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On the characteristic polynomial of (k,p)-Fibonacci sequence

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Abstract

Recently, Bednarz introduced a new two-parameter generalization of the Fibonacci sequence, which is called the (k,p)-Fibonacci sequence and denoted by $(F_{k,p}(n))_{n\geq 0}$. In this paper, we study the geometry of roots of the characteristic polynomial of this sequence.

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1 Introduction

The *Fibonacci sequence* $(F_n)_n$ is one of the most famous sequences in mathematics. This sequence is defined by the binary recurrence $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ with initial values $F_0 = 0$ and $F_1 = 1$. So, its first ten nonzero terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, and 55. A well-known nonrecursive formula for the nth Fibonacci number is called the *Binet formula*:

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $\alpha := (1+\sqrt{5})/2$ and $\beta := (1-\sqrt{5})/2$. The Fibonacci numbers have been the main object of many books (see, e.g., [1–5] and some references therein). Many generalizations of this sequence have appeared in the literature. Probably, the most known generalization is the k-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq -(k-2)}$ (also known as the *k-bonacci*, the *k-fold Fibonacci*, or *kth-order Fibonacci*) defined by

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$$

with initial values $F_{-j}^{(k)} = 0$ (for $j = 0, 1, \ldots, k-2$) and $F_1^{(k)} = 1$. Their recent wide and intensive study was started in 1960 by Miles [6]. In 1971, Miller [7] proved some basic facts on the geometry of the roots of their characteristic polynomial

$$\psi_k(x) := x^k - x^{k-1} - x^{k-2} - \cdots - x - 1,$$

which was a foundation for a systematic search for a "Binet-like" formula for $(F_n^{(k)})_n$ (see, e.g., [8–10]). The k-generalized Fibonacci sequence was further generalized; see [11–15].

Another generalization, applied in a new coding method, and defined by the recurrence

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad \text{for } n \geq p+2,$$

with $F_p(j) = 1$ for $j = 1, \ldots, p+1$, was introduced by Stakhov [16] and is called as *Fibonacci p-numbers*. Stakhov and Rozin [17, 18] studied some properties of the roots of their characteristic equation

$$x^{p+1} - x^p - 1 = 0,$$

and Kılıç [13] proved that all roots are simple and provided a "Binet-like" formula for $(F_p(n))_n$. This sequence was gradually generalized in [19–22].

In 2008, Włoch [23] studied the total number of k-independent sets in some graphs, which led her to the sequence $(P(n,k))_{n\geq 0}$ called *the generalized Pell numbers*. For $k \geq 2$, these numbers are defined by the recurrent relation

$$P(n,k) = P(n-1,k) + P(n-k+1,k) + P(n-k,k) \quad \text{for } n \geq k+3 \tag{1}$$

with initial values $P(i,k) = 2k-2$ for $3 \leq i \leq k$, $P(k+1,k) = 2k+1$, and

$$P(k+2,k) = \begin{cases} 12 & \text{if } k = 2, \\ 2k+7 & \text{if } k \geq 3. \end{cases}$$

Recently, Trojovský [24] dealt with the behavior (in the algebraic and analytic sense) of the roots of the characteristic polynomial $p_k(x) = x^k - x^{k-1} - x - 1$ of the sequence $(P(n,k))_{n\geq 0}$.

Very recently, Bednarz [25] introduced a new type of generalization of Fibonacci numbers (depending on two integer parameters $p \geq 2$ and $k \geq 3$), called the *(k,p)-Fibonacci numbers*, by the following recurrence:

$$F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k) \quad \text{for } n \geq k$$

with initial values $F_{k,p}(0) = 0$ and $F_{k,p}(j) = p^{j-1}$, $j = 1, 2, \ldots, k-1$. The characteristic polynomial of this sequence is

$$f_{k,p}(x) = x^k - px^{k-1} - (p-1)x - 1.$$

In 2020, Bednarz and Włoch [26] studied interesting interpretations of these numbers in undirected simple graphs and found some interesting identities.

In this paper, we are interested in studying the geometry of roots of the characteristic polynomial of this sequence $(F_{k,p}(n))_{n\geq 0}$. Our main result is the following:

Theorem 1 *For integers $p \geq 2$ and $k \geq 3$, the polynomial $f_{k,p}(x)$ has the following properties*:

(i) $f_{k,p}(x)$ *has a dominant root, say* $\alpha_{k,p}$ (*which is its only positive root*), *and*

$$p < \alpha_{k,p} < p + \frac{2}{p^{k-3}}$$

for all $k \geq 2$. *In particular,* $\lim_{k\to\infty} \alpha_{k,p} = p$ *and* $\lim_{p\to\infty} \alpha_{k,p} = \infty$;

(ii) $f_{k,p}(x)$ *has one negative root for any* $p \geq 2$ *and even* $k \geq 3$;

(iii) $f_{k,p}(x)$ *has two negative roots when k is odd*:

- (iii$_a$) $p = 3$ *and* $k \geq 7$,
- (iii$_b$) $p \in \{4, 5, 6\}$ *and* $k \geq 5$,
- (iii$_c$) $p \geq 7$ *and* $k \geq 3$;

(iv) *all roots of* $f_{k,p}(x)$ *are simple.*

As in all previous generalizations of Fibonacci numbers, this theorem is the basis for finding a Binet-like formula for direct calculation of terms of the sequence $F_{k,p}(n)$, but since the roots of its characteristic polynomial do not have a simple form, the existence of a certain simple formula is unlikely. We will show, however, a particular case, in which we know a little more about the roots.

Remark 1 If $k \equiv 5 \pmod 6$, then $x^2 - x + 1$ divides $f_{k,p}(x)$, that is, $f_{k,p}(\omega^j) = 0$ for $\omega = (1 + \sqrt{-3})/2$ and $j \in \{1, 2\}$. Indeed, since $\omega^3 = -1$, $\omega^4 = -\omega$, $\omega^5 = -\omega^2$, and $\omega^6 = 1$, we have (for $k = 6t + 5$, where t is a nonnegative integer)

$$\begin{aligned} f_{k,p}(\omega) &= \omega^{6t+5} - p\omega^{6t+4} - (p-1)\omega - 1 \\ &= \omega^5 - p\omega^4 - (p-1)\omega - 1 = -(\omega^2 - \omega + 1) = 0. \end{aligned}$$

The same argument can be used to deduce that $f_{k,p}(\omega^2) = 0$. Furthermore, a short calculation shows the factorization

$$\begin{aligned} f_{k,p}(x) &= (x^2 - x + 1)\left(x(x+1)(x-p)\sum_{i=0}^{(k-5)/3}(-1)^i x^{3i} - 1\right) \\ &= (x^2 - x + 1)\left(x(x-p)\frac{x^{k-2}+1}{x^2-x+1} - 1\right). \end{aligned}$$

Example 1 Using Remark 1 (and Cardano's formula), we can find the exact form for all roots of the characteristic polynomial

$$\begin{aligned} f_{5,p}(x) &= x^5 - px^4 - (p-1)x - 1 \\ &= (x^2 - x + 1)(x^3 - (p-1)x^2 - px - 1) \end{aligned} \tag{2}$$

in the following form:

$$\begin{aligned} \alpha_{1/2} &= \frac{1}{2}(1 \pm \sqrt{3}i), \\ \alpha_3 &= \frac{p-1}{3} + \frac{\sqrt[3]{2}R}{3\sqrt[3]{Q+S}} + \frac{\sqrt[3]{Q+S}}{3\sqrt[3]{2}}, \\ \alpha_{4/5} &= \frac{p-1}{3} - \frac{(1 \pm i\sqrt{3})R}{3\sqrt[3]{4}\sqrt[3]{Q+S}} - \frac{(1 \mp i\sqrt{3})\sqrt[3]{Q+S}}{6\sqrt[3]{2}}, \end{aligned}$$

where

$$Q := 2p^3 + 3p^2 - 3p + 25, \qquad R := p^2 + p + 1,$$
$$S := 3\sqrt{3\bigl(23 - 6p + 5p^2 + 2p^3 - p^4\bigr)}.$$

With respect to Theorem 1, we know that the polynomial $x^3 - (p-1)x^2 - px - 1$ (the second factor in (2)) has one positive real root α_3 and two roots $\alpha_{4/5}$, which are complex conjugate for $p = 1, 2, 3$, and for $p \geq 4$, they are negative real roots.

2 Auxiliary results

In this section, we present two results, which will be essential ingredients in the proof of our results. For clarity, we introduce some notations. As usual, $[a, b]$ denotes the set $\{a, a+1, \dots, b\}$ for integers $a < b$. Also, $B[0, 1]$ is the closed unit ball (i.e., all complex numbers z such that $|z| \leq 1$), and $\mathcal{R}_g$ is the set of all complex zeros of the polynomial $g(x)$.

The first tool is the famous *Descartes sign rule*, which gives an upper bound on the number of positive or negative real roots of a polynomial with real coefficients. For completeness, we state it as a lemma.

Lemma 1 (Descartes' sign rule) *Let* $f(x) = a_{n_1}x^{n_1} + \cdots + a_{n_k}x^{n_k}$ *be a polynomial with nonzero real coefficients and such that* $n_1 > n_2 > \cdots > n_k \geq 0$. *Set*

$$\nu := \#\bigl\{i \in [1, k-1] : a_{n_i}a_{n_{i+1}} < 0\bigr\}.$$

Then, there exists a nonnegative integer r *such that* $\#\mathcal{R}_f = \nu - 2r$ (*multiple roots of the same value are counted separately*).

As a corollary, we have that for obtaining information on the number of negative real roots, we must apply the previous rule for $f(-x)$.

Remark 2 Generally speaking, the previous result says that if the terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is equal to the number of sign differences between consecutive nonzero coefficients minus an even nonnegative integer.

A fundamental result in the theory of recurrence sequences is the following:

Lemma 2 *Let* (u_n) *be a linear recurrence sequence whose characteristic polynomial* $\psi(x)$ *splits as*

$$\psi(x) = (x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \cdots (x - \alpha_\ell)^{m_\ell},$$

where the α_j *are distinct complex numbers. Then there exist uniquely determined nonzero polynomials* $g_1, \dots, g_\ell \in \mathbb{Q}(\{\alpha_j\}_{j=1}^{\ell})[x]$, *with* $\deg g_j \leq m_j - 1$ (m_j *is the multiplicity of* α_j *as zero of* $\psi(x)$) *for* $j \in [1, \ell]$, *such that*

$$u_n = g_1(n)\alpha_1^n + g_2(n)\alpha_2^n + \cdots + g_\ell(n)\alpha_\ell^n \quad \textit{for all } n. \tag{3}$$

The proof of this result can be found in [27, Theorem C.1].

Another useful and very important result is due to Eneström and Kakeya [28, 29].

Lemma 3 (Eneström–Kakeya theorem) *Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be an n-degree polynomial with real coefficients. If $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all zeros of $f(x)$ lie in* $B[0,1]$.

Our last tool is the following:

Lemma 4 *Let $f : \mathbb{C} \to \mathbb{C}$ be the Möbius transformation*

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are real numbers with $ad - bc \neq 0$. Then $f^{-1}(\mathbb{R}) \subseteq \mathbb{R}$, that is, if $f(z)$ is a real number, then so is z.

Proof Suppose that ω is a complex number such that $f(\omega) \in \mathbb{R}$. Then $f(\omega) = \overline{f(\omega)}$ (where, as usual, $\bar{z}$ denotes the complex conjugate of z). Since $a, b, c, d \in \mathbb{R}$, we have that $\overline{f(\omega)} = f(\overline{\omega})$, and so $f(\omega) = f(\overline{\omega})$ yields

$$(a\overline{\omega} + b)(c\omega + d) = (a\omega + b)(c\overline{\omega} + d).$$

After a straightforward computation, we obtain that $ad(\overline{\omega} - \omega) = bc(\overline{\omega} - \omega)$. Since $ad \neq bc$, we have $\overline{\omega} = \omega$, that is, ω is a real number, as desired. □

Now we are ready to deal with the proof of the theorem.

3 The proof of the main theorem

3.1 Proof of item (i)

First, we use Lemma 1 to deduce that the polynomial $f_{k,p}(x) = x^k - px^{k-1} - (p-1)x - 1$ has only a positive root, say $\alpha_{k,p}$. From now on, by abuse of notation, we will write f for $f_{k,p}$ and α for $\alpha_{k,p}$. Since $\alpha^k = p\alpha^{k-1} + (p-1)\alpha + 1$, we obtain that $f(x) = (x-\alpha)g(x)$, where

$$g(x) = x^{k-1} + (\alpha - p)x^{k-2} + \alpha(\alpha - p)x^{k-3} + \cdots + \alpha^{k-3}(\alpha - p)x + \alpha^{k-1} + 1 - p - p\alpha^{k-2}.$$

We claim that if z is a root of $g(x)$, then $|z| \leq \alpha$. To prove this, it suffices to show that the roots of $h(x) := g(\alpha x)$ belong to $B[0,1]$. This holds by applying Lemma 3 to the polynomial

$$h(x) = \alpha^{k-1}x^{k-1} + \sum_{j=1}^{k-2} \alpha^{k-2}(\alpha - p)x^{k-j-1} + \alpha^{k-1} + 1 - p - p\alpha^{k-2},$$

since

$$\alpha^{k-1} > \alpha^{k-2}(\alpha - p) > \alpha^{k-1} + 1 - p - p\alpha^{k-2},$$

where the last inequality is valid because $p > 1$.

Now since α is the only positive root of $f(x)$ and $\lim_{x\to\infty} f(x) = +\infty$, we have $f(x) \geq 0$ for all $x \geq \alpha$ (also, $\alpha > p$ since $f(p) = -p(p-1) - 1$). Our second claim is that if z is a root of $f(x)$ with $\nu := |z| \geq \alpha$, then z is a real number. Indeed, since $f(\nu) \geq 0$. we have $\nu^k \geq p\nu^{k-1} + (p-1)\nu + 1$. On the other hand, the triangle inequality yields $\nu^k \leq p\nu^{k-1} + (p-1)\nu + 1$, and thus

$$\begin{aligned} 2\nu^k &= \left|z^k + pz^{k-1} + (p-1)z + 1\right| \\ &\leq |z|^k + p|z|^{k-1} + (p-1)|z| + 1 \\ &= |z|^k + p\nu^{k-1} + (p-1)\nu + 1 \leq 2\nu^k. \end{aligned}$$

Thus $|z^k + pz^{k-1} + (p-1)z + 1| = |z|^k + |pz|^{k-1} + |(p-1)z| + 1$, implying that 1, $(p-1)z$, pz^{k-1}, and z^k lie in the same ray (this follows from the fact that the equality in the *complex triangle inequality* $|\sum_{j=1}^n z_j| \leq \sum_{j=1}^n |z_j|$ occurs if and only if all nonzero z_j have the same argument, that is, $z_j = a_j\eta$ for some $(a_j, \eta) \in \mathbb{R}_{>0} \times \mathbb{C}$ with $j \in [1, n]$). So, in particular, there exists a real number t_0 such that $z^k = 1 + t_0(pz^{k-1} - 1)$. Since $z^k = pz^{k-1} + (p-1)z + 1$, we obtain that

$$t_0 = \frac{pz^{k-1} + (p-1)z}{pz^{k-1} - 1}.$$

On the other hand, the vectors $pz^{k-1} - (p-1)z$ and $pz^{k-1} - 1$ have the same direction, so that

$$t_1 := \frac{pz^{k-1} - (p-1)z}{pz^{k-1} - 1}$$

is a real number. Thus

$$t_0 + t_1 = \frac{2pz^{k-1}}{pz^{k-1} - 1}$$

is a real number, and so is z^{k-1} (by Lemma 4). From the definition of t_0 we also deduce that $z \in \mathbb{R}$.

In conclusion, we proved that if z is a root of $f(x)$ with $|z| \geq \alpha$, then z is a real number with $|z| = \alpha$. So, $z \in \{-\alpha, \alpha\}$. Suppose that $z = -\alpha$. Then since

$$f(\alpha) - f(-\alpha) = 0 \quad \text{and} \quad f(\alpha) + f(-\alpha) = 0,$$

we arrive at an absurdity as $\alpha^k = 1$ or $p\alpha^{k-1} = -1$, which contradicts that fact that $\alpha > p > 1$.

To finish the proof of this item, we must prove that

$$p < \alpha < p + \frac{2}{p^{k-3}}.$$

For that, since $f(p) < 0$, it suffices to show (by the intermediate value theorem) that $f(p + 2/p^{k-3}) > 0$. Indeed, since $f(x) = x^{k-1}(x-p) - (p-1)x - 1$, we get

$$\begin{aligned} f\left(p + \frac{2}{p^{k-3}}\right) &= \left(p + \frac{2}{p^{k-3}}\right)^{k-1} \cdot \frac{2}{p^{k-3}} - (p-1)\left(p + \frac{2}{p^{k-3}}\right) - 1 \\ &= p^{k-1}\left(1 + \frac{2}{p^{k-4}}\right)^{k-1} \cdot \frac{2}{p^{k-3}} - p^2 - \frac{2}{p^{k-2}} + p + \frac{2}{p^{k-3}} - 1 \\ &\geq 2p^2\left(1 + \frac{2(k-1)}{p^{k-4}}\right) - p^2 - \frac{2}{p^{k-2}} + p + \frac{2}{p^{k-3}} - 1 \\ &> p^2 + p - 1 > 0, \end{aligned}$$

where we used the *Bernoulli inequality* $(1+x)^n \geq 1 + nx$ for all $(n,x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{>-1}$. The proof is complete.

3.2 Proof of item (ii)

By using Lemma 1 and the equality $f(-x) = x^k + px^{k-1} + (p-1)x - 1$ (for k even) $f(x)$ has exactly one negative root.

3.3 Proof of item (iii)

In this case, $f(-x) = -x^k - px^{k-1} + (p-1)x - 1$, and so by Lemma 1 we have either zero or two negative roots. Since $f(0) = -1$ and $f(x)$ tends to $-\infty$ (when k is odd) as $x \to -\infty$, to prove the existence of two negative roots, we only need to find a real number $r < 0$ such that $f(r) > 0$ (again by the intermediate value theorem). Also, let $f(x) = x^{k-1}(x-p) - (p-1)x - 1$.

3.3.1 Proof of item (iii$_a$)

In this case, we choose $r = -3/5$, and thus

$$f\left(-\frac{3}{5}\right) = -\frac{18}{5} \cdot \left(\frac{3}{5}\right)^{k-1} + \frac{1}{5} > 0$$

whenever $k > -\log 18/\log(3/5) + 1 = 6.65823\ldots$. Since $k \geq 7$, the proof is complete.

3.3.2 Proof of item (iii$_b$)

In this case, for $r = -1/2$, we get

$$f\left(-\frac{1}{2}\right) = \frac{p+1}{2} - \frac{2p+1}{2^k} - 2.$$

Therefore

$$f\left(-\frac{1}{2}\right) = \frac{p+1}{2} - \frac{2p+1}{2^k} - 2 \geq \frac{p+1}{2} - \frac{2p+1}{32} - 2 \in \left\{\frac{7}{32}, \frac{21}{32}, \frac{35}{32}\right\}$$

for $k \geq 5$ and $p \in \{4, 5, 6\}$.

3.3.3 *Proof of item* (iii$_c$)

In this case, also for $r = -1/2$, we get

$$f\left(-\frac{1}{2}\right) = \frac{p+1}{2} - \frac{2p+1}{2^k} - 2 \geq \frac{p+1}{2} - \frac{2p+1}{8} - 2 \geq \frac{2p-13}{8} \geq \frac{1}{8}$$

for $k \geq 3$ and $p \geq 7$.

3.4 Proof of item (iv)

Note that $f''(x) = k(k-1)x^{k-2} - p(k-1)(k-2)x^{k-3}$. So, $f''(x) = 0$ if and only if $x = 0$ or $x = p(k-2)/k$. However, none of these values is a root of $f(x)$, since $f(0) = -1$ and its only positive root $\alpha > p$, whereas $p(k-2)/k \in (0,p)$. Summarizing, a possible repeated root must have multiplicity 2.

Now we claim that all real roots of $f(x)$ are simple. The only positive root α must be simple because of Lemma 1. For the negative roots, we first see that in the case of an even k, $f'(x)$ has no negative roots. Since $f'(-x) = -kx^{k-1} - p(k-1)x^{k-1} - (p-1)$ (and Lemma 1), when k is odd, we have two roots, which are distinct by the previous items, and so both must be simple (again by Lemma 1).

In conclusion, a possible double root must be a nonreal number. Note that $f(x) = 0$ and $f'(x) = 0$ imply

$$x^{k-1} = \frac{(p-1)x+1}{x-p} \quad \text{and} \quad x^{k-2} = \frac{p-1}{kx - p(k-1)},$$

respectively. By combining the previous relations, we arrive at the following quadratic equation:

$$(p-1)(k-1)x^2 + \big(p(p-1)(2-k) + k\big)x - p(k-1) = 0.$$

Since the roots are not real numbers, its discriminant must be negative. However, the discriminant is

$$\big(p(p-1)(2-k) + k\big)^2 + 4p(p-1)(k-1)^2 \geq 0.$$

This contradiction completes our proof.

4 Conclusions

In this paper, we are interested in the behavior of the so-called (k,p)-*Fibonacci numbers*, which are a kth-order two-parameter recurrence defined by $F_{k,p}(n) = pF_{k,p}(n-1) + (p-1)F_{k,p}(n-k+1) + F_{k,p}(n-k)$ with initial values $F_{k,p}(0) = 0$ and $F_{k,p}(j) = p^{j-1}$ (for $j \in [1, k-1]$). It is well known that the study of the (arithmetic and asymptotic) behavior of a sequence is closely related to the knowledge of the analytic and algebraic properties of roots of its characteristic polynomial (a kind of "Binet-like formula"). In our case, this polynomial is $f_{k,p}(x) = x^k - px^{k-1} - (p-1)x - 1$. Therefore, in this work, we provided a complete study of the roots of $f_{k,p}(x)$. For example, in our main result, we proved (among other things) the existence of a dominant root $\alpha_{k,p} \in (p, p+2)$ (together with some more accurate lower and upper bounds) for $k \geq 3$ and $p \geq 2$. Moreover, these bounds allow us to deduce that $(\alpha_{k,p})$ converges to p as $k \to \infty$ (while it is unbounded in p).

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Authors' contributions
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