Fixed Point Theory and Applications
a SpringerOpen Journal

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Generalized proximal-type methods for weak vector variational inequality problems in Banach spaces

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Abstract

In this paper, we propose a class of generalized proximal-type method by the virtue of Bregman functions to solve weak vector variational inequality problems in Banach spaces. We carry out a convergence analysis on the method and prove the weak convergence of the generated sequence to a solution of the weak vector variational inequality problems under some mild conditions. Our results extend some known results to more general cases.

MSC: 90C29; 65K10; 49M05

Keywords: vector variational inequality problems; proximal-type method; Banach spaces; Bregman functions; weak normal mapping; convergence analysis

1 Introduction

The well-known proximal point algorithm (PPA, for short) is a powerful tool for solving optimization problems and variational inequality problems, which was first introduced by Martinet [1] and its celebrated progress was attained in the work of Rockafellar [2]. The classical proximal point algorithm generated a sequence $\{z^k\} \subset H$ with an initial point z^0 through the following iteration:

$$z^{k+1} = (I + c_k T)^{-1} z^k, \tag{1.1}$$

where $\{c_k\}$ is a sequence of positive real numbers bounded away from zero. Rockafellar [2] proved that for a maximal monotone operator T, the sequence $\{z^k\}$ weakly converges to a zero of T under some mild conditions. From then on, many works have been devoted to the investigation of the proximal point algorithms, its applications, and generalizations (see [3–8] and the references therein for scalar-valued optimization problems and variational inequality problems; see [9–14] for vector-valued optimization problems). The notion of a Bregman function has its origin in [15] and the name was first used in optimization problems and its related topics by Censor and Lent in [16]. Bregman function algorithms have been extensively used for various optimization problems and variational inequality problems (*e.g.* [17–20] in finite-dimensional spaces, [21, 22] in infinite-dimensional spaces).

The concept of vector variational inequality was firstly introduced by Giannessi [23] in finite-dimensional spaces. The vector variational inequality problems have found a lot of important applications in multiobjective decision making problems, network equilibrium problems, traffic equilibrium problems, and so on. Because of these significant applications, the study of vector variational inequalities has attracted wide attention. Chen and Yang [24] investigated general vector variational inequality problems and vector complementary problems in infinite-dimensional spaces. Chen [25] considered vector variational inequality problems with a variable ordering structure. Through the last 20 years of development, existence results of solutions, duality theorems, and topological properties of solution sets of several kinds of vector variational inequalities have been derived. One can find a fairly complete review of the main results as regards the vector variational inequalities in the monograph [26]. Very recently, Chen [27] constructed a class of vector-valued proximal-type method for solving a weak vector variational inequality problem in finite-dimensional spaces. They generalized some the classical results of Rockafellar [2] from the scalar-valued case to the vector-valued case.

An important motivation for making an analysis of the convergence properties of both proximal point algorithms and Bregman function algorithms is related to *the mesh independence principle* [28–30]. The mesh independence principle relies on infinite dimensional convergence results for predicting the convergence properties of the discrete finite-dimensional method. Furthermore it can provide a theoretical foundation for the justification of refinement strategies and help to design the refinement process. As we know, many practical problems in economics and engineering are modeled in infinite-dimensional spaces, thus it is necessary to analyze the convergence property of the algorithms in abstract spaces. Motivated by [22, 27], in this paper we propose a class of generalized proximal-type methods by virtue of the Bregman function for solving weak vector variational inequality problems in Banach spaces. We carry out a convergence analysis of the method and prove convergence of the generated sequence to a solution of the weak vector variational inequality problems under some mild conditions.

The paper is organized as follows.

In Section 2, we present some basic concepts, assumptions and preliminary results. In Section 3, we introduce the proximal-type method and carry out convergence analysis on the method. In Section 4, we draw a conclusion and make some remarks.

2 Preliminaries

In this section, we present some basic definitions and propositions for the proof of our main results.

Let X be a smooth reflexive Banach space with norm $\| \cdot \|_X$, denote by X^* the dual space of X and Y is also a smooth reflexive Banach space ordered by a nontrivial, closed and convex cone C with nonempty interior $\operatorname{int} C$, which defines a partial order $\leq_C$ in Y, *i.e.*, $y_1 \leq_C y_2$ if and only if $y_2 - y_1 \in C$ for any $y_1, y_2 \in Y$ and for relation $\leq_{\operatorname{int} C}$, given by $y_1 \leq_{\operatorname{int} C} y_2$ if and only if $y_2 - y_1 \in \operatorname{int} C$ for any $y_1, y_2 \in Y$. We denote by Y^* the dual space of Y, by C^* the positive polar cone of C, *i.e.*,

$$C^* = \left\{ z \in Y^* : \langle z, y \rangle \geq 0, \forall z \in C \right\}.$$

We denote $C^{*+} = \{ z \in Y^* : \langle y, z \rangle > 0 \}$ for all $y \in C \backslash \{0\}$.

Lemma 2.1 [31] *Let $e \in \text{int } C$ be fixed and $C^0 = \{z \in C^* \mid \langle z, e \rangle = 1\}$, then C^0 is a weak*-compact subset of C^*.*

Denote by $L(X, Y)$ the set of all continuous linear operators from X to Y. Denote by $\langle \varphi, x \rangle$ the value $\varphi(x)$ of φ at $x \in X$ if $\varphi \in L(X, Y)$. Let $X_0 \subset X$ be nonempty, closed, and convex, consider the weak vector variational inequality problem of finding $\bar{x} \in X_0$ such that

$$(\text{WVVI}) \quad \langle T(\bar{x}), x - \bar{x} \rangle \nleq_{\text{int } C} 0 \quad \forall x \in X_0,$$

where $T : X_0 \to L(X, Y)$ be a mapping. Denote by $\bar{X}$ the solution set of (VVI). Let $\lambda \in C^0$ and $\lambda(T) : X_0 \to X^*$, consider the corresponding scalar-valued variational inequality problem of finding $x^* \in X_0$ such that

$$(\text{VIP}_\lambda) \quad \langle \lambda(T)(x^*), x - x^* \rangle \geq 0 \quad \forall x \in X_0.$$

Denote by $\bar{X}_\lambda$ the solution set of (VIP_λ).

Definition 2.1 [32] Let $X_0 \subset X$ be nonempty, closed, and convex, and $F : X_0 \to X^*$ be a single-valued mapping.
   (i) F is said to be *monotone* on X_0 if, for any $x_1, x_2 \in X_0$, we have

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0.$$

   (ii) F is said to be *pseudomonotone* on X_0 if, for any $x_1, x_2 \in X_0$, we have

$$\langle F(x_2), x_1 - x_2 \rangle \geq 0 \quad \Rightarrow \quad \langle F(x_1), x_1 - x_2 \rangle \geq 0.$$

Definition 2.2 [26] Let $X_0 \subset X$ be nonempty, closed, and convex. $T : X_0 \to L(X, Y)$ is a mapping, which is said to be *C-monotone* on X_0 if, for any $x_1, x_2 \in X_0$, we have

$$\langle T(x_1) - T(x_2), x_1 - x_2 \rangle \geq_C 0.$$

Proposition 2.1 [26] *Let X_0 and T be defined as in Definition 2.2, we have the following statements*:
   (i) *T is C-monotone if and only if, for any $\lambda \in C^0$, the mapping $\lambda(T) : X_0 \to X^*$ is monotone*;
   (ii) *if T is C-monotone, then for any $\lambda \in C^0$, $\lambda(T) : X_0 \to X^*$ is pseudomonotone.*

Definition 2.3 [33] Let $L \subset L(X, Y)$ be a nonempty set. The weak and strong C-polar cones of L are defined, respectively, by

$$L_C^{w_0} := \left\{ x \in X : \langle l, x \rangle \ngeq_C 0, \forall l \in L \right\} \tag{2.1}$$

and

$$L_C^{s_0} := \left\{ x \in X : \langle l, x \rangle \leq_C 0, \forall l \in L \right\}. \tag{2.2}$$

Definition 2.4 [26] Let $K \subset X$ be nonempty, closed, and convex, let $F : K \to Y$ be a vector-valued mapping. A linear operator V is said to be a strong subgradient of F at $\bar{x} \in K$ if

$$F(x) - F(\bar{x}) - \langle V, x - \bar{x} \rangle \geq_C 0 \quad \forall x \in K.$$

A linear operator V is said to be a weak subgradient of F at $\bar{x} \in K$ if

$$F(x) - F(\bar{x}) - \langle V, x - \bar{x} \rangle \nleq_{\mathrm{int}\, C} 0 \quad \forall x \in K.$$

Denote by $\partial_C^w F(\bar{x})$ the set of weak subgradients of F on K at $\bar{x}$.

Let $K \subset X$ be nonempty, closed, and convex. A vector-valued indicator function $\delta(x \mid K)$ of K at x is defined by

$$\delta(x \mid K) = \begin{cases} 0 \in Y, & \text{if } x \in K; \\ +\infty_C, & \text{if } x \notin K. \end{cases}$$

An important and special case in the theory of weak subgradients is the case that $F(x) = \delta(x \mid K)$ becomes a vector-valued indicator function of K. By Definition 2.4, we obtain $V \in \partial_C^w \delta(x^* \mid K)$ if and only if

$$\langle V, x - x^* \rangle \ngeq_{\mathrm{int}\, C} 0 \quad \forall x \in K. \tag{2.3}$$

Definition 2.5 A set $VN_K^w(x^*) \subset L(X, Y)$ is said to be a weak normality operator set to K at x^*, if for every $V \in VN_K^w(x^*)$, the inequality (2.3) holds.

Clearly, $VN_K^w(x^*) = \partial_C^w \delta(x^* \mid K)$. As for the scalar-valued case, from [34] we know that $v^* \in \partial \delta_K(x^*) = N_K(x^*)$ if and only if

$$\langle v^*, x - x^* \rangle \leq 0 \quad \forall x \in K, \tag{2.4}$$

where $\delta_K(x)$ is the scalar-valued indicator function of K. The inequality (2.4) means that v^* is normal to K at x^*.

Definition 2.6 Let $VN_K^w(\cdot) : X \rightrightarrows L(X, Y)$ be a set-valued mapping, which is said to be a weak normal mapping for K, if for any $y \in K$, $V \in VN_K^w(y)$ such that

$$\langle V, x - y \rangle \ngeq_{\mathrm{int}\, C} 0 \quad \forall x \in K. \tag{2.5}$$

$VN_K^s(\cdot)$ is said to be a strong normal mapping for K, if for any $y \in K$, $V \in VN_K^s(y)$ such that

$$\langle V, x - y \rangle \leq_C 0 \quad \forall x \in K. \tag{2.6}$$

As in [35], the normal mapping for K is a set-valued mapping, which is defined, for any $y \in K$, $v \in N_K(y)$, by

$$\langle v, x - y \rangle \leq 0 \quad \forall x \in K.$$

Next we will introduce the definition and some basic results as regards the *maximal monotone mapping.*

Definition 2.7 [32] Let a set-valued map $G : X_0 \subset X \rightrightarrows X^*$ be given, it is said to be *monotone* if

$$\langle z - \bar{z}, w - \bar{w} \rangle \geq 0$$

for all z and $\bar{z}$ in X_0, all w in $G(z)$ and $\bar{w}$ in $G(\bar{z})$. It is said to be *maximal monotone* if, in addition, the graph

$$\text{gph}(G) = \left\{ (z, w) \in X \times X^* \mid w \in G(z) \right\}$$

is not properly contained in the graph of any other monotone operator from X to X^*.

Definition 2.8 [36] An operator $A : X \to X^*$ is said to be regular if for any $u \in R(A)$ and for any $y \in \text{dom}(A)$,

$$\sup_{(z,v \in \text{grh}(A))} \langle v - u, y - z \rangle < \infty. \tag{2.7}$$

Proposition 2.2 [37] *The subdifferential of a proper lower semicontinuous convex function ϕ is a regular operator.*

Lemma 2.2 [35] *Let K be a nonempty, closed, and convex subset of X.*
  (a) *If $T_1 : X \rightrightarrows X^*$ is the normal mapping to K and $T_2 : X \to X^*$ is a single-valued monotone operator such that $\text{int} K \cap \text{dom}(T_2) \neq \emptyset$ and T_2 is continuous on K, then $T_1 + T_2$ is a maximal monotone operator.*
  (b) *If $T_1, T_2 : X \rightrightarrows X^*$ are maximal monotone operators, and $\text{int}(\text{dom} T_1) \cap \text{dom} T_2 \neq \emptyset$, then $T_1 + T_2$ is a maximal monotone operator.*

Lemma 2.3 [22] *Assume that X is a reflexive Banach space and J its normalized duality mapping. Let $T_0, T_1 : X \rightrightarrows X^*$ be maximal monotone operators such that*
  (a) *T_1 is regular.*
  (b) *$\text{dom} T_1 \cap \text{dom} T_0 \neq \emptyset$ and $R(T_1) = X^*$.*
  (c) *$T_1 + T_0$ is maximal monotone.*
Then we have $R(T_1 + T_0) = X^*$.

Proposition 2.3 [38] *Any maximal monotone operator $S : X \rightrightarrows X^*$ is demiclosed, if the conditions below hold:*
  • *If $\{x_k\}$ converges weakly to x and $\{w_k \in S(x_k)\}$ converges strongly to w, then $w \in S(x)$.*
  • *If $\{x_k\}$ converges strongly to x and $\{w_k \in S(x_k)\}$ converges weakly to w, then $w \in S(x)$.*

Definition 2.9 [22] Let X be a smooth reflexive Banach space and $f : X \to R$ a strictly convex, proper, and lower semicontinuous function with closed domain $D := \text{dom}(f)$. Assume from now on that $\text{int} D \neq \emptyset$ and that f is Gâteaux differentiable on $\text{int} D$. The Bregman

distance with respect to f is the function $B_f(z, x) : D \times \operatorname{int} D \to R$ defined by

$$B_f(z, x) := f(z) - f(x) - \langle \nabla f(x), z - x \rangle, \tag{2.8}$$

where $\nabla f(x)$ is the differential of f defined in $\operatorname{int} D$.

Consider the following set of assumptions on f.

A_1: The right level sets of $B_f(y, \cdot)$,

$$S_{y,\alpha} = \{ z \in \operatorname{int} D : B_f(y, z) \le \alpha \}, \tag{2.9}$$

are bounded for all $\alpha \ge 0$ and for all $y \in D$.

A_2: If $\{x^k\} \subset \operatorname{int} K$ and $\{y^k\} \subset \operatorname{int} K$ are bounded sequences such that $\lim_{k \to +\infty} \|x_k - y_k\| = 0$, then

$$\lim_{k \to +\infty} \left(\nabla f(x_k) - \nabla f(y_k) \right) = 0. \tag{2.10}$$

A_3: Assume $\{x^k\} \subset K$ is bounded, $\{y^k\} \subset \operatorname{int} K$ is such that $w\text{-}\lim_{k \to +\infty} y^k = y$. Further assume that $w\text{-}\lim_{k \to +\infty} B_f(x^k, y^k) = 0$, then

$$w\text{-}\lim_{k \to +\infty} x^k = y.$$

A_4: For every $y \in X^*$, there exists $x \in \operatorname{int} D$ such that $\nabla f(x) = y$.

We say that f is said to be a Bregman function if it satisfies A_1-A_3, and f is said to be a coercive Bregman function if it satisfies A_1-A_4. Without loss of generality, we assume $X_0 \subset \operatorname{dom} f$.

Definition 2.10 [39] Let $X_0 \subset X$ be a closed and convex set with $\operatorname{int} X_0 \ne \emptyset$, $f : X_0 \to R$ a convex function which is Gâteaux differentiable in $\operatorname{int} X_0$. We define the modulus of convexity of f, $v_f : \operatorname{int} X_0 \times R_+ \to R_+$ by

$$v_f(z, t) := \inf \{ B_f(x, z) : \|x - z\| = t \}, \tag{2.11}$$

where $B_f(x, z)$ is the Bregman distance with respect to f.

(a) The function f is said to be totally convex in $\operatorname{int} X_0$ if and only if $v_f(z, t) > 0$ for all $z \in \operatorname{int} X_0$ and $t > 0$.

(b) The function f is said to be uniformly totally convex if for any bounded set $K \subset \operatorname{int} X_0$ and $t > 0$, we have

$$\inf_{x \in K} v_f(x, t) > 0. \tag{2.12}$$

Next we will introduce some fundamental definitions of the asymptotic analysis in infinite spaces.

Definition 2.11 [40] Let K be a nonempty set in X. Then the asymptotic cone of the set K, denoted by K^∞, is the set of all vectors $d \in X$ that are weak limits in the direction of

the sequence $\{x_k\} \subset K$, namely

$$K^\infty = \left\{ d \in X \mid \exists t_k \to +\infty, \text{ and } x_k \in K, w\text{-}\lim_{k \to +\infty} \frac{x_k}{t_k} = d \right\}. \tag{2.13}$$

In the case that K is convex and closed, then, for any $x_0 \in K$,

$$K^\infty = \{ d \in X \mid x_0 + td \in K, \forall t > 0 \}. \tag{2.14}$$

3 Main results

Proposition 3.1 *Let $X_0 \subset X$ be nonempty, closed, and convex, and $VN_{X_0}^w(\cdot)$ be a weak normal mapping for X_0. For any $x^* \in X_0$, and $\varphi \in VN_{X_0}^w(x^*)$, there exists a $\lambda \in C^0$ such that $\lambda(\varphi) \in N_{X_0}(x^*)$.*

Proof By the definition of the weak normal mapping, we know that

$$\langle \varphi, x - x^* \rangle \not\geq_{\text{int}\, C} 0 \quad \forall x \in X_0.$$

It follows that

$$\langle \varphi, x - x^* \rangle \in Y \setminus \text{int}\, C \quad \forall x \in X_0,$$

and

$$\langle \varphi, X_0 - x^* \rangle \subset Y \setminus \text{int}\, C.$$

That is,

$$\langle \varphi, X_0 - x^* \rangle \cap \text{int}\, C = \emptyset.$$

By virtue of the convexity of X_0 and the Hahn-Banach theorem, there exists a $\lambda^* \in C^* \setminus \{0\}$ such that

$$\langle \lambda^*(\varphi), x - x^* \rangle \leq 0 \quad \forall x \in X_0.$$

Since $\|\lambda^*\| > 0$, one obtains

$$\left\langle \frac{\lambda^*}{\|\lambda^*\|}(\varphi), x - x^* \right\rangle \leq 0 \quad \forall x \in X_0.$$

Clearly, we have $\frac{\lambda^*}{\|\lambda^*\|} \in C^0$. Without loss of generality, let $\lambda = \frac{\lambda^*}{\|\lambda^*\|}$, one has

$$\langle \lambda(\varphi), x - x^* \rangle \leq 0 \quad \forall x \in X_0.$$

That is, $\lambda(\varphi) \in N_{X_0}(x^*)$. The proof is complete. $\quad\square$

We propose the following exact generalized proximal-type method (GPTM, for short) for solving the problem (WVVI):

Step (1) take $x_0 \in X_0$;

Step (2) given any $x_k \in X_0$, if $x_k \in \bar{X}$, then the algorithm stops; otherwise go to Step (3);

Step (3) take $x_k \notin \bar{X}$; we define x_{k+1} by the following expression:

$$0 \in T(x_{k+1})\lambda_k + VN^w_{X_0}(x_{k+1})\lambda_k + \varepsilon_k\big(\nabla f(x_{k+1}) - \nabla f(x_k)\big), \tag{3.1}$$

where the sequence $\lambda_k \in C^0$, $\varepsilon_k \in (0, \varepsilon]$, $\varepsilon > 0$, and $VN^w_{X_0}(\cdot)$ is the weak normal mapping to X_0, f is a coercive Bregman function; go to Step (2).

Remark 3.1 The algorithm GPTM is actually a kind of exact proximal point algorithm, where the sequence $\{\lambda_k\} \in C^0$ is called a sequence of scalarization parameters, a bounded exogenous sequence of positive real numbers $\{\varepsilon_k\}$ is called a sequence of regularization parameters. For every $x_k \notin \bar{X}$, we try to find a x_{k+1} such that $0 \in X^*$ belongs to the inclusion (3.1).

Next we will show the main results of this paper.

Theorem 3.1 *Let $X_0 \subset X$ be nonempty, closed, and convex, $T : X_0 \to L(X, Y)$ be continuous and C-monotone on X_0, if $\operatorname{dom} T \cap \operatorname{int} X_0 \neq \emptyset$. The sequence $\{x_k\}$ generated by the method (GPTM) is well defined.*

Proof Let $x_0 \in X_0$ be an initial point and suppose that the method (GPTM) reaches Step (k). We then show that the next iterate x_{k+1} does exist. By the assumptions, $T(\cdot)$ is continuous and C-monotone on X_0. By virtue of Proposition 2.1, we see that $\lambda(T)$ is monotone and continuous on X_0 for any $\lambda \in C^0$. From Proposition 3.1, there exists a $\bar{\lambda} \in C^0$ such that the mapping $VN^w_{X_0}(\cdot)\bar{\lambda}$ is a normal mapping on X_0. Thus, by the assumption $\operatorname{dom} T \cap \operatorname{int} X_0 \neq \emptyset$ and the statement (a) of Lemma 2.2, one sees that, for any $x \in X_0$, the mapping $(VN_{X_0}(x) + T(x))\bar{\lambda}$ is maximal monotone. Without loss of generality, let $\lambda_k = \bar{\lambda}$. Once again by the statement (b) of Lemma 2.2, we see that $T(\cdot)\lambda_k + VN^w_{X_0}(\cdot)\lambda_k + \varepsilon_k \nabla f(\cdot)$ is maximal monotone. Taking $\varepsilon_k \nabla f$ as T_1 in Lemma 2.3, it is easy to check that all assumptions are satisfied. It follows that $R(T(\cdot)\lambda_k + VN^w_{X_0}(\cdot)\lambda_k + \varepsilon_k \nabla f(\cdot)) = X^*$. Then we see that, for any given $\varepsilon_k \nabla f(x_k) \in X^*$, there exists a x_{k+1} such that

$$\varepsilon_k \nabla f(x_k) \in T(x_{k+1})\lambda_k + VN^w_{X_0}(x_{k+1})\lambda_k + \varepsilon_k \nabla f(x_{k+1}). \tag{3.2}$$

The proof is complete. □

Theorem 3.2 *Let the same assumptions as those in Theorem 3.1 hold. Further suppose that $X_0^\infty \cap [T(X_0)]_C^{w_0} = \{0\}$ and $\bar{X}$ is nonempty and bounded. Then the sequence $\{x_k\}$ generated by the method (GPTM) is bounded and*

$$\sum_{k=0}^{\infty} B_f(x_{k+1}, x_k) < \infty. \tag{3.3}$$

Proof From the method (GPTM), we know that if the algorithm stops at some iteration, the point x_k will be a constant thereafter. Now we assume that the sequence $\{x_k\}$ will not stop after a finite number of iterations. From Proposition 3.1, we know that there exist

$\lambda_k \in C^0$ and $\varphi_{k+1} \in VN_{X_0}^w(x_{k+1})$ such that $\lambda_k(\varphi_{k+1}) \in N_{X_0}(x_{k+1})$. From the inclusion (3.1), one has

$$0 = T(x_{k+1})\lambda_k + \lambda_k(\varphi_{k+1}) + \varepsilon_k(\nabla f(x_{k+1}) - \nabla f(x_k)).$$

By the fact that $\lambda_k(\varphi_{k+1}) \in N_{X_0}(x_{k+1})$, we obtain

$$\langle \lambda_k(\varphi_{k+1}), x - x_{k+1} \rangle \leq 0 \quad \forall x \in X_0. \tag{3.4}$$

It follows that

$$\langle T(x_{k+1})\lambda_k + \varepsilon_k(\nabla f(x_{k+1}) - \nabla f(x_k)), x - x_{k+1} \rangle \geq 0 \quad \forall x \in X_0. \tag{3.5}$$

On the other hand, from the assumption that $\bar{X}$ is nonempty and bounded, without loss of generality, let $x^* \in \bar{X}$. By virtue of Theorem 3.3 in [41], we know that x^* is also a solution of the problem (VIP_{λ_k}), where

$$(\text{VIP}_{\lambda_k}) \quad \langle T(x^*)\lambda_k, x - x^* \rangle \geq 0 \quad \forall x \in X_0.$$

It follows that

$$\langle T(x^*)\lambda_k, x^* - x_{k+1} \rangle \leq 0.$$

By the C-monotonicity of T, one has

$$\langle T(x_{k+1})\lambda_k, x^* - x_{k+1} \rangle \leq 0. \tag{3.6}$$

Combining (3.5) with (3.6), we obtain

$$\langle \varepsilon_k(\nabla f(x_{k+1}) - \nabla f(x_k)), x^* - x_{k+1} \rangle \geq 0.$$

By Property 2.1 of [22] ('three points property'), we have

$$B_f(x, x_{k+1}) = B_f(x, x_k) - B_f(x_{k+1}, x_k) + \langle \nabla f(x_k) - \nabla f(x_{k+1}), x - x_{k+1} \rangle \tag{3.7}$$

for any $x \in X_0$. Hence from the inclusion (3.1), we know there exists $\lambda_k(\varphi_{k+1}) \in VN_{X_0}(x_{k+1})$ such that

$$\frac{1}{\varepsilon_k}(T(x_{k+1})\lambda_k + \lambda_k(\varphi_{k+1})) = \nabla f(x_k) - \nabla f(x_{k+1}). \tag{3.8}$$

Combining (3.7) with (3.8), we obtain

$$B_f(x, x_{k+1}) = B_f(x, x_k) - B_f(x_{k+1}, x_k) + \frac{1}{\varepsilon_k}(T(x_{k+1})\lambda_k + \lambda_k(\varphi_{k+1})). \tag{3.9}$$

Since $\lambda_k(\varphi_{k+1}) \in N_{X_0}(x_{k+1})$, we have

$$\langle \lambda_k(\varphi_{k+1}), x^* - x_{k+1} \rangle \leq 0. \tag{3.10}$$

It follows that

$$B_f\left(x^*, x_{k+1}\right) \le B_f\left(x^*, x_k\right) - B_f(x_{k+1}, x_k) - \frac{1}{\varepsilon_k}\left\langle T(x_{k+1})\lambda_k, x_{k+1} - x^*\right\rangle. \tag{3.11}$$

Clearly, we have $\frac{1}{\varepsilon_k}\langle T(x_{k+1})\lambda_k, x_{k+1} - x^*\rangle \ge 0$. It follows that

$$B_f\left(x^*, x_{k+1}\right) \le B_f\left(x^*, x_k\right) - B_f(x_{k+1}, x_k). \tag{3.12}$$

Hence we see that the sequence $B_f(x^*, x_k)$ is decreasing as $k \to \infty$. Taking $\alpha = B_f(x^*, x_0)$, we obtain the sequence $\{x_k\} \subset S_{x^*,\alpha}$, which is a bounded set, by the assumption A_1 of the Bregman function.

As for the sequence $B_f(x^*, x_k)$, it converges to a limit l^*. From (3.12), we have

$$B_f(x_{k+1}, x_k) \le B_f\left(x^*, x_k\right) - B_f\left(x^*, x_{k+1}\right). \tag{3.13}$$

Summing up the above inequality, we obtain

$$\sum_{k=0}^{\infty} B_f(x_{k+1}, x_k) \le \sum_{k=0}^{\infty}\left\{B_f\left(x^*, x_k\right) - B_f\left(x^*, x_{k+1}\right)\right\} = B_f\left(x^*, x_0\right) - l^* < \infty. \tag{3.14}$$

The proof is complete. □

Theorem 3.3 *Let the same assumptions as those in Theorem* 3.2 *hold. If we assume that the Bregman function f is uniformly totally convex, then any weak accumulation point of* $\{x_k\}$ *is a solution of the problem* (*WVVI*).

Proof If there exists $k_0 \ge 1$ such that $x_{k_0+p} = x_{k_0}$, $\forall p \ge 1$, then it is clear that x_{k_0} is the unique weak cluster point of $\{x_k\}$ and it is also a solution of problem (WVVI). Suppose that the algorithm does not terminate finitely. Then, by Theorem 3.2, we see that $\{x_k\}$ is bounded and it has some weak cluster points. Next we show that all of weak cluster points are solutions of problem (WVVI). Let $\hat{x}$ be a weak cluster point of $\{x_k\}$ and $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$, which weakly converges to $\hat{x}$. From the inequality (3.3), we know that

$$\lim_{j \to +\infty} B_f(x_{k_j+1}, x_{k_j}) = 0. \tag{3.15}$$

Hence by the assumption A_3 of f, we obtain

$$w\text{-}\lim_{j \to +\infty} x_{k_j+1} = \hat{x}. \tag{3.16}$$

Next we will show that $\lim_{j \to +\infty}\|x_{k_j+1} - x_{k_j}\| = 0$. Suppose the contrary, without loss of generality we assume there exists a $\beta > 0$ such that

$$\|x_{k_j+1} - x_{k_j}\| > \beta, \tag{3.17}$$

for all $j \geq 0$. Let D be a bounded set which contains the sequence $\{x_{k_j}\}$. We have

$$B_f(x_{k_j+1}, x_{k_j}) \geq \inf\{B_f(z, x_{k_j}) : \|x_{k_j+1} - x_{k_j}\| = \|z - x_{k_j}\|\} = \nu_f(x_{k_j}, \|x_{k_j+1} - x_{k_j}\|). \tag{3.18}$$

From the assumption (3.17) and Property 2.2 in [22], we obtain

$$\nu_f(x_{k_j}, \|x_{k_j+1} - x_{k_j}\|) > \nu_f(x_{k_j}, \beta) \geq \inf_{x \in D} \nu_f(x, \beta). \tag{3.19}$$

Since we assume f is uniformly totally convex, it follows that $\inf_{x \in D} \nu_f(x, \beta) > 0$. One has

$$B_f(x_{k_j+1}, x_{k_j}) \geq \inf_{x \in D} \nu_f(x, \beta) > 0. \tag{3.20}$$

Clearly, there is a contradiction between (3.20) and (3.15), hence we have

$$\lim_{j \to +\infty} \|x_{k_j+1} - x_{k_j}\| = 0.$$

By the assumption A_2 of the Bregman function f, one has

$$\lim_{j \to +\infty} \left(\nabla f(x_{k_j+1}) - \nabla f(x_{k_j})\right) = 0. \tag{3.21}$$

By the inclusion (3.1), one sees that there exist $\lambda_{k_j} \in C^0$ and $\varphi_{k_j+1} \in VN_{X_0}^w(x_{k_j+1})$, such that

$$0 = T(x_{k_j+1})\lambda_{k_j} + \lambda_{k_j}(\varphi_{k_j+1}) + \varepsilon_{k_j}\left(\nabla f(x_{k_j+1}) - \nabla f(x_{k_j})\right).$$

It follows that

$$\lim_{j \to +\infty} \frac{1}{\varepsilon_{k_j}}\left\{T(x_{k_j+1})\lambda_{k_j} + \lambda_{k_j}(\varphi_{k_j+1})\right\} = 0. \tag{3.22}$$

$\{\lambda_{k_j}\} \subset C^0$, which is a weak*-compact set. Without loss of generality, we assume the sequence $w^* - \lim_{j \to +\infty} \lambda_{k_j} = \bar{\lambda}$. From Proposition 3.1, for every j we have

$$\lambda_{k_j}(\varphi_{k_j+1}) \in N_{X_0}(x_{k_j+1}).$$

By the demiclosedness of the graph $T + N_{X_0}$ (see Proposition 2.3), we conclude that

$$0 \in T(\bar{x})\bar{\lambda} + VN_{X_0}^w(\bar{x})\bar{\lambda}.$$

Meanwhile, from Proposition 3.1, we know that there exists a $\bar{\omega} \in VN_{X_0}^w(\bar{x})\bar{\lambda}$. By the definition of $N_{X_0}(\bar{x})$, we know that

$$\langle \bar{\omega}, x - \bar{x} \rangle \leq 0 \quad \forall x \in X_0.$$

That is,

$$\langle T(\bar{x})\bar{\lambda}, x - \bar{x} \rangle \geq 0 \quad \forall x \in X_0. \tag{3.23}$$

Thus

$$\langle T(\bar{x}), x - \bar{x} \rangle \notin -\operatorname{int} C \quad \forall x \in X_0. \tag{3.24}$$

We conclude that $\bar{x}$ is a solution of problem (WVVI). The proof is complete. □

Theorem 3.4 *Let the same assumptions as those in Theorem* 3.3 *hold. If* ∇f *is a weak to weak continuous mapping, then the whole sequence* $\{x_k\}$ *converges weakly to a solution of problem* (WVVI).

Proof In this part we will show $\bar{x}$ is the only solution of (WVVI). Suppose the contrary, assume that both $\hat{x}$ and $\bar{x}$ are weak cluster points of $\{x_n\}$

$$w\text{-}\lim_{j \to +\infty} x_{k_j} = \hat{x}, \qquad w\text{-}\lim_{i \to +\infty} x_{k_i} = \bar{x}$$

and $\hat{x}$ and $\bar{x}$ are also solutions of (WVVI). From the proof of Theorem 3.2, we see that the sequences $B_f(\hat{x}, x_k)$ and $B_f(\bar{x}, x_k)$ are bounded and let l_1 and l_2 be their limits, then

$$\lim_{k \to +\infty} \left(B_f(\hat{x}, x_k) - B_f(\bar{x}, x_k) \right) = l_1 - l_2 = f(\hat{x}) - f(\bar{x}) + \lim_{k \to +\infty} \langle \nabla f(x_k), \bar{x} - \hat{x} \rangle. \tag{3.25}$$

Let $\lim_{k \to +\infty} \langle \nabla f(x_k), \bar{x} - \hat{x} \rangle = l$. Taking $k = k_j$ in (3.25) and by virtue of the weak to weak continuity of $\nabla f(\cdot)$ we obtain

$$l = \langle \nabla f(\hat{x}), \bar{x} - \hat{x} \rangle. \tag{3.26}$$

Once again taking $k = k_i$ in (3.25) and by virtue of the weak to weak continuity of $\nabla f(\cdot)$ we obtain

$$l = \langle \nabla f(\bar{x}), \bar{x} - \hat{x} \rangle. \tag{3.27}$$

Combining the equality (3.26) with the equality (3.27), one has

$$\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \bar{x} - \hat{x} \rangle = 0. \tag{3.28}$$

From the strict convexity of f, we have $\bar{x} = \hat{x}$, which means $\bar{x}$ is the only solution of (WVVI). The proof is complete. □

4 Conclusions

In this paper, we proposed a class of generalized proximal-type method by virtue of the Bregman distance functions to solve weak vector variational inequality problems in Banach spaces. We carried out a convergence analysis on the method and proved weak convergence of the generated sequence to a solution of the weak vector variational inequality problems under some mild conditions. Our results extended some well-known results to more general cases.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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