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Common coupled fixed point theorems for θ-ψ-contraction mappings endowed with a directed graph

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Abstract

In this paper, we present some existence and uniqueness results for coupled coincidence point and common fixed point of θ-ψ-contraction mappings in complete metric spaces endowed with a directed graph. Our results generalize the results obtained by Kadelburg *et al.* (Fixed Point Theory Appl. 2015:27, 2015, doi:10.1007/s11590-013-0708-4). We also have an application to some integral system to support the results.

MSC: coupled fixed point; coupled coincidence point; common fixed point; Geraghty-type condition; edge preserving; metric spaces; connected graph; monotone; partially ordered set

1 Introduction and preliminaries

For $F: X \times X \to X$ and $g: X \to X$, a concept of coupled coincidence point $(x,y) \in X \times X$ such that $gx = F(x,y)$ and $gy = F(y,x)$ was first introduced by Lakshimikantham and Ćirić [2]. Their results extended the result in [3, 4]. Also, the existence and uniqueness of a coupled coincidence point for such a mapping that satisfies the mixed monotone property in a partially ordered metric space were studied. Consequently, a number of coupled fixed point and coupled coincidence point results have been shown recently. For example, see [5–17].

Choudhury and Kundu [7] give a notion of compatibility.

Definition 1.1 ([7]) Let (X,d) be a metric space, and let $g: X \to X$ and $F: X \times X \to X$. The mappings g and F are said to be *compatible* if

$$\lim_{n\to\infty} d\bigl(gF(x_n,y_n),F(gx_n,gy_n)\bigr) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\bigl(gF(y_n,x_n),F(gy_n,gx_n)\bigr) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} gy_n$.

Let Θ denote the class of all functions $\theta: [0,\infty) \times [0,\infty) \to [0,1)$ that satisfy the following conditions:

(θ_1) $\theta(s,t) = \theta(t,s)$ for all $s,t \in [0,\infty)$;

(θ_2) for any two sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative real numbers,

$$\theta(s_n, t_n) \to 1 \quad \Rightarrow \quad s_n, t_n \to 0.$$

In 2015, Kadelburg *et al.* [1] used the monotone and g-monotone properties to obtained common coupled fixed point theorems for Geraghty-type contraction with compatibility of F and g.

Let (X, d) be a metric space, Δ be a diagonal of $X \times X$, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is determined by $(V(G), E(G))$. We will use this notation of G throughout this work.

The fixed point theorem using the context of metric spaces endowed with a graph was first studied by Jachymski [18]. The result generalized the Banach contraction principle to mappings on metric spaces with a graph. Since then, many authors studied the problem of existence of fixed points for single-valued mappings and multivalued mappings in several spaces with graphs; see [19–23].

Recently, Chifu and Petrusel [24] give the concept of G-continuity for a mapping $F : X^2 \to X$ and the property A as follows.

Definition 1.2 Let (X, d) be a complete metric space, G be a directed graph, and $F : X^2 \to X$ be a mapping. Then

(i) F is called G-continuous if for all $(x^*, y^*) \in X^2$ and for any sequence $(n_i)_i \in \mathbb{N}$ of positive integers such that $F(x_{n_i}, y_{n_i}) \to x^*$, $F(y_{n_i}, x_{n_i}) \to y^*$ as $i \to \infty$ and $(F(x_{n_i}, y_{n_i}), F(x_{n_i+1}, y_{n_i+1})), (F(y_{n_i}, x_{n_i}), F(y_{n_i+1}, x_{n_i+1})) \in E(G)$, we have that

$$F\big(F(x_{n_i}, y_{n_i}), F(y_{n_i}, x_{n_i})\big) \to F\big(x^*, y^*\big) \quad \text{and}$$
$$F\big(F(y_{n_i}, x_{n_i}), F(x_{n_i}, y_{n_i})\big) \to F\big(y^*, x^*\big) \quad \text{as } i \to \infty;$$

(ii) (X, d, G) has property A if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$.

Their results generalized the result in [4] by using the context of metric spaces endowed with a directed graph.

The aim of this work is to prove some existence and uniqueness results for a coupled coincidence point and a common fixed point of θ-ψ contraction mappings in complete metric spaces endowed with a directed graph. The results generalize the results obtained by Kadelburg *et al.* [1]. An application to some integral system is provided to support the results.

2 Common coupled fixed point

We define the set $\operatorname{CcFix}(F)$ of all coupled coincidence points of mappings $F : X^2 \to X$ and $g : X \to X$ and the set $(X^2)_g^F$ as follows:

$$\operatorname{CcFix}(F) = \big\{(x, y) \in X^2 : F(x, y) = gx \text{ and } F(y, x) = gy\big\}$$

and

$$\big(X^2\big)_g^F = \big\{(x, y) \in X^2 : \big(gx, F(x, y)\big), \big(gy, F(y, x)\big) \in E(G)\big\}.$$

Now, we give some definitions that are useful for our main results.

Definition 2.1 We say that $F : X^2 \to X$ and $g : X \to X$ are G-edge preserving if

$$\big[(gx, gu), (gy, gv) \in E(G)\big] \quad \Rightarrow \quad \big[\big(F(x,y), F(u,v)\big), \big(F(y,x), F(v,u)\big) \in E(G)\big].$$

Definition 2.2 Let (X, d) be a complete metric space, and $E(G)$ be the set of the edges of the graph. We say that $E(G)$ satisfies the transitivity property if and only if, for all $x, y, a \in X$,

$$(x, a), (a, y) \in E(G) \to (x, y) \in E(G).$$

Let Ψ denote the class of all functions $\psi : [0,\infty) \to [0,\infty)$ that satisfy the following conditions:

(ψ_1) ψ is nondecreasing;
(ψ_2) $\psi(s + t) \le \psi(s) + \psi(t)$;
(ψ_3) ψ is continuous;
(ψ_4) $\psi(t) = 0 \Leftrightarrow t = 0$.

Definition 2.3 Let (X, d) be a complete metric space endowed with a directed graph G. The mappings $F : X^2 \to X$ and $g : X \to X$ are called a θ-ψ-contraction if:
(1) F and g is G-edge preserving;
(2) there exist $\theta \in \Theta$ and $\psi \in \Psi$ such that for all $x, y, u, v \in X$ satisfying $(gx, gu), (gy, gv) \in E(G)$,

$$\psi\big(d\big(F(x,y), F(u,v)\big)\big) \le \theta\big(d(gx,gu), d(gy,gv)\big)\psi\big(M(gx,gu,gy,gv)\big), \tag{1}$$

where $M(gx, gu, gy, gv) = \max\{d(gx,gu), d(gy,gv)\}$.

Lemma 2.4 *Let (X, d) be a complete metric space endowed with a directed graph G, and let $F : X^2 \to X$ and $g : X \to X$ be a θ-ψ-contraction. Assume that there exist $x_0, y_0, a_0, b_0 \in X$ and $F(X \times X) \subset g(X)$. Then*:
(i) *There exists sequences $\{x_n\}$, $\{y_n\}$, $\{a_n\}$, $\{b_n\}$ in X for which*

$$\begin{aligned} &gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}),\\ &ga_n = F(a_{n-1}, b_{n-1}) \quad \text{and} \quad gb_n = F(b_{n-1}, a_{n-1}) \quad \text{for } n = 1, 2, \ldots. \end{aligned} \tag{2}$$

(ii) *If (gx_n, ga_n) and $(gy_n, gb_n) \in E(G)$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} M(gx_n, ga_n, gy_n, gb_n) = 0.$$

Proof (i) Let $x_0, y_0, a_0, b_0 \in X$. By the assumption that $F(X \times X) \subset g(X)$ and $F(x_0, y_0)$, $F(y_0, x_0), F(a_0, b_0), F(b_0, a_0) \in X$, it easy to construct sequences $\{x_n\}$, $\{y_n\}$, $\{a_n\}$, and $\{b_n\}$ in X for which

$$\begin{aligned} &gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}),\\ &ga_n = F(a_{n-1}, b_{n-1}) \quad \text{and} \quad gb_n = F(b_{n-1}, a_{n-1}) \quad \text{for } n = 1, 2, \ldots. \end{aligned}$$

(ii) Let (gx_n, ga_n) and $(gy_n, gb_n) \in E(G)$ for all $n \in \mathbb{N}$. Using the θ-ψ-contraction (1) and (2), we obtain that

$$\begin{aligned}\psi\big(d(gx_{n+1}, ga_{n+1})\big) &= \psi\big(d\big(F(x_n, y_n), F(a_n, b_n)\big)\big)\\ &\le \theta\big(d(gx_n, ga_n), d(gy_n, gb_n)\big)\psi\big(M(gx_n, ga_n, gy_n, gb_n)\big)\end{aligned}\tag{3}$$

and

$$\begin{aligned}\psi\big(d(gy_{n+1}, gb_{n+1})\big) &= \psi\big(d\big(F(y_n, x_n), F(b_n, a_n)\big)\big)\\ &\le \theta\big(d(gy_n, gb_n), d(gx_n, ga_n)\big)\psi\big(M(gy_n, gb_n, gx_n, ga_n)\big)\\ &= \theta\big(d(gx_n, ga_n), d(gy_n, gb_n)\big)\psi\big(M(gx_n, ga_n, gy_n, gb_n)\big)\end{aligned}\tag{4}$$

for all $n \in \mathbb{N}$. From (3) and (4) we get

$$\begin{aligned}&\psi\big(M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1})\big)\\ &\quad= \psi\big(\max\big\{d(gx_{n+1}, ga_{n+1}), d(gy_{n+1}, gb_{n+1})\big\}\big)\\ &\quad\le \theta\big(d(gx_n, ga_n), d(gy_n, gb_n)\big)\psi\big(M(gx_n, ga_n, gy_n, gb_n)\big)\\ &\quad< \psi\big(M(gx_n, ga_n, gy_n, gb_n)\big)\end{aligned}\tag{5}$$

for all $n \in \mathbb{N}$, that is,

$$\psi\big(M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1})\big) < \psi\big(M(gx_n, ga_n, gy_n, gb_n)\big).$$

Regarding the properties of ψ, we conclude that

$$M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1}) < M(gx_n, ga_n, gy_n, gb_n).$$

It follows that $d_n := M(gx_n, ga_n, gy_n, gb_n)$ is decreasing. Then $d_n \to d$ as $n \to \infty$ for some $d \ge 0$. We claim that $d = 0$. Suppose not. Using (5), we have

$$\frac{\psi(M(gx_{n+1}, ga_{n+1}, gy_{n+1}, gb_{n+1}))}{\psi(M(gx_n, ga_n, gy_n, gb_n))} \le \theta\big(d(gx_n, ga_n), d(gy_n, gb_n)\big) < 1.$$

Taking the limit as $n \to \infty$, we have

$$\theta\big(d(gx_n, ga_n), d(gy_n, gb_n)\big) \to 1.$$

Since $\theta \in \Theta$,

$$d(gx_n, ga_n) \to 0 \quad \text{and} \quad d(gy_n, gb_n) \to 0$$

as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} M(gx_n, ga_n, gy_n, gb_n) = 0,$$

which is a contradiction. Hence,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} M(gx_n, ga_n, gy_n, gb_n) = 0 \qquad \square$$

Next, we will prove the main result.

Theorem 2.5 *Let (X,d) be a complete metric space endowed with a directed graph G, and let $F: X^2 \to X$ and $g: X \to X$ be a θ-ψ-contraction. Suppose that*:

(i) *g is continuous, and $g(X)$ is closed*;

(ii) *$F(X \times X) \subset g(X)$, and g and F are compatible*;

(iii) *F is G-continuous, or the tripled (X,d,G) has property A*;

(iv) *$E(G)$ satisfies the transitivity property.*

Under these conditions, $\mathrm{CcFix}(F) \neq \emptyset$ *if and only if* $(X^2)_g^F \neq \emptyset$.

Proof Let $\mathrm{CcFix}(F) \neq \emptyset$. Then there exists $(u,v) \in \mathrm{CcFix}(F)$ such that $(gu, F(u,v)) = (gu, gu)$ and $(gv, F(v,u)) = (gv, gv) \in \Delta \subset E(G)$. Thus, $(gu, F(u,v))$ and $(gv, F(v,u)) \in E(G)$. It follows that $(u,v) \in (X^2)_g^F$, so that $(X^2)_g^F \neq \emptyset$.

Now, suppose that $(X^2)_g^F \neq \emptyset$. Let $x_0, y_0 \in X$ be such that $(x_0, y_0) \in (X^2)_g^F$. Then $(gx_0, F(x_0,y_0))$ and $(gy_0, F(y_0,x_0)) \in E(G)$. From Lemma 2.4(i) we have sequences $\{x_n\}$ and $\{y_n\}$ in X for which

$$gx_n = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y_{n-1}, x_{n-1}) \quad \text{for } n = 1, 2, \ldots.$$

Since $(gx_0, F(x_0,y_0)) = (gx_0, gx_1)$ and $(gy_0, F(y_0,x_0)) = (gy_0, gy_1) \in E(G)$ and F and g are G-edge preserving, we have $(F(x_0,y_0), F(x_1,y_1)) = (gx_1, gx_2)$ and $(F(y_0,x_0), F(y_1,x_1)) = (gy_1, gy_2) \in E(G)$. By induction we shall obtain (gx_{n-1}, gx_n) and $(gy_{n-1}, gy_n) \in E(G)$ for each $n \in \mathbb{N}$. By Lemma 2.4(ii) we have

$$d_n := M(gx_{n-1}, gx_n, gy_{n-1}, gy_n) \to 0 \quad \text{as } n \to \infty. \tag{6}$$

Now, we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Applying a similar argument as in the proof of Theorem 3.1 in [1] and using (6), condition (iv), and property of ψ, it follows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. By condition (i) there exist $u, v \in g(X)$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = u \quad \text{and} \quad \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = v.$$

By the compatibility of g and F we have that

$$\lim_{n\to\infty} d\big(gF(x_n,y_n), F(gx_n, gy_n)\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(gF(y_n,x_n), F(gy_n, gx_n)\big) = 0. \tag{7}$$

Now, suppose that (a) F is G-continuous. It is easy to see that

$$d\big(gu, F(gx_n, gy_n)\big) \leq d\big(gu, gF(x_n,y_n)\big) + d\big(gF(x_n,y_n), F(gx_n, gy_n)\big).$$

Taking the limit as $n \to \infty$ and using (7), the continuity of g, and G-continuity of F, we have that $d(gu, F(u,v)) = 0$, that is, $gu = F(u,v)$. Using a similar idea, we also have that $gv = F(v,u)$. Therefore, $\mathrm{CcFix}(F) \neq \emptyset$.

Suppose now that (b) the tripled (X,d,G) with property A. Let $gx = u$ and $gy = v$ for some $x,y \in X$. Then we have (gx_n, gx) and $(gy_n, gy) \in E(G)$ for each $n \in \mathbb{N}$. By (1) we have

$$\begin{aligned}
&\psi\big(d\big(gx,F(x,y)\big)+d\big(gy,F(y,x)\big)\big)\\
&\quad\le \psi\big(d(gx,gx_{n+1})+d\big(gx_{n+1},F(x,y)\big)+d(gy,gy_{n+1})+d\big(gy_{n+1},F(y,x)\big)\big)\\
&\quad\le \psi\big(d\big(F(x_n,y_n),F(x,y)\big)\big)+\psi\big(d\big(F(y_n,x_n),F(y,x)\big)\big)\\
&\qquad+\psi\big(d(gx,gx_{n+1})\big)+\psi\big(d(gy,gy_{n+1})\big)\\
&\quad\le 2\theta\big(d(gx_n,gx),d(gy_n,gy)\big)\psi\big(M(gx_n,gx,gy_n,gy)\big)\\
&\qquad+\psi\big(d(gx,gx_{n+1})\big)+\psi\big(d(gy,gy_{n+1})\big).
\end{aligned}$$

Letting $n \to \infty$, we have $\psi(d(gx,F(x,y)) + d(gy,F(y,x))) = 0$. By properties of ψ, we can see that $d(gx,F(x,y)) + d(gy,F(y,x)) = 0$. Finally, $gx = F(x,y)$ and $gy = F(y,x)$. □

We denote by $\mathrm{CmFix}(F)$ the set of all common fixed points of mappings $F : X^2 \to X$ and $g : X \to X$, that is,

$$\mathrm{CmFix}(F) = \big\{(x,y) \in X^2 : F(x,y) = gx = x \text{ and } F(y,x) = gy = y\big\}.$$

Theorem 2.6 *In addition to hypotheses of Theorem* 2.5, *assume that*

(vi) *for any two elements* $(x,y),(u,v) \in X \times X$, *there exists* $(a,b) \in X \times X$ *such that* $(gx,ga),(gu,ga),(gy,gb),(gv,gb) \in E(G)$.

Then, $\mathrm{CmFix}(F) \neq \emptyset$ *if and only if* $(X^2)^F_g \neq \emptyset$.

Proof Theorem 2.5 implies that there exists $(x,y) \in X \times X$ such that $gx = F(x,y)$ and $gy = F(y,x)$. Suppose that there exists another $(u,v) \in X \times X$ such that $gu = F(u,v)$ and $gv = F(v,u)$. We will show that $gx = gu$ and $gy = gv$.

By condition (vi) there exists $(a,b) \in X \times X$ such that $(gx,ga),(gu,ga),(gy,gb),(gv,gb) \in E(G)$. Set $a_0 = a$, $b_0 = b$, $x_0 = x$, $y_0 = y$, $u_0 = u$, and $v_0 = v$. By Lemma 2.4(i) we have sequences $\{a_n\}$, $\{b_n\}$ $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ in X for which

$$\begin{aligned}
&ga_n = F(a_{n-1},b_{n-1}) \quad\text{and}\quad gb_n = F(b_{n-1},a_{n-1}),\\
&gx_n = F(x_{n-1},y_{n-1}) \quad\text{and}\quad gy_n = F(y_{n-1},x_{n-1}),\\
&gu_n = F(u_{n-1},v_{n-1}) \quad\text{and}\quad gv_n = F(v_{n-1},u_{n-1})
\end{aligned}$$

for $n \in \mathbb{N}$. By the properties of coincidence points, $x_n = x$, $y_n = y$ and $u_n = u$, $v_n = v$, that is,

$$gx_n = F(x,y), \qquad gy_n = F(y,x) \quad\text{and}\quad gu_n = F(u,v), \qquad gv_n = F(v,u) \quad \text{for all } n \in \mathbb{N}.$$

Since $(gx,ga),(gy,gb) \in E(G)$, we have (gx,ga_0) and $(gy,gb_0) \in E(G)$. Because F and g are G-edge preserving, we have $(F(x,y),F(a_0,b_0)) = (gx,ga_1)$ and $(F(y,x),F(b_0,a_0)) = (gy,gb_1) \in E(G)$. Similarly, (gx,ga_n) and $(gy,gb_n) \in E(G)$. By Lemma 2.4(ii) we obtain

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} M(gx,ga_n,gy,gb_n) = 0,$$

and then

$$\lim_{n\to\infty} d(gx, ga_n) = 0 \text{and} \lim_{n\to\infty} d(gy, gb_n) = 0.$$

Similarly, from $(gu, ga), (gv, gb) \in E(G)$ we have

$$\lim_{n\to\infty} d(gu, ga_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(gv, gb_n) = 0.$$

By the triangle inequality we have

$$d(gx, gu) \le d(gx, ga_n) + d(ga_n, gu) \quad \text{and} \quad d(gy, gv) \le d(gy, gb_n) + d(gb_n, gv)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in these two inequalities, we get that $d(gx, gu) = 0$ and $d(gy, gv) = 0$. Therefore, we have $gx = gu$ and $gy = gv$.

The proof of the existence and uniqueness of a common fixed point can be derived using a similar argument as in Theorem 3.7 in [1]. □

Remark 2.1 In the case where $(X, d, \preceq)$ is a partially ordered complete metric space, letting $E(G) = \{(x, y) \in X \times X : x \preceq y\}$ and $\psi(t) = t$, we obtain Theorem 3.1 and Theorem 3.7 in [1].

3 Applications

In this section, we apply our theorem to the existence theorem for a solution of the following integral system:

$$\begin{aligned} x(t) &= \int_0^T f\bigl(t, s, x(s), y(s)\bigr)\, ds + h(t), \\ y(t) &= \int_0^T f\bigl(t, s, y(s), x(s)\bigr)\, ds + h(t), \end{aligned} \tag{8}$$

where $t \in [0, T]$ with $T > 0$.

Let $X := C([0, T], \mathbb{R}^n)$ with $\|x\| = \max_{t\in[0,T]} |x(t)|$, for $x \in C([0, T], \mathbb{R}^n)$.

We define the graph G with partial order relation by

$$x, y \in X, \quad x \le y \quad \Leftrightarrow \quad x(t) \le y(t) \quad \text{for any } t \in [0, T].$$

Thus, $(X, \|\cdot\|)$ is a complete metric space endowed with a directed graph G.

Let $E(G) = \{(x, y) \in X \times X : x \le y\}$. Then $E(G)$ satisfies the transitivity property, and $(X, \|\cdot\|, G)$ has property A.

Theorem 3.1 *Consider system* (8). *Suppose that*

(i) $f : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $h : [0, T] \to \mathbb{R}^n$ *are continuous*;

(ii) *for all* $x, y, u, v \in \mathbb{R}^n$ *with* $x \le u$, $y \le v$, *we have* $f(t, s, x, y) \le f(t, s, u, v)$ *for all* $t, s \in [0, T]$;

(iii) *there exist* $0 \le k < 1$ *and* $T > 0$ *such that*

$$\bigl|f(t, s, x, y) - f(t, s, u, v)\bigr| \le \frac{k}{T}\bigl(|x - u| + |y - v|\bigr)$$

for all $t, s \in [0, T]$, $x, y, u, v \in \mathbb{R}^n$, $x \le u$, $y \le v$;

(iv) *there exists* $(x_0, y_0) \in X \times X$ *such that*

$$x_0(t) \leq \int_0^T f(t, s, x_0(s), y_0(s))\, ds + h(t) \quad \textit{and}$$
$$y_0(t) \leq \int_0^T f(t, s, y_0(s), x_0(s))\, ds + h(t),$$

where $t \in [0, T]$.

Then there exists at least one solution of the integral system (8).

Proof Let $F : X \times X \to X$, $(x, y) \mapsto F(x, y)$, where

$$F(x, y)(t) = \int_0^T f(t, s, x(s), y(s))\, ds + h(t), \quad t \in [0, T],$$

and define $g : X \to X$ by $gx(t) = \frac{x(t)}{2}$.

System (8) can be written as

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Let $x, y, u, v \in X$ be such that $gx \leq gu$ and $gy \leq gv$. We have $x \leq u$, $y \leq v$ and

$$\begin{aligned} F(x, y)(t) &= \int_0^T f(t, s, x(s), y(s))\, ds + h(t) \\ &\leq \int_0^T f(t, s, u(s), v(s))\, ds + h(t) = F(u, v)(t) \quad \text{for all } t \in [0, T] \end{aligned}$$

and

$$\begin{aligned} F(y, x)(t) &= \int_0^T f(t, s, y(s), x(s))\, ds + h(t) \\ &\leq \int_0^T f(t, s, v(s), u(s))\, ds + h(t) = F(v, u)(t) \quad \text{for all } t \in [0, T]. \end{aligned}$$

Thus, F and g are G-edge preserving.

By condition (iv) it follows that $(X^2)_g^F = \{(x, y) \in X \times X : gx \leq F(x, y) \text{ and } gy \leq F(y, x)\} \neq \emptyset$.

On the other hand,

$$\begin{aligned} &\left|F(x, y)(t) - F(u, v)(t)\right| \\ &\quad \leq \int_0^T \left|f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))\right| ds \\ &\quad = \int_0^T \left|f(t, s, x(s), y(s)) - f(t, s, u(s), v(s))\right| ds \\ &\quad \leq \frac{k}{T} \int_0^T \left(\left|x(s) - u(s)\right| + \left|y(s) - v(s)\right|\right) ds \\ &\quad \leq k\left(\frac{\|gx - gu\| + \|gy - gv\|}{2}\right) \\ &\quad \leq kM(gx, gu, gy, gv) \quad \text{for all } t \in [0, T]. \end{aligned}$$

Then, there exist $\psi(t) = t$ and $\theta \in \Theta$, where $\theta(s,t) = k$ for $s,t \in [0,\infty)$ with $k \in [0,1)$, such that

$$\psi\bigl(\bigl\|F(x,y) - F(u,v)\bigr\|\bigr) \le \theta\bigl(\|gx - gu\|, \|gy - gv\|\bigr)\psi\bigl(M(gx,gu,gy,gv)\bigr),$$

where $M(gx,gu,gy,gv) = \max\{\|gx - gu\|, \|gy - gv\|\}$. Hence, F and g are a θ-ψ-contraction.

Thus, there exists a coupled common fixed point $(x^*,y^*) \in X \times X$ of the mapping F and g, which is the solution of the integral system (8). □

Theorem 3.2 *Consider system* (8). *Suppose that*

(i) $f : [0,T] \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $h : [0,T] \to \mathbb{R}^n$ *are continuous*;

(ii) *for all* $x,y,u,v \in \mathbb{R}^n$ *with* $x \le u$, $y \le v$, *we have* $f(t,s,x,y) \le f(t,s,u,v)$ *for all* $t,s \in [0,T]$;

(iii) *for all* $t,s \in [0,T]$, $x,y,u,v \in \mathbb{R}^n$, $x \le u$, $y \le v$,

$$\bigl|f(t,s,x,y) - f(t,s,u,v)\bigr| \le \frac{1}{T}\ln\bigl(1 + \max\bigl\{|x-u|, |y-v|\bigr\}\bigr);$$

(iv) *there exists* $(x_0,y_0) \in X \times X$ *such that*

$$x_0(t) \le \int_0^T f\bigl(t,s,x_0(s),y_0(s)\bigr)\,ds + h(t),$$
$$y_0(t) \le \int_0^T f\bigl(t,s,y_0(s),x_0(s)\bigr)\,ds + h(t),$$

where $t \in [0,T]$.

Then there exists at least one solution of the integral system (8).

Proof Let $F : X \times X \to X$, $(x,y) \mapsto F(x,y)$, where

$$F(x,y)(t) = \int_0^T f\bigl(t,s,x(s),y(s)\bigr)\,ds + h(t), \quad t \in [0,T],$$

and define $g : X \to X$ by $gx(t) = x(t)$. As in Theorem 3.1, we have that F and g are G-edge preserving.

By condition (iv) it follows that $(X^2)_g^F = \{(x,y) \in X \times X : gx \le F(x,y) \text{ and } gy \le F(y,x)\} \ne \emptyset$.

On the other hand,

$$\begin{aligned}
&\bigl|F(x,y)(t) - F(u,v)(t)\bigr| \\
&\quad\le \int_0^T \bigl|f\bigl(t,s,x(s),y(s)\bigr) - f\bigl(t,s,u(s),v(s)\bigr)\bigr|\,ds \\
&\quad= \int_0^T \bigl|f\bigl(t,s,x(s),y(s)\bigr) - f\bigl(t,s,u(s),v(s)\bigr)\bigr|\,ds \\
&\quad\le \frac{1}{T}\int_0^T \ln\bigl(1 + \max\bigl\{\bigl|x(s) - u(s)\bigr|, \bigl|y(s) - v(s)\bigr|\bigr\}\bigr)\,ds \\
&\quad\le \ln\Bigl(1 + \max\Bigl\{\max_{t\in[0,T]}\bigl|x(t) - u(t)\bigr|, \max_{t\in[0,T]}\bigl|y(t) - v(t)\bigr|\Bigr\}\Bigr)
\end{aligned}$$

$$
\begin{aligned}
&\leq \ln\bigl(1 + \max\bigl\{\|x-u\|, \|y-v\|\bigr\}\bigr) \\
&= \ln\bigl(1 + M(gx,gu,gy,gv)\bigr) \quad \text{for all } t \in [0,T],
\end{aligned}
$$

where $M(gx,gu,gy,gv) = \max\{\|gx - gu\|, \|gy - gv\|\}$, which yields

$$
\begin{aligned}
&\ln\bigl(\bigl|F(x,y)(t) - F(u,v)(t)\bigr| + 1\bigr) \\
&\quad \leq \ln\bigl(\ln\bigl(1 + M(gx,gu,gy,gv)\bigr) + 1\bigr) \\
&\quad = \frac{\ln(\ln(1 + M(gx,gu,gy,gv)) + 1)}{\ln(1 + M(gx,gu,gy,gv))} \ln\bigl(1 + M(gx,gu,gy,gv)\bigr).
\end{aligned}
$$

Hence, there exist $\psi(x) = \ln(x+1)$ and $\theta \in \Theta$ defined by

$$
\theta(s,t) = \begin{cases} \frac{\ln(\ln(1+\max\{s,t\}))}{\ln(1+\max\{s,t\})}, & s > 0 \text{ or } t > 0, \\ r \in [0,1), & s = 0, t = 0, \end{cases}
$$

such that

$$
\begin{aligned}
\psi\bigl(d\bigl(F(x,y),F(u,v)\bigr)\bigr) &= \psi\bigl(\bigl\|F(x,y) - F(u,v)\bigr\|\bigr) \\
&\leq \theta\bigl(d(gx,gu), d(gy,gv)\bigr)\psi\bigl(M(gx,gu,gy,gv)\bigr).
\end{aligned}
$$

Hence, we see that F and g are a θ-ψ-contraction. Thus, there exists a coupled common fixed point $(x^*,y^*) \in X \times X$ of the mapping F and g, which is a solution for the integral system (8). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The author read and approved the final manuscript.

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