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Modified homotopy methods for generalized fractional perturbed Zakharov–Kuznetsov equation in dusty plasma

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Abstract

We propose a new modification of homotopy perturbation method (HPM) called the δ -homotopy perturbation transform method (δ -HPTM). This modification consists of the Laplace transform method, HPM, and a control parameter δ . This control convergence parameter δ in this new modification helps in adjusting and controlling the convergence region of the series solution and overcome some limitations of HPM and HPTM. The δ -HPTM and q-homotopy analysis transform method (q-HATM) are considered to study the generalized time-fractional perturbed (3 + 1)-dimensional Zakharov–Kuznetsov equation with Caputo fractional time derivative. This equation describes nonlinear dust-ion-acoustic waves in the magnetized two-ion-temperature dusty plasmas. The selection of an appropriate value of δ in δ -HPTM and the auxiliary parameters n and \hbar in q-HATM gives a guaranteed convergence of series solution, but the difference between the two techniques is that the embedding parameter p in δ -HPTM varies from zero to nonzero δ , whereas the embedding parameter q in q-HATM varies from zero to $\frac{1}{n}$, $n \geq 1$. We examine the effect of fractional order on the considered problem and present the error estimate when compared with exact solution. The outcomes reveal complete reliability and efficiency of the proposed algorithm for solving various types of physical models arising in sciences and engineering. Furthermore, we present the convergence and error analysis of the two methods.

Keywords: Laplace transform; δ -homotopy transform perturbation method; q-homotopy analysis transform method; Perturbed Zakharov–Kuznetsov equation

1 Introduction

The study of fractional partial differential equations (FPDEs) has enticed the interest of many researchers in the field of applied sciences and engineering by virtue of its enormous applications in electrodynamics, random walk, biotechnology, viscoelasticity, chaos theory, signal and image processing, nanotechnology, and many other areas [1–20]. Also, essential properties of fractional calculus were outlined by many researchers (see [21–24] for detailed discussion). Nevertheless, solving FPDEs is generally more complex than the clas-

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sical type since their operators are defined through integrals. There are many techniques proposed by many researchers to handle analytical and approximate solutions of nonlinear FPDEs such as the residual power series method [25–28], iterative Shehu transform method [29], Laplace decomposition method [30], q-homotopy analysis method [31–34], Adomian decomposition method [35], fractional reduced differential transform method [36, 37], variational iteration method [38, 39], homotopy analysis method [40], and other methods [41–44].

The homotopy perturbation method (HPM) was developed by He [45–50] by combining the perturbation and standard homotopy for solving numerous physical problems. We refer the reader to He's works for a clear understanding of HPM, where further insights can be found. Recently, an improved modification of HPM, called the parameterized homotopy perturbation method (PHPM), was proposed in [51, 52]. Another formulation, called the He–Laplace method, was proposed to obtain an exact closed approximate solution of nonlinear models [53, 54]. The HPM and well-known Laplace transformation method were combined to produce a highly effective technique, called the homotopy perturbation transform method (HPTM), for solving many nonlinear problems [55, 56]. It is worth noting that the Laplace transform method alone in some cases is insufficient in handling nonlinear problems because of the difficulties that may arise by the nonlinear terms. In this present study, we propose a new modification of HPM, called the δ -homotopy perturbation transform method (δ -HPTM), which consists of HPM, the Laplace transform method, and a control parameter δ . Similarly to the control parameters n and \hbar in q-HATM, the control parameter δ in δ -HPTM also helps in adjusting and controlling the convergence region of the series solutions and can overcome some limitations of HPM, HPTM, and He–Laplace method. It is worth mentioning that the present modification (δ -HPTM) requires neither polynomials like ADM nor Lagrange multipliers like VIM and overcomes the limitations of these methods.

To elucidate the reliability and effectiveness of the proposed modification, we consider the generalized time-fractional perturbed (3 + 1)-dimensional Zakharov–Kuznetsov (gpZK) equation given by

$$\begin{aligned} D_t^\gamma W + \beta_1 W^k \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \left(\frac{\partial^3 W}{\partial x \partial y^2} + \frac{\partial^3 W}{\partial x \partial z^2} \right) + \xi \frac{\partial^5 W}{\partial x^5} = 0, \\ 0 < \gamma \leq 1, t > 0, \end{aligned} \quad (1)$$

where W represents the electrostatic potential, k is a positive number, γ is the fractional order, ξ represents a smallness parameter, and the physical quantities β_1 , β_2 , and β_3 are constants. Zhen et al. [57] and Seadawy et al. [58, 59] have outlined these physical quantities. This equation is used to describe the nonlinear dust-ion-acoustic waves in the magnetized two-ion-temperature dusty plasmas [60, 61]. The study of ion-acoustic waves and structures in dense quantum plasmas has attracted a lot of consideration in recent years. The ZK equation comprises the nonlinear term $W \frac{\partial W}{\partial x}$ and third-order dispersion term $\frac{\partial^3 W}{\partial x^3}$:

$$\frac{\partial W}{\partial t} + \beta_1 W \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \left(\frac{\partial^3 W}{\partial x \partial y^2} + \frac{\partial^3 W}{\partial x \partial z^2} \right). \quad (2)$$

Equation (2) is limited to the waves of small amplitudes only. The width of the soliton and its velocity deviate from the predictions of this equation as the amplitude of the wave increases. The pZK equation (1) with fractional order $\gamma = 1$ and $k = 1$ includes an of extra fifth-order dispersion term $\xi \frac{\partial^5 W}{\partial x^5}$ was proposed to overcome this problem (see [57–59, 62], for more detail). The proposed δ -HPTM and q-HATM are employed to compute numerical solutions of Eq. (1). The two algorithms provide the solutions in a rapid convergent series, which can lead the solutions to a closed form. To the author’s knowledge, the approximate solutions of the gpZK (1) was not addressed in the literature before.

The rest of the paper is structured as follows. Useful notations and definitions are provided in Sect. 2. The essential idea of the two methods with convergence and error analysis are presented in Sect. 3. The applications of δ -HPTM and q-HATM on the generalized time-fractional pZK equation are detailed in Sect. 4. Numerical comparison and discussion are provided in Sect. 5. Lastly, Sect. 6 concludes the paper.

2 Preliminaries

This section contains some helpful notations and definitions.

Definition 1 Let $\omega \in \mathbb{R}$ and $m \in \mathbb{N}$. A function W is said to be in the space C_ω if there exists $\eta \in \mathbb{R}, \eta > \omega$, and $Z \in C[0, \infty)$ such that $W(t) = t^\eta Z(t) \ t \in \mathbb{R}^+$. Furthermore, $W \in C_\omega^m$ if $W^{(m)} \in C_\omega$ [63].

Definition 2 The Riemann–Liouville (RL) fractional integral of order γ of a function $W(t) \in C_\omega, \omega \geq -1$, is given as [23, 63–65]

$$J^\gamma W(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \zeta)^{\gamma-1} W(\zeta) d\zeta, \quad \gamma, t > 0, \tag{3}$$

where $J^0 W(t) = W(t)$, and Γ is the classical gamma function.

Definition 3 The fractional derivative of $W(t)$ (denoted by $D^\gamma W(t)$) in the Caputo sense for $m - 1 < \gamma < m, m \in \mathbb{N}$, is defined as [23, 65]

$$D^\gamma W(t) = \begin{cases} W^{(m)}(t), & \gamma = m, \\ J^{m-\gamma} W^{(m)}(t), & m - 1 < \gamma < m, \end{cases} \tag{4}$$

where

$$J^{m-\gamma} W^{(\gamma)}(t) = \frac{1}{\Gamma(m - \gamma)} \int_0^t (t - \zeta)^{m-\gamma-1} W^{(\gamma)}(\zeta) d\zeta, \quad \gamma, t > 0, \tag{5}$$

with the following properties:

- a. $D^\gamma (\tau_1 W(t) + \tau_2 V(t)) = \tau_1 D^\gamma W(t) + \tau_2 D^\gamma V(t), \tau_1, \tau_2 \in \mathbb{R}$,
- b. $D^\gamma J^\gamma W(t) = W(t)$,
- c. $J^\gamma D^\gamma W(t) = W(t) - \sum_{j=0}^{m-1} W_0^j(t) \frac{t^j}{j!}$.

Definition 4 The Laplace transform (denoted by \mathcal{L}) of a Riemann–Liouville fractional integral ($J_t^\gamma W(t)$) and Caputo fractional derivative ($D_t^\gamma W(t)$) of a function $W \in C_\omega (\omega \geq$

–1) are given respectively as [21, 65]

$$\begin{aligned} \mathcal{L}[J_t^\gamma W(t)] &= s^{-\gamma} \mathcal{L}[W(t)], \\ \mathcal{L}[D_t^\gamma W(t)] &= s^\gamma \mathcal{L}[W(t)] - \sum_{r=0}^{m-1} s^{\gamma-r-1} W^{(r)}(0^+), \quad m-1 < \gamma \leq m, \end{aligned} \tag{6}$$

where s is a parameter.

3 Analysis of the proposed methods

Here we give the general idea of the δ -HPTM and q-HATM. We also present some convergence and error analysis of the two methods. Consider the general nonlinear FPDE of the form

$$D_t^\gamma W + \mathcal{M}(W) + \mathcal{N}(W) = \Phi, \quad m-1 < \gamma \leq m, \tag{7}$$

with initial conditions

$$W^{(r)}(x, y, z, 0) = \frac{\partial^r W(x, y, z, 0)}{\partial t^r} = f_r(x, y, z), \quad r = 0, 1, 2, \dots, m-1, \tag{8}$$

where D_t^γ represents the Caputo fractional derivative, \mathcal{M} and \mathcal{N} denote, respectively, the linear and nonlinear differential operators, $W = W(x, y, z, t)$ specifies the unknown function, and $\Phi = \Phi(x, y, z, t)$ is the provided source term. Applying the Laplace transform (denoted by \mathcal{L}) to both sides of Eq. (7), we have

$$s^\gamma \mathcal{L}[W] - \sum_{r=0}^{m-1} s^{\gamma-r-1} W^{(r)}(x, y, z, 0) + \mathcal{L}[\mathcal{M}(W)] + \mathcal{L}[\mathcal{N}(W)] = \mathcal{L}[\Phi]. \tag{9}$$

Using the differentiation property of the Laplace transform with the initial conditions (8), upon simplification and the inverse Laplace transform (denoted by \mathcal{L}^{-1}), we obtain

$$\begin{aligned} W &= \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \left(\sum_{r=0}^{m-1} s^{\gamma-r-1} W^{(r)}(x, y, z, 0) + \mathcal{L}[\Phi] \right) \right] \\ &\quad - \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L}[\mathcal{M}(W) + \mathcal{N}(W)] \right]. \end{aligned} \tag{10}$$

3.1 The δ -homotopy perturbation transform method (δ -HPTM)

We employ the concept of HPM [45–50] to Eq. (10) as follows:

$$W = \sum_{r=0}^{\infty} p^r W_r. \tag{11}$$

We decompose the nonlinear term as

$$\mathcal{N}(W) = \sum_{r=0}^{\infty} p^r \mathcal{H}_r(W), \tag{12}$$

where $\mathcal{H}_r(W)$ are the He’s polynomials expressed in the form

$$\mathcal{H}_r(W_0, W_1, \dots, W_r) = \frac{1}{r!} \frac{\partial^r}{\partial p^r} \left[\mathcal{N} \left(\sum_{j=0}^{\infty} p^j W_j \right) \right]_{p=0}, \quad r = 0, 1, 2, \dots \tag{13}$$

In view of δ -HPM [66], we derive the propose δ -HPTM as

$$\begin{aligned} \sum_{r=0}^{\infty} p^r W_r &= \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \left(\sum_{r=0}^{m-1} s^{\gamma-r-1} W^{(r)}(x, y, z, 0) + \mathcal{L}[\Phi] \right) \right] \\ &\quad - p \left(1 - \frac{1}{\delta} \right) \left(\sum_{r=0}^{\infty} p^r W_r - W_0 \right) \\ &\quad - p \left\{ \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\mathcal{M} \left(\sum_{r=0}^{\infty} p^r W_r \right) + \sum_{r=0}^{\infty} p^r \mathcal{H}_r(W) \right] \right] \right\}. \end{aligned} \tag{14}$$

By equating the identical power terms of p in Eq. (14), we generate the sequence of δ -HPTM as

$$\begin{aligned} p^{(0)} : W_0 &= \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \left(\sum_{r=0}^{m-1} s^{\gamma-r-1} W^{(r)}(x, y, z, 0) + \mathcal{L}[\Phi] \right) \right], \\ p^{(1)} : W_1 &= -\mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L}[\mathcal{M}(W_0) + \mathcal{H}_0(W)] \right], \\ p^{(2)} : W_2 &= -\left(1 - \frac{1}{\delta} \right) W_1 - \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L}[\mathcal{M}(W_1) + \mathcal{H}_1(W)] \right], \\ &\vdots \\ p^{(r)} : W_r &= -\left(1 - \frac{1}{\delta} \right) W_{r-1} - \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L}[\mathcal{M}(W_{r-1}) + \mathcal{H}_{r-1}(W)] \right], \quad r = 2, 3, 4, \dots \end{aligned} \tag{15}$$

The solution of Eq. (7) is given as

$$W = \lim_{p \rightarrow \delta} \sum_{r=0}^{\infty} p^r W_r = \sum_{r=0}^{\infty} \mathcal{W}_r(x, y, z, t; \delta) = \sum_{r=0}^{\infty} W_r \delta^r. \tag{16}$$

Remark 1 The particular case where $\delta = 1$ is the standard HPTM [55, 56].

3.1.1 Convergence and error analysis

Theorem 1 Let $W = W(x, y, z, t)$ be defined in a Banach space \mathcal{B} [67]. Then the series solution

$$\sum_{r=0}^{\infty} \mathcal{W}_r(x, y, z, t; \delta) = \sum_{r=0}^{\infty} W_r \delta^r \tag{17}$$

is convergent for a prescribed value of δ if

$$\|\mathcal{W}_{r+1}\| \leq \frac{\varrho}{|\delta|} \|\mathcal{W}_r\|, \quad \forall \mathcal{W}_0 \in \mathcal{B}, \tag{18}$$

where $0 < \varrho < |\delta|$.

Proof Let $W_0 = \mathcal{W}_0 \in \mathcal{B}$. Define the sequence of partial sums $\{S_r\}$ of Eq. (16) as

$$\begin{aligned}
 S_0 &= W_0, \\
 S_1 &= W_0 + W_1\delta, \\
 S_2 &= W_0 + W_1\delta + W_2\delta^2, \\
 &\vdots \\
 S_r &= W_0 + W_1\delta + W_2\delta^2 + W_3\delta^3 + \dots + W_r\delta^r.
 \end{aligned}
 \tag{19}$$

We need to show that $\{S_r\}_{r=0}^\infty$ is a Cauchy sequence in the Banach space \mathcal{B} . For $\delta \neq 0$, we have

$$\|S_{r+1} - S_r\| = \|\mathcal{W}_{r+1}\| \leq \frac{\varrho}{|\delta|} \|\mathcal{W}_r\| \leq \left(\frac{\varrho}{|\delta|}\right)^2 \|\mathcal{W}_{r-1}\| \leq \dots \leq \left(\frac{\varrho}{|\delta|}\right)^{r+1} \|\mathcal{W}_0\|.
 \tag{20}$$

For all $r, k \in \mathbb{N}$ with $r \geq k$, applying the triangle inequality, we obtain

$$\begin{aligned}
 \|S_r - S_k\| &= \|(S_r - S_{r-1}) + (S_{r-1} - S_{r-2}) + \dots + (S_{k+1} - S_k)\| \\
 &\leq \|S_r - S_{r-1}\| + \|S_{r-1} - S_{r-2}\| + \dots + \|S_{k+1} - S_k\| \\
 &\leq \left(\frac{\varrho}{|\delta|}\right)^r \|\mathcal{W}_0\| + \left(\frac{\varrho}{|\delta|}\right)^{r-1} \|\mathcal{W}_0\| + \dots + \left(\frac{\varrho}{|\delta|}\right)^{k+1} \|\mathcal{W}_0\| \\
 &\leq \left(\frac{\varrho}{|\delta|}\right)^{k+1} \left(\left(\frac{\varrho}{|\delta|}\right)^{r-k-1} + \left(\frac{\varrho}{|\delta|}\right)^{r-k-2} + \dots + \left(\frac{\varrho}{|\delta|}\right) + 1 \right) \|\mathcal{W}_0\| \\
 &\leq \left(\frac{\varrho}{|\delta|}\right)^{k+1} \left(\frac{1 - \left(\frac{\varrho}{|\delta|}\right)^{r-k}}{1 - \frac{\varrho}{|\delta|}} \right) \|\mathcal{W}_0\|.
 \end{aligned}
 \tag{21}$$

Since $0 < \varrho < |\delta|$ and $\delta \neq 0$, we have $1 - \left(\frac{\varrho}{|\delta|}\right)^{r-k} < 1$. Then

$$\|S_r - S_k\| \leq \frac{\left(\frac{\varrho}{|\delta|}\right)^{k+1}}{1 - \frac{\varrho}{|\delta|}} \|\mathcal{W}_0\|.
 \tag{22}$$

Since $\|\mathcal{W}_0\| < \infty$, we have

$$\lim_{r \rightarrow \infty} \|S_r - S_k\| = 0.
 \tag{23}$$

Therefore $\{S_r\}_{r=0}^\infty$ is a Cauchy sequence in the Banach space \mathcal{B} , so the series solution Eq. (16) converges. □

Theorem 2 *If the truncated series $\sum_{r=0}^K \mathcal{W}_r(x, y, z, t; \delta) = \sum_{r=0}^K W_r \delta^r$ is employed as an approximate solution of Eq. (7), then the maximum absolute truncation error is estimated as*

$$\left\| W - \sum_{r=0}^K \mathcal{W}_r(x, y, z, t; \delta) \right\| \leq \frac{\left(\frac{\varrho}{|\delta|}\right)^{K+1}}{1 - \frac{\varrho}{|\delta|}} \|W_0\|.
 \tag{24}$$

Proof It follows from inequality (21) in Theorem 1. For $M \geq K$, we have

$$\|S_M - S_K\| \leq \left(\frac{\rho}{|\delta|}\right)^{K+1} \left(\frac{1 - (\frac{\rho}{|\delta|})^{M-K}}{1 - \frac{\rho}{|\delta|}}\right) \|\mathcal{W}_0\|. \tag{25}$$

For a prescribed value of $\delta \neq 0$, $S_M \rightarrow W$ as $M \rightarrow \infty$, and $1 - (\frac{\rho}{|\delta|})^{M-K} < 1$ (since $0 < \frac{\rho}{|\delta|} < 1$). Thus

$$\left\|W - \sum_{r=0}^K \mathcal{W}_r(x, y, z, t; \delta)\right\| \leq \frac{(\frac{\rho}{|\delta|})^{K+1}}{1 - \frac{\rho}{|\delta|}} \|W_0\|, \tag{26}$$

where $\mathcal{W}_0 = W_0$. □

3.2 The q-homotopy analysis transform method (q-HATM)

To exemplify the idea of q-HATM [68–75], we construct the zeroth-order deformation equation for $0 \leq q \leq \frac{1}{n}, n \geq 1$, as

$$(1 - nq)\mathcal{L}(\phi - W_0) = \hbar q \mathcal{H} \mathcal{N}[\phi], \tag{27}$$

where $\phi = \phi(x, y, z, t; q)$, and $\mathcal{N}[\phi]$ from Eq. (9) is defined as

$$\mathcal{N}[\phi] = \mathcal{L}[\phi] - \frac{1}{s^\gamma} \sum_{r=0}^{m-1} s^{\gamma-r-1} \phi^{(r)}(0^+) + \frac{1}{s^\gamma} (\mathcal{L}[\mathcal{M}(\phi) + \mathcal{N}(\phi) - \Phi]), \tag{28}$$

where q indicates the embedded parameter, the nonzero \hbar represents an auxiliary parameter, and $\mathcal{H} \neq 0$ is an auxiliary function. From Eq. (27) with $q = 0, \frac{1}{n}$ we get

$$\phi(x, y, z, t; 0) = W_0, \quad \phi\left(x, y, z, t; \frac{1}{n}\right) = W. \tag{29}$$

As q rises from 0 to $\frac{1}{n}$, the solutions ϕ ranges from the initial guess W_0 to the solution W . In case that W_0, \hbar , and \mathcal{H} are all selected appropriately the solutions ϕ in Eq. (27) hold for $0 \leq q \leq \frac{1}{n}$. Hence application of Taylor series expansion [76] to ϕ gives

$$\phi = W_0 + \sum_{r=1}^{\infty} W_r q^r, \tag{30}$$

where

$$W_r = \frac{1}{r!} \left. \frac{\partial^r \phi}{\partial q^r} \right|_{q=0}. \tag{31}$$

If we choose W_0, \hbar , and \mathcal{H} adequately, then Eq. (30) converges at $q = \frac{1}{n}$. From Eq. (29) we obtain

$$W = W_0 + \sum_{r=1}^{\infty} W_r \left(\frac{1}{n}\right)^r. \tag{32}$$

Differentiating Eq. (27) r times with respect to q , setting $q = 0$, and multiplying by $\frac{1}{r!}$ give

$$\mathcal{L}[W_r - \Upsilon_r^* W_{r-1}] = \hbar \mathcal{H} \mathcal{R}_r(\vec{W}_{r-1}). \tag{33}$$

The vector \vec{W}_r is expressed as

$$\vec{W}_r = \{W_0, W_1, \dots, W_r\}. \tag{34}$$

Taking the inverse LT of Eq. (33), we obtain

$$W_r = \Upsilon_r^* W_{r-1} + \hbar \mathcal{L}^{-1}[\mathcal{H} \mathcal{R}_r(\vec{W}_{r-1})], \tag{35}$$

where

$$\begin{aligned} \mathcal{R}_r(\vec{W}_{r-1}) = & \mathcal{L}[W] - \left(1 - \frac{\Upsilon_r^*}{n}\right) \left(\sum_{r=0}^{m-1} s^{\gamma-r-1} W^r(x, y, z, 0) + \frac{1}{s^\gamma} \mathcal{L}[\Phi]\right) \\ & + \frac{1}{s^\gamma} \mathcal{L}[\mathcal{M}(W) + H_{r-1}], \end{aligned} \tag{36}$$

and

$$\Upsilon_r^* = \begin{cases} 0 & r \leq 1, \\ n & \text{otherwise.} \end{cases} \tag{37}$$

In Eq. (36), H_r denotes the homotopy polynomial defined as

$$H_r = \frac{1}{r!} \frac{\partial^r \phi}{\partial q^r} \Big|_{q=0}, \quad \phi = \phi_0 + q\phi_1 + q^2\phi_2 + q^3\phi_3 + \dots \tag{38}$$

3.2.1 Convergence and error analysis

Here we present some helpful theorems with detailed proofs in [74, 75] for the purpose of completeness.

Theorem 3 (Convergence theorem [74, 75]) *Let \mathcal{B} be a Banach space, and let $F : \mathcal{B} \rightarrow \mathcal{B}$ be a nonlinear mapping. Suppose that*

$$\|F(W) - F(\widehat{W})\| \leq \varrho \|W - \widehat{W}\|, \quad \forall W, \widehat{W} \in \mathcal{B}, \tag{39}$$

where $0 < \varrho < 1$. Then ϱ has a fixed point in light of Banach’s fixed point theory [77]. Furthermore, for arbitrary choice of $W_0, \widehat{W}_0 \in \mathcal{B}$, the sequence generated by the q -HATM converges to a fixed point of ϱ , and

$$\|W_k - W_r\| \leq \frac{\varrho^r}{1 - \varrho} \|W_1 - W_0\|, \quad \forall W, \widehat{W} \in \mathcal{B}. \tag{40}$$

Theorem 4 ([75]) *Suppose that the series solution defined in Eq. (32) converges to the solution W for prescribed values of n and \hbar and that there is a real number $0 < \Theta < 1$ satisfying*

$$\|W_{j+1}\| \leq \Theta \|W_j\|, \quad \forall j. \tag{41}$$

If the truncated series

$$\mathcal{W}_{(K)} = \mathcal{W}_{(K)}(x, t; n; \hbar) = \sum_{r=0}^K W_r \left(\frac{1}{n}\right)^r \tag{42}$$

is utilized as an approximation to the solution of problem (7), then the maximum absolute truncated error is evaluated as

$$\|W - \mathcal{W}_{(K)}\| \leq \frac{\Theta^{K+1}}{1 - \Theta} \|W_0\|. \tag{43}$$

4 Application of the proposed methods

We have carefully chosen the generalized time-fractional perturbed (3 + 1)-dimensional Zakharov–Kuznetsov (gpZK) equation and apply δ -HPTM and q-HATM to obtain analytical approximate solutions in the form of convergent series. Consider

$$\begin{aligned} \mathcal{D}_t^\gamma W + \beta_1 W^k \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \left(\frac{\partial^3 W}{\partial x \partial y^2} + \frac{\partial^3 W}{\partial x \partial z^2} \right) + \xi \frac{\partial^5 W}{\partial x^5} &= 0, \\ 0 < \gamma \leq 1, t > 0, \end{aligned} \tag{44}$$

with initial condition

$$W(x, y, z, 0) = f(x, y, z). \tag{45}$$

Example 1 Consider Eq. (44) with $k = 1$ given as

$$\begin{aligned} \mathcal{D}_t^\gamma W + \beta_1 W \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \frac{\partial^3 W}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W}{\partial x \partial z^2} + \xi \frac{\partial^5 W}{\partial x^5} &= 0, \\ 0 < \gamma \leq 1, t > 0, \end{aligned} \tag{46}$$

with initial condition

$$W(x, y, z, 0) = e_0 - \frac{1680\xi p^4}{\beta_1(px + qy - (\sqrt{-\frac{\beta_2 p^2}{\beta_3} - q^2})z + \phi)^4}, \tag{47}$$

where $e_0, p, q,$ and ϕ are arbitrary constants. The exact solution for $\gamma = 1$ is given by

$$W(x, y, z, t) = e_0 - \frac{1680\xi p^4}{\beta_1(px + qy - (\sqrt{-\frac{\beta_2 p^2}{\beta_3} - q^2})z - \beta_1 e_0 p t + \phi)^4}. \tag{48}$$

δ -HPTM Solution:

Application of δ -HPTM to Eq. (44) with Eq. (45) gives

$$\begin{aligned} \sum_{r=0}^\infty p^r W_r &= \mathcal{L}^{-1} \left[\frac{1}{s} (W(x, y, z, 0)) \right] - p \left(1 - \frac{1}{\delta} \right) \left(\sum_{r=0}^\infty p^r W_r - W_0 \right) \\ &\quad - p \left\{ \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\left(\beta_2 \frac{\partial^3}{\partial x^3} + \beta_3 \frac{\partial^3}{\partial x \partial y^2} + \beta_3 \frac{\partial^3}{\partial x \partial z^2} \right) \right] \right] \right\} \end{aligned} \tag{49}$$

$$+ \xi \frac{\partial^5}{\partial x^5} \left. \left. \left. \sum_{r=0}^{\infty} p^r W_r + \beta_1 \sum_{r=0}^{\infty} p^r \sum_{i=0}^r W_i \frac{\partial W_{r-i}}{\partial x} \right] \right] \right\}.$$

By equating the identical power terms of p in Eq. (49) we generate the sequence of δ -HPTM as

$$\begin{aligned}
 p^{(0)} : W_0 &= W(x, y, z, 0), \\
 p^{(1)} : W_1 &= -\mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 W_0 \frac{\partial W_0}{\partial x} + \beta_2 \frac{\partial^3 W_0}{\partial x^3} + \beta_3 \frac{\partial^3 W_0}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_0}{\partial x \partial z^2} + \xi \frac{\partial^5 W_0}{\partial x^5} \right] \right], \\
 p^{(2)} : W_2 &= -\left(1 - \frac{1}{\delta} \right) W_1 - \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 W_0 \frac{\partial W_1}{\partial x} + \beta_1 W_1 \frac{\partial W_0}{\partial x} \right. \right. \\
 &\quad \left. \left. + \beta_2 \frac{\partial^3 W_1}{\partial x^3} + \beta_3 \frac{\partial^3 W_1}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_1}{\partial x \partial z^2} + \xi \frac{\partial^5 W_1}{\partial x^5} \right] \right], \\
 &\vdots \\
 p^{(r)} : W_r &= -\left(1 - \frac{1}{\delta} \right) W_{r-1} - \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 \sum_{i=0}^{r-1} W_i \frac{\partial W_{(r-i-1)}}{\partial x} \right. \right. \\
 &\quad \left. \left. + \beta_2 \frac{\partial^3 W_{r-1}}{\partial x^3} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial z^2} + \xi \frac{\partial^5 W_{r-1}}{\partial x^5} \right] \right], \quad r = 2, 3, 4, \dots
 \end{aligned}
 \tag{50}$$

Hence, using initial condition Eq. (45), we derive:

$$\begin{aligned}
 W_0 &= e_0 - \frac{1680\xi p^4}{\beta_1(px + qy \pm (\sqrt{-\frac{\beta_2 p^2}{\beta_3} - q^2})z + \phi)^4}, \\
 W_1 &= -\frac{6720\beta_3^{\frac{5}{2}} \xi e_0 p^5 t^\gamma}{\Gamma(\gamma + 1)(\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^5}, \\
 W_2 &= \left(\frac{1}{\delta} - 1 \right) W_1 - \frac{33,600\beta_1\beta_3^3 \xi e_0^2 p^6 t^{2\gamma}}{\Gamma(2\gamma + 1)(\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^6}, \\
 W_3 &= \left(\frac{1}{\delta} - 1 \right) W_2 - \frac{33,600\beta_1\beta_3^3 \xi e_0^2 p^6}{(\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^{11}} \\
 &\quad \times \left(\frac{6\sqrt{\beta_3}\beta_1 e_0 p t^{3\gamma} (\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^4}{\Gamma(3\gamma + 1)} \right. \\
 &\quad \left. + \frac{(1 - \delta)t^{2\gamma} (\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^5}{\delta\Gamma(2\gamma + 1)} \right. \\
 &\quad \left. + \frac{13,440\beta_3^{\frac{5}{2}} \xi p^5 t^{3\gamma}}{\Gamma(3\gamma + 1)} - \frac{6720\beta_3^{\frac{5}{2}} \xi p^5 \Gamma(2\gamma + 1)t^{3\gamma}}{\Gamma(\gamma + 1)^2\Gamma(3\gamma + 1)} \right).
 \end{aligned}$$

Accordingly, we can obtain the remaining terms $W_r, r = 4, 5, 6, \dots$

q-HATM Solution:

Implementing LT on Eq. (44) with Eq. (45), we obtain

$$\mathcal{L}[W] - \frac{1}{s} (W(x, y, z, 0))$$

$$+ \frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 W \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \frac{\partial^3 W}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W}{\partial x \partial z^2} + \xi \frac{\partial^5 W}{\partial x^5} \right] = 0. \tag{51}$$

The nonlinear operator $\mathcal{N}(\phi)$, $\phi = \phi(x, y, z, t; q)$, is given as

$$\begin{aligned} \mathcal{N}(\phi) &= \mathcal{L}[\phi] - \frac{1}{s} (W(x, y, z, 0)) \\ &+ \frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 \phi \frac{\partial \phi}{\partial x} + \beta_2 \frac{\partial^3 \phi}{\partial x^3} + \beta_3 \frac{\partial^3 \phi}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 \phi}{\partial x \partial z^2} + \xi \frac{\partial^5 \phi}{\partial x^5} \right]. \end{aligned} \tag{52}$$

Referring to Eq. (33) with $\mathcal{H} = 1$, the r th-order deformation equation is

$$\mathcal{L} [W_r - \Upsilon_r^* W_{r-1}] = \hbar \mathcal{R}_r(\vec{W}_{r-1}), \tag{53}$$

where

$$\begin{aligned} \mathcal{R}_r(\vec{W}_{r-1}) &= \mathcal{L}[W_{r-1}] - \left(1 - \frac{\Upsilon_r^*}{n}\right) \frac{1}{s} \{W(x, y, z, 0)\} \\ &+ \frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 \sum_{i=0}^{r-1} W_i \frac{\partial W_{r-i-1}}{\partial x} + \beta_2 \frac{\partial^3 W_{r-1}}{\partial x^3} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial y^2} \right. \\ &\left. + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial z^2} + \xi \frac{\partial^5 W_{r-1}}{\partial x^5} \right]. \end{aligned} \tag{54}$$

An application of the inverse LT to Eq. (53) yields

$$W_r = \Upsilon_r^* W_{r-1} + \hbar \mathcal{L}^{-1} [\mathcal{R}_r(\vec{W}_{r-1})]. \tag{55}$$

Solving Eqs. (55) using (47) and (54) for $r = 1, 2, 3, \dots$, we get:

$$\begin{aligned} W_0 &= e_0 - \frac{1680\xi p^4}{\beta_1(px + qy \pm (\sqrt{-\frac{\beta_2 p^2}{\beta_3} - q^2})z + \phi)^4}, \\ W_1 &= \frac{6720\beta_3^{\frac{5}{2}}\xi e_0 \hbar p^5 t^\gamma}{\Gamma(\gamma + 1)(\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^5}, \\ W_2 &= (n + \hbar)W_1 - \frac{33,600\beta_1\beta_3^3\xi e_0^2 \hbar^2 p^6 t^{2\gamma}}{\Gamma(2\gamma + 1)(\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^6}, \\ W_3 &= (n + \hbar)W_2 + \frac{33,600\beta_1\beta_3^3\xi e_0^2 \hbar^2 p^6}{(\sqrt{\beta_3}(px + qy + \phi) - z\sqrt{-\beta_2 p^2 - \beta_3 q^2})^{11}} \\ &\times \left(\frac{6\sqrt{\beta_3}\beta_1 e_0 \hbar p t^{3\gamma} (\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^4}{\Gamma(3\gamma + 1)} \right. \\ &- \left. \frac{(n + \hbar)t^{2\gamma} (\sqrt{\beta_3}(px + qy + \phi) - (\sqrt{-\beta_2 p^2 - \beta_3 q^2})z)^5}{\Gamma(2\gamma + 1)} \right. \\ &\left. + \frac{13,440\beta_3^{\frac{5}{2}}\xi \hbar p^5 t^{3\gamma}}{\Gamma(3\gamma + 1)} - \frac{6720\beta_3^{\frac{5}{2}}\xi \hbar p^5 \Gamma(2\gamma + 1)t^{3\gamma}}{\Gamma(\gamma + 1)^2 \Gamma(3\gamma + 1)} \right). \end{aligned}$$

Accordingly, we can derive the remaining terms.

Example 2 Consider Eq. (44) with $k = 2$ given as

$$D_t^\gamma W + \beta_1 W^2 \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \frac{\partial^3 W}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W}{\partial x \partial z^2} + \xi \frac{\partial^5 W}{\partial x^5} = 0,$$

$$0 < \gamma \leq 1, t > 0,$$
(56)

with initial condition

$$W(x, y, z, 0) = e_0 + \frac{6i\sqrt{10}\sqrt{\xi}p^2}{\sqrt{\beta_1}(px + qy - \frac{\sqrt{-i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0p^2 - \beta_2p^2 - \beta_3q^2}}{\sqrt{\beta_3}}z + \phi)^2},$$
(57)

where $e_0, p, q,$ and ϕ are arbitrary constants. The exact solution for $\gamma = 1$ is given by

$$W(x, y, z, t) = e_0 + \frac{6i\sqrt{10}\sqrt{\xi}p^2}{\sqrt{\beta_1}(px + qy - \frac{\sqrt{-i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0p^2 - \beta_2p^2 - \beta_3q^2}}{\sqrt{\beta_3}}z - \beta_1e_0^2pt + \phi)^2}.$$
(58)

δ-HPTM Solution:

Application of δ -HPTM to Eq. (56) with Eq. (57) gives

$$\sum_{r=0}^\infty p^r W_r = \mathcal{L}^{-1} \left[\frac{1}{s} (W(x, y, z, 0)) \right] - p \left(1 - \frac{1}{\delta} \right) \left(\sum_{r=0}^\infty p^r W_r - W_0 \right) - p \left\{ \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\left(\beta_2 \frac{\partial^3}{\partial x^3} + \beta_3 \frac{\partial^3}{\partial x \partial y^2} + \beta_3 \frac{\partial^3}{\partial x \partial z^2} + \xi \frac{\partial^5}{\partial x^5} \right) \sum_{r=0}^\infty p^r W_r + \beta_1 \sum_{r=0}^\infty p^r \sum_{i=0}^r \sum_{j=0}^i W_j W_{i-j} \frac{\partial W_{r-i}}{\partial x} \right] \right] \right\}.$$
(59)

By equating the identical power terms of p in Eq. (59) we generate the sequence of δ -HPTM:

$$p^{(0)} : W_0 = W(x, y, z, 0),$$

$$p^{(1)} : W_1 = -\mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 W_0^2 \frac{\partial W_0}{\partial x} + \beta_2 \frac{\partial^3 W_0}{\partial x^3} + \beta_3 \frac{\partial^3 W_0}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_0}{\partial x \partial z^2} + \xi \frac{\partial^5 W_0}{\partial x^5} \right] \right],$$

$$p^{(2)} : W_2 = -\left(1 - \frac{1}{\delta} \right) W_1 - \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 W_0 W_0 \frac{\partial W_1}{\partial x} + \beta_1 W_0 W_1 \frac{\partial W_0}{\partial x} + \beta_1 W_1 W_0 \frac{\partial W_0}{\partial x} + \beta_2 \frac{\partial^3 W_1}{\partial x^3} + \beta_3 \frac{\partial^3 W_1}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_1}{\partial x \partial z^2} + \xi \frac{\partial^5 W_1}{\partial x^5} \right] \right],$$

⋮

$$p^{(r)} : W_r = -\left(1 - \frac{1}{\delta} \right) W_{r-1} - \mathcal{L}^{-1} \left[\frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 \sum_{i=0}^{r-1} \sum_{j=0}^i W_j W_{i-j} \frac{\partial W_{r-i-1}}{\partial x} + \beta_2 \frac{\partial^3 W_{r-1}}{\partial x^3} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial z^2} + \xi \frac{\partial^5 W_{r-1}}{\partial x^5} \right] \right], \quad r = 2, 3, 4, \dots$$
(60)

Hence, using initial condition Eq. (57), we derive:

$$\begin{aligned}
 W_0 &= e_0 + \frac{6i\sqrt{10}\sqrt{\xi}p^2}{\sqrt{\beta_1}(px + qy - \frac{\sqrt{-i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0p^2 - \beta_2p^2 - \beta_3q^2}}{\sqrt{\beta_3}}z + \phi)^2}, \\
 W_1 &= \frac{12i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0^2p^3t^\gamma}{\Gamma(\gamma + 1)(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^3}, \\
 W_2 &= \left(\frac{1}{\delta} - 1\right)W_1 - \frac{36i\sqrt{10}\beta_1^{\frac{3}{2}}\sqrt{\xi}e_0^4p^4t^{2\gamma}}{\Gamma(2\gamma + 1)(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^4}, \\
 W_3 &= \left(\frac{1}{\delta} - 1\right)W_2 - \frac{36\beta_1^{\frac{3}{2}}\sqrt{\xi}e_0^4p^4}{(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^9}, \\
 &\times \left(\frac{i\sqrt{10}(1 - \delta)t^{2\gamma}(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^5}{\delta\Gamma(2\gamma + 1)}\right. \\
 &- \frac{4i\sqrt{10}\beta_1e_0^2pt^{3\gamma}(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^4}{\Gamma(3\gamma + 1)} \\
 &+ \frac{240\sqrt{\beta_1}\sqrt{\xi}e_0p^3\Gamma(2\gamma + 1)t^{3\gamma}(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^2}{\Gamma(\gamma + 1)^2\Gamma(3\gamma + 1)} \\
 &- \frac{480\sqrt{\beta_1}\sqrt{\xi}e_0p^3t^{3\gamma}(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^2}{\Gamma(3\gamma + 1)} \\
 &\left. + \frac{1920i\sqrt{10}\xi p^5\Gamma(2\gamma + 1)t^{3\gamma}}{\Gamma(\gamma + 1)^2\Gamma(3\gamma + 1)} - \frac{3840i\sqrt{10}\xi p^5t^{3\gamma}}{\Gamma(3\gamma + 1)}\right).
 \end{aligned}$$

Following this procedure, we can obtain the remaining terms.

q-HATM Solution:

Implementing LT on Eq. (56) with Eq. (57), we obtain

$$\begin{aligned}
 \mathcal{L}[W] - \frac{1}{s}(W(x, y, z, 0)) \\
 + \frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 W^2 \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \frac{\partial^3 W}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W}{\partial x \partial z^2} + \xi \frac{\partial^5 W}{\partial x^5} \right] = 0. \tag{61}
 \end{aligned}$$

The nonlinear operator $\mathcal{N}(\phi)$, $\phi = \phi(x, y, z, t; q)$, is presented as

$$\begin{aligned}
 \mathcal{N}(\phi) &= \mathcal{L}[\phi] - \frac{1}{s}(W(x, y, z, 0)) \\
 &+ \frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 \phi^2 \frac{\partial \phi}{\partial x} + \beta_2 \frac{\partial^3 \phi}{\partial x^3} + \beta_3 \frac{\partial^3 \phi}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 \phi}{\partial x \partial z^2} + \xi \frac{\partial^5 \phi}{\partial x^5} \right]. \tag{62}
 \end{aligned}$$

Referring to Eq. (55), we have

$$\mathcal{R}_r(\vec{W}_{r-1}) = \mathcal{L}[W_{r-1}] - \left(1 - \frac{\Upsilon_r^*}{n}\right) \frac{1}{s} \{W(x, y, z, 0)\}$$

$$\begin{aligned}
 & + \frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 \sum_{i=0}^{r-1} \sum_{j=0}^i W_j W_{i-j} \frac{\partial W_{r-i-1}}{\partial x} \right. \\
 & \left. + \beta_2 \frac{\partial^3 W_{r-1}}{\partial x^3} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial z^2} + \xi \frac{\partial^5 W_{r-1}}{\partial x^5} \right], \tag{63}
 \end{aligned}$$

where

$$W_r = \Upsilon_r^* W_{r-1} + \hbar \mathcal{L}^{-1} [\mathcal{R}_r(\vec{W}_{r-1})]. \tag{64}$$

Solving Eqs. (64) using (57) and (63) for $r = 1, 2, 3, \dots$, we achieve the following:

$$\begin{aligned}
 W_0 &= e_0 + \frac{6i\sqrt{10}\sqrt{\xi}p^2}{\sqrt{\beta_1}(px + qy - \frac{\sqrt{-i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0p^2 - \beta_2p^2 - \beta_3q^2}}{\sqrt{\beta_3}}z + \phi)^2}, \\
 W_1 &= -\frac{12i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0^2\hbar p^3 t^\gamma}{\Gamma(\gamma + 1)(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^3}, \\
 W_2 &= (n + \hbar)W_1 + \frac{36i\sqrt{10}\beta_1^{3/2}\sqrt{\xi}e_0^4\hbar^2 p^4 t^{2\gamma}}{\Gamma(2\gamma + 1)(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^4}, \\
 W_3 &= (n + \hbar)W_2 + \frac{36\beta_1^{\frac{3}{2}}\sqrt{\xi}e_0^4\hbar^2 p^4}{(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^9} \\
 & \times \left\{ \frac{i\sqrt{10}(n + \hbar)t^{2\gamma}(px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^5}{\Gamma(2\gamma + 1)} \right. \\
 & - \frac{4i\sqrt{10}\beta_1 e_0^2 \hbar p t^{3\gamma} (px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^4}{\Gamma(3\gamma + 1)} \\
 & + \frac{240\sqrt{\beta_1}\sqrt{\xi}e_0\hbar p^3 \Gamma(2\gamma + 1)t^{3\gamma} (px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^2}{\Gamma(\gamma + 1)^2 \Gamma(3\gamma + 1)} \\
 & - \frac{480\sqrt{\beta_1}\sqrt{\xi}e_0\hbar p^3 t^{3\gamma} (px + qy - \frac{\sqrt{-\beta_3q^2 + p^2(-\beta_2 - i\sqrt{10}\sqrt{\beta_1}\sqrt{\xi}e_0)}}{\sqrt{\beta_3}}z + \phi)^2}{\Gamma(3\gamma + 1)} \\
 & \left. + \frac{1920i\sqrt{10}\xi \hbar p^5 \Gamma(2\gamma + 1)t^{3\gamma}}{\Gamma(\gamma + 1)^2 \Gamma(3\gamma + 1)} - \frac{3840i\sqrt{10}\xi \hbar p^5 t^{3\gamma}}{\Gamma(3\gamma + 1)} \right\}.
 \end{aligned}$$

Accordingly, we can derive the remaining terms.

Example 3 Consider Eq. (44) with $k = 4$ given as

$$\begin{aligned}
 \mathcal{D}_t^\gamma W + \beta_1 W^4 \frac{\partial W}{\partial x} + \beta_2 \frac{\partial^3 W}{\partial x^3} + \beta_3 \frac{\partial^3 W}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W}{\partial x \partial z^2} + \xi \frac{\partial^5 W}{\partial x^5} &= 0, \\
 0 < \gamma \leq 1, t > 0, \tag{65}
 \end{aligned}$$

with initial condition

$$W(x, y, z, 0) = \frac{2^{\frac{3}{4}} \sqrt[4]{-15} \sqrt[4]{\xi} p}{\sqrt[4]{\beta_1}} \tan\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right), \tag{66}$$

where p and q are arbitrary constants. The exact solution for $\gamma = 1$ is given by

$$W(x, y, z, t) = \frac{2^{\frac{3}{4}} \sqrt[4]{-15} \sqrt[4]{\xi} p}{\sqrt[4]{\beta_1}} \tan\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z + 24\xi p^5 t\right). \tag{67}$$

δ-HPTM Solution

Application of δ -HPTM to Eq. (65) with Eq. (66) gives

$$\begin{aligned} \sum_{r=0}^{\infty} p^r W_r &= \mathcal{L}^{-1}\left[\frac{1}{s}(W(x, y, z, 0))\right] - p\left(1 - \frac{1}{\delta}\right)\left(\sum_{r=0}^{\infty} p^r W_r - W_0\right) \\ &\quad - p\left\{\mathcal{L}^{-1}\left[\frac{1}{s^\gamma}\mathcal{L}\left[\left(\beta_2 \frac{\partial^3}{\partial x^3} + \beta_3 \frac{\partial^3}{\partial x \partial y^2} + \beta_3 \frac{\partial^3}{\partial x \partial z^2} + \xi \frac{\partial^5}{\partial x^5}\right)\right.\right.\right. \\ &\quad \left.\left.\left.\times \sum_{r=0}^{\infty} p^r W_r + \beta_1 \sum_{r=0}^{\infty} p^r \sum_{i=0}^r \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k W_l W_{k-l} W_{j-k} W_{i-j} \frac{\partial W_{r-i}}{\partial x}\right]\right]\right\}. \end{aligned} \tag{68}$$

By equating the identical power terms of p in Eq. (68) we generate the sequence of δ -HPTM:

$$\begin{aligned} p^{(0)} : W_0 &= W(x, y, z, 0), \\ p^{(1)} : W_1 &= -\mathcal{L}^{-1}\left[\frac{1}{s^\gamma}\mathcal{L}\left[\beta_1 W_0^2 \frac{\partial W_0}{\partial x} + \beta_2 \frac{\partial^3 W_0}{\partial x^3} + \beta_3 \frac{\partial^3 W_0}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_0}{\partial x \partial z^2} + \xi \frac{\partial^5 W_0}{\partial x^5}\right]\right], \\ p^{(2)} : W_2 &= -\left(1 - \frac{1}{\delta}\right)W_1 - \mathcal{L}^{-1}\left[\frac{1}{s^\gamma}\mathcal{L}\left[\beta_1 W_0 W_0 \frac{\partial W_1}{\partial x} + \beta_1 W_0 W_1 \frac{\partial W_0}{\partial x}\right.\right. \\ &\quad \left.\left.+ \beta_1 W_1 W_0 \frac{\partial W_0}{\partial x} + \beta_2 \frac{\partial^3 W_1}{\partial x^3} + \beta_3 \frac{\partial^3 W_1}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_1}{\partial x \partial z^2} + \xi \frac{\partial^5 W_1}{\partial x^5}\right]\right], \\ &\vdots \\ p^{(r)} : W_r &= -\left(1 - \frac{1}{\delta}\right)W_{r-1} \\ &\quad - \mathcal{L}^{-1}\left[\frac{1}{s^\gamma}\mathcal{L}\left[\beta_1 \sum_{i=0}^{r-1} \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k W_l W_{k-l} W_{j-k} W_{i-j} \frac{\partial W_{(r-i-1)}}{\partial x}\right.\right. \\ &\quad \left.\left.+ \beta_2 \frac{\partial^3 W_{r-1}}{\partial x^3} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial z^2} + \xi \frac{\partial^5 W_{r-1}}{\partial x^5}\right]\right], \quad r = 2, 3, 4, \dots \end{aligned} \tag{69}$$

Hence, using initial condition Eq. (66), we derive:

$$W_0 = \frac{(2^{\frac{3}{4}} \sqrt[4]{-15} \sqrt[4]{\xi} p)}{\sqrt[4]{\beta_1}} \tan\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right),$$

$$\begin{aligned}
 W_1 &= \frac{24\sqrt[4]{-15\xi^{\frac{5}{4}}2^{\frac{3}{4}}p^6t^\gamma}}{\sqrt[4]{\beta_1}\Gamma(\gamma+1)} \sec^2\left(px+qy-\frac{\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right), \\
 W_2 &= \left(\frac{1}{\delta}-1\right)W_1 + \frac{1152\sqrt[4]{-15}2^{3/4}\xi^{9/4}p^{11}t^{2\gamma}}{\sqrt[4]{\beta_1}\Gamma(2\gamma+1)} \\
 &\quad \times \left\{ \tan\left(px+qy-\frac{\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \right. \\
 &\quad \times \left. \sec^2\left(px+qy-\frac{\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \right\}, \\
 W_3 &= \left(\frac{1}{\delta}-1\right)W_2 - \frac{36\sqrt[4]{-15}2^{\frac{3}{4}}\xi^{\frac{9}{4}}p^{11}t^{2\gamma}}{\sqrt[4]{\beta_1}\delta\Gamma(\gamma+1)^2\Gamma(2\gamma+1)\Gamma(3\gamma+1)} \\
 &\quad \times \sec^8\left(px+qy-\frac{\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \\
 &\quad \times \left\{ 16\Gamma(\gamma+1)^2\left[2(\delta-1)\Gamma(3\gamma+1)\sin\left(px+qy-\frac{\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \right. \right. \\
 &\quad \times \left. \cos^5\left(px+qy-\frac{\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \right. \\
 &\quad + 38\xi p^5\Gamma(2\gamma+1)t^\gamma\left(-569\cos\left(2px+2qy-\frac{2\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \right. \\
 &\quad + 80\cos\left(4px+4qy-\frac{4\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \\
 &\quad \left. \left. + \cos\left(6px+6qy-\frac{6\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right)+472\right) \right] \\
 &\quad + 3840\delta\xi p^5\Gamma(2\gamma+1)^2t^\gamma\sin^2\left(px+qy-\frac{\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right) \\
 &\quad \times \left. \left(2\cos\left(2px+2qy-\frac{2\sqrt{-20\xi p^4-\beta_2p^2-\beta_3q^2}}{\sqrt{\beta_3}}z\right)-5\right) \right\}.
 \end{aligned}$$

By following this procedure we can obtain other terms.

q-HATM Solution:

Implementing LT on Eq. (65) with Eq. (66), we obtain

$$\begin{aligned}
 \mathcal{L}[W] - \frac{1}{s}(W(x,y,z,0)) \\
 + \frac{1}{s^\gamma}\mathcal{L}\left[\beta_1W^4\frac{\partial W}{\partial x} + \beta_2\frac{\partial^3 W}{\partial x^3} + \beta_3\frac{\partial^3 W}{\partial x\partial y^2} + \beta_3\frac{\partial^3 W}{\partial x\partial z^2} + \xi\frac{\partial^5 W}{\partial x^5}\right] = 0.
 \end{aligned} \tag{70}$$

The nonlinear operator $\mathcal{N}(\phi)$, $\phi = \phi(x,y,z,t;q)$, is given as

$$\begin{aligned}
 \mathcal{N}(\phi) = \mathcal{L}[\phi] - \frac{1}{s}(W(x,y,z,0)) \\
 + \frac{1}{s^\gamma}\mathcal{L}\left[\beta_1\phi^4\frac{\partial\phi}{\partial x} + \beta_2\frac{\partial^3\phi}{\partial x^3} + \beta_3\frac{\partial^3\phi}{\partial x\partial y^2} + \beta_3\frac{\partial^3\phi}{\partial x\partial z^2} + \xi\frac{\partial^5\phi}{\partial x^5}\right].
 \end{aligned} \tag{71}$$

Referring to Eq. (55), we have

$$\begin{aligned} \mathcal{R}_r(\vec{W}_{r-1}) = & \mathcal{L}[W_{r-1}] - \left(1 - \frac{\Upsilon_r^*}{n}\right) \frac{1}{s} \{W(x, y, z, 0)\} \\ & + \frac{1}{s^\gamma} \mathcal{L} \left[\beta_1 \sum_{i=0}^{r-1} \sum_{j=0}^i \sum_{k=0}^j \sum_{l=0}^k W_l W_{k-l} W_{j-k} W_{i-j} \frac{\partial W_{(r-i-1)}}{\partial x} \right. \\ & \left. + \beta_2 \frac{\partial^3 W_{r-1}}{\partial x^3} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial y^2} + \beta_3 \frac{\partial^3 W_{r-1}}{\partial x \partial z^2} + \xi \frac{\partial^5 W_{r-1}}{\partial x^5} \right], \end{aligned} \tag{72}$$

where

$$W_r = \Upsilon_r^* W_{r-1} + \hbar \mathcal{L}^{-1}[\mathcal{R}_r(\vec{W}_{r-1})]. \tag{73}$$

Solving Eqs. (73) using (66) and (72) for $r = 1, 2, 3, \dots$, we get:

$$\begin{aligned} W_0 = & \frac{(2^{\frac{3}{4}} \sqrt[4]{-15} \sqrt[4]{\xi} p)}{\sqrt[4]{\beta_1}} \tan\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right), \\ W_1 = & -\frac{24 \sqrt[4]{-15} 2^{3/4} \xi^{5/4} \hbar p^6 t^\gamma \sec^2\left(px + qy - \frac{z \sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}}\right)}{\sqrt[4]{\beta_1} \Gamma(\gamma + 1)}, \\ W_2 = & (n + \hbar) W_1 + \frac{1152 \sqrt[4]{-15} 2^{3/4} \xi^{5/4} \hbar^2 p^{11} t^{2\gamma}}{\sqrt[4]{\beta_1} \Gamma(2\gamma + 1)} \\ & \times \left\{ \tan\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \right. \\ & \left. \times \sec^2\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \right\}, \\ W_3 = & (n + \hbar) W_2 + \frac{36 \sqrt[4]{-15} 2^{3/4} \xi^{5/4} \hbar^2 p^{11} t^{2\gamma}}{\sqrt[4]{\beta_1} \Gamma(\gamma + 1)^2 \Gamma(2\gamma + 1) \Gamma(3\gamma + 1)} \\ & \times \sec^8\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \\ & \times \left[16 \Gamma(\gamma + 1)^2 \left[2 \Gamma(3\gamma + 1) (n + \hbar) \sin\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \right. \right. \\ & \times \cos^5\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \\ & + 3 \xi \hbar p^5 \Gamma(2\gamma + 1) t^\gamma \left(-569 \cos\left(2px + 2qy - \frac{2 \sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \right. \\ & + 80 \cos\left(4px + 4qy - \frac{4 \sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \\ & \left. \left. + \cos\left(6px + 6qy - \frac{6 \sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) + 472 \right) \right] \\ & + 3840 \xi \hbar p^5 \Gamma(2\gamma + 1)^2 t^\gamma \sin^2\left(px + qy - \frac{\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z\right) \end{aligned}$$

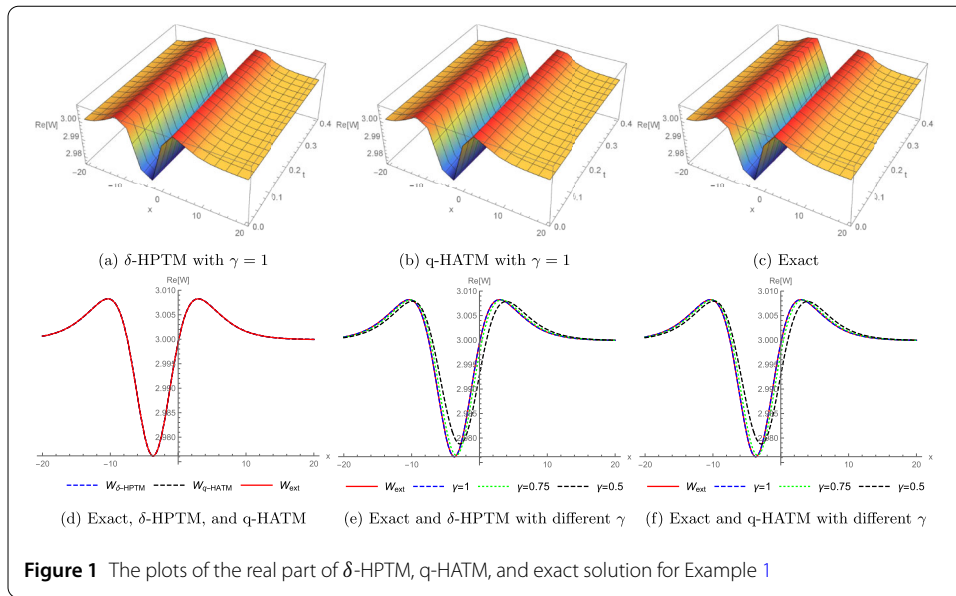


Figure 1 The plots of the real part of δ -HPTM, q-HATM, and exact solution for Example 1

$$\times \left(2 \cos \left(2px + 2qy - \frac{2\sqrt{-20\xi p^4 - \beta_2 p^2 - \beta_3 q^2}}{\sqrt{\beta_3}} z \right) - 5 \right) \Bigg\}.$$

Respectively, we can derive the remaining terms.

5 Numerical comparison

In this section, the δ -HPTM and q-HATM formulations are tested upon the generalized perturbed (3 + 1)-dimensional Zakharov–Kuznetsov (gpZK) equation with Caputo fractional derivative. The δ -HPTM solution is presented as

$$\mathcal{W}_{(3)}(x, y, z, t; \delta) = \sum_{r=0}^3 W_r \delta^r, \tag{74}$$

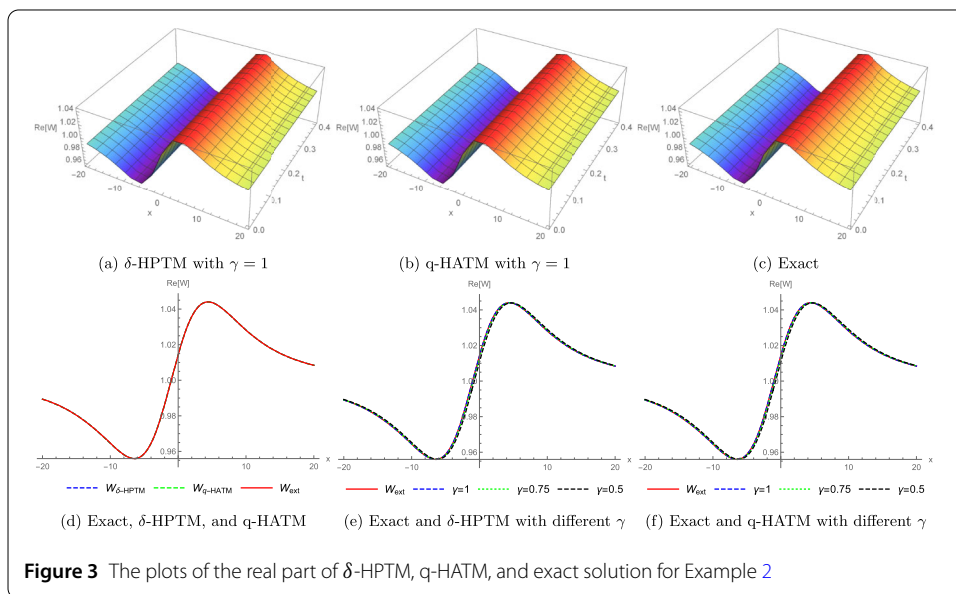
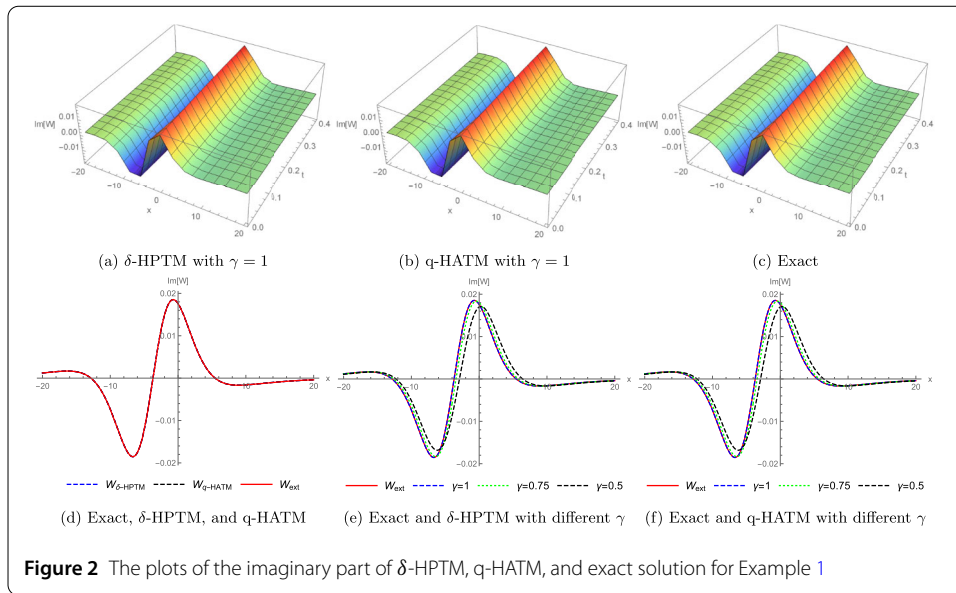
and the q-HATM solution is presented as

$$\mathcal{W}_{(3)}(x, t; n; \hbar) = W_0 + \sum_{r=1}^3 W_r \left(\frac{1}{n} \right)^r. \tag{75}$$

We observe that setting $\delta = \frac{1}{n}$ in Eq. (74) yields

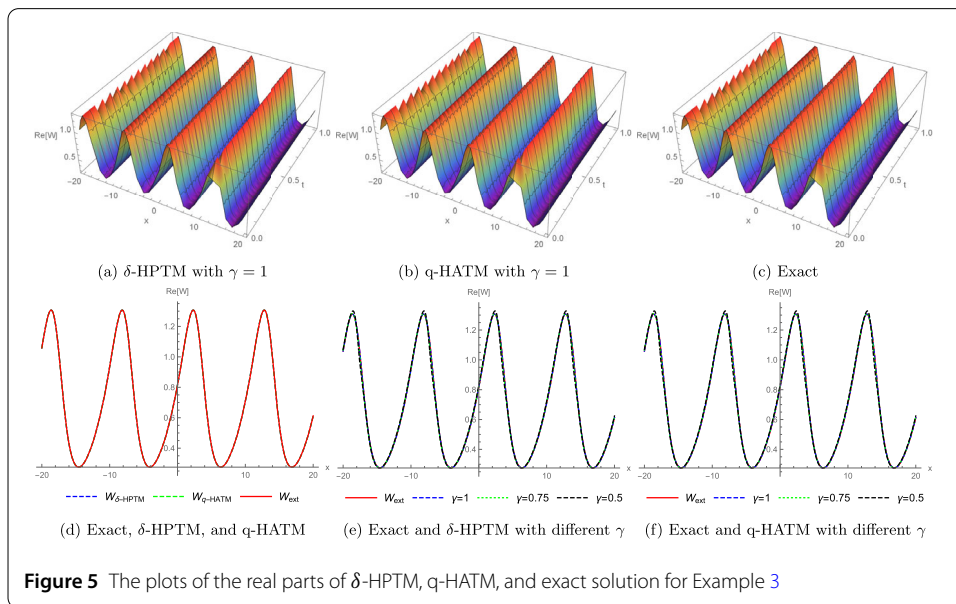
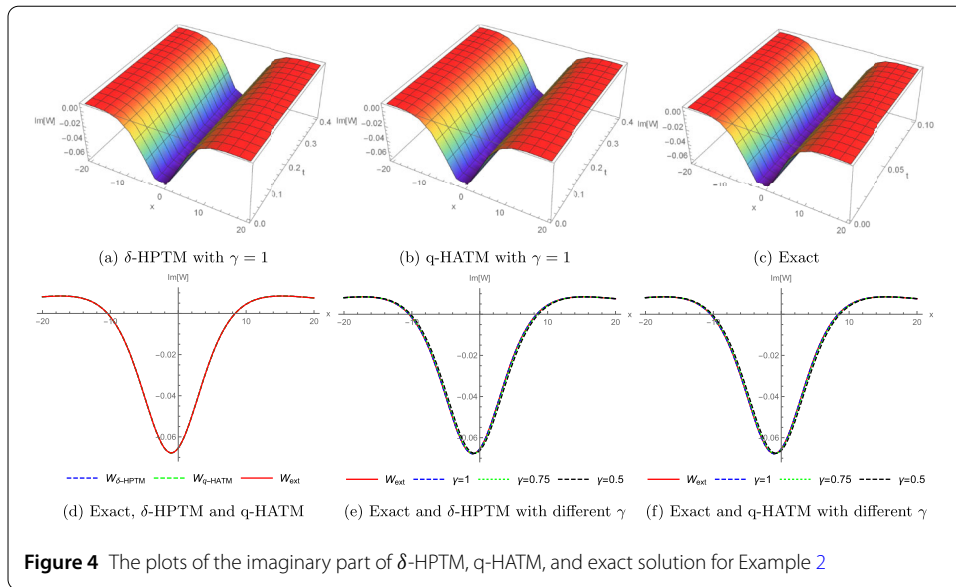
$$\mathcal{W}_{(3)} \left(x, y, z, t; \frac{1}{n} \right) = \sum_{r=0}^3 W_r \left(\frac{1}{n} \right)^r, \tag{76}$$

which is the solution of q-HATM. Thus we can conclude that this present modification (δ -HPTM) is more reliable and general. In Figs. 1–6, we present the response of the obtained solutions by the proposed methods with regard to the real and imaginary parts in terms of 2D and 3D plots. The 2D and 3D plots show the graphical comparison of the four-term approximation solutions obtain by δ -HPTM and q-HATM and their exact solutions. The 2D plots also present the effect and behavior of the distinct fractional orders on the solution profile. In addition, Figs. 1–4 exhibit different shapes of the exact and approximate



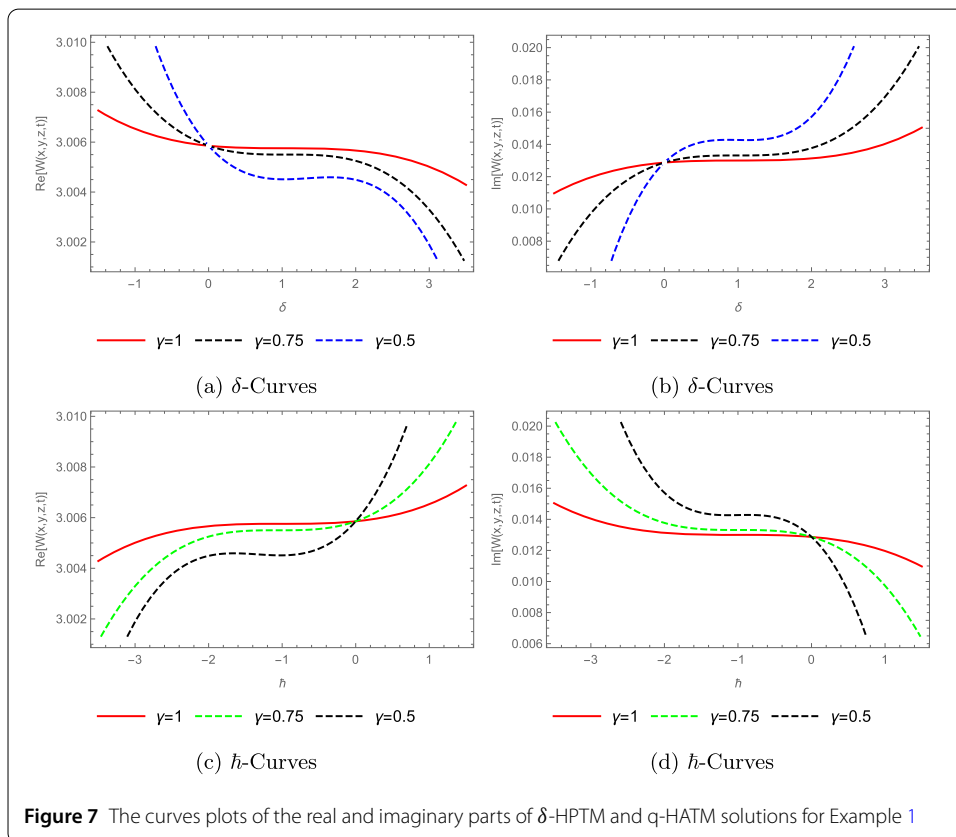
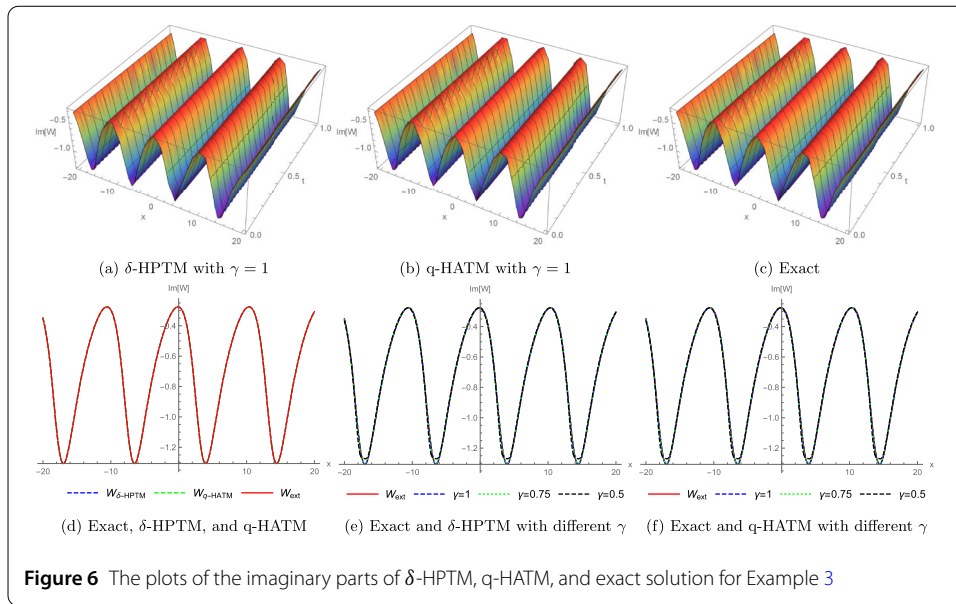
soliton-like solutions, whereas Figs. 5, 6 represent the periodic wave solutions of the gpZK equation. The dynamics of the solution profile can obviously be noted and justify why gpZK should be examined to understand the effects in real-life applications.

The selection of the auxiliary parameters δ in δ -HPTM and \hbar in q-HATM are very crucial to guarantee fast convergence of the series solutions. For this reason, in Figs. 7–9, we have provided the so-called δ -curves and \hbar -curves of the two proposed methods, which serve as a guide in our optimal selection of values in the present analysis. The horizontal line test is employed to attain the intervals containing optimal values. The comparative study for the case $\gamma = 1$ of the real and imaginary parts of the results obtained by δ -HPTM, q-HATM, and the exact solution as the benchmark are considered in Tables 1–6. From these tables and plots we can observe that the solutions obtained by the proposed methods are very accurate and in agreement with their respective exact solutions.



Remark 2 The parameter values used for Figs. 1–9 are as follows:

- Figs. 1 and 2: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, y = z = 2, \xi = 0.1, e_0 = 3, p = q = 0.5, \phi = 1, n = 1, \delta = 1, \hbar = -1,$ and $t = 0.1$.
- Figs. 3 and 4: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, y = z = 2, \xi = 0.1, e_0 = 1, p = q = 0.8, \phi = 1, n = 1, \delta = 1, \hbar = -1,$ and $t = 0.1$.
- Figs. 5 and 6: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 10, y = z = 2, \xi = 1, e_0 = 3, p = q = 0.3, n = 1, \delta = 1, \hbar = -1,$ and $t = 0.5$.
- Fig. 7: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, \xi = 0.1, e_0 = 3, p = 0.5, q = 0.5, \phi = 1, z = y = 2, n = 1, x = 1,$ and $t = 0.01$.
- Fig. 8: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, \xi = 0.1, e_0 = 1, p = 0.8, q = 0.8, \phi = 1, z = y = 2, n = 1, x = 1,$ and $t = 0.01$.



- Fig. 9: $\beta_1 = 1, \beta_2 = 2, \beta_3 = 10, \xi = 1, e_0 = 3, p = 0.3, q = 0.3, z = y = 2, n = 1, x = 1,$ and $t = 0.5$.

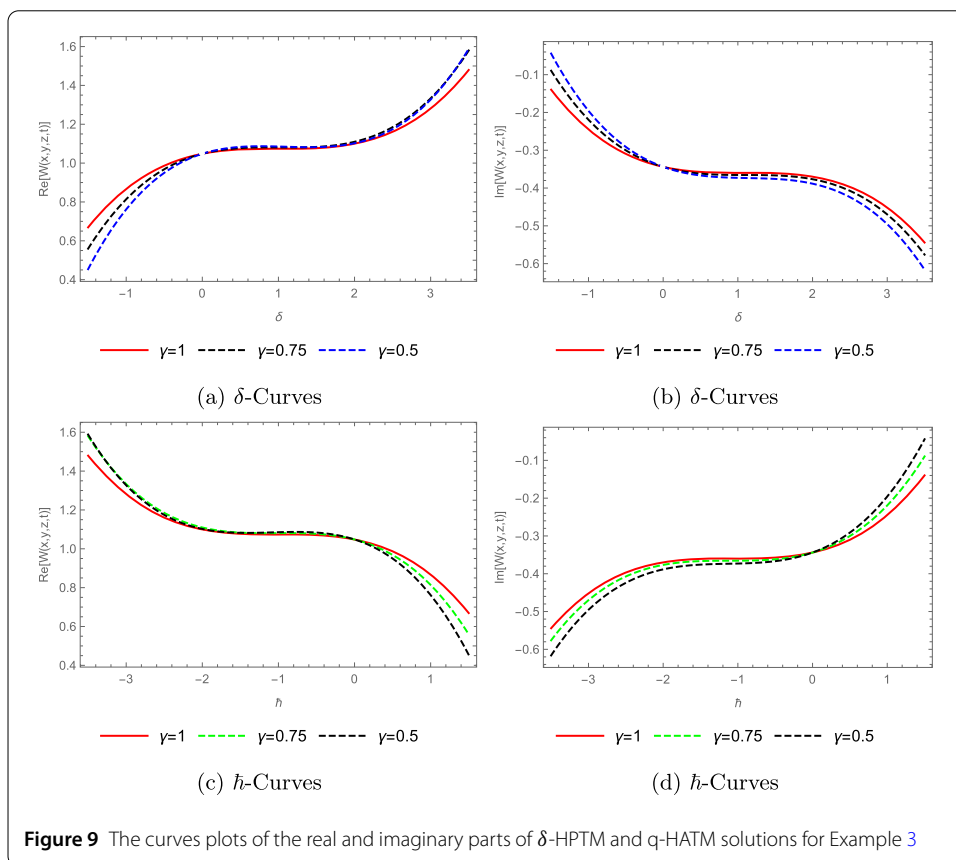
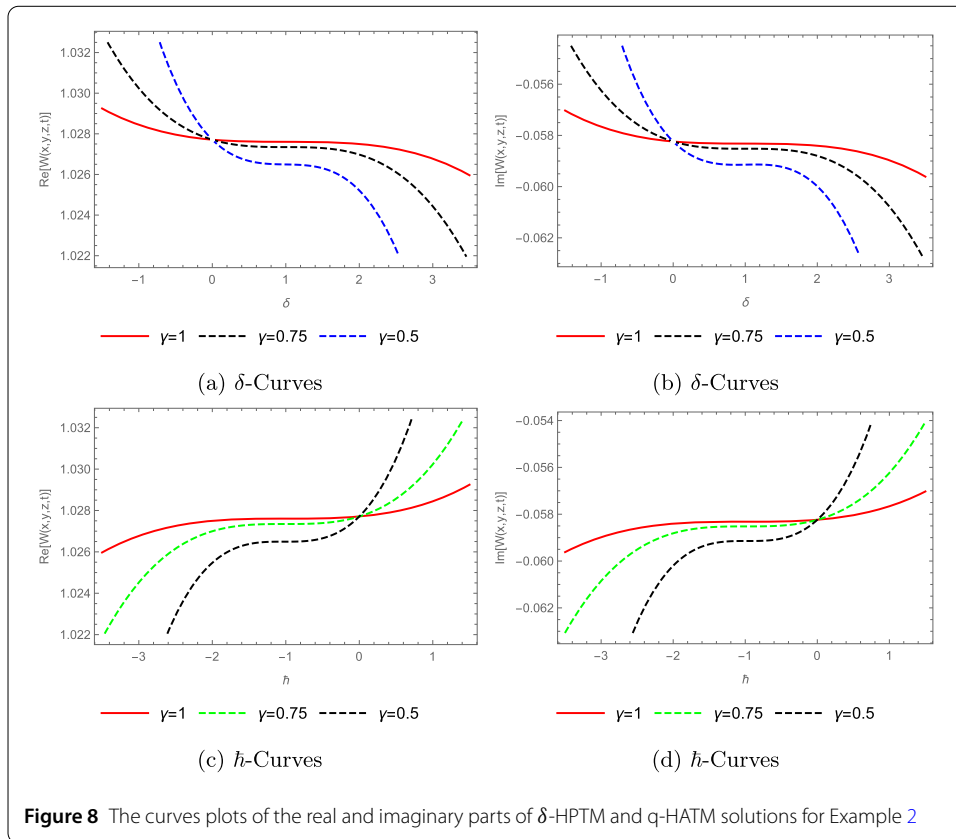


Table 1 The comparative study of $\text{Re}[\mathcal{W}_{(3)}]$ solutions of δ -HPTM, q-HATM, and exact solution for Example 1 at $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, y = 2, z = 2, \xi = 0.1, e_0 = 3, p = 0.5, q = 0.5, \phi = 1$, and $t = 0.01$

x	Exact	δ -HPTM ($\delta = 1$)		q-HPTM ($n = 1, h = 1$)	
		Approx	Absolute error	Approx	Absolute error
-15	3.0037055813	3.0037055813	2.007727×10^{-12}	3.0037055813	2.007727×10^{-12}
-10	3.00796331114	3.0079633111	1.639577×10^{-12}	3.0079633111	1.639577×10^{-12}
-5	2.9790685680	2.9790685680	5.847900×10^{-11}	2.9790685680	5.847900×10^{-11}
0	3.0010839974	3.0010839974	4.721068×10^{-11}	3.0010839974	4.721068×10^{-11}
5	3.0062055680	3.0062055680	6.424195×10^{-12}	3.0062055680	6.424195×10^{-12}
10	3.0014721223	3.0014721223	5.728751×10^{-14}	3.0014721223	5.728751×10^{-14}
15	3.0001939705	3.0001939705	1.092459×10^{-13}	3.0001939705	1.092459×10^{-13}

Table 2 The comparative study of $\text{Im}[\mathcal{W}_{(3)}]$ solutions of δ -HPTM, q-HATM, and exact solution for Example 1 at $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, y = 2, z = 2, \xi = 0.1, e_0 = 3, p = 0.5, q = 0.5, \phi = 1$, and $t = 0.01$

x	Exact	δ -HPTM ($\delta = 1$)		q-HPTM ($n = 1, h = 1$)	
		Approx	Absolute error	Approx	Absolute error
-15	0.0014301059	0.0014301059	1.789911×10^{-12}	0.0014301059	1.789911×10^{-12}
-10	-0.0084303978	-0.0084303979	2.285488×10^{-11}	-0.0084303979	2.285488×10^{-11}
-5	-0.0100504513	-0.0100504513	6.996747×10^{-11}	-0.0100504513	6.996747×10^{-11}
0	0.0168457860	0.0168457860	6.948674×10^{-12}	0.0168457860	6.948674×10^{-12}
5	0.0002672659	0.0002672659	4.853362×10^{-13}	0.0002672659	4.853362×10^{-13}
10	-0.0015748494	-0.0015748494	7.746273×10^{-13}	-0.0015748494	7.746273×10^{-13}
15	-0.0008303837	-0.0008303837	5.348478×10^{-14}	-0.0008303837	5.348478×10^{-14}

Table 3 The comparative study of $\text{Re}[\mathcal{W}_{(3)}]$ solutions of δ -HPTM, q-HATM, and exact solution for Example 2 at $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, y = 2, z = 2, \xi = 0.1, e_0 = 1, p = 0.8, q = 0.8, \phi = 1$, and $t = 0.01$

x	Exact	δ -HPTM ($\delta = 1$)		q-HPTM ($n = 1, h = 1$)	
		Approx	Absolute error	Approx	Absolute error
-15	0.9801926018	0.9801926018	5.551115×10^{-15}	0.9801926018	5.551115×10^{-15}
-10	0.9642322805	0.9642322805	6.294965×10^{-14}	0.9642322805	6.294965×10^{-14}
-5	0.9591394126	0.9591394126	1.930678×10^{-13}	0.9591394126	1.930678×10^{-13}
0	1.0156139368	1.0156139368	2.708944×10^{-13}	1.0156139368	2.708944×10^{-13}
5	1.0435482789	1.0435482789	4.729550×10^{-14}	1.0435482789	4.729550×10^{-14}
10	1.0279075447	1.0279075447	2.753353×10^{-14}	1.0279075447	2.753353×10^{-14}
15	1.0150065850	1.0150065850	2.220446×10^{-16}	1.0150065850	2.220446×10^{-16}

Table 4 The comparative study of $\text{Im}[\mathcal{W}_{(3)}]$ solutions of δ -HPTM, q-HATM, and exact solution for Example 2 at $\beta_1 = 1, \beta_2 = 2, \beta_3 = 0.1, y = 2, z = 2, \xi = 0.1, e_0 = 1, p = 0.8, q = 0.8, \phi = 1$, and $t = 0.01$

x	Exact	δ -HPTM ($\delta = 1$)		q-HPTM ($n = 1, h = 1$)	
		Approx	Absolute error	Approx	Absolute error
-15	0.0079063274	0.0079063274	1.225582×10^{-14}	0.0079063274	1.225582×10^{-14}
-10	-0.0020455489	-0.0020455489	1.109139×10^{-14}	-0.0020455489	1.109139×10^{-14}
-5	-0.0410293761	-0.0410293761	1.890016×10^{-13}	-0.0410293761	1.890016×10^{-13}
0	-0.0649954753	-0.0649954753	3.137629×10^{-13}	-0.0649954753	3.137629×10^{-13}
5	-0.0193551372	-0.0193551372	1.423792×10^{-13}	-0.0193551372	1.423792×10^{-13}
10	0.0046715148	0.0046715148	1.504092×10^{-14}	0.0046715148	1.504092×10^{-14}
15	0.0084703527	0.0084703527	7.089815×10^{-15}	0.0084703527	7.089815×10^{-15}

6 Conclusion

In this paper, we proposed a new modification of the homotopy perturbation method (HPM), called the δ -homotopy perturbation transform method (δ -HPTM), which consists of HPM, the Laplace transform method, and a control parameter δ for solving integer- and

Table 5 The comparative study of $\text{Re}[\mathcal{W}_{(3)}]$ solutions of δ -HPTM, q-HATM, and exact solution for Example 3 at $\beta_1 = 1, \beta_2 = 2, \beta_3 = 10, y = 2, z = 2, \xi = 1, e_0 = 3, p = 0.3, q = 0.3, \phi = 1$, and $t = 0.01$

x	Exact	δ -HPTM ($\delta = 1$)		q-HPTM ($n = 1, h = 1$)	
		Approx	Absolute error	Approx	Absolute error
-15	0.2924916343	0.2924916343	2.881029×10^{-14}	0.2924916343	2.881029×10^{-14}
-10	0.9093043357	0.9093043357	6.816769×10^{-14}	0.9093043357	6.816769×10^{-14}
-5	0.3436158983	0.3436158983	1.532108×10^{-14}	0.3436158983	1.532108×10^{-14}
0	0.7928167252	0.7928167252	3.141931×10^{-14}	0.7928167252	3.141931×10^{-14}
5	0.4400466777	0.4400466777	1.393330×10^{-13}	0.4400466777	1.393330×10^{-13}
10	0.6876241187	0.6876241187	1.110223×10^{-14}	0.6876241187	1.110223×10^{-14}
15	0.5904802556	0.5904802556	3.366196×10^{-13}	0.5904802556	3.366196×10^{-13}

Table 6 The comparative study of $\text{Im}[\mathcal{W}_{(3)}]$ solutions of δ -HPTM, q-HATM, and exact solution for Example 3 at $\beta_1 = 1, \beta_2 = 2, \beta_3 = 10, y = 2, z = 2, \xi = 1, e_0 = 3, p = 0.3, q = 0.3$, and $t = 0.01$

x	Exact	δ -HPTM ($\delta = 1$)		q-HPTM ($n = 1, h = 1$)	
		Approx	Absolute error	Approx	Absolute error
-15	-0.9227631870	-0.9227631870	7.371881×10^{-14}	-0.9227631870	7.371881×10^{-14}
-10	-0.2891329576	-0.2891329576	3.064215×10^{-14}	-0.2891329576	3.064215×10^{-14}
-5	-1.0474742139	-1.0474742140	1.303402×10^{-13}	-1.0474742140	1.303402×10^{-13}
0	-0.2754234881	-0.2754234881	3.419487×10^{-14}	-0.2754234881	3.419487×10^{-14}
5	-1.1689377811	-1.1689377811	1.636469×10^{-13}	-1.1689377811	1.636469×10^{-13}
10	-0.2859219243	-0.2859219243	2.742251×10^{-14}	-0.2859219243	2.742251×10^{-14}
15	-1.2662691579	-1.2662691579	5.351275×10^{-14}	-1.2662691579	5.351275×10^{-14}

noninteger-order nonlinear problems. We effectively used the proposed method and q-HATM to obtain analytical approximate solutions of the generalized fractional perturbed $(3 + 1)$ -dimensional Zakharov–Kuznetsov equation. This equation characterizes nonlinear dust-ion-acoustic waves in the magnetized two-ion-temperature dusty plasmas. In comparison to the control parameters n and h in q-HATM, the control parameter δ in δ -HPTM also helps to adjust and control the convergence region of the series solutions and can overcome some limitations of HPM, HPTM, and He–Laplace method. The two methods present series solutions in the form of recurrence relation with high exactness and minimal computations. In reality, we consider HPM, HAM, HPTM, PHPM, and He–Laplace method as particular cases of δ -HPTM and more general when compared with q-HATM (see Eq. (76)). Finally, δ -HPTM can be considered as a good refinement of the existing numerical techniques and can be employed to study strongly nonlinear mathematical models describing natural phenomena.

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Authors' contributions

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