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# Controllability of higher-order fractional damped stochastic systems with distributed delay

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## Abstract

In this paper, the controllability analysis is proposed for both linear and nonlinear higher-order fractional damped stochastic dynamical systems with distributed delay in Hilbert spaces which involve fractional Caputo derivative of different orders. Based on the properties of fractional calculus, the fixed point technique, and the construction of controllability Gramian matrix, we establish the controllability results for the considered systems. Finally, examples are constructed to illustrate the applicability of obtained results.

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**Keywords:** Controllability; Fractional damped systems; Stochastic systems; Distributed delay; Mittag-Leffler function

## 1 Introduction

Controllability theory for linear and nonlinear control systems in finite and infinite dimensional spaces has been well developed, and the particulars can be identified in the works and references [5, 8]. Besides, the deterministic models repeatedly fluctuate based on environmental noise. So, it is essential to move stochastic systems from deterministic systems for having improved performance in the models. Stochastic differential equations act a key role in formulation and analysis of applied sciences and control engineering [14, 24, 35, 38]. The inclusion of random effects in differential equations leads to several distinct classes of stochastic equations, for which the solution processes have differentiable or non-differentiable sample paths. The general theory of stochastic differential equations both finite-dimensional and infinite-dimensional and their techniques can be found in the field of many applied sciences. This theory formed a very active research topic since it provides a natural framework for mathematical modeling of many physical phenomena. Therefore, stochastic differential equations and their controllability study require attention, and some fruitful results can be found in [4, 15, 36].

The fractional-order calculus establishes the branch of mathematics dealing with differentiation and integration under an arbitrary order of the operation and has been widely applied in several fields of applied science and technology [25, 29]. So, researchers have been

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interested in studying the various phenomena in nature from the viewpoint of fractional calculus like control applications [10, 11] and so on. In recent years, research works regarding applications of fractional-order stochastic systems became a major research field for researchers and technologists since the systematic events demonstrated by fractional-order derivatives and integrals take into account the historical properties of events under consideration at each time period and estimate the fluctuations in the systems more suitably than the action of deterministic operators [22, 37].

A nonlinear dynamical system with fractional damping is of great importance in many fields such as engineering and applied sciences [2, 30, 32]. As a replacement for ordinary derivatives, the viscoelastic models containing fractional derivatives are novel research issues. In comparison with a first-order damping model, the fractional derivative damping behavior may better converge to realistic exploratory models. So, the fractional derivative model may specifically prescribe a nonlinear damping behavior than other well-known damping models. On the other hand, when the stochastic effects are applied to a fractional-order system with damping behavior, it is related to the natural situation. Especially, the model under consideration is valuable to illustrate the viscoelastic properties of beams and plates, nonlinear fractional-order oscillators with stochastic noise, and so on [1, 7, 20].

Time delay is an inevitable concept on the study of dynamical systems in the real world [16, 17, 35]. Concerning the controllability of stochastic fractional systems with delays, we point out [9, 12, 33]. The linear delay differential equations and systems with distributed delays were first studied in [26]. However, it should be pointed out that all these results were obtained only for unconstrained admissible controls, finite dimensional state space, and without delays in state or control. So, it is important to analyze the controllability of stochastic systems with distributed delay. Distributed delays have found widespread use in the modeling of aggregative property in large systems. They are often suitable in modeling processes which are irreversible and described by flow rates of objects that transfer at different rates through the given process. The distributed delay model utilized in these kind of applications is generally a time invariant one. Some interesting results on distributed delays can be found in [21, 23, 28] and the references therein. But till now, there have been few results on the problems for fractional system with distributed delays [21, 23]. In recent years, Li *et al.* [19] derived the existence and exact controllability of fractional evolution inclusions with damping in Banach spaces by utilizing an appropriate fixed point theorem. Li *et al.* [18] reported the existence and controllability for nonlinear fractional control systems with damping in Hilbert spaces. Recently, results for controllability of damped fractional differential system with impulses and state delay can be found in [27]. In [6], the authors studied the mild solutions and approximate controllability for fractional neutral differential equations with damping.

However, to the best of our knowledge, the controllability results for higher-order fractional stochastic systems with damping behavior and distributed delay are an untreated topic in the present literature. Herein, we derive the controllability result for a linear fractional stochastic damped system with Caputo fractional derivative by using the controllability Gramian matrix. We then study the controllability results for the nonlinear fractional stochastic damped system by using fixed point theory. This theory owns certain advantages of linearization for the nonlinear functional relating to the state variables. So it is more interesting and essential to study it. It should be noted that stochastic differen-

tial equations present a relationship between probability theory and the well-established fields of ordinary and partial differential equations. Based on this feature the analysis of stochastic controllability is one of the most interesting tools in practice. It is worth noting that the controllability analysis of this type of higher-order fractional damped systems involving stochastic effects with distributed delay has not been studied in the existing results.

This paper is prepared as follows. Section 2 contains definitions and preliminary results. In Sect. 3, a linear higher-order fractional damped stochastic dynamical system with distributed delay is considered and the controllability condition is established using the controllability Gramian matrix which is defined by means of a Mittag-Leffler matrix function. In Sect. 4, the corresponding nonlinear higher-order fractional damped stochastic dynamical system with distributed delay is considered, and the controllability results are examined with the natural assumption that the linear fractional system is controllable. The results are established by using the Banach fixed point theorem. In Sect. 5, examples are provided to verify the theoretical results.

**2 Preliminaries**

Let  $(\Omega, \Gamma, P)$  be a complete probability space with a probability measure  $P$  on  $\Omega$  and a filtration  $\{\Gamma_t | t \in [0, T]\}$  generated by an  $m$ -dimensional Wiener process  $w(s) : 0 \leq s \leq t$ . Let  $L_2(\Omega, \Gamma_t, \mathbb{R}^n)$  denote the Hilbert space of all  $\Gamma_t$ -measurable square integrable random variables with values in  $\mathbb{R}^n$ .  $L_2^\Gamma([0, T], \mathbb{R}^n)$  denotes the Hilbert space of all square integrable and  $\Gamma_t$ -measurable processes with values in  $\mathbb{R}^n$ .  $U_{ad} := L_2^\Gamma([0, T], \mathbb{R}^m)$  is the set of admissible controls. Let  $I(\mathcal{J}, L_2(\Omega, \Gamma_t, P))$  denote the Banach space of continuous maps from  $\mathcal{J}$  into  $L_2(\Omega, \Gamma_t, P)$  satisfying  $\sup_{t \in \mathcal{J}} \mathbb{E} \|z(t)\|^2 \leq \infty$ .  $\mathbb{E}$  denotes the mathematical expectation operator of a stochastic process with respect to the given probability measure  $P$ .

**Definition 2.1** Caputo’s fractional derivative of order  $\eta_1$  ( $0 \leq m_1 \leq \eta_1 < m_1 + 1$ ) for a function  $f : R^+ \rightarrow R$  is defined as

$${}_0^C D_t^{\eta_1} f(t) = \frac{1}{\Gamma(m_1 - \eta_1 + 1)} \int_0^t \frac{f^{(m_1+1)}(\theta)}{(t - \theta)^{\eta_1 - m_1}} d\theta.$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{ {}_0^C D_t^{\eta_1} f(t) \}(\omega) = \omega^{\eta_1} F(\omega) - \sum_{k=0}^{n-1} f^{(k)}(t) \omega^{\eta_1 - 1 - k}.$$

**Definition 2.2** The Mittag-Leffler function  $E_{\eta_1}(z)$  with  $\eta_1 > 0$  is defined by

$$E_{\eta_1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\eta_1 j + 1)}, \quad \eta_1 > 0, z \in \mathbb{C}.$$

The two-parameter Mittag-Leffler function  $E_{\eta_1, \eta_2}(z)$  with  $\eta_1, \eta_2 > 0$  is defined by

$$E_{\eta_1, \eta_2}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\eta_1 j + \eta_2)}, \quad \eta_1 > 0, z \in \mathbb{C}.$$

The Laplace transform of the Mittag-Leffler function  $E_{\eta_1, \eta_2}(z)$  is

$$\mathcal{L}\{t^{\eta_2-1} E_{\eta_1, \eta_2}(\pm at^{\eta_1})\}(\omega) = \frac{\omega^{\eta_1-\eta_2}}{\omega^{\eta_1} \mp a}.$$

For  $\eta_2 = 1$ , we have

$$\mathcal{L}\{E_{\eta_1}(\pm at^{\eta_1})\}(\omega) = \frac{\omega^{\eta_1-1}}{\omega^{\eta_1} \mp a}.$$

**Definition 2.3** Let  $f(t)$  and  $g(t)$  be two functions of  $t$ . The convolution of  $f(t)$  and  $g(t)$  is also a function of  $t$ , denoted by  $(f * g)(t)$  and is defined by the relation

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t - z)g(z) dz.$$

However, if  $f$  and  $g$  are both casual functions, then  $f(t), g(t)$  are written  $f(t)u(t)$  and  $g(t)u(t)$  respectively so that

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t - z)u(t - z)g(z)u(z) dz = \int_0^t f(t - z)g(z) dz$$

because of the properties of the step functions ( $u(t - z) = 0$  if  $z > t$  and  $u(z) = 0$  if  $z < 0$ ).

**Lemma 2.4** (Burkholder–Davis–Gundy’s inequality [24]) *For any  $r \geq 1$  and for an arbitrary  $\mathcal{L}_2^0$ -valued predictable process  $\Psi(t), t \in [0, T]$ , one has*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_0^t \Psi(\omega) d\omega \right|^{2r}\right) \leq C_r \mathbb{E}\left(\int_0^t \|\Psi(\omega)\|_{\mathcal{L}_2^0}^2 d\omega\right)^r, \tag{1}$$

where

$$C_r = (r(2r - 1))^r \left(\frac{2r}{2r - 1}\right)^{2r^2}. \tag{2}$$

Consider a higher-order fractional differential equation of the form

$$\begin{cases} {}_0^C D_t^{\eta_1} z(t) - \mathfrak{B}_0^C D_t^{\eta_2} z(t) = f(t), & t \geq 0, \\ z(0) = z_0, & z'(0) = z_1, \quad \dots, \quad z^{\rho-1} = z_{\rho-1}, \end{cases} \tag{3}$$

with  $\rho - 1 < \eta_1 \leq \rho, \mu - 1 < \eta_2 \leq \mu$ , and  $\mu \leq \rho - 1$ ,  $\mathfrak{B}$  is an  $n \times n$  matrix, and  $f: \mathcal{J} \rightarrow \mathbb{R}^n$  is a continuous function. By using the Laplace transform on both sides of (3), we get

$$\begin{aligned} \omega^{\eta_1} Z(\omega) - \omega^{\eta_1-1} z(0) - \omega^{\eta_1-2} z'(0) - \dots - \omega^{\eta_1-\rho} z^{\rho-1}(0) - \mathfrak{B} \omega^{\eta_2} Z(\omega) \\ + \mathfrak{B} \omega^{\eta_2-1} z(0) + \mathfrak{B} \omega^{\eta_2-2} z'(0) + \dots + \mathfrak{B} \omega^{\eta_2-\mu} z^{\mu-1}(0) = F(\omega), \end{aligned}$$

then

$$Z(\omega) = \frac{\omega^{\eta_1-\eta_2-1}}{\omega^{\eta_1-\eta_2} I - \mathfrak{B}} z_0 + \frac{\omega^{\eta_1-\eta_2-2}}{\omega^{\eta_1-\eta_2} I - \mathfrak{B}} z_1 + \dots + \frac{\omega^{\eta_1-\eta_2-\rho}}{\omega^{\eta_1-\eta_2} I - \mathfrak{B}} z_{\rho-1} - \mathfrak{B} \frac{\omega^{-1}}{\omega^{\eta_1-\eta_2} I - \mathfrak{B}} z_0$$

$$- \mathfrak{B} \frac{\omega^{-2}}{\omega^{\eta_1 - \eta_2} I - \mathfrak{B}} z_1 - \dots - \mathfrak{B} \frac{\omega^{-\mu}}{\omega^{\eta_1 - \eta_2} I - \mathfrak{B}} z_{\mu-1} + F(\omega) \frac{\omega^{-\eta_2}}{\omega^{\eta_1 - \eta_2} I - \mathfrak{B}}.$$

By applying the inverse Laplace transform to both sides, then substituting the Laplace transformation of Mittag-Leffler function and the Laplace convolution operator, we get

$$z(t) = \sum_{r=0}^{\rho-1} z^r(0) t^r E_{\eta_1 - \eta_2, 1+r}(\mathfrak{B} t^{\eta_1 - \eta_2}) - \sum_{r=0}^{\mu-1} z^r(0) t^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r}(\mathfrak{B} t^{\eta_1 - \eta_2}) + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) f(\omega) d\omega. \tag{4}$$

### 3 Controllability result for linear system

Consider the linear fractional-order stochastic system with damping behavior and distributed delay of the form

$${}^C D_t^{\eta_1} z(t) - \mathfrak{B} {}^C D_t^{\eta_2} z(t) = \mathfrak{C} u(t) + \int_{-h}^0 d_\lambda \mathfrak{D}(t, \lambda) u(t + \lambda) + \sigma(t) \frac{dw(t)}{dt}, \quad t \in [0, \mathcal{T}], \tag{5}$$

$$z(0) = z_0, \quad z'(0) = z_1, \quad \dots, \quad z^{\rho-1} = z_{\rho-1}, \tag{6}$$

$$u(t) = \vartheta(t), \quad \lambda \leq t \leq 0, \tag{7}$$

where  $\rho - 1 < \eta_1 \leq \rho$ ,  $\mu - 1 < \eta_2 \leq \mu$ , and  $\mu \leq \rho - 1$ ,  $z \in \mathbb{R}^n$  is a state variable, and the second integral term is in the Lebesgue–Stieltjes sense with respect to  $\lambda$ . Let  $h > 0$  be given.  $u(t) \in \mathbb{R}^m$  is a control input,  $\vartheta \in \mathbb{C}([\lambda, 0], \mathbb{R}^n)$  is the initial control function, where  $\mathbb{C}([\lambda, 0], \mathbb{R}^n)$  denotes the space of all continuous functions mapping the interval  $[\lambda, 0]$  into  $\mathbb{R}^n$ .  $\mathfrak{B} \in \mathbb{R}^{n \times n}$ ,  $\mathfrak{C} \in \mathbb{R}^{n \times m}$  are the known constant matrices,  $\mathfrak{D}(t, \lambda)$  is an  $n \times m$  dimensional matrix continuous in  $t$  for fixed  $\lambda$  and is of bounded variation in  $\lambda$  on  $[-h, 0]$  for each  $t \in \mathcal{J}$  and continuous from left in  $\lambda$  on the interval  $(-h, 0)$ .  $\lambda$  is a negative constant.  $w(t)$  is a given  $m$ -dimensional Wiener process with the filtration  $\mathcal{F}_t$  generated by  $w(\omega)$ ,  $0 \leq \omega \leq t$  and  $\sigma : [0, \mathcal{T}] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  is an appropriate continuous function.

The solution of system (5)–(7) can be expressed in the following form:

$$z(t) = \sum_{r=0}^{\rho-1} z^r(0) t^r E_{\eta_1 - \eta_2, 1+r}(\mathfrak{B} t^{\eta_1 - \eta_2}) - \sum_{r=0}^{\mu-1} z^r(0) t^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r}(\mathfrak{B} t^{\eta_1 - \eta_2}) + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} u(\omega) d\omega + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \left( \int_{-h}^0 d_\lambda \mathfrak{D}(\omega, \lambda) u(\omega + \lambda) \right) d\omega + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \left( \int_0^\eta \sigma(\theta) dw(\theta) \right) d\omega. \tag{8}$$

The fourth term in the right-hand side of (8) contains the values of the control  $u(t)$  for  $t < 0$  as well as for  $t > 0$ . To separate them, the fourth term of (8) must be transformed by changing the order of integration. Using the unsymmetric Fubini theorem, we have

$$z(t) = \sum_{r=0}^{\rho-1} z^r(0) t^r E_{\eta_1 - \eta_2, 1+r}(\mathfrak{B} t^{\eta_1 - \eta_2}) - \sum_{r=0}^{\mu-1} z^r(0) t^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r}(\mathfrak{B} t^{\eta_1 - \eta_2})$$

$$\begin{aligned}
 & + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} u(\omega) d\omega + \int_\lambda^{t+\lambda} \left[ \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1 - 1} \right. \\
 & \times E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\mathfrak{D}_\lambda \Big] d\omega \\
 & + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \left( \int_0^\eta \sigma(\theta) d\omega(\theta) \right) d\omega \\
 = & \sum_{r=0}^{\rho-1} \mathcal{Z}'(0) t^r E_{\eta_1 - \eta_2, 1+r} (\mathfrak{B} t^{\eta_1 - \eta_2}) - \sum_{r=0}^{\mu-1} \mathcal{Z}'(0) t^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r} (\mathfrak{B} t^{\eta_1 - \eta_2}) \\
 & + \int_0^0 \left[ \int_\lambda^0 (t - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - (\omega - \lambda))^{\eta_1 - \eta_2}) \right. \\
 & \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \Big] d\mathfrak{D}_\lambda \\
 & + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \left( \int_0^\eta \sigma(\theta) d\omega(\theta) \right) d\omega \\
 & + \int_0^{t+\lambda} \left[ (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 & + \left. \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right] u(\omega) d\omega \\
 & + \int_{t+\lambda}^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} u(\omega) d\omega. \tag{9}
 \end{aligned}$$

Set up the controllability Gramian matrix  $W$  as

$$\begin{aligned}
 W = & \int_0^{\mathcal{T}+\lambda} \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 & + \left. \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right] \\
 & \times \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 & + \left. \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right]^* d\omega \\
 & + \int_{\mathcal{T}+\lambda}^{\mathcal{T}} \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right] \\
 & \times \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right]^* d\omega. \tag{10}
 \end{aligned}$$

**Definition 3.1** System (5)–(7) is said to be controllable on  $\mathcal{J}$  if, for every  $z_0, z_1, z_2, \dots, z_{\rho-1}, z_{\mathcal{T}} \in \mathbb{R}^n$ , there exists a control  $u(t)$  such that the solution  $z(t)$  satisfies the conditions  $z(0) = z_0, z'(0) = z_1, \dots, z^{\rho-1}(0) = z_{\rho-1}, z(\mathcal{T}) = z_{\mathcal{T}}$ .

**Theorem 3.2** Linear system (5)–(7) is controllable on  $\mathcal{J}$  iff the  $n \times n$  Gramian matrix

$$W = \int_0^{\mathcal{T}+\lambda} \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right.$$

$$\begin{aligned}
 & + \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \Big] \\
 & \times \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 & + \left. \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right]^* d\omega \\
 & + \int_{\mathcal{T} + \lambda}^{\mathcal{T}} [(\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C}] \\
 & \times [(\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C}]^* d\omega \tag{11}
 \end{aligned}$$

is nonsingular.

*Proof* Suppose that  $W$  is nonsingular, then given  $z_0, z_1, \dots, z_{\rho-1}$ , and  $z_{\mathcal{T}}$ , we can choose the input function  $u(t)$  as

$$u(t) = \begin{cases} \mathbb{G}_1^*(\mathcal{T}, t) W^{-1}(\hat{\alpha}), & t \in [0, \mathcal{T} + \lambda], \\ \mathbb{G}_2^*(\mathcal{T}, t) W^{-1}(\hat{\alpha}), & t \in [\mathcal{T} + \lambda, \mathcal{T}], \end{cases} \tag{12}$$

where

$$\begin{aligned}
 \mathbb{G}_1(\mathcal{T}, t) & = \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 & \quad \left. + \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right], \\
 \mathbb{G}_2(\mathcal{T}, t) & = [(\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C}], \\
 \hat{\alpha} & = 1/2 \left[ z_{\mathcal{T}} - \sum_{r=0}^{\rho-1} z^r(0) \mathcal{T}^r E_{\eta_1 - \eta_2, 1+r} (\mathfrak{B} \mathcal{T}^{\eta_1 - \eta_2}) \right. \\
 & \quad + \sum_{r=0}^{\mu-1} z^r(0) \mathcal{T}^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r} (\mathfrak{B} \mathcal{T}^{\eta_1 - \eta_2}) \\
 & \quad - \int_{-h}^0 \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \right. \\
 & \quad \left. \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \right] d\mathfrak{D}_\lambda \\
 & \quad \left. - \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \left( \int_0^{\eta} \sigma(\theta) dw(\theta) \right) d\omega \right], \\
 z(\mathcal{T}) & = \sum_{r=0}^{\rho-1} z^r(0) \mathcal{T}^r E_{\eta_1 - \eta_2, 1+r} (\mathfrak{B} \mathcal{T}^{\eta_1 - \eta_2}) - \sum_{r=0}^{\mu-1} z^r(0) \mathcal{T}^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r} (\mathfrak{B} \mathcal{T}^{\eta_1 - \eta_2}) \\
 & \quad + \int_{-h}^0 \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) \right. \\
 & \quad \left. \times \vartheta(\omega) d\omega \right] d\mathfrak{D}_\lambda
 \end{aligned}$$

$$\begin{aligned}
 & + \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \left( \int_0^{\eta} \sigma(\theta) d\omega(\theta) \right) d\omega \\
 & + \int_0^{\mathcal{T} + \lambda} \mathbb{G}_1(\mathcal{T}, \omega) \mathbb{G}_1^*(\mathcal{T}, \omega) W^{-1} \hat{\alpha} d\omega + \int_{\mathcal{T} + \lambda}^{\mathcal{T}} \mathbb{G}_2(\mathcal{T}, \omega) \mathbb{G}_2^*(\mathcal{T}, \omega) W^{-1} \hat{\alpha} d\omega \\
 & = z_{\mathcal{T}}.
 \end{aligned} \tag{13}$$

Therefore, system (5)–(7) is controllable on  $\mathcal{J}$ .

On the other hand, suppose that system (5)–(7) is controllable, but for the sake of a contradiction, assume that the matrix  $W$  is singular. If  $W$  is singular, then there exists a vector  $y \neq 0$  such that

$$\begin{aligned}
 y^* W y & = y^* \int_0^{\mathcal{T} + \lambda} \mathbb{G}_1(\mathcal{T}, \omega) \mathbb{G}_1^*(\mathcal{T}, \omega) y d\omega + y^* \int_{\mathcal{T} + \lambda}^{\mathcal{T}} \mathbb{G}_2(\mathcal{T}, \omega) \mathbb{G}_2^*(\mathcal{T}, \omega) y d\omega \\
 & = 0.
 \end{aligned}$$

Thus,

$$y^* \mathbb{G}_2(\mathcal{T}, \omega) = 0 \tag{14}$$

and

$$y^* \mathbb{G}_1(\mathcal{T}, \omega) = 0. \tag{15}$$

Using (14) in (15), we get

$$y^* (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) = 0 \tag{16}$$

for  $t \in \mathcal{J}$ .

Consider the initial points  $z_0 = z_1 = \dots = z_{\rho - 1} = 0$  and the final point  $z_{\mathcal{T}} = y$ , so that system (5)–(7) is controllable. There exists a control  $u(t)$  on  $\mathcal{J}$  that steers the response from 0 to  $z_{\mathcal{T}} = y$  at  $t = \mathcal{T}$ ,

$$\begin{aligned}
 z_{\mathcal{T}} = y & = \int_{-h}^0 \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) \right. \\
 & \quad \left. \times \vartheta(\omega) d\omega \right] d\mathfrak{D}_{\lambda} \\
 & + \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \left( \int_0^{\eta} \sigma(\theta) d\omega(\theta) \right) d\omega \\
 & + \int_0^{\mathcal{T} + \lambda} \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 & \quad \left. + \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_{\lambda} \right] u(\omega) d\omega \\
 & + \int_{\mathcal{T} + \lambda}^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} u(\omega) d\omega,
 \end{aligned}$$



thus

$$\begin{aligned}
 y^*y &= \int_{-h}^0 y^* \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \right. \\
 &\quad \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \Big] d\mathfrak{D}_{\lambda} \\
 &\quad + \int_0^{\mathcal{T}} y^*(\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \left( \int_0^{\eta} \sigma(\theta) dw(\theta) \right) d\omega \\
 &\quad + \int_0^{\mathcal{T} + \lambda} y^* \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 &\quad \left. + \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_{\lambda} \right] u(\omega) d\omega \\
 &\quad + \int_{\mathcal{T} + \lambda}^{\mathcal{T}} y^*(\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} u(\omega) d\omega.
 \end{aligned}$$

Then taking into account that

$$\begin{aligned}
 &\int_{-h}^0 y^* \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \right] d\mathfrak{D}_{\lambda} \\
 &\quad + \int_0^{\mathcal{T}} y^*(\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \left( \int_0^{\eta} \sigma(\theta) dw(\theta) \right) d\omega
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^{\mathcal{T} + \lambda} y^* \left[ (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 &\quad \left. + \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_{\lambda} \right] u(\omega) d\omega \\
 &\quad + \int_{\mathcal{T} + \lambda}^{\mathcal{T}} y^*(\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} u(\omega) d\omega \tag{17}
 \end{aligned}$$

tend to zero leads to the conclusion  $y^*y = 0$ . This is a contradiction to  $y \neq 0$ . Thus the matrix  $W$  is nonsingular. □

#### 4 Controllability result for nonlinear system

In this section, we discuss the controllability criteria for the following higher-order fractional damped stochastic system with distributed delay:

$$\begin{aligned}
 {}_0^C D_t^{\eta_1} z(t) - \mathfrak{B}_0^C D_t^{\eta_2} z(t) &= \mathfrak{C}u(t) + \int_{-h}^0 d_{\lambda} \mathfrak{D}(t, \lambda) u(t + \lambda) + \mathfrak{f}(t, z(t)) \\
 &\quad + \sigma(t, z(t)) \frac{dw(t)}{dt}, \quad t \in [0, \mathcal{T}], \tag{18}
 \end{aligned}$$

$$z(0) = z_0, \quad z'(0) = z_1, \quad \dots, \quad z^{\rho - 1} = z_{\rho - 1}, \tag{19}$$

$$u(t) = \vartheta(t), \quad \lambda \leq t \leq 0, \tag{20}$$

where  $\rho - 1 < \eta_1 \leq \rho$ ,  $\mu - 1 < \eta_2 \leq \mu$ , and  $\mu \leq \rho - 1$ ,  $\mathfrak{B}$ ,  $\mathfrak{C}$ , and  $\mathfrak{D}(t, \lambda)$  are the same as defined in the previous section,  $\lambda$  is a negative constant,  $z \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $\mathfrak{f}: \mathcal{J} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,

and  $\sigma : \mathcal{J} \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ .  $w(t)$  is a given  $m$ -dimensional Wiener process with the filtration  $\mathcal{F}_t$  generated by  $w(\omega)$ . Then the solution of system (18)–(20) is given by

$$\begin{aligned}
 z(t) = & \sum_{r=0}^{\rho-1} z^r(0) t^r E_{\eta_1-\eta_2, 1+r}(\mathfrak{B} t^{\eta_1-\eta_2}) - \sum_{r=0}^{\mu-1} z^r(0) t^{\eta_1-\eta_2+r} E_{\eta_1-\eta_2, \eta_1-\eta_2+1+r}(\mathfrak{B} t^{\eta_1-\eta_2}) \\
 & + \int_{-h}^0 \left[ \int_{\lambda}^0 (t - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t - (\omega - \lambda))^{\eta_1-\eta_2}) \right. \\
 & \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \Big] d\mathfrak{D}_{\lambda} \\
 & + \int_0^t (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1-\eta_2}) f(\omega, z(\omega)) d\omega \\
 & + \int_0^t (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1-\eta_2}) \left( \int_0^{\eta} \sigma(\theta) dw(\theta) \right) d\omega \\
 & + \int_0^{t+\lambda} [(t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1-\eta_2}) \mathfrak{C} \\
 & + \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t - (\omega - \lambda))^{\eta_1-\eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_{\lambda}] u(\omega) d\omega \\
 & + \int_{t+\lambda}^t (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1-\eta_2}) \mathfrak{C} u(\omega) d\omega. \tag{21}
 \end{aligned}$$

Fix the control function

$$\begin{aligned}
 u(t) = & \begin{cases} \mathbb{G}_1^*(\mathcal{T}, t) W^{-1}(\hat{y}), & t \in [0, \mathcal{T} + \lambda], \\ \mathbb{G}_2^*(\mathcal{T}, t) W^{-1}(\hat{y}), & t \in [\mathcal{T} + \lambda, \mathcal{T}], \end{cases} \tag{22} \\
 \mathbb{G}_1(\mathcal{T}, t) = & \left[ (\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1-\eta_2}) \mathfrak{C}, \right. \\
 \mathbb{G}_1(\mathcal{T}, t) = & \left. + \int_{-h}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1-\eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_{\lambda} \right], \\
 \mathbb{G}_2(\mathcal{T}, t) = & [(\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1-\eta_2}) \mathfrak{C}], \\
 \hat{y} = & 1/2 \left[ z_{\mathcal{T}} - \sum_{r=0}^{\rho-1} z^r(0) \mathcal{T}^r E_{\eta_1-\eta_2, 1+r}(\mathfrak{B} \mathcal{T}^{\eta_1-\eta_2}) \right. \\
 & + \sum_{r=0}^{\mu-1} z^r(0) \mathcal{T}^{\eta_1-\eta_2+r} E_{\eta_1-\eta_2, \eta_1-\eta_2+1+r}(\mathfrak{B} \mathcal{T}^{\eta_1-\eta_2}) \\
 & - \int_{-h}^0 \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1-\eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) \right. \\
 & \times \vartheta(\omega) d\omega \Big] d\mathfrak{D}_{\lambda} - \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1-\eta_2}) \left( \int_0^{\eta} \sigma(\theta) dw(\theta) \right) d\omega \\
 & \left. - \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1-\eta_2}) f(\omega, z(\omega)) d\omega \right]. \tag{23}
 \end{aligned}$$

We assume the following hypotheses.

(H1) The linear fractional damped stochastic dynamical system with distributed delay (5)–(7) is controllable on  $\mathcal{J}$ .

(H2) The functions  $f$  and  $\sigma$  are continuous and satisfy the usual linear growth condition, that is, there exist positive real constants  $\tilde{N}, \tilde{L}$  such that

$$\|f(t, z)\|^2 \leq \tilde{N}(1 + \|z\|^2), \quad \|\sigma(t, z)\|^2 \leq \tilde{L}(1 + \|z\|^2).$$

(H3) The functions  $f, \sigma$  satisfy the following Lipschitz condition, and for every  $t \geq 0$  and  $z, y \in \mathbb{R}^n$ , there exist positive real constants  $N, L$  such that

$$\|f(t, z) - f(t, y)\|^2 \leq N\|z - y\|^2, \quad \|\sigma(t, z) - \sigma(t, y)\|^2 \leq L\|z - y\|^2.$$

For brevity, let us introduce the following notations:

$$\begin{aligned} u_1 &= \|t^r E_{\eta_1 - \eta_2, 1+r}(\mathfrak{B}t^{\eta_1 - \eta_2})\|^2, \\ u_2 &= \|\mathfrak{B}t^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r}(\mathfrak{B}t^{\eta_1 - \eta_2})\|^2, \\ u &= \|\vartheta(\omega)\|^2, \\ v &= \int_{\lambda}^0 \mathbb{E} \|(t - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\omega\|^2, \\ G &= \int_{-h}^0 v u d\mathfrak{D}_{\lambda}, \quad l = \|W^{-1}\|, \\ u_3 &= \|E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2})\|^2, \\ M &= \left\| \left[ (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \right. \\ &\quad \left. \left. + \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_{\lambda} \right] \right\|^2 \\ \tilde{M} &= \|(t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C}\|^2. \end{aligned} \tag{24}$$

**Theorem 4.1** *Assume that hypotheses (H1)–(H3) hold, then the nonlinear system (18)–(20) is controllable on  $\mathcal{J}$ .*

*Proof* For arbitrary initial data, we can define a nonlinear operator  $\Delta$  from  $I$  to  $I$  as follows:

$$\begin{aligned} (\Delta z)(t) &= \sum_{r=0}^{\rho-1} z^r(0) t^r E_{\eta_1 - \eta_2, 1+r}(\mathfrak{B}t^{\eta_1 - \eta_2}) - \sum_{r=0}^{\mu-1} z^r(0) t^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r}(\mathfrak{B}t^{\eta_1 - \eta_2}) \\ &\quad + \int_{-h}^0 \left[ \int_{\lambda}^0 (t - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - (\omega - \lambda))^{\eta_1 - \eta_2}) \right. \\ &\quad \left. \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \right] d\mathfrak{D}_{\lambda} \\ &\quad + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) f(\omega, z(\omega)) d\omega \\ &\quad + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \left( \int_0^{\eta} \sigma(\theta) dW(\theta) \right) d\omega \end{aligned}$$

$$\begin{aligned}
 & + \int_0^{t+\lambda} \left[ (t-\omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-\omega)^{\eta_1-\eta_2}) \mathfrak{C} \right. \\
 & + \int_{-h}^0 (t-(\omega-\lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-(\omega-\lambda))^{\eta_1-\eta_2}) \\
 & \times \mathfrak{D}(\omega-\lambda, \lambda) d\mathfrak{D}_\lambda \left. \right] u(\omega) d\omega \\
 & + \int_{t+\lambda}^t (t-\omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-\omega)^{\eta_1-\eta_2}) \mathfrak{C} u(\omega) d\omega, \tag{25}
 \end{aligned}$$

where  $u(t)$  is defined by (22).

By Theorem 3.2, control (22) transfers (21) from the initial state  $z_0$  to the final state  $z_T$  provided that the operator  $\Delta$  has a fixed point in  $I$ . So, if the operator  $\Delta$  has a fixed point, then system (18)–(20) is controllable. As mentioned before, to prove the controllability of system (18)–(20), it is enough to show that  $\Delta$  has a fixed point in  $I$ . To do this, we can employ the contraction mapping principle. In the following, we will divide the proof into two steps.

Firstly, we show that  $\Delta$  maps  $I$  into itself. From (25) we have

$$\begin{aligned}
 & \sup_{0 \leq t \leq T} \mathbb{E} \|\Delta z(t)\|^2 \\
 & = 7 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \sum_{r=0}^{\rho-1} z^r(0) t^r E_{\eta_1-\eta_2, 1+r}(\mathfrak{B} t^{\eta_1-\eta_2}) \right\|^2 \\
 & + 7 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \sum_{r=0}^{\mu-1} z^r(0) t^{\eta_1-\eta_2+r} E_{\eta_1-\eta_2, \eta_1-\eta_2+1+r}(\mathfrak{B} t^{\eta_1-\eta_2}) \right\|^2 \\
 & + 7 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_{-h}^0 \left[ \int_\lambda^0 (t-(\omega-\lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-(\omega-\lambda))^{\eta_1-\eta_2}) \right. \right. \\
 & \times \mathfrak{D}(\omega-\lambda, \lambda) \vartheta(\omega) d\omega \left. \right] d\mathfrak{D}_\lambda \left\|^2 \right. \\
 & + 7 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^t (t-\omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-\omega)^{\eta_1-\eta_2}) \mathfrak{f}(\omega, z(\omega)) d\omega \right\|^2 \\
 & + 7 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^t (t-\omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-\omega)^{\eta_1-\eta_2}) \left( \int_0^\eta \sigma(\theta) dW(\theta) \right) d\omega \right\|^2 \\
 & + 7 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_0^{t+\lambda} \left[ (t-\omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-\omega)^{\eta_1-\eta_2}) \mathfrak{C} \right. \right. \\
 & + \int_{-h}^0 (t-(\omega-\lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-(\omega-\lambda))^{\eta_1-\eta_2}) \mathfrak{D}(\omega-\lambda, \lambda) d\mathfrak{D}_\lambda \left. \right] u(\omega) d\omega \left\|^2 \right. \\
 & + 7 \sup_{0 \leq t \leq T} \mathbb{E} \left\| \int_{t+\lambda}^t (t-\omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1}(\mathfrak{B}(t-\omega)^{\eta_1-\eta_2}) \mathfrak{C} u(\omega) d\omega \right\|^2 \\
 & \triangleq \sum_{b=1}^7 \mathcal{R}_b. \tag{26}
 \end{aligned}$$

Using Holder’s inequality, Burkholder–Davis–Gundy’s inequality (here  $C_1 = 4$ ), and (24), we have the following estimates:

$$\mathcal{R}_1 \leq 7 \sum_{r=0}^{\rho-1} \mathbb{E} \left\| z_r t^r E_{\eta_1-\eta_2, 1+r} (\mathfrak{B} t^{\eta_1-\eta_2}) \right\|^2 \leq 7u_1 \sum_{r=0}^{\rho-1} \mathbb{E} \|z_r\|^2, \tag{27}$$

$$\mathcal{R}_2 \leq 7 \sum_{r=0}^{\mu-1} \mathbb{E} \left\| z_r \mathfrak{B} t^{\eta_1-\eta_2+r} E_{\eta_1-\eta_2, \eta_1-\eta_2+1+r} (\mathfrak{B} t^{\eta_1-\eta_2}) \right\|^2 \leq 7u_2 \sum_{r=0}^{\mu-1} \mathbb{E} \|z_r\|^2, \tag{28}$$

$$\begin{aligned} \mathcal{R}_3 \leq 7 \mathbb{E} \left\| \int_{-h}^0 \left[ \int_{\lambda}^0 (t - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - (\omega - \lambda))^{\eta_1-\eta_2}) \right. \right. \\ \left. \left. \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \right] d\mathfrak{D}_\lambda \right\|^2 \leq 7G, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathcal{R}_4 \leq 7 \mathbb{E} \left\| \int_0^t (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - \omega)^{\eta_1-\eta_2}) f(\omega, z(\omega)) d\omega \right\|^2 \\ \leq 7u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \tilde{N} \mathcal{T} \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) d\omega, \end{aligned} \tag{30}$$

$$\begin{aligned} \mathcal{R}_5 \leq 28 \mathbb{E} \left\| \int_0^t (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - \omega)^{\eta_1-\eta_2}) \left( \int_0^\eta \sigma(\theta, z(\theta)) d\omega(\theta) \right) d\omega \right\|^2 \\ \leq 28u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} L_\sigma \tilde{L} \int_0^{\mathcal{T}} \left( \int_0^\eta (1 + \mathbb{E} \|z(\theta)\|^2) d\theta \right) d\omega, \end{aligned} \tag{31}$$

$$\begin{aligned} \mathcal{R}_6 \leq 7 \mathbb{E} \left\| \int_0^{t+\lambda} \left[ (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - \omega)^{\eta_1-\eta_2}) \mathfrak{C} \right. \right. \\ \left. \left. + \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - (\omega - \lambda))^{\eta_1-\eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right] u(\omega) d\omega \right\|^2 \\ \leq 14 \mathbb{E} \left\| \int_0^{t+\lambda} \left[ (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - \omega)^{\eta_1-\eta_2}) \mathfrak{C} \right. \right. \\ \left. \left. + \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - (\omega - \lambda))^{\eta_1-\eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right] \right. \\ \left. \times \left[ (t - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - \omega)^{\eta_1-\eta_2}) \mathfrak{C} + \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1-1} \right. \right. \\ \left. \left. \times E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - (\omega - \lambda))^{\eta_1-\eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right]^* W^{-1} 1/2 \left[ z_{\mathcal{T}} - \sum_{r=0}^{\rho-1} z'(0) \mathcal{T}^r \right. \right. \\ \left. \left. \times E_{\eta_1-\eta_2, 1+r} (\mathfrak{B} \mathcal{T}^{\eta_1-\eta_2}) + \sum_{r=0}^{\mu-1} z'(0) \mathcal{T}^{\eta_1-\eta_2+r} E_{\eta_1-\eta_2, \eta_1-\eta_2+1+r} (\mathfrak{B} \mathcal{T}^{\eta_1-\eta_2}) \right. \right. \\ \left. \left. - \int_{-h}^0 \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (\mathcal{T} - (\omega - \lambda))^{\eta_1-\eta_2}) \right. \right. \right. \\ \left. \left. \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \right] d\mathfrak{D}_\lambda \right. \\ \left. \left. - \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (\mathcal{T} - \omega)^{\eta_1-\eta_2}) \left( \int_0^\eta \sigma(\theta) d\omega(\theta) \right) d\omega \right. \right. \end{aligned}$$

$$\begin{aligned}
 & - \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) f(\omega, z(\omega)) \, d\omega \Big\| ^2 \\
 \leq & 84M^2 l^2 (\mathcal{T} + \lambda) \left\{ 1/2 \left[ \mathbb{E} \|z_{\mathcal{T}}\|^2 + u_1 \sum_{r=0}^{\rho-1} \mathbb{E} \|z_r\|^2 + u_2 \sum_{r=0}^{\mu-1} \mathbb{E} \|z_r\|^2 \right. \right. \\
 & + G + u_3 \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \tilde{N} \mathcal{T} \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) \, d\omega \\
 & \left. \left. + 4u_3 L_{\sigma} \tilde{L} \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \int_0^{\mathcal{T}} \left( \int_0^{\eta} (1 + \mathbb{E} \|z(\theta)\|^2) \, d\theta \right) \, d\omega \right] \right\} \\
 \leq & 42M^2 l^2 (\mathcal{T} + \lambda) \left[ \mathbb{E} \|z_{\mathcal{T}}\|^2 + u_1 \sum_{r=0}^{\rho-1} \mathbb{E} \|z_r\|^2 + u_2 \sum_{r=0}^{\mu-1} \mathbb{E} \|z_r\|^2 \right. \\
 & + G + u_3 \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \tilde{N} \mathcal{T} \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) \, d\omega \\
 & \left. + 4u_3 L_{\sigma} \tilde{L} \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \int_0^{\mathcal{T}} \left( \int_0^{\eta} (1 + \mathbb{E} \|z(\theta)\|^2) \, d\theta \right) \, d\omega \right], \tag{32}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{R}_7 \leq & 7 \mathbb{E} \left\| \left[ \int_{t+\lambda}^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right] \right. \\
 & \times \left[ \int_{t+\lambda}^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right]^* \\
 & \times W^{-1} 1/2 \left[ z_{\mathcal{T}} - \sum_{r=0}^{\rho-1} z^r(0) \mathcal{T}^r E_{\eta_1 - \eta_2, 1+r} (\mathfrak{B} \mathcal{T}^{\eta_1 - \eta_2}) \right. \\
 & + \sum_{r=0}^{\mu-1} z^r(0) \mathcal{T}^{\eta_1 - \eta_2 + r} E_{\eta_1 - \eta_2, \eta_1 - \eta_2 + 1+r} (\mathfrak{B} \mathcal{T}^{\eta_1 - \eta_2}) \\
 & - \int_{-h}^0 \left[ \int_{\lambda}^0 (\mathcal{T} - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - (\omega - \lambda))^{\eta_1 - \eta_2}) \right. \\
 & \left. \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) \, d\omega \right] d\mathfrak{D}_{\lambda} \\
 & - \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \left( \int_0^{\eta} \sigma(\theta) \, d\omega(\theta) \right) \, d\omega \\
 & \left. - \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) f(\omega, z(\omega)) \, d\omega \right\| ^2 \\
 \leq & 21 \tilde{M}^2 l^2 \left[ \mathbb{E} \|z_{\mathcal{T}}\|^2 + u_1 \sum_{r=0}^{\rho-1} \mathbb{E} \|z_r\|^2 + u_2 \sum_{r=0}^{\mu-1} \mathbb{E} \|z_r\|^2 \right. \\
 & + G + u_3 \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \tilde{N} \mathcal{T} \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) \, d\omega \\
 & \left. + 4u_3 L_{\sigma} \tilde{L} \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \int_0^{\mathcal{T}} \left( \int_0^{\eta} (1 + \mathbb{E} \|z(\theta)\|^2) \, d\theta \right) \, d\omega \right] \tag{33}
 \end{aligned}$$

From (27)–(33), we have

$$\begin{aligned}
 \sup_{0 \leq t \leq T} \mathbb{E} \|\Delta z(t)\|^2 &\leq 7u_1 \sum_{r=0}^{\rho-1} \mathbb{E} \|z_r\|^2 + 7u_2 \sum_{r=0}^{\mu-1} \mathbb{E} \|z_r\|^2 \\
 &+ 7G + 7u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \tilde{N}\mathcal{T} \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) d\omega \\
 &+ 28u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} L_\sigma \tilde{L} \int_0^{\mathcal{T}} \left( \int_0^\eta (1 + \mathbb{E} \|z(\theta)\|^2) d\theta \right) d\omega \\
 &+ 42M^2 l^2 (\mathcal{T} + \lambda) \left[ \mathbb{E} \|z_{\mathcal{T}}\|^2 + u_1 \sum_{r=0}^{\rho-1} \mathbb{E} \|z_r\|^2 + u_2 \sum_{r=0}^{\mu-1} \mathbb{E} \|z_r\|^2 \right. \\
 &+ G + u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \tilde{N}\mathcal{T} \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) d\omega \\
 &\left. + 4u_3 L_\sigma \tilde{L} \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \left( \int_0^\eta (1 + \mathbb{E} \|z(\theta)\|^2) d\theta \right) d\omega \right] \\
 &+ 21\tilde{M}^2 l^2 [\mathbb{E} \|z_{\mathcal{T}}\|^2 + u_1 \sum_{r=0}^{\rho-1} \mathbb{E} \|z_r\|^2 + u_2 \sum_{r=0}^{\mu-1} \mathbb{E} \|z_r\|^2 \\
 &+ G + u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \tilde{N}\mathcal{T} \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) d\omega \\
 &+ 4u_3 L_\sigma \tilde{L} \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \left( \int_0^\eta (1 + \mathbb{E} \|z(\theta)\|^2) d\theta \right) d\omega] \\
 &\leq C(1 + \mathcal{T}) \left[ \int_0^{\mathcal{T}} (1 + \mathbb{E} \|z(\omega)\|^2) d\omega \right] \\
 &\leq C \left( 1 + \mathcal{T} \sup_{0 \leq t \leq \mathcal{T}} \mathbb{E} \|z(\omega)\|^2 \right) \tag{34}
 \end{aligned}$$

for all  $t \in [0, \mathcal{T}]$ , where  $C$  is constant. This implies that  $\Delta$  maps  $I$  into itself.

Secondly, we prove that  $\Delta$  is a contraction mapping on  $I$ , for any  $z, y \in I$ ,

$$\begin{aligned}
 &\mathbb{E} \|\Delta z(t) - \Delta y(t)\|^2 \\
 &\leq 6 \sup_{0 \leq t \leq \mathcal{T}} \mathbb{E} \left\| \mathbb{G}_1(\mathcal{T}, t) \mathbb{G}_1^{\mathcal{T}}(\mathcal{T}, t) W^{-1} \right. \\
 &\quad \times \left[ \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1-\eta_2}) \right. \\
 &\quad \times \left( \int_0^\eta [\sigma(\theta, z(\theta)) - \sigma(\theta, y(\theta))] dw(\theta) \right) d\omega \\
 &\quad \left. + \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1-\eta_2}) (f(\omega, z(\omega)) - f(\omega, y(\omega))) d\omega \right] \\
 &\quad \left. + \mathbb{G}_2(\mathcal{T}, t) \mathbb{G}_2^{\mathcal{T}}(\mathcal{T}, t) W^{-1} \right. \\
 &\quad \times \left[ \int_0^{\mathcal{T}} (\mathcal{T} - \omega)^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1-\eta_2}) \right.
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( \int_0^\eta [\sigma(\theta, z(\theta)) - \sigma(\theta, y(\theta))] d\omega(\theta) \right) d\omega \\
 & + \int_0^T (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) (\mathfrak{f}(\omega, z(\omega)) - \mathfrak{f}(\omega, y(\omega))) d\omega \Big] \\
 & + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \left( \int_0^\eta [\sigma(\theta, z(\theta)) - \sigma(\theta, y(\theta))] d\omega(\theta) \right) d\omega \\
 & + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) (\mathfrak{f}(\omega, z(\omega)) - \mathfrak{f}(\omega, y(\omega))) d\omega \Big\|^2 \\
 \leq & 6M^2 l^2 \left\{ \int_0^T (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \right. \\
 & \times \left[ \int_0^\eta (\sigma(\theta, z(\theta)) - \sigma(\theta, y(\theta))) d\omega(\theta) \right] d\omega \Big\|^2 \\
 & + \left\| \int_0^T (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) [\mathfrak{f}(\omega, z(\omega)) - \mathfrak{f}(\omega, y(\omega))] d\omega \right\|^2 \Big\} \\
 & + 6\tilde{M}^2 l^2 \left\{ \int_0^T (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \right. \\
 & \times \left[ \int_0^\eta (\sigma(\theta, z(\theta)) - \sigma(\theta, y(\theta))) d\omega(\theta) \right] d\omega \Big\|^2 \\
 & + \left\| \int_0^T (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) [\mathfrak{f}(\omega, z(\omega)) - \mathfrak{f}(\omega, y(\omega))] d\omega \right\|^2 \Big\} \\
 & + 6\mathbb{E} \left\| \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \right. \\
 & \times \left[ \int_0^\eta (\sigma(\theta, z(\theta)) - \sigma(\theta, y(\theta))) d\omega(\theta) \right] d\omega \Big\|^2 \\
 & + 6\mathbb{E} \left\| \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) [\mathfrak{f}(\omega, z(\omega)) - \mathfrak{f}(\omega, y(\omega))] d\omega \right\|^2 \\
 \triangleq & 6M^2 l^2 \sum_{b=1}^2 \mathcal{S}_b + 6\tilde{M}^2 l^2 \sum_{b=3}^4 \mathcal{S}_b + \sum_{b=5}^6 \mathcal{S}_b. \tag{35}
 \end{aligned}$$

Using the Lipschitz condition, we have the following estimates:

$$\begin{aligned}
 \mathcal{S}_1 & \leq 24M^2 l^2 \left\| \int_0^T (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) \right. \\
 & \times \left[ \int_0^\eta (\sigma(\theta, z(\theta)) - \sigma(\theta, y(\theta))) d\omega(\theta) \right] d\omega \Big\|^2 \\
 & \leq 24M^2 l^2 u_3 L_\sigma L \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \int_0^T \left( \int_0^\eta (\mathbb{E} \|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega, \tag{36}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{S}_2 & \leq 6M^2 l^2 \left\| \int_0^T (\mathcal{T} - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1} (\mathfrak{B}(\mathcal{T} - \omega)^{\eta_1 - \eta_2}) [\mathfrak{f}(\omega, z(\omega)) - \mathfrak{f}(\omega, y(\omega))] d\omega \right\|^2 \\
 & \leq 6M^2 l^2 u_3 \mathcal{T} N \frac{\mathcal{T}^{2\eta_1 - 1}}{2\eta_1 - 1} \int_0^T \mathbb{E} \|z(\omega) - y(\omega)\|^2 d\omega, \tag{37}
 \end{aligned}$$



$$S_3 \leq 24\tilde{M}^2 l^2 u_3 L_\sigma L \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \left( \int_0^\eta (\mathbb{E}\|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega, \tag{38}$$

$$S_4 \leq 6\tilde{M}^2 l^2 u_3 \mathcal{T}N \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \mathbb{E}\|z(\omega) - y(\omega)\|^2 d\omega, \tag{39}$$

$$S_5 \leq 24u_3 L_\sigma L \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \left( \int_0^\eta (\mathbb{E}\|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega, \tag{40}$$

$$S_6 \leq 6u_3 \mathcal{T}N \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \mathbb{E}\|z(\omega) - y(\omega)\|^2 d\omega. \tag{41}$$

Together with inequalities (36)–(41), we get

$$\begin{aligned} & \mathbb{E}\|(\Delta z)(t) - (\Delta y)(t)\|^2 \\ & \leq 24M^2 l^2 u_3 L_\sigma L \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \left( \int_0^\eta (\mathbb{E}\|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega \\ & \quad + 6M^2 l^2 u_3 \mathcal{T}N \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \mathbb{E}\|z(\omega) - y(\omega)\|^2 d\omega \\ & \quad + 24\tilde{M}^2 l^2 u_3 L_\sigma L \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \left( \int_0^\eta (\mathbb{E}\|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega \\ & \quad + 6\tilde{M}^2 l^2 u_3 \mathcal{T}N \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \mathbb{E}\|z(\omega) - y(\omega)\|^2 d\omega \\ & \quad + 24u_3 L_\sigma L \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \left( \int_0^\eta (\mathbb{E}\|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega \\ & \quad + 6u_3 \mathcal{T}N \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} \int_0^{\mathcal{T}} \mathbb{E}\|z(\omega) - y(\omega)\|^2 d\omega \\ & \leq 24u_3 L_\sigma L \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} (M^2 l^2 + \tilde{M}^2 l^2 + 1) \int_0^{\mathcal{T}} \left( \int_0^\eta (\mathbb{E}\|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega \\ & \quad + 6u_3 \mathcal{T}N \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} (M^2 l^2 + \tilde{M}^2 l^2 + 1) \int_0^{\mathcal{T}} \mathbb{E}\|z(\omega) - y(\omega)\|^2 d\omega \\ & \leq 6u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} (M^2 l^2 + \tilde{M}^2 l^2 + 1) \left[ 4L_\sigma L \int_0^{\mathcal{T}} \left( \int_0^\eta (\mathbb{E}\|z(\theta) - y(\theta)\|^2) d\theta \right) d\omega \right. \\ & \quad \left. + \mathcal{T}N \int_0^{\mathcal{T}} \mathbb{E}\|z(\omega) - y(\omega)\|^2 d\omega \right] \\ & \leq 6u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} (M^2 l^2 + \tilde{M}^2 l^2 + 1) (4L_\sigma L + \mathcal{T}N) \sup_{0 \leq t \leq \mathcal{T}} \mathbb{E}\|z(t) - y(t)\|^2 d\omega. \tag{42} \end{aligned}$$

Therefore we conclude that if  $6u_3 \frac{\mathcal{T}^{2\eta_1-1}}{2\eta_1-1} (M^2 l^2 + \tilde{M}^2 l^2 + 1) (4L_\sigma L + \mathcal{T}N) \leq 1$ , then  $\Delta$  is a contraction mapping on  $I$ , which implies that the mapping  $\Delta$  has a unique fixed point.

Hence we have

$$\begin{aligned} z(t) &= \sum_{r=0}^{\rho-1} z^r(0) t^r E_{\eta_1-\eta_2, 1+r} (\mathfrak{B} t^{\eta_1-\eta_2}) - \sum_{r=0}^{\mu-1} z^r(0) t^{\eta_1-\eta_2+r} E_{\eta_1-\eta_2, \eta_1-\eta_2+1+r} (\mathfrak{B} t^{\eta_1-\eta_2}) \\ & \quad + \int_{-h}^0 \left[ \int_\lambda^0 (t - (\omega - \lambda))^{\eta_1-1} E_{\eta_1-\eta_2, \eta_1} (\mathfrak{B} (t - (\omega - \lambda))^{\eta_1-\eta_2}) \right. \end{aligned}$$

$$\begin{aligned}
 & \times \mathfrak{D}(\omega - \lambda, \lambda) \vartheta(\omega) d\omega \Big] d\mathfrak{D}_\lambda \\
 & + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{f}(\omega, z(\omega)) d\omega \\
 & + \int_0^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \\
 & \times \left( \int_0^\eta \sigma(\theta) dW(\theta) \right) d\omega + \int_0^{t+\lambda} \left[ (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} \right. \\
 & \left. + \int_{-h}^0 (t - (\omega - \lambda))^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - (\omega - \lambda))^{\eta_1 - \eta_2}) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right] u(\omega) d\omega \\
 & + \int_{t+\lambda}^t (t - \omega)^{\eta_1 - 1} E_{\eta_1 - \eta_2, \eta_1}(\mathfrak{B}(t - \omega)^{\eta_1 - \eta_2}) \mathfrak{C} u(\omega) d\omega. \tag{43}
 \end{aligned}$$

Thus  $z(t)$  is the solution of system (18)–(20), and it is easy to verify that  $z(\mathcal{T}) = z_{\mathcal{T}}$ . Further the control function  $u(t)$  steers system (18)–(20) from the initial state to the final state  $z_{\mathcal{T}}$  on  $\mathcal{J}$ . Hence system (18)–(20) is controllable on  $\mathcal{J}$ . □

*Remark 4.2* If  $\eta_1 \in (1, 2]$ ,  $\eta_2 \in (0, 1]$ ,  $\rho = 2$ , and  $\sigma = 0$ , then system (5)–(7) reduces to the system which was discussed in [3]. When  $\eta_1 \in (1, 2]$ ,  $\eta_2 \in (0, 1]$ ,  $\rho = 2$ , and  $C = \sigma = 0$ , system (18)–(20) reduces to the system studied in [4]. Controllability of the linear system is obtained by the Gramian matrix. Further, under the assumption that the linear control system is controllable and by using the successive approximation technique, the controllability of nonlinear systems was obtained. If we choose  $\eta_1 \in (0, 1]$ ,  $\eta_2 = 0$ ,  $\mathfrak{D} = 0$ , and  $\rho = 1$  in (18)–(20), Theorem 3.3 in [31] can be regarded as a special case of our result.

*Remark 4.3* It should be noted that the results in [21] have been derived for linear fractional-order systems, and the results in [6, 18] have been obtained for nonlinear fractional-order systems. However, in comparison with [13, 32], the results proposed in this paper are more general ones, as the results presented are applicable for higher-order fractional damped stochastic systems with distributed delays.

### 5 Examples

Two examples are provided to demonstrate the controllability results for the proposed criteria.

*Example 5.1* The problem of linear fractional damped stochastic dynamical system with distributed delay is as follows:

$${}_0^C D_t^{\eta_1} z(t) - \mathfrak{B}_0^C D_t^{\eta_2} z(t) = \mathfrak{C} u(t) + \int_{-h}^0 d_\lambda \mathfrak{D}(t, \lambda) u(t + \lambda) + \sigma(t) \frac{dw(t)}{dt}, \tag{44}$$

$$z(0) = z_0, \quad z'(0) = z_1, \quad z''(0) = z_2, \quad z'''(0) = z_3, \tag{45}$$

where  $3 < \eta_1 \leq 4$ ,  $2 < \eta_2 \leq 3$ ,

$$\mathfrak{B} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathfrak{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathfrak{D} = \begin{bmatrix} e^\lambda \\ 0 \\ e^\lambda \end{bmatrix}, \quad \text{and} \quad \sigma(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}.$$

To show that linear system (44) is controllable, it is enough to show that the controllability Gramian matrix  $W$  is positive definite by Theorem 3.2.

The Mittag-Leffler matrix of the given system is

$$E_{\eta_1-\eta_2,\eta_1}(\mathfrak{B}(t-\omega)^{\eta_1-\eta_2}) = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{2k(\eta_1-\eta_2)}}{\Gamma(2k(\eta_1-\eta_2)+\eta_1)} & \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{(2k+1)(\eta_1-\eta_2)}}{\Gamma((2k+1)(\eta_1-\eta_2)+\eta_1)} & \frac{1}{\Gamma(\eta_1)} \\ -\sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{(2k+1)(\eta_1-\eta_2)}}{\Gamma((2k+1)(\eta_1-\eta_2)+\eta_1)} & \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{2k(\eta_1-\eta_2)}}{\Gamma(2k(\eta_1-\eta_2)+\eta_1)} & \frac{1}{\Gamma(\eta_1)} \\ \frac{1}{\Gamma(\eta_1)} & \frac{1}{\Gamma(\eta_1)} & \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{k(\eta_1-\eta_2)}}{\Gamma(k(\eta_1-\eta_2)+\eta_1)} \end{bmatrix},$$

$$E_{1,3.5}(\mathfrak{B}(t-\omega)) = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{2k}}{\Gamma(2k+3.5)} & \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{(2k+1)}}{\Gamma[2k+4.5]} & \frac{1}{\Gamma(3.5)} \\ -\sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{(2k+1)}}{\Gamma[2k+4.5]} & \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^{2k}}{\Gamma(2k+3.5)} & \frac{1}{\Gamma(3.5)} \\ \frac{1}{\Gamma(3.5)} & \frac{1}{\Gamma(3.5)} & \sum_{k=0}^{\infty} \frac{(-1)^k(t-\omega)^k}{\Gamma(k+3.5)} \end{bmatrix},$$

$$(\mathcal{T}-\omega)^{2.5}E_{1,3.5}(\mathfrak{B}(\mathcal{T}-\omega)) = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{2k+2.5}}{\Gamma(2k+3.5)} & \sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{2k+3.5}}{\Gamma[2k+4.5]} & \frac{(\mathcal{T}-\omega)^{2.5}}{\Gamma(3.5)} \\ -\sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{2k+3.5}}{\Gamma[2k+4.5]} & \sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{2k+2.5}}{\Gamma(2k+3.5)} & \frac{(\mathcal{T}-\omega)^{2.5}}{\Gamma(3.5)} \\ \frac{(\mathcal{T}-\omega)^{2.5}}{\Gamma(3.5)} & \frac{(\mathcal{T}-\omega)^{2.5}}{\Gamma(3.5)} & \sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{k+2.5}}{\Gamma(k+3.5)} \end{bmatrix},$$

$$(\mathcal{T}-\omega)^{2.5}E_{1,3.5}(\mathfrak{B}(\mathcal{T}-\omega))\mathfrak{C} = \begin{bmatrix} P_1 & P_2 & P_3 \\ P_2 & P_1 & P_3 \\ P_3 & P_3 & P_9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} P_2 \\ P_1 \\ P_3 \end{bmatrix},$$

where

$$P_1 = P_5 = \sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{2k+2.5}}{\Gamma(2k+3.5)},$$

$$P_2 = -P_4 = \sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{2k+3.5}}{\Gamma[2k+4.5]},$$

$$P_9 = \sum_{k=0}^{\infty} \frac{(-1)^k(\mathcal{T}-\omega)^{k+2.5}}{\Gamma(k+3.5)},$$

$$P_3 = P_6 = P_7 = P_8 = \frac{(\mathcal{T}-\omega)^{2.5}}{\Gamma(3.5)},$$

$$[(\mathcal{T}-\omega)^{2.5}E_{1,3.5}(\mathfrak{B}(\mathcal{T}-\omega))\mathfrak{C}][(\mathcal{T}-\omega)^{2.5}E_{1,3.5}(\mathfrak{B}(\mathcal{T}-\omega))\mathfrak{C}]^*$$

$$= \begin{bmatrix} P_2 \\ P_1 \\ P_3 \end{bmatrix} \begin{bmatrix} P_2 & P_1 & P_3 \end{bmatrix}$$

$$= \begin{bmatrix} P_2^2 & P_1P_2 & P_2P_3 \\ P_1P_2 & P_1^2 & P_1P_3 \\ P_2P_3 & P_1P_3 & P_3^2 \end{bmatrix}.$$

Similarly,

$$\int_{-1}^0 (\mathcal{T} - (\omega - \lambda))^{2.5} E_{1,3.5}(\mathfrak{B}(\mathcal{T} - (\omega - \lambda))) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda$$

$$= \begin{bmatrix} Q_1 & Q_2 & Q_3 \\ Q_2 & Q_1 & Q_3 \\ Q_3 & Q_3 & Q_9 \end{bmatrix} \begin{bmatrix} e^\lambda \\ 0 \\ e^\lambda \end{bmatrix} = \begin{bmatrix} Q_1 e^\lambda + Q_3 e^\lambda \\ Q_2 e^\lambda + Q_3 e^\lambda \\ Q_3 e^\lambda + Q_9 e^\lambda \end{bmatrix},$$

where

$$Q_1 = Q_5 = \sum_{k=0}^\infty \frac{(-1)^k (\mathcal{T} - (\omega - \lambda))^{2k+2.5}}{\Gamma(2k + 3.5)},$$

$$Q_2 = -Q_4 = \sum_{k=0}^\infty \frac{(-1)^k (\mathcal{T} - (\omega - \lambda))^{2k+3.5}}{\Gamma[2k + 4.5]},$$

$$Q_9 = \sum_{k=0}^\infty \frac{(-1)^k (\mathcal{T} - (\omega - \lambda))^{k+2.5}}{\Gamma(k + 3.5)},$$

$$Q_3 = Q_6 = Q_7 = Q_8 = \frac{(\mathcal{T} - (\omega - \lambda))^{2.5}}{\Gamma(3.5)},$$

$$\begin{aligned} & \left[ (\mathcal{T} - \omega)^{2.5} E_{1,3.5}(\mathfrak{B}(\mathcal{T} - \omega)) \mathfrak{C} \right. \\ & \quad \left. + \int_{-1}^0 (\mathcal{T} - (\omega - \lambda))^{2.5} E_{1,3.5}(\mathfrak{B}(\mathcal{T} - (\omega - \lambda))) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right] \\ & = \begin{bmatrix} P_2 \\ P_1 \\ P_3 \end{bmatrix} + \begin{bmatrix} Q_1 e^\lambda + Q_3 e^\lambda \\ Q_2 e^\lambda + Q_3 e^\lambda \\ Q_3 e^\lambda + Q_9 e^\lambda \end{bmatrix} = \begin{bmatrix} P_2 + Q_1 e^\lambda + Q_3 e^\lambda \\ P_1 + Q_2 e^\lambda + Q_3 e^\lambda \\ P_3 + Q_3 e^\lambda + Q_9 e^\lambda \end{bmatrix}, \\ & \left[ (\mathcal{T} - \omega)^{2.5} E_{1,3.5}(\mathfrak{B}(\mathcal{T} - \omega)) \mathfrak{C} \right. \\ & \quad \left. + \int_{-1}^0 (\mathcal{T} - (\omega - \lambda))^{2.5} E_{1,3.5}(\mathfrak{B}(\mathcal{T} - (\omega - \lambda))) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right] \\ & \quad \times \left[ (\mathcal{T} - \omega)^{2.5} E_{1,3.5}(\mathfrak{B}(\mathcal{T} - \omega)) \mathfrak{C} \right. \\ & \quad \left. + \int_{-1}^0 (\mathcal{T} - (\omega - \lambda))^{2.5} E_{1,3.5}(\mathfrak{B}(\mathcal{T} - (\omega - \lambda))) \mathfrak{D}(\omega - \lambda, \lambda) d\mathfrak{D}_\lambda \right]^* \\ & = \begin{bmatrix} P_2 + Q_1 e^\lambda + Q_3 e^\lambda \\ P_1 + Q_2 e^\lambda + Q_3 e^\lambda \\ P_3 + Q_3 e^\lambda + Q_9 e^\lambda \end{bmatrix} \begin{bmatrix} P_2 + Q_1 e^\lambda + Q_3 e^\lambda & P_1 + Q_2 e^\lambda + Q_3 e^\lambda & P_3 + Q_3 e^\lambda + Q_9 e^\lambda \end{bmatrix} \\ & = \begin{bmatrix} (P_2 + Q_1 e^\lambda + Q_3 e^\lambda)^2 & \alpha_1 & \alpha_2 \\ \alpha_1 & (P_1 + Q_2 e^\lambda + Q_3 e^\lambda)^2 & \alpha_3 \\ \alpha_2 & \alpha_3 & (P_3 + Q_3 e^\lambda + Q_9 e^\lambda)^2 \end{bmatrix}, \end{aligned}$$

where

$$\alpha_1 = (P_2 + Q_1 e^\lambda + Q_3 e^\lambda)(P_1 + Q_2 e^\lambda + Q_3 e^\lambda),$$

$$\alpha_2 = (P_2 + Q_1e^\lambda + Q_3e^\lambda)(P_3 + Q_3e^\lambda + Q_9e^\lambda),$$

$$\alpha_3 = (P_1 + Q_2e^\lambda + Q_3e^\lambda)(P_3 + Q_3e^\lambda + Q_9e^\lambda).$$

Hence, the controllability matrix  $W$  for the system is found by

$$W = \int_0^{\mathcal{T}+\lambda} \begin{bmatrix} (P_2 + Q_1e^\lambda + Q_3e^\lambda)^2 & \alpha_1 & \alpha_2 \\ \alpha_1 & (P_1 + Q_2e^\lambda + Q_3e^\lambda)^2 & \alpha_3 \\ \alpha_2 & \alpha_3 & (P_3 + Q_3e^\lambda + Q_9e^\lambda)^2 \end{bmatrix} ds$$

$$+ \int_{\mathcal{T}+\lambda}^{\mathcal{T}} \begin{bmatrix} P_2^2 & P_1P_2 & P_2P_3 \\ P_1P_2 & P_1^2 & P_1P_3 \\ P_2P_3 & P_1P_3 & P_3^2 \end{bmatrix} ds$$

$$> 0,$$

which is positive definite for any  $\mathcal{T} > 0$ . Therefore the corresponding linear higher-order fractional damped stochastic system with distributed delay is controllable.

*Example 5.2* The problem of nonlinear fractional damped stochastic dynamical system with distributed delay is as follows:

$${}^C_0D_t^{\eta_1} z(t) - \mathfrak{B}^C_0D_t^{\eta_2} z(t) = \mathfrak{C}u(t) + \int_{-h}^0 d_\lambda \mathfrak{D}(t, \lambda)u(t + \lambda) + \mathfrak{f}(t, z(t)) + \sigma(t) \frac{dw(t)}{dt}, \tag{46}$$

$$z(0) = z_0, \quad z'(0) = z_1, \quad z''(0) = z_2, \quad z'''(0) = z_3, \tag{47}$$

where  $3 < \eta_1 \leq 4, 2 < \eta_2 \leq 3$ ,

$$\mathfrak{B} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathfrak{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathfrak{D} = \begin{bmatrix} e^\lambda \\ 0 \\ e^\lambda \end{bmatrix},$$

$$f(t, z(t)) = \begin{bmatrix} 7z_1^5 \\ 9z_3^3 \\ z_1^2 + z_2^2 \end{bmatrix} \quad \text{and} \quad \sigma(t, z(t)) = \begin{bmatrix} \sin(z_1) \\ z_2 \operatorname{cosec}(z_2) \\ \tanh(z_3) \end{bmatrix}.$$

Since the linear system is controllable and the nonlinear functions  $f$  and  $\sigma$  satisfy the Lipschitz condition and the linear growth condition, nonlinear system (46)–(47) is controllable on  $[0, \mathcal{T}]$ .

### 6 Conclusion

In this paper, the controllability Gramian matrix is used to derive the controllability results for a linear fractional stochastic system with damping behavior and distributed delays. New sufficient conditions for controllability of the nonlinear damped fractional stochastic system with distributed delays have been deduced by using the fixed point theory and stochastic analysis techniques. Finally, examples are given to demonstrate the effectiveness of the obtained results. Further, the method and technique presented in this paper can be applied to solve other fractional-order systems containing various effects like impulsive behavior and several types of delays.

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#### Authors' contributions

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