s_n$ and $p(x_n,z) \leq t_n$ for all $n \in \mathbb{N}$, then the sequence $\{y_n\}$ converges to z*;

(3) *if $p(x_n,x_m) \leq s_n$ for all $n,m \in \mathbb{N}$ with $m > n$, then the sequence $\{x_n\}$ is a Cauchy sequence*;

(4) *if $p(y,x_n) \leq s_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Let (X,d) be a metric space and let g be a function of X into $(-\infty,\infty] = \mathbb{R} \cup \{\infty\}$. Then g is *proper* if there exists $x \in X$ such that $g(x) < \infty$. A function g is *lower semicontinuous* if for any $t \in \mathbb{R}$, the set $\{x \in X : g(x) \leq t\}$ is closed. A function g is *bounded below* if there exists $K \in \mathbb{R}$ such that

$$K \leq g(x), \quad \forall x \in X.$$

Kada *et al.* [4] improved Caristi's fixed point theorem [2] as follows; see also [8], Theorem 2.2.8.

Theorem 2.2 ([4]) *Let (X,d) be a complete metric space, $p \in W(X)$, and let $\phi : X \to (\infty,\infty]$ be a proper, bounded below, and lower semicontinuous function. Let $T : X \to X$ be a mapping such that for each $x \in X$,*

$$p(x,Tx) + \phi(Tx) \le \phi(x).$$

Then there exists $z \in X$ such that $Tz = z$ and $p(z,z) = 0$.

A mean μ is called a *Banach limit* on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$ for all $\{x_n\} \in \ell^\infty$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f = \{x_n\} \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = \{x_n\} \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [8].

3 Existence theorems for single-valued mappings

In this section, using means and w-distances, we first prove an existence theorem for mappings in metric spaces which generalizes Takahashi *et al.* [7].

Theorem 3.1 *Let (X,d) be a complete metric space, let $p \in W(X)$ and let $\{x_n\}$ be a sequence in X such that $\{p(x_n,w)\}$ and $\{p(w,x_n)\}$ are bounded for some $w \in X$. Let μ be a mean on ℓ^∞ and let $\phi : X \to (-\infty,\infty]$ be a proper, bounded below, and lower semicontinuous function. Let $S : X \to X$ be a mapping. Suppose that there exist $l,m \in \mathbb{N} \cup \{0\}$ such that*

$$\mu_n p\bigl(x_n,S^l y\bigr) + \mu_n p\bigl(S^m y,x_n\bigr) + \phi(Sy) \le \phi(y) \tag{3.1}$$

for all $y \in X$. Then there exists $x_0 \in X$ such that

(1) *x_0 is a unique fixed point of S in $\{x \in X : \phi(x) < \infty\}$;*
(2) *$x_0 = \lim_{k\to\infty} S^k y$ for all $y \in X$ with $\phi(y) < \infty$;*
(3) *$\phi(x_0) = \inf_{v\in X} \phi(v)$.*

Proof Since $\{p(x_n,w)\}$ is bounded for some $w \in X$, we have, for any $y \in X$, $\{p(x_n,y)\}$ is bounded. In fact, we have, for any $n \in \mathbb{N}$,

$$p(x_n,y) \le p(x_n,w) + p(w,y) \le \sup_{k\in\mathbb{N}} p(x_k,w) + p(w,y).$$

Furthermore, since $\{p(w,x_n)\}$ is bounded, we see that $\{p(z,x_n)\}$ is bounded for all $z \in X$. In fact, we have, for any $n \in \mathbb{N}$,

$$p(z,x_n) \le p(z,w) + p(w,x_n) \le p(z,w) + \sup_{k\in\mathbb{N}} p(w,x_k).$$

We have from (3.1)

$$\mu_n p(x_n, S^l y) + \phi(Sy) \le \phi(y) \quad \text{and} \quad \mu_n p(S^m y, x_n) + \phi(Sy) \le \phi(y) \tag{3.2}$$

for all $y \in X$. For $y \in X$ with $\phi(y) < \infty$, we have from (3.2) $\phi(S^k y) < \infty$ for all $k \in \mathbb{N} \cup \{0\}$ and hence

$$\mu_n p(x_n, S^l S^k y) \le \phi(S^k y) - \phi(S^{k+1} y) \tag{3.3}$$

and

$$\mu_n p(S^m S^k y, x_n) \le \phi(S^k y) - \phi(S^{k+1} y). \tag{3.4}$$

Then we see that $\{\phi(S^k y)\}$ is a decreasing sequence which is bounded below. Hence $\lim_{k\to\infty} \phi(S^k y)$ exists. Put $s = \lim_{k\to\infty} \phi(S^k y)$. Since

$$\mu_n p(x_n, S^{l+k} y) \le \phi(S^k y) - \phi(S^{k+1} y) \le \phi(S^k y) - s$$

and

$$\mu_n p(S^{m+k} y, x_n) \le \phi(S^k y) - \phi(S^{k+1} y) \le \phi(S^k y) - s$$

for all $k \in \mathbb{N}$, we have

$$\limsup_{k\to\infty} \mu_n p(x_n, S^{l+k} y) \le 0 \quad \text{and} \quad \limsup_{k\to\infty} \mu_n p(S^{m+k} y, x_n) \le 0.$$

Then we have

$$\lim_{k\to\infty} \mu_n p(x_n, S^{l+k} y) = 0 \quad \text{and} \quad \lim_{k\to\infty} \mu_n p(S^{m+k} y, x_n) = 0. \tag{3.5}$$

We have, for any $k, n \in \mathbb{N}$,

$$p(S^{l+m+k} y, S^{l+m+k+1} y) \le p(S^{l+m+k} y, x_n) + p(x_n, S^{l+m+k+1} y).$$

Since μ is a mean on ℓ^∞, we have from (3.3) and (3.4), for any $k \in \mathbb{N}$,

$$\begin{aligned} p(S^{l+m+k} y, S^{l+m+k+1} y) &\le \mu_n p(S^{l+m+k} y, x_n) + \mu_n p(x_n, S^{l+m+k+1} y) \\ &\le \phi(S^{l+k} y) - \phi(S^{l+k+1} y) + \phi(S^{m+k+1} y) - \phi(S^{m+k+2} y). \end{aligned} \tag{3.6}$$

We have from (3.6), for any $h, k \in \mathbb{N}$ with $k > h$,

$$\begin{aligned} p(S^{l+m+h} y, S^{l+m+k} y) &\le p(S^{l+m+h} y, S^{l+m+h+1} y) \\ &\quad + p(S^{l+m+h+1} y, S^{l+m+h+2} y) + \cdots + p(S^{l+m+k-1} y, S^{l+m+k} y) \\ &\le \phi(S^{l+h} y) - \phi(S^{l+h+1} y) + \phi(S^{m+h+1} y) - \phi(S^{m+h+2} y) \end{aligned}$$

$$\begin{aligned}
&+\phi\big(S^{l+h+1}y\big)-\phi\big(S^{l+h+2}y\big)+\phi\big(S^{m+h+2}y\big)-\phi\big(S^{m+h+3}y\big)+\cdots\\
&+\phi\big(S^{l+k-1}y\big)-\phi\big(S^{l+k}y\big)+\phi\big(S^{m+k}y\big)-\phi\big(S^{m+k+1}y\big)\\
&=\phi\big(S^{l+h}y\big)-\phi\big(S^{l+k}y\big)+\phi\big(S^{m+h+1}y\big)-\phi\big(S^{m+k+1}y\big)\\
&\le\phi\big(S^{l+h}y\big)-s+\phi\big(S^{m+h+1}y\big)-s\\
&\le\phi\big(S^{l+h}y\big)-s+\phi\big(S^{m+h}y\big)-s\\
&=\alpha_h-s+\beta_h-s,
\end{aligned}\tag{3.7}$$

where $\alpha_h=\phi(S^{l+h}y)$ and $\beta_h=\phi(S^{m+h}y)$. Since $\alpha_h-s+\beta_h-s\to 0$ as $h\to\infty$, we see from Lemma 2.1 that $\{S^{l+m+k}y\}$ is a Cauchy sequence in X. Since X is complete, there exists $y_0\in X$ such that $\lim_{k\to\infty}S^{l+m+k}y=y_0$. We know from the definition of p that, for any $n\in\mathbb{N}$, $y\mapsto p(x_n,y)$ is lower semicontinuous. Using this and following the technique of [7], we have, for any $n\in\mathbb{N}$,

$$p(x_n,y_0)\le\liminf_{k\to\infty}p\big(x_n,S^{l+m+k}y\big)$$

and hence

$$\mu_n p(x_n,y_0)\le\mu_n\Big(\liminf_{k\to\infty}p\big(x_n,S^{l+m+k}y\big)\Big).\tag{3.8}$$

On the other hand, we have from (3.7), for any $h,k,n\in\mathbb{N}$ with $k>h$,

$$p\big(x_n,S^{l+m+k}y\big)\le p\big(x_n,S^{l+m+h}y\big)+p\big(S^{l+m+h}y,S^{l+m+k}y\big)\le p\big(x_n,S^{l+m+h}y\big)+\alpha_h-s+\beta_h-s$$

and hence

$$\limsup_{k\to\infty}p\big(x_n,S^{l+m+k}y\big)\le p\big(x_n,S^{l+m+h}y\big)+\alpha_h-s+\beta_h-s.$$

Applying μ to both sides of the inequality, we have

$$\mu_n\Big(\limsup_{k\to\infty}p\big(x_n,S^{l+m+k}y\big)\Big)\le\mu_n p\big(x_n,S^{l+m+h}y\big)+\alpha_h-s+\beta_h-s.$$

Letting $h\to\infty$, we get from (3.5) that

$$\begin{aligned}
\mu_n\Big(\limsup_{k\to\infty}p\big(x_n,S^{l+m+k}y\big)\Big)&\le\liminf_{h\to\infty}\mu_n p\big(x_n,S^{l+m+h}y\big)+0\\
&=\lim_{h\to\infty}\mu_n p\big(x_n,S^{l+m+h}y\big)\\
&=0.
\end{aligned}\tag{3.9}$$

Then we have from (3.8) and (3.9)

$$\begin{aligned}
\mu_n p(x_n,y_0)&\le\mu_n\Big(\liminf_{k\to\infty}p\big(x_n,S^{l+m+k}y\big)\Big)\\
&\le\mu_n\Big(\limsup_{k\to\infty}p\big(x_n,S^{l+m+k}y\big)\Big)
\end{aligned}$$

$$\begin{aligned} &\le \lim_{k\to\infty} \mu_n p\bigl(x_n, S^{l+m+k}y\bigr) \\ &= 0. \end{aligned} \tag{3.10}$$

This implies that

$$\mu_n p(x_n, y_0) = 0.$$

Similarly, for another $u \in X$ with $\phi(u) < \infty$, there exists $u_0 \in X$ such that $\lim_{k\to\infty} S^{l+m+k}u = u_0$ and $\mu_n p(x_n, u_0) = 0$. We also have, for $k, n \in \mathbb{N}$,

$$p\bigl(S^{l+m+k}y, y_0\bigr) \le p\bigl(S^{l+m+k}y, x_n\bigr) + p(x_n, y_0)$$

and hence

$$\begin{aligned} p\bigl(S^{l+m+k}y, y_0\bigr) &\le \mu_n p\bigl(S^{l+m+k}y, x_n\bigr) + \mu_n p(x_n, y_0) \\ &= \mu_n p\bigl(S^{l+m+k}y, x_n\bigr) + 0 \\ &= \mu_n p\bigl(S^{l+m+k}y, x_n\bigr). \end{aligned} \tag{3.11}$$

Furthermore, we have, for $k, n \in \mathbb{N}$,

$$p\bigl(S^{l+m+k}y, u_0\bigr) \le p\bigl(S^{l+m+k}y, x_n\bigr) + p(x_n, u_0)$$

and hence

$$\begin{aligned} p\bigl(S^{l+m+k}y, u_0\bigr) &\le \mu_n p\bigl(S^{l+m+k}y, x_n\bigr) + \mu_n p(x_n, u_0) \\ &= \mu_n p\bigl(S^{l+m+k}y, x_n\bigr) + 0 \\ &= \mu_n p\bigl(S^{l+m+k}y, x_n\bigr). \end{aligned} \tag{3.12}$$

We know that $\mu_n p(S^{l+m+k}y, x_n) \to 0$ as $k \to \infty$. Thus, we have from (3.11), (3.12), and Lemma 2.1 $y_0 = u_0$. Therefore we have $x_0 = \lim_{k\to\infty} S^k z$ for all $z \in X$ with $\phi(z) < \infty$. Since ϕ is lower semicontinuous and $\lim_{k\to\infty} S^k z = x_0$ for all $z \in X$ with $\phi(z) < \infty$, we have

$$\phi(x_0) \le \liminf_{k\to\infty} \phi\bigl(S^k z\bigr) = \lim_{k\to\infty} \phi\bigl(S^k z\bigr) = \inf_{k\in\mathbb{N}\cup\{0\}} \phi\bigl(S^k z\bigr) \le \phi(z).$$

This implies that

$$\phi(x_0) = \inf_{y\in X} \phi(y). \tag{3.13}$$

We finally prove that x_0 is a unique fixed point of S in $\{x \in X : \phi(x) < \infty\}$. Since, from (3.13),

$$0 \le \mu_n p\bigl(x_n, S^l x_0\bigr) \le \phi(x_0) - \phi(Sx_0) \le 0,$$

we have $\mu_n p(x_n, S^l x_0) = 0$. We also know $\mu_n p(x_n, x_0) = 0$. For $k, n \in \mathbb{N}$, we have

$$p(S^k S^m y, S^l x_0) \le p(S^k S^m y, x_n) + p(x_n, S^l x_0)$$

and

$$p(S^k S^m y, x_0) \le p(S^k S^m y, x_n) + p(x_n, x_0).$$

Then, as in the above argument, we have

$$\begin{aligned} p(S^k S^m y, S^l x_0) &\le \mu_n p(S^k S^m y, x_n) + \mu_n p(x_n, S^l x_0) \\ &= \mu_n p(S^k S^m y, x_n) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} p(S^k S^m y, x_0) &\le \mu_n p(S^k S^m y, x_n) + \mu_n p(x_n, x_0) \\ &= \mu_n p(S^k S^m y, x_n). \end{aligned} \tag{3.15}$$

We also know from (3.5) that $\mu_n p(S^{m+k} y, x_n) \to 0$ as $k \to \infty$. Therefore, from (3.14), (3.15), and Lemma 2.1 $S^l x_0 = x_0$. Using $S^l x_0 = x_0$, we have from (3.13)

$$\begin{aligned} 0 &\le \mu_n p(x_n, Sx_0) = \mu_n p(x_n, S^{l+1} x_0) \\ &\le \phi(Sx_0) - \phi(S^2 x_0) \\ &\le \phi(x_0) - \phi(S^2 x_0) \le 0 \end{aligned}$$

and hence $\mu_n p(x_n, Sx_0) = 0$. Since, for $k, n \in \mathbb{N}$,

$$p(S^k S^m y, Sx_0) \le p(S^k S^m y, x_n) + p(x_n, Sx_0),$$

we have

$$\begin{aligned} p(S^k S^m y, Sx_0) &\le \mu_n p(S^k S^m y, x_n) + \mu_n p(x_n, Sx_0) \\ &= \mu_n p(S^k S^m y, x_n). \end{aligned} \tag{3.16}$$

We have from (3.15), (3.16), and Lemma 2.1 $Sx_0 = x_0$. We show that x_0 is a unique fixed point of S in $\{x \in X : \phi(x) < \infty\}$. Indeed, if z_0 is a fixed point of S with $\phi(z_0) < \infty$, then

$$0 \le \mu_n p(x_n, z_0) = \mu_n p(x_n, S^l z_0) \le \phi(z_0) - \phi(Sz_0) = \phi(z_0) - \phi(z_0) = 0$$

and hence $\mu_n p(x_n, z_0) = 0$. Since, for $k, n \in \mathbb{N}$,

$$p(S^k S^m y, z_0) \le p(S^k S^m y, x_n) + p(x_n, z_0),$$

we have

$$p(S^k S^m y, z_0) \le \mu_n p(S^k S^m y, x_n) + \mu_n p(x_n, z_0) = \mu_n p(S^k S^m y, x_n). \tag{3.17}$$

Since $\mu_n p(S^{m+k}y, x_n) \to 0$ as $k \to \infty$, from (3.15), (3.17), and Lemma 2.1, we have $z_0 = x_0$. Therefore x_0 is a unique fixed point of S in $\{y \in X : \phi(y) < \infty\}$. This completes the proof. □

Using Theorem 3.1, we can obtain the following result proved by Takahashi *et al.* [7].

Theorem 3.2 ([7]) *Let (X, d) be a complete metric space, let $p \in W_0(X)$ and let $\{x_n\}$ be a sequence in X such that $\{p(x_n, x)\}$ is bounded for some $x \in X$. Let μ be a mean on ℓ^∞ and let $\psi : X \to (-\infty, \infty]$ be a proper, bounded below, and lower semicontinuous function. Let $T : X \to X$ be a mapping. Suppose that there exists $m \in \mathbb{N} \cup \{0\}$ such that*

$$\mu_n p\bigl(x_n, T^m y\bigr) + \psi(Ty) \le \psi(y), \quad \forall y \in X. \tag{3.18}$$

Then there exists $\bar{x} \in X$ such that

(a) $\bar{x} = \lim_{k\to\infty} T^k y$ *for all $y \in X$ with $\psi(y) < \infty$;*
(b) $\psi(\bar{x}) = \inf_{u\in X} \psi(u)$;
(c) $\bar{x}$ *is a unique fixed point of T in $\{x \in X : \psi(x) < \infty\}$.*

Proof Since $\{x_n\}$ is a bounded sequence in X such that $\{p(x_n, x)\}$ is bounded for some $x \in X$, we see from $p \in W_0(X)$ that $\{p(x, x_n)\}$ is bounded. Putting $S = T$, $l = m$, and $\phi = 2\psi$ in Theorem 3.1, we have

$$2\mu_n p\bigl(T^m y, x_n\bigr) + 2\psi(Ty) \le 2\psi(y), \quad \forall y \in X$$

and hence

$$\mu_n p\bigl(T^m y, x_n\bigr) + \psi(Ty) \le \psi(y), \quad \forall y \in X.$$

Thus we have the desired result from Theorem 3.1. □

Using Theorem 3.1 and the generalized Caristi's fixed point theorem (Theorem 2.2), we also have a unique fixed point theorem of Caristi's type [2] with lower semicontinuous functions and w-distances.

Theorem 3.3 *Let (X, d) be a complete metric space and let $p \in W(X)$ such that $p(x, x) = 0$ for all $x \in X$. Let $\phi : X \to (-\infty, \infty]$ be a proper, bounded below, and lower semicontinuous function. Let $S : X \to X$ be a mapping. Suppose that there exists $\alpha \in \mathbb{R}$ such that*

$$\alpha\bigl(p(Sx, y) + p(y, Sx)\bigr) + (1 - \alpha)\bigl(p(x, y) + p(y, x)\bigr) + \phi(Sy) \le \phi(y), \quad \forall x, y \in X. \tag{3.19}$$

Then there exists $x_0 \in X$ such that

(1) x_0 *is a unique fixed point of S in $\{x \in X : \phi(x) < \infty\}$;*
(2) $x_0 = \lim_{k\to\infty} S^k y$ *for all $y \in X$ with $\phi(y) < \infty$;*
(3) $\phi(x_0) = \inf_{v\in X} \phi(v)$.

Proof Let us first consider $\alpha > 0$. Putting $y = x$ in (3.19), we have from $p(x, x) = 0$

$$\alpha\bigl(p(Sx, x) + p(x, Sx)\bigr) + \phi(Sx) \le \phi(x), \quad \forall x \in X$$

and hence

$$\alpha p(x,Sx) + \phi(Sx) \le \phi(x), \quad \forall x \in X.$$

By Theorem 2.2, there exists $u_0 \in X$ such that $Su_0 = u_0$. Putting $x = u_0$ in (3.19) again, we have, for any $y \in X$,

$$\alpha\bigl(p(Su_0,y) + p(y,Su_0)\bigr) + (1-\alpha)\bigl(p(u_0,y) + p(y,u_0)\bigr) + \phi(Sy) \le \phi(y).$$

Since $Su_0 = u_0$, we have, for any $y \in X$,

$$p(u_0,y) + p(y,u_0) + \phi(Sy) \le \phi(y).$$

By Theorem 3.1, we see that x_0 is a unique fixed point of S in $\{x \in X : \phi(x) < \infty\}$ such that $\phi(x_0) = \inf_{u\in X}\phi(u)$ and $x_0 = \lim_{k\to\infty} S^k z$ for all $z \in X$ with $\phi(z) < \infty$.

Next let us consider the case of $\alpha = 0$. Then we have

$$p(x,y) + p(y,x) + \phi(Sy) \le \phi(y), \quad \forall x,y \in X. \tag{3.20}$$

Replacing x and y by Sx and x in (3.20), respectively, we have

$$p(Sx,x) + p(x,Sx) + \phi(Sx) \le \phi(x), \quad \forall x \in X$$

and hence

$$p(x,Sx) + \phi(Sx) \le \phi(x), \quad \forall x \in X.$$

We also see from Theorem 2.2 that there exists $u_0 \in X$ such that $Su_0 = u_0$. Putting $x = u_0$ in (3.19), we have also

$$p(u_0,y) + p(y,u_0) + \phi(Sy) \le \phi(y), \quad \forall y \in X.$$

By Theorem 3.1, we see that x_0 is a unique fixed point of S in $\{x \in X : \phi(x) < \infty\}$ such that $\phi(x_0) = \inf_{u\in X}\phi(u)$ and $x_0 = \lim_{k\to\infty} S^k z$ for all $z \in X$ with $\phi(z) < \infty$.

In the case of $\alpha < 0$, we have $1 - \alpha > 0$. Furthermore, replacing y by Sx in (3.19), we have from $p(Sx,Sx) = 0$

$$(1-\alpha)\bigl(p(x,Sx) + p(Sx,x)\bigr) + \phi\bigl(S^2x\bigr) \le \phi(Sx), \quad \forall x \in X \tag{3.21}$$

and hence

$$(1-\alpha)p(x,Sx) + \phi\bigl(S^2x\bigr) \le \phi(Sx), \quad \forall x \in X.$$

Take $x \in X$ with $\phi(x) < \infty$. Then we have, for any $n \in \mathbb{N}$,

$$(1-\alpha)p(x,Sx) + \phi\bigl(S^2x\bigr) \le \phi(Sx),$$

$$(1-\alpha)p(Sx,S^2x)+\phi(S^3x)\le\phi(S^2x),$$

$$\vdots$$

$$(1-\alpha)p(S^{n-1}x,S^nx)+\phi(S^{n+1}x)\le\phi(S^nx).$$

Adding these inequalities, we have

$$(1-\alpha)\{p(x,Sx)+p(Sx,S^2x)+\cdots+p(S^{n-1}x,S^nx)\}\le\phi(Sx)-\phi(S^{n+1}x).$$

Since $\{\phi(S^nx)\}$ is a decreasing sequence and bounded below, we see that there exists $s = \lim_{n\to\infty}\phi(S^nx)$. Thus we have, for any $n\in\mathbb{N}$,

$$\begin{aligned}(1-\alpha)p(x,S^nx)&\le(1-\alpha)\{p(x,Sx)+p(Sx,S^2x)+\cdots+p(S^{n-1}x,S^nx)\}\\&\le\phi(Sx)-\phi(S^{n+1}x)\\&\le\phi(Sx)-s<\infty.\end{aligned}$$

Then $\{p(x,S^nx)\}$ is bounded. Furthermore, from (3.21) we have

$$(1-\alpha)p(Sx,x)+\phi(S^2x)\le\phi(Sx),\quad\forall x\in X.$$

As in the above argument, we have, for any $n\in\mathbb{N}$,

$$(1-\alpha)p(S^nx,x)\le\phi(Sx)-s<\infty.$$

Then $\{p(S^nx,x)\}$ is bounded. Replacing x by S^nx in (3.19), we have, for any $n\in\mathbb{N}$,

$$\begin{aligned}&\alpha\bigl(p(S^{n+1}x,y)+p(y,S^{n+1}x)\bigr)\\&\quad+(1-\alpha)\bigl(p(S^nx,y)+p(y,S^nx)\bigr)+\phi(Sy)\le\phi(y),\quad\forall y\in X.\end{aligned}$$

Applying a Banach limit μ to the both sides of this inequality, we have

$$\begin{aligned}&\alpha\bigl(\mu_np(S^{n+1}x,y)+\mu_np(y,S^{n+1}x)\bigr)\\&\quad+(1-\alpha)\bigl(\mu_np(S^nx,y)+\mu_np(y,S^nx)\bigr)+\phi(Sy)\le\phi(y),\quad\forall y\in X.\end{aligned}$$

Since $\mu_np(S^{n+1}x,y)+\mu_np(y,S^{n+1}x)=\mu_np(S^nx,y)+\mu_np(y,S^nx)$, we get

$$\mu_n\bigl(p(S^nx,y)+p(y,S^nx)\bigr)+\phi(Sy)\le\phi(y),\quad\forall y\in X.\tag{3.22}$$

By Theorem 3.1, S has a unique fixed point x_0 in $\{x\in X:\phi(x)<\infty\}$ such that $\phi(x_0)=\inf_{u\in X}\phi(u)$ and $x_0=\lim_{k\to\infty}S^kz$ for all $z\in X$ with $\phi(z)<\infty$. □

4 Existence theorems for set-valued mappings

Using w-distances, we have the following existence theorem for set-valued mappings in a complete metric space. Let (X,d) be a metric space and let $P(X)$ be the class of all nonempty subsets of X. A mapping of X into $P(X)$ is called a *set-valued mapping*, or a *multi-valued mapping*.

Theorem 4.1 *Let (X,d) be a complete metric space, let $p \in W(X)$, and let $\{x_n\}$ be a sequence in X such that $\{p(x_n,w)\}$ and $\{p(w,x_n)\}$ are bounded for some $w \in X$. Let μ be a mean on ℓ^∞ and let $\phi : X \to (-\infty,\infty]$ be a proper, bounded below, and lower semicontinuous function. Let $S : X \to P(X)$ be a set-valued mapping such that for each $x \in X$, there exists $y \in Sx$ satisfying*

$$\mu_n p(x_n,x) + \mu_n p(x,x_n) + \phi(y) \le \phi(x). \tag{4.1}$$

Then there exists $x_0 \in X$ such that

(1) $x_0 \in Sx_0$;
(2) $\phi(x_0) = \inf_{y\in X} \phi(y)$;
(3) *for any $z \in X$ with $\phi(z) < \infty$, there exists a sequence $\{z_m\} \subset X$ such that $z_{m+1} \in Sz_m$, $m \in \mathbb{N} \cup \{0\}$ and $z_m \to x_0$ as $m \to \infty$.*

Proof For each $z_1 = z \in X$ with $\phi(z) < \infty$, there exists $z_2 \in Sz_1$ such that

$$\mu_n p(x_n,z_1) + \mu_n p(z_1,x_n) \le \phi(z_1) - \phi(z_2).$$

Repeating this process, we get a sequence $\{z_m\}$ in X such that $z_{m+1} \in Sz_m$ and

$$\mu_n p(x_n,z_m) + \mu_n p(z_m,x_n) \le \phi(z_m) - \phi(z_{m+1}) \tag{4.2}$$

for each $m \in \mathbb{N}$. Clearly, $\{\phi(z_m)\}$ is a decreasing sequence which is bounded below. Hence $\lim_{m\to\infty} \phi(z_m)$ exists. Put $s = \lim_{m\to\infty} \phi(z_m)$. We have from (4.2)

$$\lim_{m\to\infty} \mu_n p(x_n,z_m) = 0 \quad \text{and} \quad \lim_{m\to\infty} \mu_n p(z_m,x_n) = 0. \tag{4.3}$$

We have, for any $m,n \in \mathbb{N}$,

$$p(z_m,z_{m+1}) \le p(z_m,x_n) + p(x_n,z_{m+1}).$$

Since μ is a mean on ℓ^∞, we have, for any $m \in \mathbb{N}$,

$$\begin{aligned} p(z_m,z_{m+1}) &\le \mu_n p(z_m,x_n) + \mu_n p(x_n,z_{m+1}) \\ &\le \phi(z_m) - \phi(z_{m+1}) + \phi(z_{m+1}) - \phi(z_{m+2}) \\ &= \phi(z_m) - \phi(z_{m+2}). \end{aligned} \tag{4.4}$$

We have from (4.4), for any $l,m \in \mathbb{N}$ with $m > l$,

$$\begin{aligned} p(z_l,z_m) &\le p(z_l,z_{l+1}) + p(z_{l+1},z_{l+2}) + \cdots + p(z_{m-1},z_m) \\ &\le \phi(z_l) - \phi(z_{l+2}) + \phi(z_{l+1}) - \phi(z_{l+3}) \\ &\quad + \cdots + \phi(z_{m-1}) - \phi(z_{m+1}) \\ &= \phi(z_l) + \phi(z_{l+1}) - \phi(z_m) - \phi(z_{m+1}) \\ &\le \phi(z_l) + \phi(z_{l+1}) - s - s \end{aligned}$$

$$\begin{aligned} &\le \phi(z_l) + \phi(z_l) - s - s \\ &= 2\phi(z_l) - 2s \end{aligned} \tag{4.5}$$

and $2\phi(z_l) - 2s \to 0$ as $l \to \infty$. We see from Lemma 2.1 that $\{z_m\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x_0 \in X$ such that $\lim_{m\to\infty} z_m = x_0$. We know from the definition of p that, for any $n \in \mathbb{N}$, $y \mapsto p(x_n, y)$ is lower semicontinuous. Using this and following the technique of [7], we have, for any $n \in \mathbb{N}$,

$$p(x_n, x_0) \le \liminf_{m\to\infty} p(x_n, z_m)$$

and hence

$$\mu_n p(x_n, x_0) \le \mu_n \Bigl(\liminf_{m\to\infty} p(x_n, z_m)\Bigr). \tag{4.6}$$

On the other hand, we have from (4.5), for any $l, k, n \in \mathbb{N}$ with $m > l$,

$$\begin{aligned} p(x_n, z_m) &\le p(x_n, z_l) + p(z_l, z_m) \\ &\le p(x_n, z_l) + 2\phi(z_l) - 2s \end{aligned}$$

and hence

$$\limsup_{m\to\infty} p(x_n, z_m) \le p(x_n, z_l) + 2\phi(z_l) - 2s.$$

Applying μ to both sides of the inequality, we have

$$\mu_n \Bigl(\limsup_{m\to\infty} p(x_n, z_m)\Bigr) \le \mu_n p(x_n, z_l) + 2\phi(z_l) - 2s.$$

Letting $l \to \infty$, we get

$$\mu_n \Bigl(\limsup_{m\to\infty} p(x_n, z_m)\Bigr) \le \liminf_{l\to\infty} \mu_n p(x_n, z_l). \tag{4.7}$$

We have from (4.3), (4.6), and (4.7)

$$\begin{aligned} \mu_n p(x_n, x_0) &\le \mu_n \Bigl(\liminf_{m\to\infty} p(x_n, z_m)\Bigr) \\ &\le \mu_n \Bigl(\limsup_{m\to\infty} p(x_n, z_m)\Bigr) \\ &\le \liminf_{m\to\infty} \mu_n p(x_n, z_m) \\ &= \lim_{m\to\infty} \mu_n p(x_n, z_m) = 0. \end{aligned} \tag{4.8}$$

This implies that

$$\mu_n p(x_n, x_0) = 0.$$

Doing the same argument as above for each $y_1 = y \in X$ with $\phi(y) < \infty$, we can construct a sequence $\{y_m\}$ in X such that $\{\phi(y_m)\}$ is a decreasing sequence, $\lim_{m\to\infty} y_m = y_0$ for some $y_0 \in X$, and $\mu_n p(x_n, y_0) = 0$. We show that $x_0 = y_0$. We have, for any $m, n \in \mathbb{N}$,

$$p(z_m, x_0) \le p(z_m, x_n) + p(x_n, x_0).$$

Then, we have

$$\begin{aligned} p(z_m, x_0) &\le \mu_n p(z_m, x_n) + \mu_n p(x_n, x_0) \\ &= \mu_n p(z_m, x_n). \end{aligned} \tag{4.9}$$

Furthermore, we have, for any $m, n \in \mathbb{N}$,

$$p(z_m, y_0) \le p(z_m, x_n) + p(x_n, y_0)$$

and hence

$$\begin{aligned} p(z_m, y_0) &\le \mu_n p(z_m, x_n) + \mu_n p(x_n, y_0) \\ &= \mu_n p(z_m, x_n). \end{aligned} \tag{4.10}$$

We know from (4.3) that $\mu_n p(z_m, x_n) \to 0$ as $m \to \infty$. Therefore, from (4.9), (4.10), and Lemma 2.1 $x_0 = y_0$. Since ϕ is lower semicontinuous,

$$\phi(x_0) = \phi(y_0) \le \liminf_{m\to\infty} \phi(y_m) = \lim_{m\to\infty} \phi(y_m) = \inf_{m\in\mathbb{N}} \phi(y_m) \le \phi(y_1).$$

Since y_1 is any point of X with $\phi(y_1) < \infty$, we have

$$\phi(x_0) = \inf_{y\in X} \phi(y). \tag{4.11}$$

Using (4.1), we have $u_0 \in X$ such that $u_0 \in Sx_0$ and

$$\mu_n p(x_n, x_0) + \mu_n p(x_0, x_n) \le \phi(x_0) - \phi(u_0). \tag{4.12}$$

Furthermore, repeating this process, we have $v_0 \in X$ such that $v_0 \in Su_0$ and

$$\mu_n p(x_n, u_0) + \mu_n p(u_0, x_n) \le \phi(u_0) - \phi(v_0).$$

Using (4.11), we have

$$\mu_n p(x_n, u_0) + \mu_n p(u_0, x_n) \le \phi(u_0) - \phi(v_0) \le \phi(u_0) - \phi(x_0). \tag{4.13}$$

Then we have from (4.12) and (4.13)

$$\mu_n p(x_n, u_0) + \mu_n p(u_0, x_n) + \mu_n p(x_n, u_0) + \mu_n p(u_0, x_n) \le 0.$$

This implies that

$$\mu_n p(x_n, u_0) = 0.$$

Since $p(z_m, u_0) \leq p(z_m, x_n) + p(x_n, u_0)$ for $m, n \in \mathbb{N}$, we have

$$\begin{aligned} p(z_m, u_0) &\leq \mu_n p(z_m, x_n) + \mu_n p(x_n, u_0) \\ &= \mu_n p(z_m, x_n). \end{aligned} \tag{4.14}$$

We know from (4.3) that $\mu_n p(z_m, x_n) \to 0$ as $m \to \infty$. Therefore, from (4.9), (4.14), and Lemma 2.1 $x_0 = u_0$. Since $u_0 \in Sx_0$, we have $x_0 \in Sx_0$. This completes the proof. □

Let (X, d) be a metric space. Then $S : X \to P(X)$ is called a *multi-valued weakly Picard operator* [10] if for each $x \in X$ and each $y \in Sx$, there exists a sequence $\{x_n\}$ in X such that

(1) $x_0 = x$, $x_1 = y$;

(2) $x_{n+1} \in Sx_n$, $n \in \mathbb{N} \cup \{0\}$;

(3) $\{x_n\}$ is convergent and its limit is a fixed point of S.

Using Theorem 4.1, we can get the following result proved by Takahashi *et al.* [7].

Theorem 4.2 ([7]) *Let (X, d) be a complete metric space, let $p \in W_0(X)$ and let $\{x_n\}$ be a sequence in X such that $\{p(x_n, x)\}$ is bounded for some $x \in X$. Let μ be a mean on ℓ^∞ and let $\psi : X \to (-\infty, \infty)$ be a bounded below and lower semicontinuous function. Let $T : X \to P(X)$ be a set-valued mapping such that for each $u \in X$, there exists $v \in Tu$ satisfying*

$$\mu_n p(x_n, u) + \psi(v) \leq \psi(u).$$

Then T is a multi-valued weakly Picard operator.

Proof Putting $S = T$ and $\phi = 2\psi$ in Theorem 4.1, we see that, for each $x \in X$, there exists $y \in Tx$ such that

$$2\mu_n p(x_n, x) + 2\psi(y) \leq 2\psi(x)$$

and hence

$$\mu_n p(x_n, x) + \psi(y) \leq \psi(x).$$

For each $x \in X$ and each $y \in Tx$, put $u_0 = x$ and $u_1 = y$. Then we can take $u_2 \in Tu_1$ such that

$$\mu_n p(x_n, u_1) + \psi(u_2) \leq \psi(u_1).$$

Repeating this process, we get a sequence $\{u_m\}$ in X such that $u_{m+1} \in Tu_m$ and

$$\mu_n p(x_n, u_m) \leq \psi(u_m) - \psi(u_{m+1}) \tag{4.15}$$

for each $m \in \mathbb{N} \cup \{0\}$. Thus we have the desired result from Theorem 4.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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