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Oscillation of nonlinear third-order difference equations with mixed neutral terms

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Abstract

In this paper, new oscillation results for nonlinear third-order difference equations with mixed neutral terms are established. Unlike previously used techniques, which often were based on Riccati transformation and involve limsup or liminf conditions for the oscillation, the main results are obtained by means of a new approach, which is based on a comparison technique. Our new results extend, simplify, and improve existing results in the literature. Two examples with specific values of parameters are offered.

MSC: 34N05; 39A10

Keywords: Oscillation; Comparison; Nonlinear third-order difference equations; Mixed neutral terms

1 Introduction and preliminaries

Oscillation of solutions for third-order difference equations has received comparably little attention, although such equations are of importance in many fields of science such as economics, physics, mathematical biology, and other areas of mathematics [3, 4, 6, 7, 10, 11, 13, 14, 24, 27, 31, 32, 35–37]. It is worth to mention that third-order difference equations may have totally different behavior from corresponding third-order differential equations; see the explicit example in [9]. On the other hand, oscillation of solutions for difference equations of first and second order have been extensively investigated in the literature; see the monographs [1, 2, 8] and the papers [5, 12, 15–19, 21, 23, 25, 26, 30, 33].

In this study, we consider a nonlinear third-order difference equation with mixed nonlinear neutral terms. We obtain conditions guaranteeing oscillation of solutions of this equation. The main results are proved by using a comparison technique with first-order equations. Such an approach was effectively used for other types of equations in [20, 22]. To demonstrate this, we present two examples, which cannot be discussed using any of the previously established results.

We consider the equation

$$\Delta\bigl(p_1(t)\bigl(\Delta^2 y(t)\bigr)^{\alpha_1}\bigr) = p_2(t)x^{\alpha_2}(t-m+1) + p_3(t)x^{\alpha_3}\bigl(t+m^*\bigr), \tag{1.1}$$

where $y(t) = x(t) + p_4(t)x^{\alpha_4}(t-k) - p_5(t)x^{\alpha_5}(t-k)$, and subject to the assumptions:

(i) $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are the ratios of positive odd integers, $\alpha_1 \geq 1$,

(ii) $p_1, p_2, p_3, p_4, p_5 : \mathbb{Z} \to (0, \infty)$ are sequences,

(iii) $m, m^*, k \in \mathbb{N}$ are such that $m > 2$, $m^* > 2$, $k < m - 1$.

A solution of (1.1) is called *oscillatory* if it is neither eventually negative nor eventually positive. We call (1.1) oscillatory provided that all its solutions are oscillatory.

The objective of this paper is to offer conditions ensuring oscillation of (1.1) whenever

$$\alpha_4 < 1 < \alpha_5 \quad \text{or} \quad \alpha_4 < \alpha_5 \leq 1$$

and subject to the assumption

$$P_1(t, t_1) \to \infty \quad \text{as } t \to \infty, \quad \text{where } P_1(\nu, u) := \sum_{\tau=u}^{\nu-1} \frac{1}{p_1^{\frac{1}{\alpha_1}}(\tau)}. \tag{1.2}$$

In view of the results established in the literature and to the best of our observations, there are no oscillation results for (1.1).

This paper is organized as follows: In Sect. 2, we give some auxiliary results and introduce some notation. Sect. 3 features the main results of the paper. We present our investigations under two cases for (1.1). The first case is when $\alpha_4 < 1 < \alpha_5$, and the other case is when $\alpha_4 < \alpha_5 \leq 1$. Our approach is based on a comparison technique with first-order difference equations. In Sect. 4, two examples are provided in order to illustrate our main theorems.

2 Auxiliary results and notations

We start with the following fundamental result. See [22, Lemma 1], and for the proof of (I), see [29, Lemma 2.2].

Lemma 2.1

(I) *If the first-order delay difference inequality*

$$\Delta y(t) + p_2(t)y^{\gamma}(t - m + 1) \leq 0$$

has an eventually positive solution, then so does the corresponding delay difference equation.

(II) *If the first-order advanced difference inequality*

$$\Delta y(t) - p_2(t)y^{\gamma}\left(t + m^*\right) \geq 0$$

has an eventually positive solution, then so does the corresponding advanced difference equation.

We also need the following lemmas.

Lemma 2.2 (see [28]) *If* $X, Y \geq 0$, *then*

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda XY^{\lambda-1} \geq 0 \quad \textit{for } \lambda > 1 \tag{2.1}$$

and

$$X^{\lambda} - (1-\lambda)Y^{\lambda} - \lambda XY^{\lambda-1} \leq 0 \quad \textit{for } 0 < \lambda < 1. \tag{2.2}$$

Lemma 2.3 *Assume* (1.2). *Then*

$$\Delta Y(t) > 0 \quad \textit{eventually, where } Y := p_1\left(\Delta^2 y\right)^{\alpha_1} \tag{2.3}$$

implies that eventually one of the following four situations occur:

$$\begin{aligned}
&\textit{Case PPP.} \quad y(t) > 0, \quad \Delta y(t) > 0, \quad \Delta^2 y(t) > 0;\\
&\textit{Case PPN.} \quad y(t) > 0, \quad \Delta y(t) > 0, \quad \Delta^2 y(t) < 0;\\
&\textit{Case NNN.} \quad y(t) < 0, \quad \Delta y(t) < 0, \quad \Delta^2 y(t) < 0;\\
&\textit{Case NPN.} \quad y(t) < 0, \quad \Delta y(t) > 0, \quad \Delta^2 y(t) < 0.
\end{aligned}$$

Proof By (2.3), there exists $t_0 \in \mathbb{N}_0$ satisfying

$$\Delta Y(t) > 0 \quad \text{for all } t \geq t_0. \tag{2.4}$$

We first assume that

$$\text{there exists } t_1 \geq t_0 \quad \text{with } Y(t_1) > 0. \tag{2.5}$$

Then, for all $t \geq t_1$, we have

$$Y(t) = Y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta Y(\tau) \overset{(2.4)}{\geq} Y(t_1) \overset{(2.5)}{>} 0.$$

Hence,

$$\Delta^2 y(t) > 0 \quad \text{for all } t \geq t_1. \tag{2.6}$$

Now, for $t \geq t_1$, we get

$$\begin{aligned}
\Delta y(t) &= \Delta y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta^2 y(\tau) = \Delta y(t_1) + \sum_{\tau=t_1}^{t-1} \frac{Y^{\frac{1}{\alpha_1}}(\tau)}{p_1^{\frac{1}{\alpha_1}}(\tau)}\\
&\overset{(2.4)}{\geq} \Delta y(t_1) + \sum_{\tau=t_1}^{t-1} \frac{Y^{\frac{1}{\alpha_1}}(t_1)}{p_1^{\frac{1}{\alpha_1}}(t_1)} = \Delta y(t_1) + Y^{\frac{1}{\alpha_1}}(t_1) P_1(t,t_1) \overset{(2.5)}{\to} \infty \quad \text{as } t \to \infty,
\end{aligned}$$

due to (1.2). Thus,

$$\text{there exists } t_2 \geq t_1 \quad \text{with } \Delta y(t) > 0 \quad \text{for all } t \geq t_2. \tag{2.7}$$

Hence, for $t \geq t_2$, we obtain

$$\begin{aligned} y(t) &= y(t_2) + \sum_{\tau=t_2}^{t-1} \Delta y(\tau) \overset{(2.6)}{\geq} y(t_2) + \sum_{\tau=t_2}^{t-1} \Delta y(t_2) \\ &= y(t_2) + (t - \tau_2)\Delta y(t_2) \overset{(2.7)}{\to} \infty \quad \text{as } t \to \infty. \end{aligned}$$

Therefore,

$$\text{there exists } t_3 \geq t_2 \quad \text{with } y(t) > 0 \quad \text{for all } t \geq t_3. \tag{2.8}$$

By (2.6), (2.7), and (2.8), we have

$$y(t) > 0, \quad \Delta y(t) > 0, \quad \Delta^2 y(t) > 0 \quad \text{for all } t \geq t_3,$$

so Case PPP holds. Next, if (2.5) does not hold, then the only other possibility is

$$Y(t) < 0 \quad \text{for all } t \geq t_0,$$

and hence

$$\Delta^2 y(t) < 0 \quad \text{for all } t \geq t_0. \tag{2.9}$$

We assume that

$$\text{there exists } t_1 \geq t_0 \quad \text{with } \Delta y(t_1) < 0. \tag{2.10}$$

Then, for all $t \geq t_1$, we have

$$\Delta y(t) = \Delta y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta^2 y(\tau) \overset{(2.9)}{\leq} \Delta y(t_1) \overset{(2.10)}{<} 0.$$

Thus,

$$\Delta y(t) < 0 \quad \text{for all } t \geq t_1. \tag{2.11}$$

Now, for $t \geq t_1$, we get

$$\begin{aligned} y(t) &= y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta y(\tau) \overset{(2.9)}{\leq} y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta y(t_1) \\ &= y(t_1) + (t - t_1)\Delta y(t_1) \overset{(2.10)}{\to} -\infty \quad \text{as } t \to \infty. \end{aligned}$$

Hence,

$$\text{there exists } t_2 \geq t_1 \quad \text{with } y(t) < 0 \quad \text{for all } t \geq t_2. \tag{2.12}$$

By (2.9), (2.11), and (2.12), we have

$$y(t) < 0, \qquad \Delta y(t) < 0, \qquad \Delta^2 y(t) < 0 \quad \text{for all } t \geq t_2,$$

so Case NNN holds. Next, if (2.10) does not hold, then the only other possibility is

$$\Delta y(t) > 0 \quad \text{for all } t \geq t_0. \tag{2.13}$$

We assume that

$$\text{there exists } t_1 \geq t_0 \quad \text{with } y(t_1) > 0. \tag{2.14}$$

Then, for all $t \geq t_1$, we have

$$y(t) = y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta y(\tau) \overset{(2.13)}{\geq} y(t_1) \overset{(2.14)}{>} 0 \quad \text{for all } t \geq t_1. \tag{2.15}$$

By (2.9), (2.13), and (2.15), we have

$$y(t) > 0, \qquad \Delta y(t) > 0, \qquad \Delta^2 y(t) < 0 \quad \text{for all } t \geq t_1,$$

so Case PPN holds. Finally, if (2.14) does not hold, then the only other possibility is

$$y(t) < 0 \quad \text{for all } t \geq t_0. \tag{2.16}$$

By (2.9), (2.13), and (2.16), we have

$$y(t) < 0, \qquad \Delta y(t) > 0, \qquad \Delta^2 y(t) < 0 \quad \text{for all } t \geq t_1,$$

so Case NPN holds. There are no other cases. See Table 1 for an illustration of the proof. □

Throughout the remainder of the paper, we suppose that

$$k_0, k_1, k_2, k_3 \in \mathbb{N} \quad \text{satisfy } 2k_0 < m^*, k_1 < m+1, \text{ and } k_2 < k_3 < m+1-k. \tag{2.17}$$

For convenience, we introduce the notations

$$\xi_0(t) := t + m^* - 2k_0, \qquad \xi_1(t) := t - m + k_1 - 1, \qquad \xi_2(t) := t - m + k - 1,$$

Table 1 Illustration of the proof of Lemma 2.3

	$\Delta^2 y(t)$		$\Delta y(t)$		$y(t)$
(2.5)	P		P		P
(2.9)	N	(2.10)	N		N
		(2.13)	P	(2.14)	P
				(2.16)	N

$$
\begin{aligned}
&\xi_3(t) := \xi_2(t) + k_3, \qquad \Lambda_0(t) := \sum_{\tau=t-k_0}^{t-1}\left(\frac{1}{p_1(\tau)}\sum_{s=\tau-k_0}^{\tau-1} p_3(s)\right)^{\frac{1}{\alpha_1}},\\
&\Lambda_1(t) := p_2(t)\bigl((t-m+1)P_1(t-m+k_1, t-m)\bigr)^{\alpha_2},\\
&\Lambda_2(t) := Q(t)\left(\sum_{\tau=t_1}^{t-m+k} P_1(\tau, t_1)\right)^{\frac{\alpha_2}{\alpha_5}},\\
&\Lambda_3(t) := k_2^{\frac{\alpha_2}{\alpha_5}} Q(t) P_1^{\frac{\alpha_2}{\alpha_5}}(t-m+k+k_3, t-m+k+k_2), \quad Q(t) := \frac{p_2(t)}{p_5^{\frac{\alpha_2}{\alpha_5}}(t-m+k+1)}.
\end{aligned}
$$

Remark 2.4

1. Note that, due to the assumptions $m > 2$, $m^* > 2$, and $k < m - 1$, it is always possible to find k_0, k_1, k_2, k_3 such that (2.17) holds, e.g., one may pick

$$k_0 = k_1 = k_2 = 1 \quad \text{and} \quad k_3 = 2.$$

2. Note that $\xi_0(t) > t$ holds always since $m^* - 2k_0 > 0$. Hence, equations involving ξ_0 are of advanced type. Moreover, $\xi_1(t) < t$, $\xi_2(t) < t$, and $\xi_3(t) < t$ always since $m + 1 - k_1 > 0$, $m + 1 - k > 0$, and $m + 1 - k - k_3 > 0$. Hence, equations involving ξ_1, ξ_2, ξ_3 are of delay type.

3 Main results

Now we present our first oscillation result.

Theorem 3.1 *Let $\alpha_4 < 1 < \alpha_5$. Suppose that* (i)–(iii), (1.2), *and* (2.17) *hold. Assume that there exists $p : \mathbb{Z} \to (0, \infty)$ such that*

$$
\begin{aligned}
&\lim_{t\to\infty}\bigl(g_1(t) + g_2(t)\bigr) = 0, \quad \textit{where}\\
&\quad g_1(t) := (1-\alpha_4)\alpha_4^{\frac{\alpha_4}{1-\alpha_4}} p^{\frac{\alpha_4}{\alpha_4-1}}(t) p_4^{\frac{1}{1-\alpha_4}}(t) \textit{ and}\\
&\quad g_2(t) := (\alpha_5-1)\alpha_5^{\frac{\alpha_5}{1-\alpha_5}} p^{\frac{\alpha_5}{\alpha_5-1}}(t) p_5^{\frac{1}{1-\alpha_5}}(t).
\end{aligned} \tag{3.1}
$$

Let $\theta_0, \theta_1 \in (0, 1)$. If the first-order advanced difference equation

$$\Delta y(t) = \theta_0 \Lambda_0(t) y^{\frac{\alpha_3}{\alpha_1}}\bigl(\xi_0(t)\bigr) \tag{3.2}$$

and the first-order delay difference equations

$$\Delta Z(t) + \theta_1 \Lambda_1(t) Z^{\frac{\alpha_2}{\alpha_1}}\bigl(\xi_1(t)\bigr) = 0, \tag{3.3}$$

$$\Delta Z(t) + \Lambda_2(t) Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\bigl(\xi_2(t)\bigr) = 0, \tag{3.4}$$

and

$$\Delta Z(t) + \Lambda_3(t) Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\bigl(\xi_3(t)\bigr) = 0 \tag{3.5}$$

are oscillatory, then so is (1.1).

Proof Assume that x is a nonoscillatory solution of (1.1), say

$$x(t) > 0, \qquad x(t-k) > 0, \qquad x(t-m+1) > 0, \qquad x(t+m^*) > 0$$

eventually. It follows from (1.1) that, eventually,

$$\Delta\big(p_1(t)\big(\Delta^2 y(t)\big)^{\alpha_1}\big) = p_2(t)x^{\alpha_2}(t-m+1) + p_3(t)x^{\alpha_3}\big(t+m^*\big) > 0. \tag{3.6}$$

Hence (2.3) is satisfied, and thus, by Lemma 2.3, only the four Cases PPP, PPN, NNN, and NPN are possible. We now discuss each of these possible cases.

Cases PPP and PPN. Applying (2.1) with

$$\lambda := \alpha_5 > 1, \qquad X := p_5^{\frac{1}{\alpha_5}}(t)x(t-k), \qquad Y := \left(\frac{1}{\alpha_5}p(t)p_5^{-\frac{1}{\alpha_5}}(t)\right)^{\frac{1}{\alpha_5-1}},$$

we obtain

$$p(t)x(t-k) - p_5(t)x^{\alpha_5}(t-k) \le g_2(t),$$

while applying (2.2) with

$$\lambda := \alpha_4 \in (0,1), \qquad X := p_4^{\frac{1}{\alpha_4}}(t)x(t-k), \qquad Y := \left(\frac{1}{\alpha_4}p(t)p_4^{-\frac{1}{\alpha_4}}(t)\right)^{\frac{1}{\alpha_4-1}},$$

we get

$$-\big(p(t)x(t-k) - p_4(t)x^{\alpha_4}(t-k)\big) \le g_1(t).$$

Using these two inequalities, we have

$$\begin{aligned} x(t) &= y(t) - \big(p(t)x(t-k) - p_5(t)x^{\alpha_5}(t-k)\big) + \big(p(t)x(t-k) - p_4(t)x^{\alpha_4}(t-k)\big) \\ &\ge y(t) - g_1(t) - g_2(t) = \left[1 - \frac{g_1(t)+g_2(t)}{y(t)}\right]y(t). \end{aligned}$$

Since y in both Cases PPP and PPN is positive and nondecreasing, there exists $C > 0$ satisfying $y(t) \ge C$, and so we have

$$x(t) \ge \left[1 - \frac{g_1(t)+g_2(t)}{C}\right]y(t).$$

Next, due to (3.1), there exists $\kappa \in (0,1)$ such that

$$x(t) \ge \kappa y(t) \quad \text{eventually}. \tag{3.7}$$

Thus, we have

$$\Delta\big(p_1(t)\big(\Delta^2 y(t)\big)^{\alpha_1}\big) \ge \kappa^{\alpha_2}p_2(t)y^{\alpha_2}(t-m+1) + \kappa^{\alpha_3}p_3(t)y^{\alpha_3}\big(t+m^*\big) \ge 0. \tag{3.8}$$

Case PPP. By (3.8), we get

$$\Delta\big(p_1(t)\big(\Delta^2 y(t)\big)^{\alpha_1}\big) \geq \kappa^{\alpha_3} p_3(t) y^{\alpha_3}\big(t+m^*\big). \tag{3.9}$$

Summing (3.9) from $t-k_0$ to $t-1$, we get

$$\begin{aligned} p_1(t)\big(\Delta^2 y(t)\big)^{\alpha_1} &= p_1(t-k_0)\big(\Delta^2 y(t-k_0)\big)^{\alpha_1} + \sum_{\tau=t-k_0}^{t-1} \Delta\big(p_1(\tau)\big(\Delta^2 y(\tau)\big)^{\alpha_1}\big) \\ &\overset{(3.9)}{\geq} \kappa^{\alpha_3} \sum_{\tau=t-k_0}^{t-1} p_3(\tau) y^{\alpha_3}\big(\tau+m^*\big) \\ &\geq \kappa^{\alpha_3} y^{\alpha_3}\big(t+m^*-k_0\big) \sum_{\tau=t-k_0}^{t-1} p_3(\tau). \end{aligned}$$

Therefore, we have

$$\Delta^2 y(t) \geq \kappa^{\frac{\alpha_3}{\alpha_1}} y^{\frac{\alpha_3}{\alpha_1}}\big(t+m^*-k_0\big)\left(\frac{1}{p_1(t)} \sum_{\tau=t-k_0}^{t-1} p_3(\tau)\right)^{\frac{1}{\alpha_1}}. \tag{3.10}$$

Summing (3.10) again from $t-k_0$ to $t-1$, we obtain

$$\begin{aligned} \Delta y(t) &= \Delta y(t-k_0) + \sum_{\tau=t-k_0}^{t-1} \Delta^2 y(\tau) \\ &\overset{(3.10)}{\geq} \kappa^{\frac{\alpha_3}{\alpha_1}} \sum_{\tau=t-k_0}^{t-1} y^{\frac{\alpha_3}{\alpha_1}}\big(\tau+m^*-k_0\big)\left(\frac{1}{p_1(\tau)} \sum_{s=\tau-k_0}^{\tau-1} p_3(s)\right)^{\frac{1}{\alpha_1}} \\ &\geq \kappa^{\frac{\alpha_3}{\alpha_1}} y^{\frac{\alpha_3}{\alpha_1}}\big(t+m^*-2k_0\big) \sum_{\tau=t-k_0}^{t-1}\left(\frac{1}{p_1(\tau)} \sum_{s=\tau-k_0}^{\tau-1} p_3(s)\right)^{\frac{1}{\alpha_1}}. \end{aligned}$$

In summary, y is a positive and increasing solution of

$$\Delta y(t) - \kappa^{\frac{\alpha_3}{\alpha_1}} \Lambda_0(t) y^{\frac{\alpha_3}{\alpha_1}}\big(\xi_0(t)\big) \geq 0.$$

Employing Lemma 2.1 (II), (3.2) also has an eventually positive solution, which is a contradiction.

Case PPN. We introduce

$$Z := -p_1\big(\Delta^2 y\big)^{\alpha_1} > 0 \quad \text{eventually}. \tag{3.11}$$

By (3.8), we obtain

$$-\Delta Z(t) \geq \kappa^{\alpha_2} p_2(t) y^{\alpha_2}(t-m+1). \tag{3.12}$$

First, we see that, eventually,

$$\begin{aligned} y(t) &= y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta y(\tau) \geq \sum_{\tau=t_1}^{t-1} \Delta y(\tau) \\ &\geq \sum_{\tau=t_1}^{t-1} \Delta y(t-1) = (t-t_1)\Delta y(t-1) = t\Delta y(t-1)\left(1 - \frac{t_1}{t}\right). \end{aligned}$$

Since $t_1/t \to 0$ as $t \to \infty$, there exists $\theta \in (0,1)$ so that

$$y(t) \geq \theta t \Delta y(t-1) \quad \text{eventually.} \tag{3.13}$$

Now, we put

$$u := t - m \quad \text{and} \quad v := u + k_1 > u.$$

Then, eventually,

$$\begin{aligned} 0 &\leq \Delta y(v) = \Delta y(u) + \sum_{\tau=u}^{v-1} \Delta^2 y(\tau) \\ &\overset{(3.11)}{=} \Delta y(u) - \sum_{\tau=u}^{v-1} \frac{Z^{\frac{1}{\alpha_1}}(\tau)}{p_1^{\frac{1}{\alpha_1}}(\tau)} \\ &\overset{(3.6)}{\leq} \Delta y(u) - \sum_{\tau=u}^{v-1} \frac{Z^{\frac{1}{\alpha_1}}(v-1)}{p_1^{\frac{1}{\alpha_1}}(\tau)} \\ &= \Delta y(u) - Z^{\frac{1}{\alpha_1}}(v-1) P_1(v,u), \end{aligned}$$

and hence

$$\Delta y(u) \geq Z^{\frac{1}{\alpha_1}}(v-1) P_1(v,u) \quad \text{eventually.} \tag{3.14}$$

Altogether, eventually,

$$\begin{aligned} -\Delta Z(t) &\overset{(3.12)}{\geq} \kappa^{\alpha_2} p_2(t) y^{\alpha_2}(t-m+1) \\ &\overset{(3.13)}{\geq} (\theta\kappa)^{\alpha_2} p_2(t)(t-m+1)^{\alpha_2} \big(\Delta y(t-m)\big)^{\alpha_2} \\ &= (\theta\kappa)^{\alpha_2} p_2(t)(t-m+1)^{\alpha_2} \big(\Delta y(u)\big)^{\alpha_2} \\ &\overset{(3.14)}{\geq} (\theta\kappa)^{\alpha_2} p_2(t)(t-m+1)^{\alpha_2} \big(Z^{\frac{1}{\alpha_1}}(v-1) P_1(v,u)\big)^{\alpha_2} \\ &= (\theta\kappa)^{\alpha_2} p_2(t)(t-m+1)^{\alpha_2} \big(Z^{\frac{1}{\alpha_1}}\big(\xi_1(t)\big) P_1(t-m+k_1, t-m)\big)^{\alpha_2}. \end{aligned}$$

In summary, Z is a positive and decreasing solution of

$$\Delta Z(t) + (\theta\kappa)^{\alpha_2} \Lambda_1(t) Z^{\frac{1}{\alpha_1}}\big(\xi_1(t)\big) \leq 0.$$

Employing Lemma 2.1 (I), (3.3) also has an eventually positive solution, which is a contradiction.

Cases NNN and NPN. Throughout the remainder of the proof, we introduce Z again by (3.11). First note that, eventually,

$$y(t) = x(t) + p_4(t)x^{\alpha_4}(t-k) - p_5(t)x^{\alpha_5}(t-k) \geq -p_5(t)x^{\alpha_5}(t-k).$$

Hence, eventually,

$$x(t-k) \geq -\left(\frac{y(t)}{p_5(t)}\right)^{\frac{1}{\alpha_5}}. \tag{3.15}$$

Thus, eventually,

$$\begin{aligned} -\Delta Z(t) &\overset{(1.1)}{=} p_2(t)x^{\alpha_2}(t-m+1) + p_3(t)x^{\alpha_5}\left(t+m^*\right) \\ &\geq p_2(t)x^{\alpha_2}(t-m+1) \overset{(3.15)}{\geq} -p_2(t)\left(\frac{y(t-m+k+1)}{p_5(t-m+k+1)}\right)^{\frac{\alpha_2}{\alpha_5}} \\ &= -Q(t)y^{\frac{\alpha_2}{\alpha_5}}(t-m+k+1). \end{aligned} \tag{3.16}$$

Case NNN. First note that, eventually,

$$\begin{aligned} \Delta y(t) &= \Delta y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta^2 y(\tau) \leq \sum_{\tau=t_1}^{t-1} \Delta^2 y(\tau) \\ &\overset{(3.11)}{=} -\sum_{\tau=t_1}^{t-1} \frac{Z^{\frac{1}{\alpha_1}}(\tau)}{p_1^{\frac{1}{\alpha_1}}(\tau)} \\ &\overset{(3.6)}{\leq} -\sum_{\tau=t_1}^{t-1} \frac{Z^{\frac{1}{\alpha_1}}(t-1)}{p_1^{\frac{1}{\alpha_1}}(\tau)} \\ &= -Z^{\frac{1}{\alpha_1}}(t-1)P_1(t,t_1), \end{aligned}$$

and therefore, eventually,

$$\begin{aligned} y(t) &= y(t_1) + \sum_{\tau=t_1}^{t-1} \Delta y(\tau) \\ &\leq \sum_{\tau=t_1}^{t-1} \Delta y(\tau) \leq -\sum_{\tau=t_1}^{t-1} Z^{\frac{1}{\alpha_1}}(\tau-1)P_1(\tau,t_1) \\ &\overset{(3.6)}{\leq} -Z^{\frac{1}{\alpha_1}}(t-2)\sum_{\tau=t_1}^{t-1} P_1(\tau,t_1), \end{aligned}$$

and thus, eventually,

$$\begin{aligned} -\Delta Z(t) &\overset{(3.16)}{\geq} -Q(t)y^{\frac{\alpha_2}{\alpha_5}}(t-m+k+1) \\ &\geq Q(t)\left(Z^{\frac{1}{\alpha_1}}(t-m+k-1)\sum_{\tau=t_1}^{t-m+k} P_1(\tau,t_1)\right)^{\frac{\alpha_2}{\alpha_5}} \\ &= Q(t)\left(\sum_{\tau=t_1}^{t-m+k} P_1(\tau,t_1)\right)^{\frac{\alpha_2}{\alpha_5}} Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_2(t)\big). \end{aligned}$$

In summary, Z is a positive and decreasing solution of

$$\Delta Z(t) + \Lambda_2(t)Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_2(t)\big) \leq 0.$$

Employing Lemma 2.1 (I), (3.4) also has an eventually positive solution, which is a contradiction.

Case NPN. We let

$$u := t-m+k+1, \qquad v := u+k_2 > u, \quad \text{and} \quad w := u+k_3-1 > v-1.$$

First, we have, eventually,

$$\begin{aligned} 0 \geq y(v) &= y(u) + \sum_{\tau=u}^{v-1} \Delta y(\tau) \\ &\geq y(u) + \sum_{\tau=u}^{v-1} \Delta y(v-1) = y(u) + (v-u)\Delta y(v-1), \end{aligned}$$

so,

$$-y(u) \geq (v-u)\Delta y(v-1). \tag{3.17}$$

Next, we have, eventually,

$$\begin{aligned} 0 &\leq \Delta y(w) = \Delta y(v-1) + \sum_{\tau=v-1}^{w-1} \Delta^2 y(\tau) \\ &\overset{(3.11)}{=} \Delta y(v-1) - \sum_{\tau=v-1}^{w-1} \frac{Z^{\frac{1}{\alpha_1}}(\tau)}{p_1^{\frac{1}{\alpha_1}}(\tau)} \\ &\overset{(3.6)}{\leq} \Delta y(v-1) - \sum_{\tau=v-1}^{w-1} \frac{Z^{\frac{1}{\alpha_1}}(w-1)}{p_1^{\frac{1}{\alpha_1}}(\tau)} \\ &= \Delta y(v-1) - Z^{\frac{1}{\alpha_1}}(w-1)P_1(w,v-1), \end{aligned}$$

so,

$$\Delta y(v-1) \geq Z^{\frac{1}{\alpha_1}}(w-1)P_1(w,v-1). \tag{3.18}$$

Thus, we see that

$$\begin{aligned}
-\Delta Z(t) &\overset{(3.16)}{\geq} -Q(t)y^{\frac{\alpha_2}{\alpha_5}}(t-m+k+1) = -Q(t)y^{\frac{\alpha_2}{\alpha_5}}(u) \\
&\overset{(3.17)}{\geq} Q(t)\big((v-u)\Delta y(v-1)\big)^{\frac{\alpha_2}{\alpha_5}} = Q(t)k_2^{\frac{\alpha_2}{\alpha_5}}\big(\Delta y(v-1)\big)^{\frac{\alpha_2}{\alpha_5}} \\
&\overset{(3.18)}{\geq} Q(t)k_2^{\frac{\alpha_2}{\alpha_5}}\big(Z^{\frac{1}{\alpha_1}}(w-1)P_1(w,v-1)\big)^{\frac{\alpha_2}{\alpha_5}} \\
&= Q(t)k_2^{\frac{\alpha_2}{\alpha_5}}P_1^{\frac{\alpha_2}{\alpha_5}}(t-m+k+k_3,t-m+k+k_2)Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_3(t)\big).
\end{aligned}$$

In summary, Z is a positive and decreasing solution of

$$\Delta Z(t) + \Lambda_3(t)Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_3(t)\big) \leq 0.$$

Employing Lemma 2.1 (I), (3.5) also has an eventually positive solution, which is a contradiction. □

We now prove the following consequence of Theorem 3.1.

Theorem 3.2 *Let $\alpha_4 < 1 < \alpha_5$. Suppose that* (i)–(iii), (1.2), (2.17), *and* (3.1) *hold. Let $\theta_0, \theta_1 \in (0,1)$. If the first-order advanced difference equation* (3.2) *and the first-order delay difference equations* (3.3) *and*

$$\Delta Z(t) + \min\big\{\Lambda_2(t),\Lambda_3(t)\big\}Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_3(t)\big) = 0 \tag{3.19}$$

are oscillatory, then so is (1.1).

Proof We claim that oscillation of (3.19) implies oscillation of both (3.4) and (3.5). As all the other assumptions are the same as in Theorem 3.1, the statement then follows from Theorem 3.1. So assume that (3.19) is oscillatory. First, suppose that (3.4) is not oscillatory, say, there exists eventually positive Z satisfying

$$0 = \Delta Z(t) + \Lambda_2(t)Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_2(t)\big) \geq \Delta Z(t) + \min\big\{\Lambda_2(t),\Lambda_3(t)\big\}Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_2(t)\big). \tag{3.20}$$

From the equality in (3.20), we see that Z is eventually decreasing, and since

$$\xi_3(t) = \xi_2(t) + k_3 > \xi_2(t),$$

we obtain $Z(\xi_3(t)) \leq Z(\xi_2(t))$ eventually. Using this in (3.20), we get

$$0 \geq \Delta Z(t) + \min\big\{\Lambda_2(t),\Lambda_3(t)\big\}Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_2(t)\big) \geq \Delta Z(t) + \min\big\{\Lambda_2(t),\Lambda_3(t)\big\}Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_3(t)\big).$$

By Lemma 2.1 (I), (3.19) also has an eventually positive solution, a contradiction, showing that (3.4) is indeed oscillatory. Next, suppose that (3.5) is not oscillatory, say, there exists eventually positive Z satisfying

$$0 = \Delta Z(t) + \Lambda_3(t)Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_3(t)\big) \geq \Delta Z(t) + \min\big\{\Lambda_2(t),\Lambda_3(t)\big\}Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}\big(\xi_3(t)\big).$$

By Lemma 2.1 (I), (3.19) also has an eventually positive solution, a contradiction, showing that (3.5) is indeed oscillatory as well. Thus, the proof is complete. □

Remark 3.3 Observe that the minimum occurring in (3.19) may be calculated as follows:

$$\min\{\Lambda_2(t),\Lambda_3(t)\} = Q(t)\left(\min\left\{\sum_{\tau=t_1}^{t-m+k} P_1(\tau,t_1), k_2P_1(t-m+k+k_3,t-m+k+k_2)\right\}\right)^{\frac{\alpha_2}{\alpha_5}}.$$

The following theorem gives some further criteria for a special case.

Theorem 3.4 *Let $\alpha_4 < 1 < \alpha_5$ and $\alpha_2 \leq \alpha_1 \leq \alpha_3$. Suppose that* (i)–(iii), (1.2), (2.17), *and* (3.1) *hold. If*

$$\limsup_{t\to\infty} \sum_{\tau=t}^{\xi_0(t)-1} \Lambda_0(\tau) = \infty, \tag{3.21}$$

$$\limsup_{t\to\infty} \sum_{\tau=\xi_1(t)}^{t} \Lambda_1(\tau) = \infty, \tag{3.22}$$

$$\limsup_{t\to\infty} \sum_{\tau=\xi_2(t)}^{t} \Lambda_2(\tau) = \infty, \tag{3.23}$$

and

$$\limsup_{t\to\infty} \sum_{\tau=\xi_3(t)}^{t} \Lambda_3(\tau) = \infty, \tag{3.24}$$

then (1.1) *is oscillatory.*

Proof We claim that under the additional assumption $\alpha_2 \leq \alpha_1 \leq \alpha_3$, (3.21), (3.22), (3.23), and (3.24) imply oscillation of (3.2), (3.3), (3.4), and (3.5), respectively. As all the other assumptions are the same as in Theorem 3.1, the statement then follows from Theorem 3.1. First, suppose that (3.2) is not oscillatory, say, there exists eventually positive y satisfying (3.2). As can be seen from (3.2), y is eventually increasing and thus bounded below by some $C > 0$. Summing (3.2) from $\tau = t$ to $\tau = \xi_0(t) - 1$, we obtain, eventually,

$$y(\xi_0(t)) \geq y(\xi_0(t)) - y(t) \overset{(3.2)}{=} \sum_{\tau=t}^{\xi_0(t)-1} \theta_0\Lambda_0(\tau)y^{\frac{\alpha_3}{\alpha_1}}(\xi_0(\tau)) \geq \sum_{\tau=t}^{\xi_0(t)-1} \theta_0\Lambda_0(\tau)y^{\frac{\alpha_3}{\alpha_1}}(\xi_0(t)),$$

and thus, as $\alpha_3 \geq \alpha_1$,

$$\theta_0 \sum_{\tau=t}^{\xi_0(t)-1} \Lambda_0(\tau) \leq \frac{1}{y^{\frac{\alpha_3-\alpha_1}{\alpha_1}}(\xi_0(t))} \leq \frac{1}{C^{\frac{\alpha_3-\alpha_1}{\alpha_1}}},$$

contradicting (3.21). Next, suppose that (3.3) is not oscillatory, say, there exists eventually positive Z satisfying (3.3). As can be seen from (3.3), Z is eventually decreasing and thus bounded above by some $C > 0$. Summing (3.3) from $\tau = \xi_1(t)$ to $\tau = t$, we obtain, eventually,

$$Z(\xi_1(t)) \geq Z(\xi_1(t)) - Z(t+1) \overset{(3.3)}{=} \sum_{\tau=\xi_1(t)}^{t} \theta_1 \Lambda_1(\tau) Z^{\frac{\alpha_2}{\alpha_1}}(\xi_1(t)),$$

and thus, as $\alpha_1 \geq \alpha_2$,

$$\theta_1 \sum_{\tau=\xi_1(t)}^{t} \Lambda_1(\tau) \leq Z^{\frac{\alpha_1-\alpha_2}{\alpha_1}}(\xi_1(t)) \leq C^{\frac{\alpha_1-\alpha_2}{\alpha_1}},$$

contradicting (3.22). Next, suppose that (3.4) is not oscillatory, say, there exists eventually positive Z satisfying (3.4). As can be seen from (3.4), Z is eventually decreasing and thus bounded above by some $C > 0$. Summing (3.4) from $\tau = \xi_2(t)$ to $\tau = t$, we obtain, eventually,

$$\begin{aligned}
Z(\xi_2(t)) &\geq Z(\xi_2(t)) - Z(t+1) \overset{(3.4)}{=} \sum_{\tau=\xi_2(t)}^{t} \Lambda_2(\tau) Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}(\xi_2(\tau)) \\
&\geq \sum_{\tau=\xi_2(t)}^{t} \Lambda_2(\tau) Z^{\frac{\alpha_2}{\alpha_1\alpha_5}}(\xi_2(t)),
\end{aligned}$$

and thus, as $\alpha_1\alpha_5 \geq \alpha_1 \geq \alpha_2$,

$$\sum_{\tau=\xi_2(t)}^{t} \Lambda_2(\tau) \leq Z^{\frac{\alpha_1\alpha_5-\alpha_2}{\alpha_1\alpha_5}}(\xi_2(t)) \leq C^{\frac{\alpha_1\alpha_5-\alpha_2}{\alpha_1\alpha_5}},$$

contradicting (3.23). The proof that (3.24) implies oscillation of (3.5) follows exactly like the proof that (3.23) implies oscillation of (3.4), only with ξ_2 and Λ_2 replaced by ξ_3 and Λ_3, respectively. □

Remark 3.5 Note that Tang and Liu [34] have developed other oscillation criteria for sublinear delay difference equations, which could be used in place of (3.22), (3.23), and (3.24) to examine the oscillation of (3.3), (3.4), and (3.5). Similar criteria for superlinear advanced difference equations could not be found in the literature.

Now, we focus on the case $\alpha_4 < \alpha_5 \leq 1$.

Theorem 3.6 *Let* $\alpha_4 < \alpha_5 \leq 1$. *Suppose that* (i)–(iii), (1.2), *and* (2.17) *hold. Assume*

$$\lim_{t\to\infty} P(t) = 0, \quad \textit{where } P(t) := \frac{\alpha_5 - \alpha_4}{\alpha_4} \left(\frac{\alpha_4}{\alpha_5} p_4(t)\right)^{\frac{\alpha_5}{\alpha_5-\alpha_4}} (p_5(t))^{\frac{\alpha_4}{\alpha_4-\alpha_5}}. \tag{3.25}$$

Let $\theta_0, \theta_1 \in (0,1)$. *If* (3.2), (3.3), (3.4), *and* (3.5) *are oscillatory, then so is* (1.1).

Proof Inspecting the proof of Theorem 3.1, we see that $\alpha_5 > 1$ was needed only in the discussions under the headline "Cases PPP and PPN" leading to (3.7). The rest of the proof does not use $\alpha_5 > 1$ and remains unaffected. Thus, while we used (3.1) in Theorem 3.1 to show (3.7), we will now use (3.25) to show (3.7), hence completing the proof. Applying (2.1) with

$$\lambda := \frac{\alpha_5}{\alpha_4} > 1, \qquad X := x^{\alpha_4}(t-k), \qquad Y := \left(\frac{\alpha_4 p_4(t)}{\alpha_5 p_5(t)}\right)^{\frac{\alpha_4}{\alpha_5-\alpha_4}},$$

we obtain

$$x^{\alpha_5}(t-k) + \frac{\alpha_5-\alpha_4}{\alpha_4}\left(\frac{\alpha_4 p_4(t)}{\alpha_5 p_5(t)}\right)^{\frac{\alpha_5}{\alpha_5-\alpha_4}} \geq \frac{\alpha_5}{\alpha_4} x^{\alpha_4}(t-k)\frac{\alpha_4 p_4(t)}{\alpha_5 p_5(t)},$$

and thus

$$\begin{aligned} x(t) &= y(t) - \left(p_4(t)x^{\alpha_4}(t-k) - p_5(t)x^{\alpha_5}(t-k)\right) \\ &\geq y(t) - P(t) = \left(1 - \frac{P(t)}{y(t)}\right)y(t). \end{aligned}$$

Since y is positive and nondecreasing, there exists $C > 0$ such that $y(t) \geq C$, and so we have

$$x(t) \geq \left(1 - \frac{P(t)}{C}\right)y(t).$$

Next, due to (3.25), there exists $\kappa \in (0,1)$ such that (3.7) is satisfied. This completes the proof. □

As before in Theorem 3.2 and Theorem 3.4, we now obtain the following results.

Theorem 3.7 *Let $\alpha_4 < \alpha_5 \leq 1$. Suppose that* (i)–(iii), (1.2), (2.17), *and* (3.25) *hold. Let $\theta_0, \theta_1 \in (0,1)$. If* (3.2), (3.3), *and* (3.19) *are oscillatory, then so is* (1.1).

Theorem 3.8 *Let $\alpha_4 < \alpha_5 \leq 1$ and $\alpha_2/\alpha_5 \leq \alpha_1 \leq \alpha_3$. Suppose that* (i)–(iii), (1.2), (2.17), *and* (3.25) *hold. If* (3.21), (3.22), (3.23), *and* (3.24) *hold, then* (1.1) *is oscillatory.*

4 Examples

We conclude this paper by giving two examples, illustrating our theoretical findings.

Example 4.1 We consider the equation

$$\begin{aligned} &\Delta\left((t+1)^3\left(\Delta^2\left(x(t) + \frac{1}{t}x^{\frac{1}{3}}(t-1) - x^3(t-1)\right)\right)^3\right) \\ &\quad = t^2 x(t-2) + (t+2)^4 x(t+6). \end{aligned} \tag{4.1}$$

Then (4.1) is in the form (1.1), where

$$\alpha_1 = \alpha_3 = \alpha_5 = 3, \qquad \alpha_2 = 1, \qquad \alpha_4 = \frac{1}{3}, \quad k = 1, m = 3, m^* = 6,$$

$$p_1(t) = (t+1)^3, \qquad p_2(t) = t^2, \quad p_3(t) = (t+2)^4, \qquad p_4(t) = \frac{1}{t}, \qquad p_5(t) = 1.$$

Next, (i)–(iii) are satisfied, and so is (1.2) due to

$$P_1(\nu,u)=\sum_{\tau=u}^{\nu-1}\left(\frac{1}{p_1(\tau)}\right)^{\frac{1}{3}}=\sum_{\tau=u}^{\nu-1}\frac{1}{\tau+1}=\sum_{\tau=u+1}^{\nu}\frac{1}{\tau}\to\infty.$$

Now we may pick (see Remark 2.4)

$$k_0=k_1=k_2=1 \quad \text{and} \quad k_3=2,$$

and then (2.17) is satisfied, and we have

$$\xi_0(t)=t+4, \qquad \xi_1(t)=\xi_2(t)=t-3, \qquad \xi_3(t)=t-1.$$

Moreover, we have $\alpha_4<1<\alpha_5$ and $\alpha_2<\alpha_1=\alpha_3$, so we will apply Theorem 3.4. We pick $p=p_4$, and then

$$\begin{aligned} g_1(t)+g_2(t)&=\frac{2}{3}\left(\frac{1}{3}\right)^{\frac{1}{2}}\big(p(t)\big)^{-\frac{1}{2}}\big(p_4(t)\big)^{\frac{3}{2}}+2\cdot 3^{-\frac{3}{2}}\big(p(t)\big)^{\frac{3}{2}}\big(p_5(t)\big)^{-\frac{1}{2}}\\ &=\frac{2}{3\sqrt{3t}}\left(1+\frac{1}{\sqrt{t}}\right)\to 0 \quad \text{as } t\to\infty, \end{aligned}$$

and thus, (3.1) is satisfied. We also calculate

$$\begin{aligned} \Lambda_0(t)&=\sum_{\tau=t-1}^{t-1}\left(\frac{1}{p_1(\tau)}\sum_{s=\tau-1}^{\tau-1}p_3(s)\right)^{\frac{1}{3}}=\left(\frac{p_3(t-2)}{p_1(t-1)}\right)^{\frac{1}{3}}=\left(\frac{t^4}{t^3}\right)^{\frac{1}{3}}=t^{\frac{1}{3}},\\ \Lambda_1(t)&=p_2(t)(t-2)P_1(t-2,t-3)=p_2(t)=t^2,\\ \Lambda_2(t)&=p_2(t)\sum_{\tau=t_1}^{t-2}P_1(\tau,t_1)\geq\frac{t^2}{t-2}\geq t, \end{aligned}$$

and

$$\Lambda_3(t)=p_2(t)\big(P_1(t,t-1)\big)^{\frac{1}{3}}=p_2(t)\left(\frac{1}{t}\right)^{\frac{1}{3}}=t^{\frac{5}{3}}.$$

Hence, (3.21), (3.22), (3.23), and (3.24) hold. Now all the conditions of Theorem 3.4 are fulfilled, and thus, (4.1) is oscillatory.

Example 4.2 We consider the equation

$$\begin{aligned} &\Delta\left((t+1)^3\left(\Delta^2\left(x(t)+\frac{1}{t}x^{\frac{1}{3}}(t-1)-x^{\frac{2}{3}}(t-1)\right)\right)^3\right)\\ &\quad=t^2x(t-2)+(t+2)^4x(t+6). \end{aligned} \tag{4.2}$$

Note that all data in (4.2) are the same as in (4.1), except

$$\alpha_5=\frac{2}{3}.$$

Moreover, we have $\alpha_4 < \alpha_5 < 1$ and $\alpha_2/\alpha_5 < \alpha_1 = \alpha_3$, so we will apply Theorem 3.8. We calculate

$$P(t) = \left(\frac{1}{2}p_4(t)\right)^2 \left(p_5(t)\right)^{-1} = \frac{1}{4t^2} \to 0 \quad \text{as } t \to \infty,$$

and thus, (3.25) is satisfied. The fulfillment of all other conditions of Theorem 3.8 follows in the same way as in Example 4.1, and hence (4.2) is oscillatory.

Remark 4.3 The results of this paper may be extended to higher-order difference equations of the form

$$\Delta\left(p_1(t)\left(\Delta^{n-1}y(t)\right)^{\alpha_1}\right) = p_2(t)x^{\alpha_2}(t-m+1) + p_3(t)x^{\alpha_3}\left(t+m^*\right),$$

where $y(t) = x(t) + p_4(t)x^{\alpha_4}(t-k) - p_5x^{\alpha_5}(t-k)$. We leave the details for future consideration.

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