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On highly efficient derivative-free family of numerical methods for solving polynomial equation simultaneously

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Abstract
A highly efficient new three-step derivative-free family of numerical iterative schemes for estimating all roots of polynomial equations is presented. Convergence analysis proved that the proposed simultaneous iterative method possesses 12th-order convergence locally. Numerical examples and computational cost are given to demonstrate the capability of the method presented.

Keywords: Numerical scheme; Polynomials; Computational efficiency; CPU-time; Convergence order

1 Introduction

A lot of engineering and physical problems can be formulated as a nonlinear polynomial equation

$$f(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_0 = \prod_{j=1}^{n}(r-\zeta_j) = (r-\zeta_i)\prod_{\substack{j=1\\ j\neq i}}^{n}(r-\zeta_j), \tag{1}$$

where $\zeta_1 \cdots \zeta_n$ denote all the simple or complex roots of (1). Classical Newton's method has local quadratic convergence given as

$$s^{(t)} = r^{(t)} - \frac{f(r^{(t)})}{f'(r^{(t)})} \quad (t = 0, 1, \ldots, n). \tag{2}$$

But method (2) has a major drawback, i.e., it requires evaluation of derivative at each step, which requires high computational cost. To overcome this, using forward difference approximation of $f'(r^{(t)})$

$$f'\left(r^{(t)}\right) \cong \frac{f(r^{(t)} + f(r^{(t)})) - f(r^{(t)})}{f(r^{(t)})}, \tag{3}$$

in (2), we get Steffensen's iterative method [1] of convergence order 2:

$$s^{(t)} = r^{(t)} - \frac{(f(r^{(t)}))^2}{f(r^{(t)} + f(r^{(t)})) - f(r^{(t)})}. \tag{4}$$

Later, Farooq et al. [2] presented the following derivative-free method having local quadratic convergence:

$$s^{(t)} = r^{(t)} - \frac{\alpha(f(r^{(t)}))^2}{f(r^{(t)} + \alpha f(r^{(t)})) - f(r^{(t)})}, \tag{5}$$

where $\alpha \in \mathbb{R}$. In the last few years, a lot of work has been done on those numerical iterative methods which approximate single root at one time of (1) (see, e.g., [3–7]). Besides these single root estimating methods in literature, we found another class of derivative-free iterative schemes which approximate all roots of (1) simultaneously. Iterative methods for approximating all roots of (1) have been very popular in recent years due to their global convergence and parallel implementation on computer (see, e.g., Weierstrass [8], Kanno [9], Proinov [10], Petković [11], Mir [12], Nourein [13], Aberth [14] and the references cited therein [15–23]).

Among derivative-free simultaneous methods, Weierstrass–Dochive [24] method is the most attractive method given by

$$s_i^{(t)} = r_i^{(t)} - w\bigl(r_i^{(t)}\bigr), \tag{6}$$

where

$$w\bigl(r_i^{(t)}\bigr) = \frac{f(r_i^{(t)})}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (r_i^{(t)} - r_j^{(t)})} \quad (i, j = 1, 2, 3, \ldots, n)$$

is Weierstrass correction. Method (6) has local quadratic convergence.

Nedzibove et al. [25] presented the following simultaneous method having a local quadratic convergence:

$$z_i^{(t)} = y_i^{(t)} - \frac{f(r_i^{(t)}) w(r_i^{(t)})}{f(r_i^{(t)}) - f(y_i^{(t)})}, \tag{7}$$

where $y_i^{(t)} = r_i^{(t)} - w(r_i^{(t)})$.

Petkovic et al. [11] escalated the convergence order of Ehrlich iterative numerical schemes from three to ten (abbreviated as NIM10):

$$s_i^{(t)} = r_i^{(t)} - \frac{1}{\frac{1}{N_i(r_i^{(t)})} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{(r_i^{(t)} - \overset{*}{v}_j^{(t)})}}, \tag{8}$$

where $\overset{*}{v}_j^{(t)} = u_j^{(t)} - \frac{(y_j^{(t)} - u_j^{(t)}) f(u_j^{(t)}) (\frac{f(r_j^{(t)})}{f'(r_j^{(t)})})}{(f(r_j^{(t)}) - f(u_j^{(t)}))^2} [f(y_j^{(t)}) - \frac{(f(r_j^{(t)}))^2}{f(y_j^{(t)}) - f(u_j^{(t)})}]$, $u_j^{(t)} = y_j^{(t)} - \frac{f(r_j^{(t)}) f(y_j^{(t)}) (\frac{f(r_j^{(t)})}{f'(r_j^{(t)})})}{(f(r_j^{(t)}) - f(y_j^{(t)}))^2}$, $y_j^{(t)} = r_j^{(t)} - \frac{f(r_j^{(t)})}{f'(r_j^{(t)})}$.

The main aim of this paper is to construct a high order efficient derivative-free family of methods among all existing simultaneous methods in the literature.

2 Construction of simultaneous method

Consider well-known three-step Newton methods [26] of convergence order eight as follows:

$$v^{(t)} = u^{(t)} - \frac{f(u^{(t)})}{f'(u^{(t)})}, \tag{9}$$

where $u^{(t)} = s^{(t)} - \frac{f(s^{(t)})}{f'(s^{(t)})}$ and $s^{(t)} = r^{(t)} - \frac{f(r^{(t)})}{f'(r^{(t)})}$. Taking Weierstrass correction [24]

$$\frac{f(r_i^{(t)})}{f'(r_i^{(t)})} = w\bigl(r_i^{(t)}\bigr) = \frac{f(r_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(r_i^{(t)} - r_j^{(t)})}, \tag{10}$$

and replacing $r_j^{(t)} = \overset{*}{s}{}_j^{(t)}$ in (10), we have

$$\frac{f(r_i^{(t)})}{f'(r_i^{(t)})} = \frac{f(r_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(r_i^{(t)} - \overset{*}{s}{}_j^{(t)})}, \tag{11}$$

where $\overset{*}{s}{}_j^{(t)} = r_j^{(t)} - \frac{\alpha(f(r_j^{(t)}))^2}{f(r_j^{(t)}+\alpha f(r_j^{(t)}))-f(r_j^{(t)})}$. Using $\frac{f(r_i^{(t)})}{f'(r_i^{(t)})} = \frac{f(r_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(r_i^{(t)}-\overset{*}{s}{}_j^{(t)})}$, $\frac{f(s_i^{(t)})}{f'(s_i^{(t)})} = \frac{f(s_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(s_i^{(t)}-s_j^{(t)})}$, and $\frac{f(u_i^{(t)})}{f'(u_i^{(t)})} = \frac{f(u_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(u_i^{(t)}-u_j^{(t)})}$ in (9), we have

$$v_i^{(t)} = u_i^{(t)} - \frac{f(u_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(u_i^{(t)} - u_j^{(t)})}, \tag{12}$$

where $u_i^{(t)} = s_i^{(t)} - \frac{f(s_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(s_i^{(t)}-s_j^{(t)})}$ and $s_i^{(t)} = r_i^{(t)} - \frac{f(r_i^{(t)})}{\prod_{\substack{j=1\\ j\neq i}}^{n}(r_i^{(t)}-\overset{*}{s}{}_j^{(t)})}$.

Thus, we have constructed a new simultaneous iterative method (12), which is abbreviated as NIM12.

2.1 Convergence aspect

In this section, we prove that method NIM12 has local convergence order 12.

Theorem 1 *Let $\zeta_1, \dots, \zeta_n$ be the n simple roots of (1). If $r_1^{(0)}, \dots, r_n^{(0)}$ are the initial estimates of the roots respectively and sufficiently close to actual roots, then NIM12 has a convergence order 12.*

Proof Let $\epsilon_i = r_i^{(t)} - \zeta_i$, $\epsilon_i' = s_i^{(t)} - \zeta_i$, $\epsilon_i'' = u_i^{(t)} - \zeta_i$, and $\epsilon_i''' = v_i^{(t)} - \zeta_i$ be the errors in r_i, s_i, u_i, and v_i, respectively. From (12), the first step of NIM12, we have

$$s_i^{(t)} - \zeta_i = r_i^{(t)} - \zeta_i - w_i\bigl(r_i^{(t)}\bigr),$$

$$\begin{aligned}
\epsilon_i' &= \epsilon_i - \epsilon_i \frac{w_i(r_i^{(t)})}{\epsilon_i}, \\
\epsilon_i' &= \epsilon_i(1 - E_i),
\end{aligned} \tag{13}$$

where

$$E_i = \frac{w_i(r_i^{(t)})}{\epsilon_i} = \prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(r_i^{(t)} - \zeta_j)}{(r_i^{(t)} - \overset{*}{s}{}_j^{(t)})}, \tag{14}$$

$$\frac{r_i^{(t)} - \zeta_j}{r_i^{(t)} - \overset{*}{s}{}_j^{(t)}} = 1 + \frac{\overset{*}{s}{}_j^{(t)} - \zeta_j}{r_i^{(t)} - \overset{*}{s}{}_j^{(t)}} = 1 + O(\epsilon^2),$$

and $\overset{*}{s}{}_j^{(t)} - \zeta_j = O(\epsilon^2)$ see [2]. For a simple root ζ and small enough ϵ, $|r_i^{(t)} - \overset{*}{s}{}_j^{(t)}|$ is bounded away from zero, and so

$$\begin{aligned}
&\prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(r_i^{(t)} - \zeta_j)}{(r_i^{(t)} - \overset{*}{s}{}_j^{(t)}))} = \bigl(1 + O(\epsilon^2)\bigr)^{n-1} = 1 + (n-1)O(\epsilon^2) = 1 + O(\epsilon^2), \\
&E_i = 1 + O(\epsilon^2), \\
&E_i - 1 = O(\epsilon^2).
\end{aligned}$$

Thus, (13) gives

$$\epsilon_i' = O(\epsilon)^3. \tag{15}$$

From the second step of NIM12, we have

$$\begin{aligned}
u_i^{(t)} - \zeta_i &= s_i^{(t)} - \zeta_i - w_i\bigl(s_i^{(t)}\bigr), \\
\epsilon_i'' &= \epsilon_i' - \epsilon_i' \frac{w_i(s_i^{(t)})}{\epsilon_i'}, \\
\epsilon_i'' &= \epsilon_i'(1 - U_i),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
U_i &= \frac{w_i(s_i^{(t)})}{\epsilon_i'} = \prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(s_i^{(t)} - \zeta_j)}{(s_i^{(t)} - s_j^{(t)})}, \\
\frac{s_i^{(t)} - \zeta_j}{s_i^{(t)} - s_j^{(t)}} &= 1 + \frac{s_i^{(t)} - \zeta_j}{s_i^{(t)} - s_j^{(t)}} = 1 + O(\epsilon_i').
\end{aligned}$$

For a simple root ζ and small enough ϵ, $|s_i^{(t)} - \overset{(t)}{s}{}_j|$ is bounded away from zero, so

$$\prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(s_i^{(t)} - \zeta_j)}{(s_i^{(t)} - s_j^{(t)})} = \bigl(1 + O(\epsilon')\bigr)^{n-1} = 1 + (n-1)O(\epsilon') = 1 + O(\epsilon'),$$

$$U_i = 1 + O(\epsilon'),$$
$$1 - U_i = O(\epsilon'),$$
$$\epsilon_i'' = O(\epsilon')^2.$$

Since from (15), $\epsilon_i' = O\,(\epsilon)^3$. Thus,

$$\epsilon_i'' = O\big((\epsilon)^3\big)^2,$$
$$\epsilon_i'' = O(\epsilon)^6. \tag{17}$$

From the third step of NIM12, we have

$$v_i^{(t)} - \zeta_i = u_i^{(t)} - \zeta_i - w_i\big(u_i^{(t)}\big),$$
$$\epsilon_i''' = \epsilon_i'' - \epsilon_i'' \frac{w_i(u_i^{(t)})}{\epsilon_i''},$$
$$\epsilon_i''' = \epsilon_i''(1 - G_i), \tag{18}$$

where

$$G_i = \frac{w_i(u_i)}{\epsilon_i''} = \prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(u_i^{(t)} - \zeta_j)}{(u_i^{(t)} - u_j^{(t)})},$$
$$\frac{u_i^{(t)} - \zeta_j}{u_i^{(t)} - u_j^{(t)}} = 1 + \frac{u_j^{(t)} - \zeta_j}{u_i^{(t)} - u_j^{(t)}} = 1 + O\big(\epsilon_i''\big). \tag{19}$$

With the same argument used in (16), we have

$$\prod_{\substack{j \neq i \\ j=1}}^{n} \frac{(u_i^{(t)} - \zeta_j)}{(u_i^{(t)} - u_j^{(t)})} = \big(1 + O(\epsilon'')\big)^{n-1} = 1 + (n-1)O(\epsilon'') = 1 + O(\epsilon'').$$

Therefore,

$$G_i = 1 + O(\epsilon''),$$
$$1 - G_i = O(\epsilon''),$$
$$\epsilon_i''' = O(\epsilon'')^2. \tag{20}$$

Since from (17) ϵ_i'' =O$(\epsilon)^6$, we obtain

$$\epsilon_i''' = O\big((\epsilon)^6\big)^2,$$
$$\epsilon_i''' = O(\epsilon)^{12}. \tag{21}$$

Hence, (21) proves 12th order convergence. □

Table 1 Number of operations (real arithmetic)

Methods	AS_m	M_m	D_m
NIM12	$7m^2 + O(m)$	$5m^2 + O(m)$	$2m^2 + O(m)$
NIM10	$22m^2 + O(m)$	$18m^2 + O(m)$	$2m^2 + O(m)$

3 Computational aspect

In this section, we compare the computational efficiencies of methods NIM10 and NIM12. As presented in [11], we can formulate the efficiency indices as follows:

$$\rho(\mathrm{NIM12}, \mathrm{NIM10}) = \left(\frac{E(\mathrm{NIM12})}{E(\mathrm{NIM10})} - 1\right) \times 100 \tag{22}$$

or

$$\rho(\mathrm{NIM10}, \mathrm{NIM12})) = \left(\frac{E(\mathrm{NIM10})}{E((\mathrm{NIM12}))} - 1\right) \times 100, \tag{23}$$

where

$$E(IN) = \frac{\log \mathbf{r}}{\mathbf{Q}}. \tag{24}$$

The cost of computation is represented by $\mathbf{Q}$ [11] and convergence order by $\mathbf{r}$ given as

$$\mathbf{Q} = \mathbf{Q}(m) = w_{as}AS_m + w_m M_m + w_d D_m. \tag{25}$$

Using the expression of $\mathbf{Q}$ in (24), we have

$$EL(m) = \left(\frac{\log \mathbf{r}}{w_{as}AS_m + w_m M_m + w_d D_m}\right). \tag{26}$$

The number of operations of real arithmetic of a complex polynomial with real and complex roots reduces to operations of real arithmetic as given in Table 1.

Figure 1(a)–(b) shows the percentage ratios of NIM10 and NIM12. It is evident from Fig. 1(a)–(b) that NIM12 is much better than NIM10.

Figure 1(a)–(b) shows the computational efficiency of simultaneous method NIM12 and NIM10 with respect to each other. Figure 1(a)–(b) clearly shows the dominance efficiency of our newly constructed method NIM12 over NIM10.

4 Numerical results

For numerical calculations, we use the following stopping criteria to terminate the computer programme using Maple 18 with 125-digit floating point arithmetic:

$$e_i^{(t)} = \left\| r_i^{(t+1)} \text{-} r_i^{(t)} \right\|_2 < 10^{-30},$$

where $e_i^{(t)}$ represents the absolute error. In all the tables, CPU means computational time in seconds. In all numerical calculations, we take $\alpha = 12/130$.

Application in engineering

In this section, we also discuss some applications from engineering.

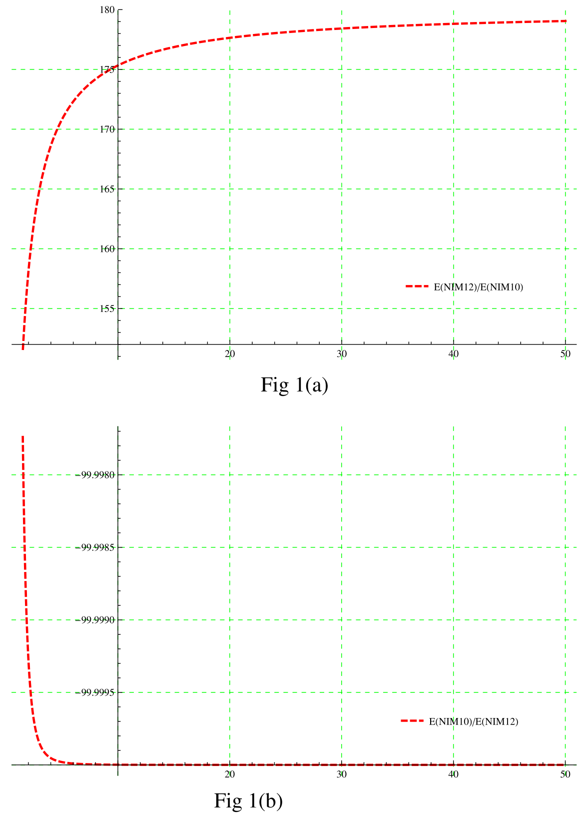

Figure 1 Percentage computational efficiency of simultaneous method NIM12 w.r.t NIM10

Table 2 Residual errors of simultaneous methods NIM10 and NIM12 for finding all roots of $f_1(r)$

Method	CPU	$e_1^{(3)}$	$e_2^{(3)}$	$e_3^{(3)}$	$e_4^{(3)}$	$e_5^{(3)}$	$e_6^{(3)}$	$e_7^{(3)}$	$e_8^{(3)}$
NIM10	0.766	6.9e–10	2.3e–12	6.1e–11	5.0e–11	2.7e–9	1.7e–11	7.6e–9	1.0e–13
NIM12	0.250	2.7e–25	4.5e–25	1.1e–23	1.6e–23	5.0e–24	1.2e–23	4.5e–24	8.0e–25

Example 1 ([27]) Consider

$$f_1(r) = (r+1)(r+2)\left(r^2-2r+2\right)\left(r^2+1\right)(r-2)(r+2-i),$$

with exact roots

$$\zeta_1 = -1, \qquad \zeta_2 = -2, \qquad \zeta_{3,4} = 1 \pm i, \qquad \zeta_{5,6} = \pm i, \qquad \zeta_7 = 2, \qquad \zeta_8 = -2+i.$$

The initial guessed values have been taken as follows:

$$\overset{(0)}{r}_1 = -1.3+0.2i, \qquad \overset{(0)}{r}_2 = -2.2-0.3i, \qquad \overset{(0)}{r}_3 = 1.3+1.2i, \qquad \overset{(0)}{r}_4 = 0.7-1.2i,$$
$$\overset{(0)}{r}_5 = -0.2+0.8i, \qquad \overset{(0)}{r}_6 = 0.2-1.3i, \qquad \overset{(0)}{r}_7 = 2.2-0.3i, \qquad \overset{(0)}{r}_8 = -2.2+0.7i.$$

Table 2 evidently illustrates the supremacy behavior of NIM12 over NIM10 in estimated absolute error and in CPU time on the same number of iterations $n = 3$ for guesstimating all roots of the nonlinear polynomial equation used in Example 1.

Table 3 Residual errors of simultaneous methods NIM10 and NIM12 for finding all roots of $f_2(r)$

Method	CPU	$e_1^{(4)}$	$e_2^{(4)}$	$e_3^{(4)}$	$e_4^{(4)}$
NIM10	0.071	1.3e–25	1.4e–25	4.5e–28	1.2e–28
NIM12	0.031	0.0	0.0	1.3e–37	1.3e–38

Example 2 ([28] Fractional conversion) The expression described in [29, 30]

$$f_2(r) = r^4 - 7.79075r^3 + 14.7445r^2 + 2.511r - 1.674 \tag{27}$$

is the fractional conversion of nitrogen, hydrogen feed at 250 atm. and 227k.

The exact roots of (27) are:

$$\zeta_1 = 3.9485 + 0.3161i, \qquad \zeta_2 = 3.9485 - 0.3161i, \qquad \zeta_3 = -0.3841, \qquad \zeta_4 = 0.2778.$$

The initial calculated values of (27) have been taken as follows:

$$\overset{(0)}{r_1} = 3.5 + 0.3i, \qquad \overset{(0)}{r_2} = 3.5 - 0.3i, \qquad \overset{(0)}{r}_3 = -0.3 + 0.01i, \qquad \overset{(0)}{r}_4 = 1.8 + 0.01i.$$

Table 3 evidently illustrates the supremacy behavior of NIM12 over NIM10 in estimated absolute error and in CPU time on the same number of iterations $n = 4$ for guesstimating all roots of the nonlinear polynomial equation used in Example 2.

Example 3 ([27] Continuous stirred tank reactor (CSTR)) An isothermal CSTR is considered here. Items E_1 and E_2 are fed to the reactor at rates of R and q-R, respectively. Complex chain reactions are developed in the reactor given as follows:

$$E_1 + E_2 \longrightarrow E_3, \qquad E_3 + E_2 \longrightarrow E_4, \qquad E_4 + E_2 \longrightarrow E_5, \qquad E_4 + E_2 \longrightarrow E_6.$$

This problem was first tested by Douglas (see [31]), and the following equation of transfer function of the rector was found:

$$H_c \frac{2.98(r + 2.25)}{(r + 1.45)(r + 2.85)^2(r + 4.35)} = -1, \tag{28}$$

H_c being the gain of the proportional controller. This transfer function yields the following nonlinear equation by taking $H_c = 0$:

$$f_3(r) = r^4 + 11.50r^3 + 47.49r^2 + 83.06325r + 51.23266875 = 0. \tag{29}$$

The transfer function has four negative real roots, i.e., $r_1 = -1.45$, $r_2 = -2.85$, $r_3 = -2.85$, $r_4 = -4.45$.

The initial calculated values of (29) have been taken as follows:

$$\overset{(0)}{r_1} = -1.0, \qquad \overset{(0)}{r_2} = -1.1, \qquad \overset{(0)}{r}_3 = -2.2, \qquad \overset{(0)}{r}_4 = -3.9.$$

Table 4 evidently illustrates the supremacy behavior of NIM12 over NIM10 in estimated absolute error and in CPU time on the same number of iterations $n = 4$ for guesstimating all roots of the nonlinear polynomial equation used in Example 3.

Table 4 Residual errors of simultaneous methods NIM10 and NIM12 for finding all roots of $f_3(r)$

Method	CPU	$e_1^{(4)}$	$e_2^{(4)}$	$e_3^{(4)}$	$e_4^{(4)}$
NIM10	0.016	0.1	0.05	0.20	1.9e–3
NIM12	0.015	1.1e–8	1.1e–8	8.0e–32	2.7e–32

5 Conclusion

We have developed here a family of three-step simultaneous methods of order 12 which is the highest order derivative-free simultaneous iterative method among existing methods in the literature. From Tables 1–4 and Fig. 1(a), (b), we observe that our family of derivative-free simultaneous methods NIM12 is admirable in terms of efficiency, CPU time, and residual errors as compared to the NIM10 method.

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