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Approximation by a power series summability method of Kantorovich type Szász operators including Sheffer polynomials

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Abstract
The main purpose of this paper is to use a power series summability method to study some approximation properties of Kantorovich type Szász–Mirakyan operators including Sheffer polynomials. We also establish Voronovskaya type result.

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1 Introduction and background

Let $\mathcal{K}_m = \{i \le m : i \in \mathcal{K} \subseteq \mathbb{N}\}$. Then the natural density of $\mathcal{K}$ is defined by $\sigma(\mathcal{K}) = \lim_m \frac{1}{m}|\mathcal{K}_m|$ provided the limit exists, where $|\mathcal{K}_m|$ denotes the cardinality of $\mathcal{K}_m$. A sequence $\eta = (\eta_i)$ is "statistically convergent" (see [9]) to $\mathfrak{s}$ if for every $\epsilon > 0$

$$\lim_m \frac{1}{m}\left|\left\{i \le m : |\eta_i - \mathfrak{s}| \ge \epsilon\right\}\right| = 0$$

and we write $st - \lim_m \eta_m = \mathfrak{s}$.

Let $\mathfrak{T} = (\mathfrak{d}_{ij})$ be an infinite matrix. It is said to be regular if it transforms a convergent sequence into a convergent one with the same limit.

Let $\mathfrak{T} = (\mathfrak{d}_{ij})$ be regular matrix. A sequence $\zeta = (\zeta_j)$ is said to be $\mathfrak{T}$-statistically convergent (see [10]) to the number $\mathfrak{s}$ if, for any $\epsilon > 0$, $\lim_i \sum_{j:|\eta_j - \mathfrak{s}| \ge \epsilon} \mathfrak{d}_{ij} = 0$, and denote $st_{\mathfrak{T}} - \lim \eta = \mathfrak{s}$. If

$$\mathfrak{d}_{ij} = \begin{cases} \frac{1}{j}, & i \le j, \\ 0; & i > j. \end{cases} \tag{1.1}$$

Then it reduces to statistical convergence.

For a sequence of positive real numbers $(\mathfrak{p}_j)$, denote the corresponding power series $\mathfrak{p}(\mathfrak{y}) = \sum_{j=1}^{\infty} \mathfrak{p}_j \mathfrak{y}^{j-1}$ which has radius of convergence $R > 0$. A sequence $\eta = (\eta_j)$ is convergent in the sense of power series method (see [12, 21]) if $\lim_{\mathfrak{y}\to R^-} \frac{1}{\mathfrak{p}(\mathfrak{y})} \sum_{j=1}^{\infty} \eta_j \mathfrak{p}_j \mathfrak{y}^{j-1} = \mathcal{L}$ for all $\mathfrak{y} \in (0, R)$. Moreover, the power series method is regular if and only if $\lim_{\mathfrak{y}\to R^-} \frac{\mathfrak{p}_j \mathfrak{y}^{j-1}}{\mathfrak{p}(\mathfrak{y})} = 0$ holds for each $j \in \{1, 2, \ldots\}$ (see [2]). The power series method is more effective than the ordinary convergence (see [22, 23]). For more summability methods, see [3–5, 7, 13, 15–19].

We study a Korovkin type theorem for the Kantorovich type generalization of Szász operators involving Sheffer polynomials via power series method. We determine the rate of convergence for these operators. Furthermore, we give a Voronovskaya type theorem for $\mathfrak{T}$-statistical convergence.

The multiple Sheffer polynomials $\{S_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ are defined as follows. The generating function is

$$A(t_1, t_2) e^{xH(t_1,t_2)} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} S_{k_1,k_2}(x) \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}, \tag{1.2}$$

where $A(t_1, t_2)$ and $H(t_1, t_2)$ have series expansions of the form

$$A(t_1, t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} a_{k_1,k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!} \tag{1.3}$$

and

$$H(t_1, t_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} h_{k_1,k_2} \frac{t_1^{k_1} t_2^{k_2}}{k_1! k_2!}, \tag{1.4}$$

respectively, with the conditions

$$A(0,0) = a_{0,0} \neq 0 \quad \text{and} \quad H(0,0) = h_{0,0} \neq 0.$$

In [1], one defined the positive linear operators involving multiple Sheffer polynomials for $x \in [0, \infty)$ as follows:

$$G_n(f, x) = \frac{e^{-\frac{nx}{2} H(1,1)}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1! k_2!} f\left(\frac{k_1 + k_2}{n}\right), \tag{1.5}$$

provided that the right-hand side of the above series converge, under conditions that:

(1) $S_{k_1,k_2}(x) \geq 0, k_1, k_2 \in \mathbb{N}$,
(2) $A(1,1) \neq 0, H_{t_1}(1,1) = 1, H_{t_2}(1,1) = 1$,
(3) Series (1.2), (1.3) and (1.4) are convergent for $|t_1| < R$, $|t_2| < R$ and $(R_1, R_2) > 1$.

In [6] one defined the Kantorovich variant of Szász operators induced by multiple Sheffer polynomials as follows:

$$K_n^{(S)}(f, x) = \frac{ne^{-\frac{nx}{2} H(1,1)}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1! k_2!} \int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}} f(t)\, dt, \quad x \in [0, \infty), \tag{1.6}$$

provided that the right-hand side of the above relation exists.

Example 1.1 of [6], gives us the following expressions for moments of the Kantorovich variant of Szász operators induced by multiple Sheffer polynomials:

$$K_n^{(S)}(1,x) = 1,$$

$$K_n^{(S)}(t,x) = \frac{1}{2}\cdot\frac{2aa_{0,1}+2aa_{1,0}+aa_{0,0}}{aa_{0,0}n} + x,$$

$$K_n^{(S)}\left(t^2,x\right) = \frac{1}{3}\cdot\frac{3aa_{0,2}+3aa_{0,1}+3aa_{0,2}+6aa_{1,1}+3aa_{1,0}+aa_{0,0}}{aa_{0,0}n^2} + \frac{x}{2}\cdot\frac{aa_{0,0}hh_{0,2}+2aa_{0,0}hh_{1,1}+aa_{0,0}hh_{2,0}+2aa_{0,0}+4aa_{0,1}+4aa_{1,0}}{aa_{0,0}n} + x^2.$$

The central moments of the Kantorovich variant of Szász operators induced by multiple Sheffer polynomials are [6]

$$K_n^{(S)}(t-x,x) = (2\tilde{a}_{0,1}+2\tilde{a}_{1,0}+\tilde{a}_{0,0})\frac{1}{2\tilde{a}_{0,0}n},$$

$$K_n^{(S)}\left((t-x)^2,x\right) = (3\tilde{a}_{0,2}+6\tilde{a}_{1,1}+3\tilde{a}_{1,0}+3\tilde{a}_{2,0}+3\tilde{a}_{0,1}+\tilde{a}_{0,0})\frac{1}{3\tilde{a}_{0,0}n^2} + (\tilde{h}_{0,2}+2\tilde{h}_{1,1}+\tilde{h}_{2,0})\frac{x}{2n},$$

$$\begin{aligned}K_n^{(S)}&\left((t-x)^3,x\right)\\ &= (6\tilde{a}_{0,2}+4\tilde{a}_{3,0}+6\tilde{a}_{2,0}+4\tilde{a}_{1,0}+4\tilde{a}_{0,3}+12\tilde{a}_{2,1}+\tilde{a}_{0,0}+4\tilde{a}_{0,1}+12\tilde{a}_{1,2}\\ &\quad+12\tilde{a}_{1,1})\frac{1}{4\tilde{a}_{0,0}n^3} + (3\tilde{a}_{0,0}\tilde{h}_{0,2}+2\tilde{a}_{0,0}\tilde{h}_{0,3}+6\tilde{a}_{0,0}\tilde{h}_{1,1}+6\tilde{a}_{0,0}\tilde{h}_{1,2}+3\tilde{a}_{0,0}\tilde{h}_{2,0}\\ &\quad+6\tilde{a}_{0,0}\tilde{h}_{2,1}+2\tilde{a}_{0,0}\tilde{h}_{3,0}+6\tilde{a}_{0,1}\tilde{h}_{0,2}+12\tilde{a}_{0,1}\tilde{h}_{1,1}+6\tilde{a}_{0,1}\tilde{h}_{2,0}+6\tilde{a}_{1,0}\tilde{h}_{0,2}\\ &\quad+12\tilde{a}_{1,0}\tilde{h}_{1,1}+6\tilde{a}_{1,0}\tilde{h}_{2,0})\frac{x}{4\tilde{a}_{0,0}n^2}.\end{aligned}$$

Similarly, there exist constants C_{di} (dependent only on $\tilde{a}_{i,j}$ and $\tilde{h}_{i,j}$) such that

$$K_n^{(S)}\left((t-x)^4,x\right) = \frac{C_{44}}{4\tilde{a}_{0,0}n^4} + \frac{xC_{43}}{2\tilde{a}_{0,0}n^3} + \frac{3x^2C_{42}}{4n^2},$$

$$K_n^{(S)}\left((t-x)^5,x\right) = \frac{C_{55}}{6\tilde{a}_{0,0}n^5} + \frac{xC_{54}}{12\tilde{a}_{0,0}n^4} + \frac{5x^2C_{53}}{8\tilde{a}_{0,0}n^3},$$

$$K_n^{(S)}\left((t-x)^6,x\right) = \frac{C_{66}}{7\tilde{a}_{0,0}n^6} + \frac{xC_{65}}{2\tilde{a}_{0,0}n^5} + \frac{5x^2C_{64}}{4\tilde{a}_{0,0}n^4}. + \frac{15x^2C_{63}}{8n^3}.$$

As a consequence of the above relations, we obtain

$$\lim_{n\to\infty} nK_n^{(S)}(t-x,x) = \frac{2\tilde{a}_{0,1}+2\tilde{a}_{1,0}+\tilde{a}_{0,0}}{2\tilde{a}_{0,0}},$$

$$\lim_{n\to\infty} nK_n^{(S)}\left((t-x)^2,x\right) = \frac{(\tilde{h}_{0,2}+2\tilde{h}_{1,1}+\tilde{h}_{2,0})x}{2},$$

$$\lim_{n\to\infty} n^2K_n^{(S)}\left((t-x)^3,x\right) = E_3x, \qquad \lim_{n\to\infty} n^2K_n^{(S)}\left((t-x)^4,x\right) = E_4x^2,$$

$$\lim_{n\to\infty} n^3K_n^{(S)}\left((t-x)^5,x\right) = E_5x^3, \qquad \lim_{n\to\infty} n^3K_n^{(S)}\left((t-x)^6,x\right) = E_6x^3,$$

where E_3, E_4, E_5, E_6 are constant dependent on the derivatives of $A(t_1,t_2)$ and $H(t_1,t_2)$ up to order three at the point $(t_1,t_2) = (1,1)$.

2 Korovkin type results

The statistical form of Korovkin's theorem was studied in [11] and the A-statistical version was considered in [8] (see also [13, 17] for other summability methods).

Let $B[0,\infty)$ ($C[0,\infty)$) be "the space of all bounded (continuous) functions" on the interval $[0,\infty)$.

Theorem 2.1 *Let $\mathfrak{T} = (\mathfrak{d}_{ij})$ be regular matrix and $K_n^{(S)}(f,x)$ be as in (1.6) on $[0,M]$, for any finite M. If*

$$st_{\mathfrak{T}} - \lim_n \left\| K_n^{(S)}(f,x)e_i - e_i \right\| = 0 \quad (i = 1,2),$$

then

$$st_{\mathfrak{T}} - \lim_n \left\| K_n^{(S)}(f,x)\mathfrak{h} - \mathfrak{h} \right\| = 0,$$

$\mathfrak{h} \in C([0,M])$, *where* $\|\mathfrak{h}\| = \sup_{t\in[0,M]} |\mathfrak{h}(t)|$.

Proof From Example 1.1 of [6], we have $st_{\mathfrak{T}} - \lim_n \|K_n^{(S)}e_0 - e_0\| = 0$. Now

$$\left\| K_n^{(S)}e_1 - e_1 \right\| \leq \left\| \frac{1}{2} \cdot \frac{2aa_{0,1} + 2aa_{1,0} + aa_{0,0}}{aa_{0,0}n} \right\|.$$

Also $\lim_{n\to\infty} \|K_n^{(S)}e_1 - e_1\| = 0$. Moreover,

$$\begin{aligned} &\left\| K_n^{(S)}e_2 - e_2 \right\| \\ &\quad = \left\| \frac{1}{3} \cdot \frac{3aa_{0,2} + 3aa_{0,1} + 3aa_{0,2} + 6aa_{1,1} + 3aa_{1,0} + aa_{0,0}}{aa_{0,0}n^2} \right. \\ &\qquad \left. + \frac{x}{2} \cdot \frac{aa_{0,0}hh_{0,2} + 2aa_{0,0}hh_{1,1} + aa_{0,0}hh_{2,0} + 2aa_{0,0} + 4aa_{0,1} + 4aa_{1,0}}{aa_{0,0}n} \right\| \to 0, \end{aligned}$$

as $n \to \infty$. Now the proof follows directly from the statistical version of the Korovkin theorem [11]. □

Example 2.2 ([14]) Under the conditions given in Theorem 2.1, set

$$N_n(h,x) = (1 + u_n)K_n^{(S)}(h,x),$$

where

$$u_n = \begin{cases} 1; & m^2 - m \leq n \leq m^2 - 1, \\ \frac{1}{m^4}; & n = m^2; m \in \mathbb{N} \setminus \{1\}, \\ 0; & \text{otherwise,} \end{cases}$$

then

$$N_n(e_0,x) = (1+u_n),$$
$$N_n(e_1,x) = (1+u_n)\left(\frac{1}{2}\cdot\frac{2aa_{0,1}+2aa_{1,0}+aa_{0,0}}{aa_{0,0}n}+x\right),$$
$$N_n(e_2,x) = (1+u_n)\left(\frac{1}{3}\cdot\frac{3aa_{0,2}+3aa_{0,1}+3aa_{0,2}+6aa_{1,1}+3aa_{1,0}+aa_{0,0}}{aa_{0,0}n^2}\right.$$
$$\left.+\frac{x}{2}\cdot\frac{aa_{0,0}hh_{0,2}+2aa_{0,0}hh_{1,1}+aa_{0,0}hh_{2,0}+2aa_{0,0}+4aa_{0,1}+4aa_{1,0}}{aa_{0,0}n}+x^2\right).$$

If the matrix $\mathfrak{T}$ is as in (1.1), then, by Theorem 2.1 we obtain $st_{\mathfrak{T}}-\lim_n\|N_nh-h\|=0$, but the operators $N_n(h,x)$, do not satisfy the conditions of the theorem in [11].

In the following result we use a power series method as in [20, 24]; the Abel summability method was used.

Theorem 2.3 *Let* $(K_n^{(S)})$ *be a sequence of positive linear operators from* $C[0,M]$ *into* $B[0,M]$ $(0<M<\infty)$ *such that*

$$\lim_{t\to R^-}\frac{1}{\mathfrak{p}(t)}\left\|\sum_{n=0}^{\infty}\left(K_n^{(S)}e_i-e_i\right)\mathfrak{p}_nt^n\right\|=0,\quad i=0,1,2. \tag{2.1}$$

Then, for $\mathfrak{h}\in C[0,M]$,

$$\lim_{t\to R^-}\frac{1}{\mathfrak{p}(t)}\left\|\sum_{n=0}^{\infty}\left(K_n^{(S)}\mathfrak{h}-\mathfrak{h}\right)\mathfrak{p}_nt^n\right\|=0. \tag{2.2}$$

Proof Clearly, from (2.2) follows (2.1). Now we show the converse that (2.1) implies (2.2). Let $\mathfrak{h}\in C[0,M]$. Then there exists a constant $K>0$ such that $|\mathfrak{h}(u)|\le K$ for all $u\in[0,M]$. Therefore

$$\left|\mathfrak{h}(u)-\mathfrak{h}(x)\right|\le 2K,\quad u\in[0,M]. \tag{2.3}$$

For every given $\epsilon>0$, there exists a $\delta>0$ such that

$$\left|\mathfrak{h}(u)-\mathfrak{h}(x)\right|\le\epsilon \tag{2.4}$$

whenever $|u-x|<\delta$ for all $u\in[0,M]$. Define $\psi\equiv\psi(u,x)=(u-x)^2$. If $|u-x|\ge\delta$, then

$$\left|\mathfrak{h}(u)-\mathfrak{h}(x)\right|\le\frac{2K}{\delta^2}\psi(u,x). \tag{2.5}$$

From (2.3)–(2.5), we have $|\mathfrak{h}(u)-\mathfrak{h}(x)|<\epsilon+\frac{2K}{\delta^2}\psi(u,x)$, namely,

$$-\epsilon-\frac{2K}{\delta^2}\psi(u,x)<\mathfrak{h}(t)-\mathfrak{h}(x)<\frac{2K}{\delta^2}\psi(u,x)+\epsilon.$$

By applying the operator $K_n^{(S)}(1,x)$, $K_n^{(S)}(1,x)$ is a monotone and linear operator, we obtain

$$K_n^{(S)}(1,x)\left(-\epsilon-\frac{2K}{\delta^2}\psi\right)<K_n^{(S)}(1,x)\big(\mathfrak{h}(u)-\mathfrak{h}(x)\big)<K_n^{(S)}(1,x)\left(\frac{2K}{\delta^2}\psi+\epsilon\right),$$

which implies

$$\begin{aligned}-\epsilon K_n^{(S)}(1,x)-\frac{2K}{\delta^2}K_n^{(S)}\big(\psi(u),x\big)&<K_n^{(S)}(\mathfrak{h},x)-\mathfrak{h}(x)K_n^{(S)}(1,x)\\&<\frac{2K}{\delta^2}K_n^{(S)}\big(\psi(u),x\big)+\epsilon K_n^{(S)}(1,x).\end{aligned}\tag{2.6}$$

On the other hand

$$K_n^{(S)}(\mathfrak{h},x)-\mathfrak{h}(x)=K_n^{(S)}(\mathfrak{h},x)-\mathfrak{h}(x)K_n^{(S)}(1,x)+\mathfrak{h}(x)\big[K_n^{(S)}(1,x)-1\big].\tag{2.7}$$

From (2.6) and (2.7) we get

$$K_n^{(S)}(\mathfrak{h},x)-\mathfrak{h}(x)<\frac{2K}{\delta^2}K_n^{(S)}\big(\psi(u),x\big)+\epsilon K_n^{(S)}(1,x)+\mathfrak{h}(x)\big[K_n^{(S)}(1,x)-1\big].\tag{2.8}$$

Now we estimate the following expression:

$$\begin{aligned}K_n^{(S)}\big(\psi(u),x\big)&=K_n^{(S)}\big((x-u)^2,x\big)=K_n^{(S)}\big(\big(x^2-2xu+u^2\big),x\big)\\&=x^2K_n^{(S)}(1,x)-2xK_n^{(S)}(u,x)+K_n^{(S)}\big(u^2,x\big).\end{aligned}$$

By (2.8), we obtain

$$\begin{aligned}K_n^{(S)}(\mathfrak{h},x)-\mathfrak{h}(x)&<\frac{2K}{\delta^2}\big\{x^2\big[K_n^{(S)}(1,x)-1\big]-2x\big[K_n^{(S)}(u,x)-x\big]\\&\quad+\big[K_n^{(S)}\big(u^2,x\big)-x^2\big]\big\}+\epsilon K_n^{(S)}(1,x)+f(x)\big[K_n^{(S)}(1,x)-1\big]\\&=\epsilon+\epsilon\big[K_n^{(S)}(1,x)-1\big]+\mathfrak{h}(x)\big[K_n^{(S)}(1,x)-1\big]\\&\quad+\frac{2K}{\delta^2}\big\{x^2\big[K_n^{(S)}(1,x)-1\big]-2x\big[K_n^{(S)}(u,x)-x\big]+\big[K_n^{(S)}\big(u^2,x\big)-x^2\big]\big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\big|K_n^{(S)}(\mathfrak{h},x)-\mathfrak{h}(x)\big|&\le\epsilon+\left(\epsilon+K+\frac{2KM^2}{\delta^2}\right)\big|K_n^{(S)}(1,x)-1\big|\\&\quad+\frac{4KM}{\delta^2}\big|K_n^{(S)}(u,x)-x\big|+\frac{2K}{\delta^2}\big|K_n^{(S)}\big(u^2,x\big)-x^2\big|.\end{aligned}$$

From the above relations and the linearity of $K_n^{(S)}$, we obtain

$$\begin{aligned}&\frac{1}{\mathfrak{p}(\nu)}\left\|\sum_{n=0}^{\infty}\big(U_{n,p}(\mathfrak{h};x)-\mathfrak{h}(x)\big)\mathfrak{p}_n\nu^n\right\|\\&\quad\le\epsilon+\left(\epsilon+K+\frac{2KM^2}{\delta^2}\right)\frac{1}{\mathfrak{p}(t)}\left\|\sum_{n=0}^{\infty}\big(K_n^{(S)}(1;x)-1\big)\mathfrak{p}_nt^n\right\|\end{aligned}$$

$$+\frac{4KM}{\delta^2}\frac{1}{\mathfrak{p}(\nu)}\left\|\sum_{n=0}^{\infty}\left(K_n^{(S)}(u;x)-x\right)\mathfrak{p}_n\nu^n\right\|+\frac{2K}{\delta^2}\frac{1}{\mathfrak{p}(\nu)}\left\|\sum_{n=0}^{\infty}\left(K_n^{(S)}\left(u^2;x\right)-x^2\right)\mathfrak{p}_n\nu^n\right\|.$$

Hence, (2.2) follows from the last relation and (2.1). □

3 Rate of convergence

The modulus of continuity is defined by

$$\omega(\mathfrak{h},\delta)=\sup_{|h|<\delta}\left|\mathfrak{h}(x+h)-\mathfrak{h}(x)\right|,\quad \mathfrak{h}(x)\in C[0,M]\cap E.$$

Note that

$$\left|\mathfrak{h}(x)-\mathfrak{h}(y)\right|\le\omega(\mathfrak{h},\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{3.1}$$

Theorem 3.1 *Let $\mathfrak{T}=(\mathfrak{a}_{ij})$ be regular and $\mathfrak{h}\in C[0,M]$. If (α_n) is a sequence of positive real numbers such that $\omega(\mathfrak{h},\delta_n)=st_{\mathfrak{T}}-0(\alpha_n)$, then*

$$\left\|K_n^{(S)}\mathfrak{h}-\mathfrak{h}\right\|=st_{\mathfrak{T}}-0(\alpha_n),$$

where

$$\begin{aligned}\delta_n=\Bigg\{\frac{1}{3}\cdot\left\|\frac{3aa_{0,2}+3aa_{0,1}+3aa_{0,2}+6aa_{1,1}+3aa_{1,0}+aa_{0,0}}{aa_{0,0}n^2}\right\|\\ +M\left[\left\|\frac{1}{2}\cdot\frac{aa_{0,0}hh_{0,2}+2aa_{0,0}hh_{1,1}+aa_{0,0}hh_{2,0}+2aa_{0,0}+4aa_{0,1}+4aa_{1,0}}{aa_{0,0}n}\right\|\right.\\ \left.+\left\|\frac{2aa_{0,1}+2aa_{1,0}+aa_{0,0}}{aa_{0,0}n}\right\|\right]\Bigg\}^2,\end{aligned}$$

for any positive integer n.

Proof By (3.1), we see

$$\begin{aligned}&\left|K_n^{(S)}(\mathfrak{h};x)-\mathfrak{h}\right|\\ &\quad\le K_n^{(S)}\left(\left|\mathfrak{h}(t)-\mathfrak{h}(x)\right|;x\right)\\ &\quad\le\frac{ne^{-\frac{nx}{2}H(1,1)}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}\omega(\mathfrak{h},\delta)\left(1+\frac{|t-x|}{\delta}\right)dt\\ &\quad\le\omega(\mathfrak{h},\delta)\left[1+\frac{1}{\delta}\frac{ne^{-\frac{nx}{2}H(1,1)}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{S_{k_1,k_2}(\frac{nx}{2})}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}\left(|t-x|\right)dt\right],\quad\text{see [6]}\\ &\quad=\omega(\mathfrak{h},\delta)\left[1+\frac{1}{\delta}K_n^{(S)}\left(|t-x|;x\right)\right].\end{aligned}$$

By applying the Cauchy–Schwartz inequality, we have

$$\left|K_n^{(S)}(\mathfrak{h};x)-\mathfrak{h}\right|\le\omega(\mathfrak{h},\delta)\left[1+\frac{1}{\delta}\left(K_n^{(S)}\left(|t-x|^2;x\right)\right)^{\frac{1}{2}}\right].$$

From Example 1.1 of [6], we obtain

$$\begin{aligned}
&K_n^{(S)}\big((u-x)^2;x\big)\\
&\quad = K_n^{(S)}(e_2;x) - 2xK_n^{(S)}(e_1;x) + x^2K_n^{(S)}(e_0;x)\\
&\quad \leq \frac{1}{3}\cdot\left\| \frac{3aa_{0,2}+3aa_{0,1}+3aa_{0,2}+6aa_{1,1}+3aa_{1,0}+aa_{0,0}}{aa_{0,0}n^2}\right\|\\
&\qquad + M\left[\left\|\frac{1}{2}\cdot\frac{aa_{0,0}hh_{0,2}+2aa_{0,0}hh_{1,1}+aa_{0,0}hh_{2,0}+2aa_{0,0}+4aa_{0,1}+4aa_{1,0}}{aa_{0,0}n}\right\|\right.\\
&\qquad \left.+\left\|\frac{2aa_{0,1}+2aa_{1,0}+aa_{0,0}}{aa_{0,0}n}\right\|\right].
\end{aligned}$$

By taking

$$\begin{aligned}
\delta_n = \Bigg\{&\frac{1}{3}\cdot\left\| \frac{3aa_{0,2}+3aa_{0,1}+3aa_{0,2}+6aa_{1,1}+3aa_{1,0}+aa_{0,0}}{aa_{0,0}n^2}\right\|\\
&+ M\left[\left\|\frac{1}{2}\cdot\frac{aa_{0,0}hh_{0,2}+2aa_{0,0}hh_{1,1}+aa_{0,0}hh_{2,0}+2aa_{0,0}+4aa_{0,1}+4aa_{1,0}}{aa_{0,0}n}\right\|\right.\\
&\left.+\left\|\frac{2aa_{0,1}+2aa_{1,0}+aa_{0,0}}{aa_{0,0}n}\right\|\right]\Bigg\}^2,
\end{aligned}$$

we get $\|K_n^{(S)}\mathfrak{h}-\mathfrak{h}\| \leq 2\cdot\omega(\mathfrak{h},\delta_n)$. Therefore, for every $\epsilon>0$, we have

$$\frac{1}{\alpha_n}\sum_{\|K_n^{(S)}\mathfrak{h}-\mathfrak{h}\|\geq\epsilon}\mathfrak{t}_{nj} \leq \frac{1}{\alpha_n}\sum_{2\cdot\omega(f,\delta_n)\geq\epsilon}\mathfrak{t}_{nj}.$$

From the conditions that are given in the theorem, we have $\|K_n^{(S)}\mathfrak{h}-\mathfrak{h}\| = st_{\mathfrak{T}} - 0(\alpha_i)$, as claimed. □

Now, we obtain the rate of convergence for our method.

Theorem 3.2 *Let* $\mathfrak{h}\in C[0,M]$ *and let* ϕ *be a positive real function defined on* $(0,M)$. *If* $\omega(\mathfrak{h},\psi(u)) = O(\phi(u))$, *as* $\nu\to R^-$, *then we have*

$$\frac{1}{\mathfrak{p}(\nu)}\left\|\sum_{n=0}^{\infty}\big(K_n^{(S)}e_i - e_i\big)\mathfrak{p}_n\nu^n\right\| = O\big(\phi(\nu)\big),$$

where the function $\psi:(0,M)\to\mathbb{R}$ *is defined by the relation*

$$\psi(u) = \left\{\sup_{\substack{x\in(0,M)\\ n\in\mathbb{N}}}\left\{K_n^{(S)}\big((u-x)^2;x\big)\right\}\right\}^{\frac{1}{2}}.$$

Proof For any $u\in(0,R)$, $x\in(0,M)$ and $\delta>0$, we have

$$\left|\sum_{n=0}^{\infty}\big[K_n^{(S)}(\mathfrak{h};x)-\mathfrak{h}(x)\big]\mathfrak{p}_n\nu^n\right|$$

$$\leq \sum_{n=0}^{\infty} K_n^{(S)}\big(\big|\mathfrak{h}(u)-\mathfrak{h}(x)\big|;x\big)\mathfrak{p}_n \nu^n$$

$$\leq \sum_{n=0}^{\infty} K_n^{(S)}\left(\omega\left(\mathfrak{h},\frac{|u-x|}{\delta}\delta\right);x\right)\mathfrak{p}_n\nu^n \leq \sum_{n=0}^{\infty} K_n^{(S)}\left(\left(1+\left[\left|\frac{u-x}{\delta}\right|\right]\right)\omega(\mathfrak{h},\delta);x\right)\mathfrak{p}_n\nu^n$$

$$\leq \omega(\mathfrak{h},\delta)\sum_{n=0}^{\infty} K_n^{(S)}\left(1+\frac{(u-x)^2}{\delta^2};x\right)\mathfrak{p}_n\nu^n$$

$$\leq \omega(\mathfrak{h},\delta)\sum_{n=0}^{\infty} K_n^{(S)}\big(e_0(u);x\big)\mathfrak{p}_n\nu^n + \frac{\omega(\mathfrak{h},\delta)}{\delta^2}\sum_{n=0}^{\infty} K_n^{(S)}\big((u-x)^2;x\big)\mathfrak{p}_n\nu^n$$

$$= p(\nu)\omega(\mathfrak{h},\delta) + \frac{\omega(\mathfrak{h},\delta)}{\delta^2}\sup_{\substack{0\leq x\leq 1\\ n\in\mathbb{N}}}\left\{K_n^{(S)}\big((u-x)^2;x\big)\right\}\sum_{n=0}^{\infty}\mathfrak{p}_n\nu^n,$$

which leads to

$$\left|\sum_{n=0}^{\infty}\left[K_n^{(S)}(f;x)-f(x)\right]\mathfrak{p}_n\nu^n\right| \leq \mathfrak{p}(\nu)\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta^2}\sup_{0\leq x\leq 1}\left\{K_n^{(S)}\big((u-x)^2;x\big)\right\}\mathfrak{p}(\nu).$$

If we set $\delta = \psi(u)$, then from the last inequality we have

$$0 \leq \frac{1}{\mathfrak{p}(\nu)}\left\|\sum_{n=0}^{\infty}\left(K_n^{(S)}\mathfrak{h}-\mathfrak{h}\right)\mathfrak{p}_n\nu^n\right\| \leq 2\omega(\mathfrak{h},\delta),$$

as required. □

4 Voronovskaya type theorems

It is well known that there is a Voronovskaya type theorem for the Kantorovich type generalization of Szász operators involving Sheffer type polynomials and it is stated as follows.

Theorem 4.1 ([6]) *For $f \in C_B[0,\infty)$,*

$$\lim_{n\to\infty} n\left[K_n^{(S)}\big(f(t),x\big)-f(x)\right] = f'(x)\left[\frac{2\tilde{a}_{0,1}+2\tilde{a}_{1,0}+\tilde{a}_{0,0}}{2\tilde{a}_{0,0}}\right]+\frac{f''(x)}{2}\left[(\tilde{h}_{0,2}+2\tilde{h}_{1,1}+\tilde{h}_{2,0})\frac{x}{2}\right],$$

for every $x \in [0,M]$ and any finite M.

We extend the Voronovskaya type theorem for the $\mathfrak{T}$-statistical method for these operators. Let us consider the following operators.

Example 4.2 Define the operators

$$NB_n(h,x) = (1+u_n)K_n^{(S)}(h,x),$$

where

$$u_n = \begin{cases} \frac{1}{m^3} & m^2-m \leq n \leq m^2-1, \\ \frac{1}{m^4} & n = m^2; m \in \mathbb{N}\setminus\{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.3 *Let* $\mathfrak{h} \in C[0,M]$ *such that* $\mathfrak{h}', \mathfrak{h}'' \in C[0,M]$, $x \in [0,M]$. *Then we obtain*

$$n^2 NB_n^{(S)}\big((y-x)^4; x\big) \sim E_4 x^2 (st_{\mathfrak{T}}) \quad on\ [0,M].$$

Proof It follows directly from Remark 2.6 given in [6]. □

Theorem 4.4 *Let* $\mathfrak{h} \in C[0,M]$ *such that* $\mathfrak{h}', \mathfrak{h}'' \in C[0,M]$, $x \in [0,M]$, *for any finite* M. *Then*

$$n\big[NB_n^{(S)}(\mathfrak{h};x) - \mathfrak{h}(x)\big] \sim \mathfrak{h}'(x)\left[\frac{2\tilde{a}_{0,1} + 2\tilde{a}_{1,0} + \tilde{a}_{0,0}}{2\tilde{a}_{0,0}}\right] + \frac{\mathfrak{h}''(x)}{2}\left(\frac{x(\tilde{h}_{0,2} + 2\tilde{h}_{1,1} + \tilde{h}_{2,0})}{2}\right)(st_T),$$

on $[0,M]$.

Proof Taylor's formula gives

$$\mathfrak{h}(y) = \mathfrak{h}(x) + (y-x)\mathfrak{h}'(x) + \frac{1}{2}(y-x)^2\mathfrak{h}''(x) + (y-x)^2\psi(y-x), \tag{4.1}$$

where $\psi(y-x) \to 0$, as $y-x \to 0$. After applying $NB_n^{(S)}$ on both sides of Eq. (4.1), we obtain

$$\begin{aligned}
NB_n^{(S)}(\mathfrak{h}) &= (1+u_n)\mathfrak{h}(x) + (1+u_n)\mathfrak{h}'(x)\left((2\tilde{a}_{0,1} + 2\tilde{a}_{1,0} + \tilde{a}_{0,0})\frac{1}{2\tilde{a}_{0,0}n}\right) \\
&\quad + (1+u_n)\frac{\mathfrak{h}''(x)}{2}\left((3\tilde{a}_{0,2} + 6\tilde{a}_{1,1} + 3\tilde{a}_{1,0} + 3\tilde{a}_{2,0} + 3\tilde{a}_{0,1} + \tilde{a}_{0,0})\frac{1}{3\tilde{a}_{0,0}n^2}\right. \\
&\quad \left. + (\tilde{h}_{0,2} + 2\tilde{h}_{1,1} + \tilde{h}_{2,0})\frac{x}{2n}\right) + (1+x_n)NB_n^{(S)}\big(\Phi^2\psi(y-x);x\big).
\end{aligned}$$

This yields

$$\begin{aligned}
nNB_n^{(S)}(\mathfrak{h}) &= n(1+u_n)\mathfrak{h}(x) + n(1+u_n)\mathfrak{h}'(x)\left((2\tilde{a}_{0,1} + 2\tilde{a}_{1,0} + \tilde{a}_{0,0})\frac{1}{2\tilde{a}_{0,0}n}\right) \\
&\quad + n(1+u_n)\frac{\mathfrak{h}''(x)}{2}\left((3\tilde{a}_{0,2} + 6\tilde{a}_{1,1} + 3\tilde{a}_{1,0} + 3\tilde{a}_{2,0} + 3\tilde{a}_{0,1} + \tilde{a}_{0,0})\frac{1}{3\tilde{a}_{0,0}n^2}\right. \\
&\quad \left. + (\tilde{h}_{0,2} + 2\tilde{h}_{1,1} + \tilde{h}_{2,0})\frac{x}{2n}\right) + n(1+u_n)NB_n^{(S)}\big(\Phi^2\psi(y-x);x\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left| n[NB_n^{(S)}(\mathfrak{h};x) - \mathfrak{h}(x) - \mathfrak{h}'(x)\left[(2\tilde{a}_{0,1} + 2\tilde{a}_{1,0} + \tilde{a}_{0,0})\frac{1}{2\tilde{a}_{0,0}}\right]\right. \\
&\quad \left. - \frac{\mathfrak{h}''(x)}{2}\left((\tilde{h}_{0,2} + 2\tilde{h}_{1,1} + \tilde{h}_{2,0})\frac{x}{2}\right)\right| \\
&\leq nKu_n + nK_1u_n\left|\frac{2\tilde{a}_{0,1} + 2\tilde{a}_{1,0} + \tilde{a}_{0,0}}{2\tilde{a}_{0,0}n}\right| \\
&\quad + n\frac{K_2}{2}\left|\frac{3\tilde{a}_{0,2} + 6\tilde{a}_{1,1} + 3\tilde{a}_{1,0} + 3\tilde{a}_{2,0} + 3\tilde{a}_{0,1} + \tilde{a}_{0,0}}{3\tilde{a}_{0,0}n^2}\right| \\
&\quad + nu_n\frac{K_2}{2}\left|\frac{3\tilde{a}_{0,2} + 6\tilde{a}_{1,1} + 3\tilde{a}_{1,0} + 3\tilde{a}_{2,0} + 3\tilde{a}_{0,1} + \tilde{a}_{0,0}}{3\tilde{a}_{0,0}n^2} + (\tilde{h}_{0,2} + 2\tilde{h}_{1,1} + \tilde{h}_{2,0})\frac{x}{2n}\right|
\end{aligned}$$

$$+ n\left|NB_n^{(S)}\left((y-x)^2\psi(y-x);x\right)\right| + u_n n\left|NB_n^{(S)}\left((y-x)^2\psi(y-x);x\right)\right|,$$

where $K = \sup_{x\in[0,M]} |\mathfrak{h}(x)|$, $K_1 = \sup_{x\in[0,M]} |\mathfrak{h}'(x)|$ and $K_2 = \sup_{x\in[0,M]} |\mathfrak{h}''(x)|$.

Now we have to prove that

$$\lim_{n\to\infty} n\left|NB_n^{(S)}\left((y-x)^2\psi(y-x);x\right)\right| = 0.$$

By applying the Cauchy–Schwartz inequality, we obtain

$$n\left|NB_n^{(S)}\left((y-x)^2\psi(y-x);x\right)\right| \le \left[n^2 NB_n^{(S)}\left((y-x)^4;x\right)\right]^{\frac{1}{2}} \cdot \left[NB_n^{(S)}\left(\psi^2;x\right)\right]^{\frac{1}{2}}. \tag{4.2}$$

Also, by setting $\eta_x(y) = (\psi(y-x))^2$, we have $\eta_x(x) = 0$ and $\eta_x(\cdot) \in C[0,M]$. So

$$NB_n^{(S)}(\eta_x) \to 0(st_{\mathfrak{T}}) \quad \text{on } [0,M]. \tag{4.3}$$

Now from the previous relation, (4.2), (4.3), and Lemma 4.3, we obtain

$$n^2 NB_n^{(S)}\left((y-x)^2\psi(y-x);x\right) \to 0(st_{\mathfrak{T}}) \quad \text{on } [0,M]. \tag{4.4}$$

From the construction of (u_n), it follows that $nu_n \to 0(st_{\mathfrak{T}})$ on $[0,M]$.

For a given $\epsilon > 0$, we define the sets

$$\begin{aligned}
A &= \left|\{n : |n[NB_n^{(S)}(\mathfrak{h};x) - \mathfrak{h}(x) - \mathfrak{h}'(x)\left[(2\tilde{a}_{0,1} + 2\tilde{a}_{1,0} + \tilde{a}_{0,0})\frac{1}{2\tilde{a}_{0,0}}\right]\right. \\
&\quad \left. - \frac{\mathfrak{h}''(x)}{2}\left((\tilde{h}_{0,2} + 2\tilde{h}_{1,1} + \tilde{h}_{2,0})\frac{x}{2}\right)\right|, \\
A_1 &= \left|\left\{n : |nu_n| \ge \frac{\epsilon}{3K}\right\}\right|, \\
A_2 &= \left|\left\{n : \left|nNB_n^{(S)}\left((y-x)^2\psi(y-x);x\right)\right| \ge \frac{\epsilon}{3}\right\}\right|, \\
A_3 &= \left|\left\{n : \left|nu_n NB_n^{(S)}\left((y-x)^2\psi(y-x);x\right)\right| \ge \frac{\epsilon}{3}\right\}\right|.
\end{aligned}$$

From these relations we obtain $A \le A_1 + A_2 + A_3$. Hence the result follows. □

Theorem 4.5 *Let* $\mathfrak{h}, \mathfrak{h}', \mathfrak{h}'' \in C[0,\infty)$. *Then*

$$\begin{aligned}
&\left|n\left(K_n^{(S)}(\mathfrak{h},x) - \mathfrak{h}(x)\right) - \mathfrak{h}'(x)\left((2\tilde{a}_{0,1} + 2\tilde{a}_{1,0} + \tilde{a}_{0,0})\frac{1}{2\tilde{a}_{0,0}n}\right)\right. \\
&\quad - \frac{\mathfrak{h}''(x)}{2} \cdot \left[(3\tilde{a}_{0,2} + 6\tilde{a}_{1,1} + 3\tilde{a}_{1,0} + 3\tilde{a}_{2,0} + 3\tilde{a}_{0,1} + \tilde{a}_{0,0})\frac{1}{3\tilde{a}_{0,0}n^2}\right. \\
&\quad \left.\left. + (\tilde{h}_{0,2} + 2\tilde{h}_{1,1} + \tilde{h}_{2,0})\frac{x}{2n}\right]\right| \\
&= 0(1)\omega\left(\mathfrak{h}'', n^{-\frac{1}{2}}\right),
\end{aligned}$$

as $n \to \infty$, *and for every* $x \in [0,M]$, *for any finite* M.

Proof From Taylor's theorem, we have

$$\mathfrak{h}(u) = \mathfrak{h}(x) + \mathfrak{h}'(x)(u-x) + \frac{\mathfrak{h}''(x)}{2}(u-x)^2 + R(u,x),$$

where $R(u,x) = \frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2}(u-x)^2$, for $\theta \in (u,x)$. Now we obtain

$$\left|K_n^{(S)}(\mathfrak{h},x) - \mathfrak{h}(x) - \mathfrak{h}'(x)K_n^{(S)}\big((u-x);x\big) - \frac{\mathfrak{h}''(x)}{2}K_n^{(S)}\big((u-x)^2;x\big)\right| \le K_n^{(S)}\big(\big|R(u,x)\big|,x\big).$$

From this we get

$$\begin{aligned}&\left|n\big(K_n^{(S)}(\mathfrak{h},x) - \mathfrak{h}(x)\big) - \mathfrak{h}'(x)\left(\frac{2\tilde{a}_{0,1}+2\tilde{a}_{1,0}+\tilde{a}_{0,0}}{2\tilde{a}_{0,0}}\right)\right.\\ &\quad - \frac{\mathfrak{h}''(x)}{2}\cdot\Big[(3\tilde{a}_{0,2}+6\tilde{a}_{1,1}+3\tilde{a}_{1,0}+3\tilde{a}_{2,0}+3\tilde{a}_{0,1}+\tilde{a}_{0,0})\frac{1}{3\tilde{a}_{0,0}n}\\ &\quad \left.+ (\tilde{h}_{0,2}+2\tilde{h}_{1,1}+\tilde{h}_{2,0})\frac{x}{2}\right|\\ &\quad \le n\cdot K_n^{(S)}\big(\big|R(u,x)\big|,x\big).\end{aligned}$$

By the properties of the continuity modulus, we have

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right| \le \frac{1}{2!}\left(1+\frac{|\theta-x|}{\delta}\right)\omega\big(\mathfrak{h}'',\delta\big).$$

On the other hand

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right| \le \begin{cases}\omega(\mathfrak{h}'',\delta); & |u-x|\le\delta,\\ \frac{(t-x)^4}{\delta^4}\omega(\mathfrak{h}'',\delta); & |u-x|\ge\delta.\end{cases}$$

For $0<\delta<1$, we obtain

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right| \le \omega\big(\mathfrak{h}'',\delta\big)\left(1+\frac{(u-x)^4}{\delta^4}\right),$$

which gives

$$\big|R(u,x)\big| \le \omega\big(\mathfrak{h}'',\delta\big)\left(1+\frac{(u-x)^4}{\delta^4}\right)(u-x)^2 = \omega\big(\mathfrak{h}'',\delta\big)\left((u-x)^2+\frac{(u-x)^6}{\delta^4}\right).$$

By the linearity of $K_n^{(S)}$ and the above relation we obtain

$$K_n^{(S)}\big(\big|R(u,x)\big|,x\big) \le \omega\big(\mathfrak{h}'',\delta\big)\left(K_n^{(S)}\big((u-x)^2,x\big)+\frac{1}{\delta^4}K_n^{(S)}\big((u-x)^6,x\big)\right).$$

Taking into consideration Remark 2.6 in [6], for every $x\in[0,M]$, we have

$$K_n^{(S)}\big(\big|R(u,x)\big|,x\big) \le \omega\big(\mathfrak{h}'',\delta\big)\left(O\left(\frac{1}{n}\right)+\frac{1}{\delta^4}O\left(\frac{1}{n^3}\right)\right) = O\left(\frac{1}{n}\right)\omega\big(\mathfrak{h}'',\delta\big).$$

For $\delta = n^{-\frac{1}{2}}$, we complete the proof. □

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