Abdalla et al. *Advances in Difference Equations* (2021) 2021:450
https://doi.org/10.1186/s13662-021-03612-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Certain fractional formulas of the extended k-hypergeometric functions

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Abstract
In this article, we aim to investigate various formulae for the (p,k)-analogues of Gauss hypergeometric functions, including the integral transforms and the operators of fractional calculus. All the outcomes presented here are of general attractiveness and can yield a number of previous works as special cases.

MSC: 33B15; 33C20; 33C90; 33E20

Keywords: (p,k)-hypergeometric functions; Integral transforms; Fractional calculus

1 Overture

Throughout this work, $\mathbb{N} := \{1,2,3,\dots\}$ denotes the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}^- := \{-1,-2,-3,\dots\}$ denotes the set of negative integers, $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$, $\mathbb{R}^+$ denotes the set of positive real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

In 1813, Gauss first summarized his investigations of hypergeometric functions, which has been of great significance in the mathematical modeling of physical phenomena and other applications. Recently, various developments and expansions of the hypergeometric functions have been proposed and discussed (for example, see [1–12]).

In [13], Diaz and Pariguan introduced an interesting extensions of the gamma, beta, Pochhammer, and hypergeometric functions as follows.

Definition 1.1 For $k \in \mathbb{R}^+$, the k-gamma function $\Gamma^k(y)$ is defined by

$$\Gamma^k(y) = \int_0^\infty u^{y-1} e^{-\frac{u^k}{k}}\, du, \tag{1.1}$$

where $y \in \mathbb{C} \setminus k\mathbb{Z}^-$. We note that $\Gamma^k(y) \to \Gamma(y)$ for $k \to 1$ where $\Gamma(y)$ is the classical Euler's gamma function and $(y)_{m,k}$ is the k-Pochhammer symbol given by

$$(y)_{n,k} = \frac{\Gamma^k(y+nk)}{\Gamma^k(y)} = \begin{cases} y(y+k)\dots(y+(n-1)k), & n \in \mathbb{N}, y \in \mathbb{C}, \\ 1, & n = 0, k \in \mathbb{R}^+, y \in \mathbb{C} \setminus \{0\}, \end{cases} \tag{1.2}$$

the relation between the $\Gamma^k(y)$ and the usual gamma function $\Gamma(y)$ follows easily as

$$\Gamma^k(y) = k^{\frac{y}{k}-1}\Gamma\left(\frac{y}{k}\right) \quad \text{or} \quad \Gamma(w) = k^{1-w}\Gamma^k(kw).$$

Definition 1.2 The k-beta function $\mathbf{B}^k(s,t)$ is defined by

$$\mathbf{B}^k(s,t) = \begin{cases} \frac{1}{k}\int_0^1 y^{\frac{s}{k}-1}(1-y)^{\frac{t}{k}-1}\,dy, & (k \in \mathbb{R}^+, \min\{\mathrm{Re}(s),\mathrm{Re}(t)\} > 0), \\ \frac{\Gamma^k(s)\Gamma^k(t)}{\Gamma^k(s+t)}, & (k \in \mathbb{R}^+, s,t \in \mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} \tag{1.3}$$

Clearly, the case $k = 1$ in (1.3) reduces to the known beta function $\mathbb{B}(s,t)$.

$$\mathbb{B}(s,t) = \int 1_0 y^{s-1}(1-y)^{t-1}\,dy.$$

Also, the relation between the k-beta function $\mathbf{B}^k(s,t)$ and the original beta function $\mathbb{B}(s,t)$ is

$$\mathbf{B}^k(s,t) = \frac{1}{k}\mathbb{B}\left(\frac{s}{k},\frac{t}{k}\right).$$

Definition 1.3 Let $k \in \mathbb{R}^+$ and $\alpha_1, \alpha_2, y \in \mathbb{C}$ and $\alpha_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-$, then k-hypergeometric series is defined by the form

$${}_2H_1^k\left[\begin{matrix}(\alpha_1;k),(\alpha_2;k) \\ (\alpha_3;k);\end{matrix}\ y\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}.\frac{y^n}{n!}, \quad |y| < \frac{1}{k}, \tag{1.4}$$

where $(\alpha_1)_{n,k}$ is the k-Pochhammer symbol given in (1.2).

Indeed, in their special case when $k = 1$, Eq. (1.4) is reduced to the Gauss hypergeometric function ${}_2H_1(\cdot)$. The ${}_2H_1(\cdot)$ is the special case of the generalized hypergeometric functions ${}_mH_n(\cdot)$ of m numerator and n denominator parameters defined by (see, e.g., [14, Sect. 1.5]):

$${}_mH_n\left[\begin{matrix}\alpha_1\ldots\alpha_m; \\ \delta_1\ldots\delta_n;\end{matrix}\ y\right] = \sum_{j=0}^{\infty}\frac{(\alpha_1)_j\ldots(\alpha_m)_j}{(\delta_1)_j\ldots(\delta_n)_j}.\frac{y^j}{j!}, \tag{1.5}$$

where

$$(\alpha_1)_n = \frac{\Gamma(\alpha_1+n)}{\Gamma(\alpha_1)} = \begin{cases}\alpha_1(\alpha_1+1)\ldots(\alpha_1+n-1), & i \in \mathbb{N}, \alpha_1 \in \mathbb{C}, \\ 1, & i = 0; \alpha_1 \in \mathbb{C}\setminus\{0\},\end{cases} \tag{1.6}$$

is the usual Pochhammer symbol (or the shifted factorial) and $\Gamma(\cdot)$ is the standard gamma function (see, e.g., [14, Sect. 1.1]).

Currently, several different outcomes concerning the k-analogue of special functions have been archived, the interested reader may refer to the monographs by many researchers (see, e.g., [15–22] and the references cited therein).

Recently, Abdalla and Hidan [23] and Hidan et al. [24] introduced and studied several properties of the following (p,k)-analogues of Gauss hypergeometric functions:

$$ {}_2\mathbf{H}_1^{(p,k)}\left[\begin{array}{c}(\alpha_1;k),(\alpha_2;k)\\(\alpha_3;k)\end{array};w\right]=\sum_{j=0}^{\infty}\frac{(\alpha_1)_{j,k}(\alpha_2)_{j,k}}{(\alpha_3)_{j,k}}\cdot\frac{w^j}{(pj)!}, \tag{1.7}$$

which is an entire function for $p>1$, where $k\in\mathbb{R}^+$ and $\alpha_1,\alpha_2,w\in\mathbb{C}$ and $\alpha_3\in\mathbb{C}\setminus\mathbb{Z}_0^-$, and $(\alpha)_{j,k}$ is the k-Pochhammer symbol defined in (1.2).

Remark 1.1 Among the important special cases of ${}_2\mathbf{H}_1^{(p,k)}$ are equations of type (1.4) and (1.5). Further, at $k=1$, we obtain the p-extended Gauss hypergeometric functions in the following form (cf. [25]):

$$ {}_2\mathbf{H}_1^{(p)}\left[\begin{array}{c}\alpha_1,\alpha_2\\\alpha_3\end{array};w\right]=\sum_{j=0}^{\infty}\frac{(\alpha_1)_{j}(\alpha_2)_{j}}{(\alpha_3)_{j}}\cdot\frac{w^j}{(pj)!},\quad p\in\mathbb{N}, \tag{1.8}$$

which is also an entire function for $p>1$.

The purpose of this work is to continue the investigation of new formulae like integral transforms and fractional calculus operators on the (p,k)-analogues of Gauss hypergeometric functions ${}_2\mathbf{H}_1^{(p,k)}$. In Sects. 2 and 3, respectively, we introduce several integral transforms and image formulae for the (p,k)-analogues of Gauss hypergeometric functions ${}_2\mathbf{H}_1^{(p,k)}$ by applying a certain integral transform (like Laplace transform and fractional Fourier transform) and diverse fractional operators. Also, some special cases and significance of our main outcomes are considered.

2 Integral transforms

In this section, we prove two theorems, which exhibit the connection between integral transforms like the Laplace transform and the fractional Fourier transform for the ${}_2\mathbf{H}_1^{(p,k)}$ given in (1.7). We recall the Laplace transform and the fractional Fourier transform, respectively.

Definition 2.1 (Laplace transform) Let $f(\xi)$ be a function of $\xi>0$. Then the Laplace transform of $f(\xi)$ is defined by

$$\mathbf{F}(s)=\mathcal{L}\{f(\xi):s\}=\int_0^{\infty}e^{-s\xi}f(\xi)\,d\xi,\quad \Re(s)>0, \tag{2.1}$$

provided that the improper integral exists, $e^{-\lambda\xi}$ is the kernel of the transformation, and the function $f(\xi)$ is called the inverse Laplace transform of $\mathbf{F}(\lambda)$ (see [26, Chap. 3]).

Definition 2.2 (Fractional Fourier transform) Assume that φ is a function belonging to Lizorkin space $\psi(R)$. The fractional Fourier transform (FFT) of order $\beta, 0<\beta\leq 1$, is defined as (cf. [26, 27])

$$\varphi_\beta(\omega)=\mathfrak{F}_\beta[\varphi](\omega)=\int_R e^{i\omega^{\frac{1}{\beta}}\xi}\varphi(\xi)\,d\xi,\quad i=\sqrt{-1}. \tag{2.2}$$

Remark 2.1 It may by observed that when $\omega > 0$ it reduces to the FFT introduced by Luchko, Martinez, and Trujillo (see, e.g., [27, pp. 225–240] for details). The relationship between the two Fourier transforms, the classical and FFT, is given by the following relation:

$$\mathfrak{F}_\beta[\varphi](\omega) = \mathfrak{F}[\varphi](s) \quad \text{for } s = \omega^{\frac{1}{\beta}}, 0 < \beta \le 1.$$

Theorem 2.1 *The Laplace transform for the* ${}_2\mathbf{H}_1^{(p,k)}$ *given by* (1.7) *is in the form*

$$\begin{aligned} &\mathcal{L}\left\{\xi^{\frac{\delta}{k}-1}{}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k) \\ (\alpha_3,k);\end{matrix}\quad u\xi\right]\right\} \\ &\quad = \frac{k\Gamma^k(\delta)}{(ks)^{\frac{\delta}{k}}}{}_3\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k)(\delta,k) \\ (\alpha_3,k);\end{matrix}\quad \frac{u}{ks}\right],\end{aligned} \tag{2.3}$$

($\alpha_1,\alpha_2,u,\xi \in \mathbb{C}$, $\alpha_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $\operatorname{Re}(\alpha_1) > 0$, $\operatorname{Re}(\alpha_2) > 0$, $\operatorname{Re}(s) > 0$, $|\frac{u}{ks}| < 1$, $k \in \mathbb{R}^+$ *and* $p \in \mathbb{N}$).

Proof Taking the left-hand side of Eq. (2.3) by $\mathfrak{I}$ and upon using (1.7), we have

$$\begin{aligned}\mathfrak{I} &= \int_0^\infty e^{-s\xi}\xi^{\frac{\delta}{k}-1}\sum_{n=0}^\infty \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}(u\xi)^n}{(\alpha_3)_{n,k}(pn)!}\,d\xi \\ &= \sum_{n=0}^\infty \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}\frac{u^n}{(pn)!}\left\{\int_0^\infty e^{-s\xi}\xi^{\frac{\delta}{k}+n-1}\,d\xi\right\}.\end{aligned}$$

Putting $s\xi = \frac{v^k}{k}$, we have

$$\begin{aligned}\mathfrak{I} &= \sum_{n=0}^\infty \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}\frac{u^n}{(np)!}\int_0^\infty e^{\frac{-v^k}{k}}\left(\frac{v^k}{ks}\right)^{\frac{\delta}{k}+n-1}\frac{v^{k-1}}{s}\,dv \\ &= \sum_{n=0}^\infty \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}\frac{u^n}{(np)!}\int_0^\infty \frac{e^{\frac{-v^k}{k}}v^{\delta+nk-1}}{k^{\frac{\delta}{k}+n-1}s^{\frac{\delta}{k}+n}} \\ &= \frac{k\Gamma^k(\delta)}{(ks)^{\frac{\delta}{k}}}\sum_{n=0}^\infty\left(\frac{u}{ks}\right)^n\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}(\delta)_{n,k}}{(\alpha_3)_{n,k}}\frac{1}{(pn)!}.\end{aligned}$$

We thus obtain the required result. □

Theorem 2.2 *For* $\alpha_1,\alpha_2,w \in \mathbb{C}$, $\alpha_3 \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $\operatorname{Re}(\alpha_1) > 0$, $\operatorname{Re}(\alpha_2) > 0$, $k \in \mathbb{R}^+$, $p \in \mathbb{N}$, *and* $0 < \beta \le 1$, *the following fractional Fourier transform* (*FFT*) *holds true*:

$$\begin{aligned}&\mathfrak{F}_\beta\left\{{}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k) \\ (\alpha_3,k)\end{matrix};w\right]\right) \\ &\quad = \sum_{n=0}^\infty \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}(\omega)^{-(\frac{n+1}{\beta})}(-1)^n(i)^{(n-1)} \\ &\qquad \times \frac{n!}{(pn)!}.\end{aligned} \tag{2.4}$$

Proof For convenience, let the left-hand side of (2.4) be denoted by T. Applying the fractional Fourier transform (2.2) to (1.7) when $w < 0$, we observe that

$$\begin{aligned}
\mathrm{T} &= \int_R e^{i\omega^{\frac{1}{\beta}} w} \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}(np)!} w^n \, dw \\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}} \int_{-\infty}^{0} e^{i\omega^{\frac{1}{\beta}} w} w^n \, dw.
\end{aligned}$$

Letting $-t = i\omega^{\frac{1}{\beta}} w$, we obtain

$$\begin{aligned}
\mathrm{T} &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}(np)!} \int_0^{\infty} e^{-t} \left(\frac{-t}{i\omega^{\frac{1}{\beta}}}\right)^n \left(\frac{dt}{i\omega^{\frac{1}{\beta}}}\right) \\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}(np)!} (-1)^n (i)^{-(n+1)} (\omega)^{\frac{-(n+1)}{\beta}} \Gamma(n+1),
\end{aligned}$$

which yields our required result (2.4). □

Corollary 2.1 *For $p = 1$, the FFT of k-Gauss hypergeometric function of order β is (see [28])*

$$\mathfrak{F}_\beta\left[{}_2\mathbf{H}_1^{(k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k)\\(\alpha_3,k)\end{matrix};w\right]\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}} (-1)^n (i)^{-(n+1)} (\omega)^{\frac{-(n+1)}{\beta}}.$$

Further, for $p = 1$ and $k = 1$, we get the FFT of Gauss hypergeometric function

$$\mathfrak{F}_\beta\left[{}_2\mathbf{H}_1\left[\begin{matrix}(\alpha_1)(\alpha_2)\\(\alpha_3)\end{matrix};w\right]\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n}(\alpha_2)_{n}}{(\alpha_3)_{n}} (-1)^n (i)^{-(n+1)} (\omega)^{\frac{-(n+1)}{\beta}}.$$

3 k-fraction calculus of the ${}_2\mathrm{H}_1^{(p,k)}$

Nowadays, computations of images of k-analogues of special functions under operators of k-fractional calculus have found significant importance and applications by many references (for instance, see [15–17, 28–40]).

The k-Riemann–Liouville fractional integral using k-gamma function is defined in [31] as follows:

$$\left(\mathbf{I}_k^{\upsilon} f(\tau)\right)(x) = \frac{1}{k\Gamma^k(\upsilon)} \int_0^x f(\tau)(x-\tau)^{\frac{\upsilon}{k}-1} \, d\tau, \quad \upsilon, k \in \mathbb{R}^+. \tag{3.1}$$

Therefore, the k-Riemann–Liouville fractional derivative of order υ is introduced in [29, 31] by

$$\mathbf{D}_k^{\upsilon}\left\{f(\eta)\right\} = D\left(\mathbf{I}_k^{(1-\upsilon)} f(\eta)\right); \quad 0 < \upsilon \le 1, D = \frac{d}{d\eta}. \tag{3.2}$$

Let $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{C}(\mathrm{Re}(\eta) > 0)$ and $x > 0$, then the generalized fractional calculus operator (the Marichev–Saigo–Maeda operator) is defined by (see [33, 41, 42])

$$\begin{aligned}\left(\mathbf{I}_{0,x}^{\alpha,\beta,\gamma,\delta,\eta}\right)f(x) &= \frac{x^{-\alpha}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\beta} \\ &\quad \times F_3 \left[\begin{matrix}\alpha & \beta & \gamma & \delta \\ & \eta & & \end{matrix}; 1-\frac{t}{x}, 1-\frac{x}{t}\right] f(t)\,dt,\end{aligned} \tag{3.3}$$

where F_3 denotes the Appell third function, known also as Horn's F_3-function for $(\max\{|z| < 1, |w|\} < 1)$ defined by the series

$$F_3 \left[\begin{matrix}\alpha & \beta & \gamma & \delta \\ & \eta & & \end{matrix}; z, w\right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_m (\delta)_n}{(\eta)_{m+n} m! n!} z^m w^n,$$

which reduces to the Gauss hypergeometric function as follows:

$$\begin{aligned}{}_2\mathbf{H}_1 \left[\begin{matrix}\alpha, \beta \\ \eta\end{matrix}; w\right] &= F \left[\begin{matrix}\alpha & \beta & \gamma & \delta \\ & \eta & & \end{matrix}; z, 0\right] = F_3 \left[\begin{matrix}\alpha & 0 & \gamma & \delta \\ & \eta & & \end{matrix}; z, w\right] \\ &= F_3 \left[\begin{matrix}\alpha & \beta & \gamma & 0 \\ & \eta & & \end{matrix}; z, w\right].\end{aligned}$$

The following image formula, which is required in the sequel, can be easily derived from the direct application of the fractional integral operator (3.3), (see, e.g., [41, 42]):

$$\begin{aligned}&\left(\mathbf{I}_{0,x}^{\alpha,\beta,\gamma,\delta,\eta} t^{\theta-1}\right)(x) \\ &\quad = \frac{\Gamma(\theta)\Gamma(\theta+\eta-\alpha-\beta-\gamma)\Gamma(\theta+\delta-\beta)}{\Gamma(\theta+\delta)\Gamma(\theta+\eta-\alpha-\beta)\Gamma(\theta+\eta-\beta-\gamma)} x^{\theta+\eta-\alpha-\beta-1},\end{aligned} \tag{3.4}$$

where $\mathrm{Re}(\eta) > 0$, $\mathrm{Re}(\theta) > \max\{0, \mathrm{Re}(\alpha+\beta+\gamma-\eta), \mathrm{Re}(\beta-\delta)\}$.

Here, we aim at establishing certain new image formulas for the (p,k)-analogues of Gauss hypergeometric functions by applying the k-fractional derivative by (3.2) and left-sided operator of Marichev–Saigo–Maeda fractional integral defined by (3.3). On account of the general nature of the hypergeometric functions, a number of known formulas can easily be found as special cases of our main outcomes.

Theorem 3.1 *For* $\alpha_1, \alpha_2, \nu, u \in \mathbb{C}$, $\alpha_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathrm{Re}(\alpha_1) > 0$, $\mathrm{Re}(\alpha_2) > 0$, $k \in \mathbb{R}^+$, $p \in \mathbb{N}$, *and* $0 < \Re(\nu) \leq 1$, *we have*

$$\begin{aligned}&\mathbf{D}_k^{\nu} \left\{ u^{\frac{\delta}{k}} {}_2\mathbf{H}_1^{(p,k)} \left[\begin{matrix}(\alpha_1,k)(\alpha_2,k) \\ (\alpha_3,k)\end{matrix}; u\right]\right\} \\ &\quad = \frac{\lambda\Gamma^k(\lambda)}{k\Gamma^k(1-\nu+\delta)} u^{\frac{1-\nu+\delta}{k}-1} {}_3\mathbf{H}_2^{(p,k)} \left[\begin{matrix}(\alpha_1,k)(\alpha_2,k)(\delta+k,k) \\ (\alpha_3,k)(1-\nu+\delta,k)\end{matrix}; u\right].\end{aligned} \tag{3.5}$$

Proof From (1.7) and (3.2), we observe that

$$
\begin{aligned}
&\mathbf{D}_k^{\nu}\left[u^{\frac{\delta}{k}}{}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k)\\(\alpha_3,k)\end{matrix};u\right]\right]\\
&=\frac{d}{du}\left[\mathbf{I}_k^{1-\nu}u^{\frac{\delta}{k}}{}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k)\\(\alpha_3,k)\end{matrix};t\right]\right]dt\\
&=\frac{d}{du}\frac{1}{k\Gamma^k(1-\nu)}\int_0^u(u-t)^{\frac{1-\nu}{k}-1}t^{\frac{\delta}{k}}{}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k)\\(\alpha_3,k)\end{matrix};t\right]]\,dt.
\end{aligned}
$$

Putting $t=ux$ in the above equation and after simple computations, we arrive at

$$
\begin{aligned}
&\mathbf{D}_k^{\nu}\left[u^{\frac{\delta}{k}}{}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1,k)(\alpha_2,k)\\(\alpha_3,k)\end{matrix};u\right]\right]\\
&=\frac{1}{k\Gamma^k(1-\nu)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}(np)!}\frac{d}{du}\int_0^1(u-ux)^{\frac{1-\nu}{k}-1}(ux)^{n+\frac{\delta}{k}}u\,dx\\
&=\frac{1}{k\Gamma^k(1-\nu)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}(np)!}\frac{d}{du}\int_0^1(1-x)^{\frac{1-\nu}{k}-1}(x)^{n+\frac{\delta}{k}}u^{\frac{1-\nu+\delta+nk}{k}}\,dx\\
&=\frac{1}{k\Gamma^k(1-\nu)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}(np)!}\frac{d}{du}u^{\frac{1-\nu+\delta+nk}{k}}\mathbb{B}\left(\frac{1-\nu}{k},n+\frac{\delta}{k}+1\right)\\
&=\frac{1}{k\Gamma^k(1-\nu)}\sum_{n=0}^{\infty}\left[\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}(np)!}\left(\frac{1-\nu+\delta+nk}{k}\right)u^{\frac{1-\nu+\delta+nk}{k}-1}\frac{\Gamma(\frac{1-\nu}{k})\Gamma(\frac{nk+\delta+k}{k})}{(\frac{1-\nu+\lambda+nk+}{k})}\right]\\
&=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}}\frac{\Gamma^k(nk+\delta+k)}{k\Gamma^k(1-\nu+\delta+nk)}\frac{u^{\frac{1-\nu+\delta+nk}{k}-1}}{(np)!}\\
&=u^{\frac{1-\nu+\delta}{k}-1}\frac{\delta\Gamma^k(\delta)}{k\Gamma^k(1-\nu+\delta)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}(\delta+k)_{n,k}}{(\alpha_3)_{n,k}(1-\nu+\delta)_{n,k}}\frac{u^n}{(np)!}.
\end{aligned}
$$

This completes the proof of Theorem 3.1. □

Theorem 3.2 *Assume that* $\alpha,\beta,\gamma,\delta,\eta,\vartheta,\alpha_1,\alpha_2\in\mathbb{C}$, $\alpha_3\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $x>0$, $k\in\mathbb{R}^+$, *and* $p\in\mathbb{N}$ *such that* $\Re(\frac{\vartheta}{k})>\max\{0,\Re(\beta-\delta),\Re(\alpha+\beta+\gamma-\eta)\}$, *then we have*

$$
\begin{aligned}
&\left(\mathbf{I}_{0,x}^{\alpha,\beta,\gamma,\delta,\eta}w^{\frac{\vartheta}{k}-1}{}_2\mathbf{H}_1^{(p,k)}\left[\begin{matrix}(\alpha_1;k),(\alpha_2;k)\\(\alpha_3;k)\end{matrix};w\right]\right)(x)\\
&=k^{\eta}x^{-\alpha-\beta+\eta+\frac{\vartheta}{k}-1}\frac{\Gamma^k(\vartheta)\Gamma^k(\vartheta-k\beta+k\delta)\Gamma^k(\vartheta-k\alpha-k\beta-k\gamma+k\eta)}{\Gamma^k(\vartheta+k\delta)\Gamma^k(\vartheta-k\alpha-k\beta+k\eta)\Gamma^k(\vartheta-k\beta-k\gamma+k\eta)}\\
&\times{}_5\mathbf{H}_4^{(p,k)}\left[\begin{matrix}(\alpha_1;k) & (\alpha_2;k) & (\vartheta;k) & (\vartheta-k\beta+k\delta;k) & (\vartheta-k\alpha-k\beta-k\gamma+k\eta;k)\\ & (\alpha_3;k) & (\vartheta+k\delta;k) & (\vartheta-k\alpha-k\beta+k\eta;k) & (\vartheta-k\beta-k\gamma+k\eta;k)\end{matrix};x\right].
\end{aligned}
\tag{3.6}
$$

Proof We indicate the left-hand side of(3.6) by Υ, and invoking to Eqs. (3.3) and (1.7), we find

$$\begin{aligned}\Upsilon &= \left(\mathbf{I}_{0,x}^{\alpha,\beta,\gamma,\delta,\eta} w^{\frac{\vartheta}{k}-1} \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}} \frac{w^n}{(pn)!}\right)(x) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}} \frac{1}{(pn)!} \left(\mathbf{I}_{0,x}^{\alpha,\beta,\gamma,\delta,\eta} w^{\frac{\vartheta}{k}+n-1}\right)(x) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}} \frac{\Gamma(\frac{\vartheta}{k}+n)}{(pn)!\Gamma(\delta+\frac{\vartheta}{k}+n)} \\ &\quad\times \frac{\Gamma(-\beta+\delta+\frac{\vartheta+nk}{k})\Gamma(-\alpha-\beta-\gamma+\eta+\frac{\vartheta+nk}{k})}{\Gamma(-\alpha-\beta+\eta+\frac{\vartheta+nk}{k})\Gamma(-\beta-\gamma+\eta+\frac{\vartheta+nk}{k})} x^{-\alpha-\beta+\eta+\frac{\vartheta+nk}{k}-1}.\end{aligned}$$

Upon using (3.4) and after a simplification, we get the following expression:

$$\begin{aligned}\Upsilon &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k}(\alpha_2)_{n,k}}{(\alpha_3)_{n,k}} \frac{x^n k^\eta}{(pn)!} \frac{(\vartheta)_{n,k}\Gamma^k(\vartheta)}{(\vartheta+k\delta)_{n,k}\Gamma^k(\vartheta+k\delta)} \\ &\quad\times \frac{(\vartheta-k\beta+k\delta)_{n,k}\Gamma^k(-k\beta+k\delta+\vartheta)}{(\vartheta-k\alpha-k\beta+k\eta)_{n,k}\Gamma^k(\vartheta-k\alpha-k\beta+k\eta)} \\ &\quad\times \frac{(\vartheta-k\alpha-k\beta+k\eta)_{n,k}}{(\vartheta-k\beta-k\gamma+k\eta)_{n,k}} \cdot \frac{\Gamma^k(\vartheta-k\alpha-k\beta-k\gamma+k\eta)}{\Gamma^k(\vartheta-k\beta-k\gamma+k\eta)} x^{-\alpha-\beta+\eta+\frac{\vartheta}{k}-1},\end{aligned}$$

whose last summation, in view of (1.2), is easily seen to arrive at the expression in (3.6). This completes the proof of Theorem 3.2. $\square$

4 Conclusion

Recently, the applications and importance of integral transforms and fractional calculus operators involving a variety of special functions have received more attention in various fields like mathematical analysis, survival analysis, physics, statistics, and engineering. In fact, this manuscript is a continuation of the recent authors' articles [23, 24], where we have introduced the (p,k)-analogues of hypergeometric functions and their various properties and applications. In this line of research, we have derived integral transforms and image formulas for the (p,k)-analogues of hypergeometric functions. We also have considered that by setting $p \to 1$, the various outcomes considered in this manuscript reduce to the corresponding outcomes (see [28, 31, 36]). Also, for $k \to 1$, we obtain many interesting new outcomes for the p-extended hypergeometric functions. Further, if we take both $k \to 1$ and $p \to 1$, then the obtained results reduce to the results analogous to the usual hypergeometric functions. This approach allows the related research work to be reported in further articles. Additionally, all the outcomes presented here are expected to find some applications in control theory and to the solutions of fractional-order systems, for instance, see [43–47] and the references cited therein.

Acknowledgements
The third and fourth authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Group Project under Grant No. (R.G.P-1/95/42).

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. They have all read and approved the final manuscript.

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Publisher's Note

Received: 16 August 2021 Accepted: 29 September 2021 Published online: 14 October 2021

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