Shammakh et al. *Advances in Difference Equations* (2020) 2020:645
https://doi.org/10.1186/s13662-020-03104-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

On multi-term proportional fractional differential equations and inclusions

Wafa Shammakh[1], Hadeel Z. Alzumi[1*] and Zahra Albarqi[1]

*Correspondence:
hzalzumi@uj.edu.sa
[1]Mathematics Department, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia

Abstract

The aim of this paper is to study new nonlocal boundary value problems of fractional differential equations and inclusions supplemented with slit-strips integral boundary conditions. Based on the functional analysis tools, the existence results for a nonlinear boundary value problem involving a proportional fractional derivative are presented. In addition to that, the extension of the problem at hand to its inclusion case is discussed. The obtained results are very interesting and are well illustrated with examples.

Keywords: Generalized fractional integral; Generalized proportional derivative; Fixed point theorem

1 Introduction

Problems of fractional differential equations arise in mathematical modeling of systems occurring in many scientific and engineering disciplines. Especially, multi-term fractional differential equations have been used to model many types of visco-elastic damping [1]. Bagley–Torvik [2] and Basset equations [3] are important examples of this class of equations. In addition to that, several methods have been suggested in the literature to solve these problems, for example, piecewise polynomial collocation [4], Haar wavelet method [5], Legendre wavelet method [6, 7], second kind Chebyshev wavelet method [8], spectral tau and collocation methods [9], and spline collocation method. For recent works on multi-term fractional differential equations, we refer the reader to [10–13].

For some recent development on this topic, a variety of initial and boundary conditions (BCs), such as classical, nonlocal, multipoint, periodic/ non-periodic, and integral boundary conditions, have been investigated. The concept of slits-strips conditions was introduced by Ahmad et al. [14, 15]. It was a new idea and had useful applications in imaging via strip-detectors [16] and acoustics [17]. For examples of boundary value problems for nonlinear differential equations, one can see [18–21].

Later on, in [22, 23], Anderson suggested a newly defined local derivative that tended to the original function as the order ρ tended to zero, and hence improved the conformable derivatives. Following this trend, some authors came up with new types of fractional derivatives and differences that allow the appearance of exponential function [24, 25] or the Mittag-Leffler function [26] in the kernel of the operators. Nevertheless, the new non-singular kernel type fractional derivatives have the disadvantage that their corresponding integral operators do not possess a semigroup property, which makes it uneasy to solve certain complicated fractional systems in their frames.

Inspired by the above works and based on a special case of the proportional- derivative, Jarad et al. [27] generated Caputo and Riemann–Liouville generalized proportional fractional derivatives involving exponential functions in their kernels. The advantage of the newly defined derivatives, which made them distinctive, was their corresponding proportional fractional integrals possessing a semigroup property and they provided an undeviating generalization to the existing Caputo and Riemann–Liouville fractional derivatives and integrals.

In this paper, we study the existence of solutions for nonlinear fractional differential equations and inclusions of order $\alpha \in (0,1)$. Precisely, we consider the following problems:

$$\begin{cases} a_1\,{}^C D_{0^+}^{\alpha+2,\rho} u(t) + a_2\,{}^C D_{0^+}^{\alpha+1,\rho} u(t) + a_3\,{}^C D_{0^+}^{\alpha,\rho} u(t) = f(t,u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ \sum_{i=1}^{m} \delta_i u(\xi_i) = \frac{1}{\rho^\beta \Gamma(\beta)} [a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} u(s)\,ds \\ \qquad + a_2 \int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} u(s)\,ds], \end{cases} \tag{1.1}$$

where ${}^C D_{0^+}^{\alpha,\rho}$ denotes the generalized proportional fractional (GPF) derivative of Caputo type, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha \in (0,1)$, $\beta \in (0,1)$, $\rho \in (0,1]$, a_i $(i = 1,2,3)$, $0 < \eta_1 < \xi_1 < \xi_2 < \cdots < \xi_m < \eta_2 < 1$, δ_i $(i = 1,2,\ldots,m)$ are real constants, and

$$\begin{cases} a_1\,{}^C D_{0^+}^{\alpha+2,\rho} u(t) + a_2\,{}^C D_{0^+}^{\alpha+1,\rho} u(t) + a_3\,{}^C D_{0^+}^{\alpha,\rho} u(t) \in F(t,u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ \sum_{i=1}^{m} \delta_i u(\xi_i) = \frac{1}{\rho^\beta \Gamma(\beta)} [a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} u(s)\,ds \\ \qquad + a_2 \int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} u(s)\,ds], \end{cases} \tag{1.2}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and the other quantities are the same as defined in problem (1.1).

Our problems are modeled by multi-term fractional differential equations equipped by slit-strips integral boundary conditions, and the fractional derivative is of proportional type. This makes the problems at hand very important from an application point of view.

This paper is organized as follows. In Sect. 2, we present some basic definitions and properties of GPF integrals and derivatives. In Sect. 3, based on the Leray–Schauder and Krasnoselskii's fixed point theorems, we prove the existence results of solutions for boundary value problem (1.1). In addition, some examples are presented to illustrate the main results. In Sect. 4, we prove the existence results for multivalued problem (1.2). The first result for problem (1.2), associated with the convex-valued multivalued map, is derived with the aid of Leray–Schauder nonlinear alternative for multivalued maps, while the result for a nonconvex-valued map for problem (1.2) is proved by applying a fixed point theorem due to Covitz and Nadler.

2 Preliminaries

For convenience of the reader, we present here some definitions and lemmas that will be used in the proof of our main results. For basic notions of GPF integrals and derivatives, one can see [27]. In what follows, let $f(t) \in AC^n[a,b]$.

Definition 1 (GPF integral) For $\rho \in (0,1]$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, we define the left GPF integral of f starting by a:

$$\left({}_aI^{\alpha,\rho}f\right)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}f(s)\,ds.$$

Definition 2 (GPF derivative of Caputo type) For $\rho \in (0,1]$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, we define the left GPF derivative of Caputo type starting by a:

$$\begin{aligned}\left({}_a^CD^{\alpha,\rho}f\right)(t) &= {}_aI^{n-\alpha,\rho}\left(D^{n,\rho}f\right)(t) \\ &= \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{n-\alpha-1}\left(D^{n,\rho}f\right)(s)\,ds,\end{aligned}$$

where $n = [\Re(\alpha)] + 1$.

Theorem 3 *Let $f \in L_1(a,b)$ and ${}_aI^{\alpha,\rho}f(t) \in AC^n[a,b]$. For $\rho \in (0,1]$ and $n = [\Re(\alpha)] + 1$, we have*

$${}_aI^{\alpha,\rho}\left({}_a^CD^{\alpha,\rho}f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(D^{k,\rho}f)(a)}{\rho^k k!}(t-a)^k e^{\frac{\rho-1}{\rho}(t-a)}.$$

Proposition 4 *For any $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\rho \in (0,1]$, $n = [\Re(\alpha)] + 1$, we have*

$$\left({}_a^CD^{\alpha,\rho}f\right)(t) = \left({}_aD^{\alpha,\rho}f\right)(t) - \sum_{k=0}^{n-1} \frac{\rho^{\alpha-k}}{\Gamma(k+1-\alpha)}(t-a)^{k-\alpha} e^{\frac{\rho-1}{\rho}(t-a)}\left(D^{k,\rho}f\right)(a).$$

The following fixed point theorems play a crucial role in our main results.

Theorem 5 (Krasnoselskii's fixed point theorem [28]) *Let $\mathcal{N}$ be a closed, convex, bounded, and nonempty subset of a Banach space X. Let T_1, T_2 be operators such that*

(i) *$T_1(u_1) + T_2(u_2)$ belong to $\mathcal{N}$ whenever $u_1, u_2 \in \mathcal{N}$.*

(ii) *T_1 is compact and continuous and T_2 is a contraction mapping.*

Then there exists $u_0 \in \mathcal{N}$ such that $u_0 = T_1(u_0) + T_2(u_0)$.

Theorem 6 (Nonlinear alternative of Leray–Schauder type [29]) *Let C be a closed and convex subset of a Banach space E and U be an open subset of C with $0 \in U$. Suppose that $\mathcal{V} : \overline{U} \to C$ is a continuous, compact (that is, $\mathcal{V}(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *$\mathcal{V}$ has a fixed point in $\overline{U}$, or*

(ii) *there are $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda\mathcal{V}(u)$.*

For computational convenience, we introduce the notations:

$$\begin{aligned}&A_1 = a_1\rho^2, \qquad A_2 = 2a_1\rho(1-\rho) + a_2\rho, \\ &A_3 = a_1(1-\rho)^2 + a_2(1-\rho) + a_3, \qquad c_0 = \left({}^CD_{0^+}^{2,\rho}u\right)(0), \\ &C_0 = c_0(a_1 + a_2 + a_3). \end{aligned}\tag{2.1}$$

Lemma 7 *Let a_1, a_2, a_3 be positive constants such that $A_2^2 - 4A_1A_3 > 0$ and*

$$\Omega = \sum_{i=1}^{m} \delta_i \psi(\xi_i) - (\Omega_1 + \Omega_2) \neq 0,$$

then the solution of the linear multi-term fractional differential equations

$$\begin{aligned} &a_1\, {}^C D_{0^+}^{\alpha+2,\rho} u(t) + a_2\, {}^C D_{0^+}^{\alpha+1,\rho} u(t) + a_3\, {}^C D_{0^+}^{\alpha,\rho} u(t) \\ &\quad = \mathfrak{h}(\mathrm{t}), \quad \alpha \in (0,1), \rho \in (0,1], t \in [0,1], \end{aligned} \tag{2.2}$$

supplemented with BCs (1.1) is given by

$$\begin{aligned} u(t) = {} & \frac{1}{A_1(\chi_2 - \chi_1)\rho^\alpha \Gamma(\alpha)} \biggl\{ \int_0^t w(t) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} \mathfrak{h}(\tau)\, d\tau \biggr) ds \\ & + \frac{\psi(t)}{\Omega} \biggl[\frac{1}{\rho^\beta \Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau - y)^{\alpha-1} \mathfrak{h}(y)\, dy \biggr) d\tau\, ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} \mathfrak{h}(y)\, dy \biggr) d\tau\, ds \biggr) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} \mathfrak{h}(\tau)\, d\tau \biggr) ds \biggr] \biggr\}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} & w(r) = e^{\chi_2(r-s)} - e^{\chi_1(r-s)}, \quad r = t, s, \xi_i, \qquad w(s) = e^{\chi_2(s-\tau)} - e^{\chi_1(s-\tau)}, \\ & \chi_1 = \frac{-A_2 - \sqrt{A_2^2 - 4A_1A_3}}{2A_1}, \qquad \chi_2 = \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}, \\ & \psi(t) = \rho \biggl[\frac{(e^{\frac{\rho-1}{\rho}t} - e^{\chi_2 t})}{(\rho - 1 - \rho\chi_2)} - \frac{(e^{\frac{\rho-1}{\rho}t} - e^{\chi_1 t})}{(\rho - 1 - \rho\chi_1)} \biggr], \\ & \Omega_1 = \frac{a_1}{\rho^\beta \Gamma(\beta)} \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} \psi(s)\, ds, \\ & \Omega_2 = \frac{a_2}{\rho^\beta \Gamma(\beta)} \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \psi(s)\, ds. \end{aligned} \tag{2.4}$$

Proof By Theorem 3 (with $a = 0$), the general solution of the multi-term fractional differential equation (2.2) can be written as

$$a_1\, {}^C D_{0^+}^{2,\rho} u(t) + a_2\, {}^C D_{0^+}^{1,\rho} u(t) + a_3 u(t) = I^{\alpha,\rho} \mathfrak{h}(\mathrm{t}) + C_0 e^{\frac{\rho-1}{\rho}t},$$

where C_0 is an unknown arbitrary constant.

Using Proposition 4, we get

$$A_1 u''(t) + A_2 u'(t) + A_3 u(t) = I^{\alpha,\rho} \mathfrak{h}(t) + C_0 e^{\frac{\rho-1}{\rho}t}. \tag{2.5}$$

Now, by the method of variation of parameters, the solution of (2.5) can be written as

$$\begin{aligned} u(t) &= C_1 e^{\chi_1 t} + C_2 e^{\chi_2 t} + \frac{1}{A_1(\chi_2-\chi_1)}\biggl[\int_0^t \bigl(e^{\chi_2(t-s)} - e^{\chi_1(t-s)}\bigr) \\ &\quad \cdot \biggl(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\mathfrak{h}(\tau)\,d\tau\biggr)\,ds + C_0\psi(t)\biggr]. \end{aligned} \tag{2.6}$$

Using the boundary conditions $u(0) = u(0)' = 0$, we get $C_1 = 0$, $C_2 = 0$. Thus (2.6) takes the form

$$\begin{aligned} u(t) &= \frac{1}{A_1(\chi_2-\chi_1)}\biggl[\int_0^t \bigl(e^{\chi_2(t-s)} - e^{\chi_1(t-s)}\bigr) \\ &\quad \cdot \biggl(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\mathfrak{h}(\tau)\,d\tau\biggr)\,ds + C_0\psi(t)\biggr]. \end{aligned} \tag{2.7}$$

Now, using the last condition in (1.1), we get

$$\begin{aligned} C_0 &= \frac{1}{\Omega}\biggl\{\frac{a_1}{\rho^\beta\Gamma(\beta)}\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s) \\ &\quad \cdot \biggl(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\mathfrak{h}(y)\,dy\biggr)\,d\tau\,ds \\ &\quad + \frac{a_2}{\rho^\beta\Gamma(\beta)}\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\ &\quad \cdot \biggl(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\mathfrak{h}(y)\,dy\biggr)\,d\tau\,ds \\ &\quad - \sum_{i=1}^m \delta_i\int_0^{\xi_i} w(\xi_i)\biggl(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\mathfrak{h}(\tau)\,d\tau\biggr)\,ds\biggr\}. \end{aligned} \tag{2.8}$$

Substituting the value of C_0 in (2.7), we obtain solution (2.3). Conversely, we can establish this direction by immediate computation. Applying the GPF derivative of Caputo type ${}^C D_{0^+}^{2,\rho}$ on both sides of (2.7) and using Proposition 4, we get

$$\begin{aligned} &a_1\,{}^C D_{0^+}^{2,\rho}u(t) + a_2\,{}^C D_{0^+}^{1,\rho}u(t) + a_3 u(t) \\ &\quad = A_1 u''(t) + A_2 u'(t) + A_3 u(t) \\ &\quad = \frac{1}{A_1(\chi_2-\chi_1)}\biggl[A_1(\chi_2-\chi_1)I^{\alpha,\rho}\mathfrak{h}\bigl(t,u(t)\bigr) \\ &\qquad + A_1\int_0^t \bigl(\chi_2^2 e^{\chi_2(t-s)} - \chi_1^2 e^{\chi_1(t-s)}\bigr)I^{\alpha,\rho}\mathfrak{h}\bigl(s,u(s)\bigr)\,ds \\ &\qquad + A_2\int_0^t \bigl(\chi_2 e^{\chi_2(t-s)} - \chi_1 e^{\chi_1(t-s)}\bigr)I^{\alpha,\rho}\mathfrak{h}\bigl(s,u(s)\bigr)\,ds \\ &\qquad + A_3\int_0^t \bigl(e^{\chi_2(t-s)} - e^{\chi_1(t-s)}\bigr)I^{\alpha,\rho}\mathfrak{h}\bigl(s,u(s)\bigr)\,ds \end{aligned}$$

$$+ C_0\big(A_1\psi''(t) + A_2\psi'(t) + A_3\psi(t)\big)\Bigg]$$

$$= I^{\alpha,\rho}\mathfrak{h}\big(t,u(t)\big) + C_0\frac{a_3 e^{\frac{\rho-1}{\rho}t}}{a_1(\rho-1-\rho\chi_1)(\rho-1-\rho\chi_2)}, \tag{2.9}$$

where A_1, A_2, and A_3 are defined in (2.1). Now, by applying ${}^{C}D^{\alpha,\rho}_{0^+}$ on both sides of (2.9), we get

$$a_1\,{}^{C}D^{\alpha+2,\rho}_{0^+}u(t) + a_2\,{}^{C}D^{\alpha+1,\rho}_{0^+}u(t) + a_3\,{}^{C}D^{\alpha,\rho}_{0^+}u(t) = \mathfrak{h}\big(t,u(t)\big),$$

which shows that the obtained solution satisfies the given differential equation. Also, we can prove easily that the solution satisfies the boundary conditions. □

Remark 8 We confirm Lemma 7 in the case where A_1, A_2, and A_3 fulfill the condition $A_2^2 - 4A_1A_3 > 0$. The other cases are $A_2^2 - 4A_1A_3 = 0$ and $A_2^2 - 4A_1A_3 < 0$, which are solved in the same way as above. We include here the solutions and omit the details. Thus, in Lemma 7, assume that

$$\bar{\Omega} = \sum_{i=1}^{m}\delta_i\bar{\psi}(\xi_i) - (\bar{\Omega}_1 + \bar{\Omega}_2) \neq 0$$

and

$$\bar{\bar{\Omega}} = \sum_{i=1}^{m}\delta_i\bar{\bar{\psi}}(\xi_i) - (\bar{\bar{\Omega}}_1 + \bar{\bar{\Omega}}_2) \neq 0.$$

If A_1, A_2, and A_3 satisfy the condition:

(i) $A_2^2 - 4A_1A_3 = 0$, then the solution of (2.2) is

$$\begin{aligned} u(t) = \frac{1}{A_1\rho^\alpha\Gamma(\alpha)}\Bigg\{&\int_0^t \bar{w}(t)\bigg(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\mathfrak{h}(\tau)\,d\tau\bigg)\,ds \\ &+ \frac{\bar{\psi}(t)}{\bar{\Omega}}\Bigg[\frac{1}{\rho^\beta\Gamma(\beta)}\bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s \bar{w}(s) \\ &\cdot\bigg(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\mathfrak{h}(y)\,dy\bigg)\,d\tau\,ds \\ &+ a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s \bar{w}(s) \\ &\cdot\bigg(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\mathfrak{h}(y)\,dy\bigg)\,d\tau\,ds\bigg) \\ &- \sum_{i=1}^m\delta_i\int_0^{\xi_i}\bar{w}(\xi_i)\bigg(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\mathfrak{h}(\tau)\,d\tau\bigg)\,ds\Bigg]\Bigg\}, \end{aligned} \tag{2.10}$$

where

$$\bar{w}(r) = (r-s)e^{\chi(r-s)}, \quad r = t, s, \xi_i, \qquad \bar{w}(s) = (s-\tau)e^{\chi(s-\tau)},$$

$$
\begin{aligned}
&\chi = -p, \quad p = \frac{A_2}{2A_1},\\
&\bar{\psi}(t) = te^{xt}\left(\frac{\rho}{(\rho-1-\rho\chi)}\right) + \left(\frac{\rho}{(\rho-1-\rho\chi)}\right)^2\left(e^{t\frac{\rho-1}{\rho}} - e^{\chi t}\right),\\
&\bar{\Omega}_1 = \frac{a_1}{\rho^\beta\Gamma(\beta)}\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\bar{\psi}(s)\,ds,\\
&\bar{\Omega}_2 = \frac{a_2}{\rho^\beta\Gamma(\beta)}\int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\bar{\psi}(s)\,ds.
\end{aligned}
\tag{2.11}
$$

(ii) $A_2^2 - 4A_1A_3 < 0$, then the solution of (2.2) is

$$
\begin{aligned}
u(t) = \frac{1}{A_1\beta\rho^\alpha\Gamma(\alpha)}\Bigg\{&\int_0^t \bar{\bar{w}}(t)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\mathfrak{h}(\tau)\,d\tau\right)ds\\
&+ \frac{\bar{\bar{\psi}}(t)}{\bar{\bar{\Omega}}}\Bigg[\frac{1}{\rho^\beta\Gamma(\beta)}\Bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s \bar{\bar{w}}(s)\\
&\cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\mathfrak{h}(y)\,dy\right)d\tau\,ds\\
&+ a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s \bar{\bar{w}}(s)\\
&\cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\mathfrak{h}(y)\,dy\right)d\tau\,ds\Bigg)\\
&- \sum_{i=1}^m \delta_i\int_0^{\xi_i} \bar{\bar{w}}(\xi_i)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\mathfrak{h}(\tau)\,d\tau\right)ds\Bigg]\Bigg\},
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
&\bar{\bar{w}}(r) = \sin\mathfrak{b}(r-s)e^{-\gamma(r-s)}, \quad r = t, s, \xi_i, \qquad \bar{\bar{w}}(s) = \sin\mathfrak{b}(s-\tau)e^{-\gamma(s-\tau)},\\
&\chi_{1,2} = -\gamma \pm \mathfrak{b}i, \quad \gamma = \frac{A_2}{2A_1}, \mathfrak{b} = \frac{\sqrt{4A_1A_3 - A_2^2}}{2A_1},\\
&\bar{\bar{\psi}}(t) = \frac{1}{\mathfrak{b}}\left[e^{-\gamma t}\cos\left(\mathfrak{b}t - \left(\frac{\rho-1}{\rho} - \gamma\right)\right) - e^{t\frac{\rho-1}{\rho}}\cos\left(\frac{\rho-1}{\rho} - \gamma\right)\right],\\
&\bar{\bar{\Omega}}_1 = \frac{a_1}{\rho^\beta\Gamma(\beta)}\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\bar{\bar{\psi}}(s)\,ds,\\
&\bar{\bar{\Omega}}_2 = \frac{a_2}{\rho^\beta\Gamma(\beta)}\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\bar{\bar{\psi}}(s)\,ds.
\end{aligned}
\tag{2.13}
$$

3 Main results

Denote by $\mathcal{C} = \{u(t) : u(t) \in C([0,1], \mathbb{R})\}$ the Banach space of all continuous functions defined on $[0,1]$ into $\mathbb{R}$ endowed with the norm

$$
\|u\| = \sup\{|u(t)|, t \in [0,1]\}.
$$

In view of Lemma 7, problem (1.1) can be transformed into the fixed point problem as follows.

Case I: For $A_2^2 - 4A_1A_3 > 0$ as $u = \mathcal{L}u$, we define an operator $\mathcal{L} : \mathcal{C} \longrightarrow \mathcal{C}$ by the following formula:

$$\begin{aligned}
(\mathcal{L}u)(t) = {}& \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\Biggl\{\int_0^t w(t) \\
&\cdot\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1} f\bigl(\tau,u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&+ \frac{\psi(t)}{\Omega}\Biggl[\frac{1}{\rho^\beta\Gamma(\beta)}\biggl(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot\biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1} f\bigl(y,u(y)\bigr)\,dy\biggr)\,d\tau\,ds \\
&+ a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot\biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1} f\bigl(y,u(y)\bigr)\,dy\biggr)\,d\tau\,ds\biggr) \\
&- \sum_{i=1}^m \delta_i\int_0^{\xi_i} w(\xi_i)\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1} f\bigl(\tau,u(\tau)\bigr)\,d\tau\biggr)\,ds\Biggr]\Biggr\}.
\end{aligned} \tag{3.1}$$

For the sake of computational convenience, we set

$$K = \frac{2}{A_1(\chi_2-\chi_1)\rho^\alpha}\left[\frac{1}{\Gamma(\alpha+2)} + \frac{\Phi}{|\Omega|}\left(\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^\beta\Gamma(\alpha+\beta+2)} - \sum_{i=1}^m \frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right], \tag{3.2}$$

$$\hat{K} = \frac{2\Phi}{A_1(\chi_2-\chi_1)\rho^\alpha|\Omega|}\left[\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^\beta\Gamma(\alpha+\beta+2)} - \sum_{i=1}^m \frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right], \tag{3.3}$$

$$\Phi = \max_{t\in[0,1]}\left|\rho\left[\frac{(e^{\frac{\rho-1}{\rho}t} - e^{\chi_2 t})}{(\rho-1-\rho\chi_2)} - \frac{(e^{\frac{\rho-1}{\rho}t} - e^{\chi_1 t})}{(\rho-1-\rho\chi_1)}\right]\right|. \tag{3.4}$$

Theorem 9 *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following conditions hold*:

(H_1) $|f(t,u)-f(t,v)| \le L\|u-v\|$, $\forall t\in[0,1]$, $L>0$, $u,v\in\mathbb{R}$,

(H_2) $|f(t,u)| \le \mu(t)$, $\forall (t,u)\in[0,1]\times\mathbb{R}$ *and* $\mu\in C([0,1],\mathbb{R}^+)$ *with*

$$\|\mu\| = \sup_{t\in[0,1]}\bigl|\mu(t)\bigr|.$$

Then there exists at least one solution for problem (1.1) *with* $A_2^2 - 4A_1A_3 > 0$ *on* $[0,1]$ *if*

$$L\hat{K} < 1. \tag{3.5}$$

Proof We consider a closed ball $B_r = \{u\in\mathcal{C} : \|u\| \le r\}$ with $r \ge K\|\mu\|$. We introduce the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ on B_r as follows:

$$(\mathcal{L}_1u)(t) = \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\int_0^t w(t)\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1} f\bigl(\tau,u(\tau)\bigr)\,d\tau\biggr)\,ds,$$

$$\begin{aligned}
(\mathcal{L}_2u)(t) = {}& \frac{\psi(t)}{A_1(\chi_2-\chi_1)\Omega}\Bigg[\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\Bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot\Bigg(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,u(y)\big)\,dy\Bigg)\,d\tau\,ds \\
&+ a_2\int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot\Bigg(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,u(y)\big)\,dy\Bigg)\,d\tau\,ds\Bigg) \\
&- \sum_{i=1}^{m}\frac{\delta_i}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\xi_i} w(\xi_i)\Bigg(\int_0^{s} e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\Bigg)\,ds\Bigg].
\end{aligned}$$

Notice that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$. For $u, v \in B_r$, we have

$$\begin{aligned}
\|\mathcal{L}_1u + \mathcal{L}_2v\| = {}& \sup_{t\in[0,1]}\Bigg\{\Bigg|\frac{1}{A_1(\chi_2-\chi_1)\rho^{\alpha}\Gamma(\alpha)}\int_0^t w(t) \\
&\cdot\Bigg(\int_0^{s} e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\Bigg)\,ds \\
&+ \frac{\psi(t)}{A_1(\chi_2-\chi_1)\Omega}\Bigg[\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\cdot\Bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1} \\
&\cdot\int_0^s w(s)\Bigg(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,v(y)\big)\,dy\Bigg)\,d\tau\,ds \\
&+ a_2\int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot\Bigg(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,v(y)\big)\,dy\Bigg)\,d\tau\,ds\Bigg) \\
&- \sum_{i=1}^{m}\frac{\delta_i}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\xi_i} w(\xi_i) \\
&\cdot\Bigg(\int_0^{s} e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,v(\tau)\big)\,d\tau\Bigg)\,ds\Bigg]\Bigg|\Bigg\} \\
\le {}& \|\mu\|\sup_{t\in[0,1]}\Bigg\{\Bigg|\frac{1}{A_1(\chi_2-\chi_1)\rho^{\alpha}\Gamma(\alpha)}\int_0^t w(t)\bigg(\frac{s^{\alpha}}{\alpha}\bigg)\,ds \\
&+ \frac{\psi(t)}{A_1(\chi_2-\chi_1)\Omega}\Bigg[\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\cdot\Bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1} \\
&\cdot\int_0^s w(s)\bigg(\frac{\tau^{\alpha}}{\alpha}\bigg)\,d\tau\,ds \\
&+ a_2\int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s)\bigg(\frac{\tau^{\alpha}}{\alpha}\bigg)\,d\tau\,ds\Bigg) \\
&- \sum_{i=1}^{m}\frac{\delta_i}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\xi_i} w(\xi_i)\bigg(\frac{s^{\alpha}}{\alpha}\bigg)\,ds\Bigg]\Bigg|\Bigg\}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\|\mathcal{L}_1u+\mathcal{L}_2v\| &\le \|\mu\| \sup_{t\in[0,1]}\left\{\frac{2}{A_1(\chi_2-\chi_1)\rho^{\alpha}}\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{|\psi(t)|}{|\Omega|}\left(\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^{\beta}\Gamma(\alpha+\beta+2)}\right.\right.\right.\\ &\quad\left.\left.\left.-\sum_{i=1}^{m}\frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right]\right\}\\ &\le \|\mu\|K\le r.\end{aligned}$$

Thus, $\mathcal{L}_1u+\mathcal{L}_2v\in B_r$. Using assumption (H_1), we obtain

$$\begin{aligned}\|\mathcal{L}_2u-\mathcal{L}_2v\| &= \sup_{t\in[0,1]}\left\{\left|\frac{\psi(t)}{A_1(\chi_2-\chi_1)\Omega}\left[\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\right.\right.\right.\\ &\quad\cdot\left(a_1\int_0^{\eta_1}e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^{s}w(s)\right.\\ &\quad\cdot\left(\int_0^{\tau}e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\big[f\big(y,u(y)\big)-f\big(y,v(y)\big)\big]\,dy\right)d\tau\,ds\\ &\quad+a_2\int_{\eta_2}^{1}e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^{s}w(s)\\ &\quad\left.\cdot\left(\int_0^{\tau}e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\big[f\big(y,u(y)\big)-f\big(y,v(y)\big)\big]\,dy\right)d\tau\,ds\right)\\ &\quad-\sum_{i=1}^{m}\frac{\delta_i}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\xi_i}w(\xi_i)\\ &\quad\left.\left.\left.\cdot\left(\int_0^{s}e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\big[f\big(\tau,u(\tau)\big)-f\big(\tau,v(\tau)\big)\big]\,d\tau\right)ds\right]\right|\right\}\\ &\le L\sup_{t\in[0,1]}\left\{\frac{2|\psi(t)|}{A_1(\chi_2-\chi_1)\rho^{\alpha}|\Omega|}\left[\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^{\beta}\Gamma(\alpha+\beta+2)}\right.\right.\\ &\quad\left.\left.-\sum_{i=1}^{m}\frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right\}\|u-v\|\\ &\le L\left\{\frac{2\Phi}{A_1(\chi_2-\chi_1)\rho^{\alpha}|\Omega|}\left[\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^{\beta}\Gamma(\alpha+\beta+2)}-\sum_{i=1}^{m}\frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right]\right\}\|u-v\|.\end{aligned}$$

Therefore,

$$\|\mathcal{L}_2u-\mathcal{L}_2v\| = L\hat{K}\|u-v\|,$$

which, in view of condition (3.5), shows that $\mathcal{L}_2$ is a contraction.

Next, we show that $\mathcal{L}_1$ is compact and continuous. Notice that the continuity of f implies that the operator $\mathcal{L}_1$ is continuous. Also, $\mathcal{L}_1$ is uniformly bounded on B_r as

$$\|\mathcal{L}_1u\| = \sup_{t\in[0,1]}\left\{\left|\frac{1}{A_1(\chi_2-\chi_1)\rho^{\alpha}\Gamma(\alpha)}\int_0^{t}w(t)\right.\right.$$

$$
\begin{aligned}
&\cdot \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f\big(\tau, u(\tau)\big)\, d\tau \right) ds \Bigg| \Bigg\} \\
&\le \frac{2\|\mu\|}{A_1(\chi_2-\chi_1)\rho^\alpha \Gamma(\alpha+2)}.
\end{aligned}
$$

Let us fix $\sup_{(t,u)\in[0,1]\times B_r} |f(t,u(t))| = \bar{f}$, and take $0 < t_1 < t_2 < 1$. Then

$$
\begin{aligned}
\big|(\mathcal{L}_1 u)(t_2) - (\mathcal{L}_1 u)(t_1)\big| &= \left| \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha \Gamma(\alpha)} \left[\int_0^{t_1} \big[w(t_2) - w(t_1)\big] \right.\right. \\
&\quad \cdot \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f\big(\tau, u(\tau)\big)\, d\tau \right) ds \\
&\quad \left.\left. + \int_{t_1}^{t_2} w(t_2) \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f\big(\tau, u(\tau)\big)\, d\tau \right) \right] \right| \\
&\le \frac{2\bar{f}}{A_1(\chi_2-\chi_1)\rho^\alpha \Gamma(\alpha+2)} \big|t_2^{\alpha+1} - t_1^{\alpha+1}\big| \to 0
\end{aligned}
$$

as $(t_2 - t_1) \to 0$, independently of $u \in B_r$. This implies that $\mathcal{L}_1$ is relatively compact on B_r, it follows by the Arzelá–Ascoli theorem that the operator $\mathcal{L}_1$ is compact on B_r. By using Krasnoselskii's fixed point theorem, there exists at least one solution on $[0,1]$. □

Now we apply the Leray–Schauder nonlinear alternative to prove the existence of solutions for problem (1.1) with $A_2^2 - 4A_1A_3 > 0$.

Theorem 10 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_3) *There exist a function $q \in C([0,1], \mathbb{R}^+)$ and a nondecreasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,u)| \le q(t)g(\|u\|)$, $\forall (t,u) \in [0,1] \times \mathbb{R}$.*

(H_4) *There exists a constant $M > 0$ such that*

$$
\frac{M}{\|q\| g(M) K} > 1.
$$

Then problem (1.1) with $A_2^2 - 4A_1A_3 > 0$ has at least one solution on $[0,1]$.

Proof We consider the operator $\mathcal{L} : \mathcal{C} \longrightarrow \mathcal{C}$ defined by (3.1). We show that $\mathcal{L}$ maps bounded sets into bounded sets in $\mathcal{C}$. For a positive number r, let $B_r = \{u \in \mathcal{C} : \|u\| \le r\}$ be a bounded set in $\mathcal{C}$. Then we have

$$
\begin{aligned}
\|\mathcal{L}u\| = \sup_{t\in[0,1]} &\left| \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha \Gamma(\alpha)} \left\{ \int_0^t w(t) \right.\right. \\
&\cdot \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f\big(\tau, u(\tau)\big)\, d\tau \right) ds \\
&+ \frac{\psi(t)}{\Omega} \left[\frac{1}{\rho^\beta \Gamma(\beta)} \cdot \left(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)} (\eta_1-s)^{\beta-1} \int_0^s w(s) \right.\right. \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f\big(y, u(y)\big)\, dy \right) d\tau\, ds \\
&+ a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\left(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,u(y)\big)\,dy\right)d\tau\,ds\right) \\
&-\sum_{i=1}^{m}\delta_i\int_0^{\xi_i} w(\xi_i)\left(\int_0^{s} e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\right)ds\Bigg]\Bigg\}\Bigg|, \\
\le{}& \|q\|g\big(\|u\|\big)\sup_{t\in[0,1]}\left\{\frac{2}{A_1(\chi_2-\chi_1)\rho^{\alpha}}\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{|\psi(t)|}{|\Omega|}\right.\right. \\
&\left.\left.\cdot\left(\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^{\beta}\Gamma(\alpha+\beta+2)}-\sum_{i=1}^{m}\frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right]\right\}, \\
\le{}& \|q\|g\big(\|u\|\big)K\le\|q\|g(r)K.
\end{aligned}
$$

Next, we show that $\mathcal{L}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $u \in B_r$, where B_r is a bounded set of $\mathcal{C}$. Then we obtain

$$
\begin{aligned}
\big|(\mathcal{L}u)(t_2)-(\mathcal{L}u)(t_1)\big| ={}& \Bigg|\frac{1}{A_1(\chi_2-\chi_1)\rho^{\alpha}\Gamma(\alpha)}\Bigg\{\int_0^{t_1}\big[w(t_2)-w(t_1)\big] \\
&\cdot\left(\int_0^{s} e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\right)ds \\
&+\int_{t_1}^{t_2} w(t_2)\left(\int_0^{s} e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\right)ds\Bigg| \\
&+\frac{|\psi(t_2)-\psi(t_1)|}{|\Omega|}\Bigg[\frac{1}{\rho^{\beta}\Gamma(\beta)}\cdot\Bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1} \\
&\cdot\int_0^{s} w(s)\left(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,u(y)\big)\,dy\right)d\tau\,ds \\
&+a_2\int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^{s} w(s) \\
&\cdot\left(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,u(y)\big)\,dy\right)d\tau\,ds\Bigg) \\
&-\sum_{i=1}^{m}\delta_i\int_0^{\xi_i} w(\xi_i) \\
&\cdot\left(\int_0^{s} e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\right)ds\Bigg]\Bigg\} \\
\le{}& \frac{2\|q\|g(r)}{A_1(\chi_2-\chi_1)\rho^{\alpha}}\Bigg[\frac{|t_2^{\alpha+1}-t_1^{\alpha+1}|}{\Gamma(\alpha+2)} \\
&+\frac{|\psi(t_2)-\psi(t_1)|}{|\Omega|}\left(\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^{\beta}\Gamma(\alpha+\beta+2)}-\sum_{i=1}^{m}\frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right)\Bigg],
\end{aligned}
$$

which tends to zero independently of $u \in B_r$ as $t_2 - t_2 \to 0$. As $\mathcal{L}$ satisfies the above assumptions, it follows by the Arzelá–Ascoli theorem that $\mathcal{L} : \mathcal{C} \longrightarrow \mathcal{C}$ is completely continuous. It remains to show the boundedness of the set of solutions of $u = \lambda\mathcal{L}u$ for $\lambda \in [0,1]$. Indeed, let u be a solution. Then, for $t \in [0,1]$, we have

$$\begin{aligned} |u(t)| &= |\lambda \mathcal{L}u(t)| \\ &\leq |\mathcal{L}u(t)|, \end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|u\| \leq \|q\| g(r) K,$$

and then

$$\frac{\|u\|}{\|q\| g(r) K} \leq 1.$$

In view of (H_4), there exists M such that $\|u\| \neq M$. Let us set

$$U = \{u \in \mathcal{C} : \|u\| < M\}.$$

Note that the operator $\mathcal{L} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda \mathcal{L}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type, we deduce that $\mathcal{L}$ has a fixed point $u \in \overline{U}$, which is a solution of problem (1.1) with $A_2^2 - 4A_1A_3 > 0$. □

Case II: For $A_2^2 - 4A_1A_3 = 0$ as $u = \mathcal{J}u$, we define an operator $\mathcal{J} : \mathcal{C} \longrightarrow \mathcal{C}$ by the following formula:

$$\begin{aligned} (\mathcal{J}u)(t) = \frac{1}{A_1\rho^\alpha \Gamma(\alpha)} \Bigg\{ & \int_0^t \bar{w}(t) \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f(\tau, u(\tau))\, d\tau \right) ds \\ & + \frac{\bar{\psi}(t)}{\bar{\Omega}} \Bigg[\frac{1}{\rho^\beta \Gamma(\beta)} \Bigg(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} \int_0^s \bar{w}(s) \\ & \cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f(y, u(y))\, dy \right) d\tau\, ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s \bar{w}(s) \\ & \cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f(y, u(y))\, dy \right) d\tau\, ds \Bigg) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} \bar{w}(\xi_i) \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f(\tau, u(\tau))\, d\tau \right) ds \Bigg] \Bigg\}. \end{aligned} \tag{3.6}$$

For the sake of computational convenience, we set

$$Q = \frac{2}{A_1\rho^\alpha} \left[\frac{1}{\Gamma(\alpha+3)} + \frac{\bar{\Phi}}{|\bar{\Omega}|} \left(\frac{(a_1\eta_1^{\beta+\alpha+2} + a_2)}{\rho^\beta \Gamma(\alpha+\beta+3)} - \sum_{i=1}^m \frac{\delta_i \xi_i^{\alpha+2}}{\Gamma(\alpha+3)} \right) \right], \tag{3.7}$$

$$\hat{Q} = \frac{2\bar{\Phi}}{A_1\rho^\alpha |\bar{\Omega}|} \left(\frac{(a_1\eta_1^{\beta+\alpha+2} + a_2)}{\rho^\beta \Gamma(\alpha+\beta+3)} - \sum_{i=1}^m \frac{\delta_i \xi_i^{\alpha+2}}{\Gamma(\alpha+3)} \right), \tag{3.8}$$

$$\bar{\Phi} = \max_{t\in[0,1]}\left| te^{\chi t}\left(\frac{\rho}{(\rho-1-\rho\chi)}\right) + \left(\frac{\rho}{(\rho-1-\rho\chi)}\right)^2 \left(e^{t\frac{\rho-1}{\rho}} - e^{\chi t}\right)\right|. \tag{3.9}$$

Corollary 11 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying conditions (H_1) and (H_2). Then problem (1.1) with $A_2^2-4A_1A_3=0$ has at least one solution on $[0,1]$ if*

$$L\hat{Q}<1.$$

Corollary 12 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Then problem (1.1) with $A_2^2-4A_1A_3=0$ has at least one solution on $[0,1]$, if (H_3) and the condition*

(H_5) *There exists a constant $M_1>0$ such that*

$$\frac{M_1}{\|q\|g(M_1)Q}>1,$$

where Q is defined by (3.7), are satisfied.

Case III: For $A_2^2-4A_1A_3<0$ as $u=\mathcal{G}u$, we define an operator $\mathcal{G}:\mathcal{C}\longrightarrow\mathcal{C}$ by the following formula:

$$\begin{aligned}(\mathcal{G}u)(t) = \frac{1}{A_1\mathfrak{b}\rho^{\alpha}\Gamma(\alpha)}\Biggl\{&\int_0^t \bar{\bar{w}}(t)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\right)ds \\ &+ \frac{\bar{\bar{\psi}}(t)}{\bar{\bar{\Omega}}}\Biggl[\frac{1}{\rho^{\beta}\Gamma(\beta)}\biggl(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s \bar{\bar{w}}(s) \\ &\cdot\left(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,u(y)\big)\,dy\right)d\tau\,ds \\ &+ a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s \bar{\bar{w}}(s) \\ &\cdot\left(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f\big(y,u(y)\big)\,dy\right)d\tau\,ds\biggr) \\ &- \sum_{i=1}^m \delta_i\int_0^{\xi_i}\bar{\bar{w}}(\xi_i)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f\big(\tau,u(\tau)\big)\,d\tau\right)ds\Biggr]\Biggr\}.\end{aligned} \tag{3.10}$$

For the sake of computational convenience, we set

$$H = \frac{1}{A_1\mathfrak{b}\rho^{\alpha}}\left[\frac{1}{\Gamma(\alpha+2)} + \frac{\bar{\bar{\Phi}}}{|\bar{\bar{\Omega}}|}\left(\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^{\beta}\Gamma(\alpha+\beta+2)} - \sum_{i=1}^m \frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right], \tag{3.11}$$

$$\hat{H} = \frac{\bar{\bar{\bar{\Phi}}}}{A_1\mathfrak{b}\rho^{\alpha}|\bar{\bar{\Omega}}|}\left(\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^{\beta}\Gamma(\alpha+\beta+2)} - \sum_{i=1}^m \frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\right), \tag{3.12}$$

$$\bar{\bar{\Phi}} = \max_{t\in[0,1]}\left|\frac{1}{\mathfrak{b}}\left[-e^{t\frac{\rho-1}{\rho}}\cos\left(\frac{\rho-1}{\rho}-\gamma\right) + e^{-\gamma t}\cos\left(\mathfrak{b}t - \left(\frac{\rho-1}{\rho}-\gamma\right)\right)\right]\right|. \tag{3.13}$$

Corollary 13 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying conditions (H_1) and (H_2). Then problem (1.1) with $A_2^2-4A_1A_3<0$ has at least one solution on $[0,1]$ if*

$$L\hat{H}<1.$$

Corollary 14 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then problem (1.1) with $A_2^2 - 4A_1A_3 < 0$ has at least one solution on $[0,1]$ if (H_3) and the condition*

(H_6) *There exists a constant $M_2 > 0$ such that*

$$\frac{M_2}{\|q\|g(M_2)H} > 1,$$

where H is defined by (3.11), are satisfied.

We conclude this section with some examples showing the applicability of our main results.

Example 1 Consider the following problem:

$$\begin{cases} {}^CD_{0^+}^{\frac{5}{2},\frac{1}{4}}u(t) + 3\,{}^CD_{0^+}^{\frac{3}{2},\frac{1}{4}}u(t) + 2\,{}^CD_{0^+}^{\frac{1}{2},\frac{1}{4}}u(t) = \frac{e^t}{\sqrt{63+t}} + \frac{\tan^{-1}u(t)}{2t^5+40}, \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ \sum_{i=1}^2 \delta_i u(\xi_i) = \frac{1}{\rho^\beta\Gamma(\beta)}[a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}u(s)\,ds \\ \qquad + a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}u(s)\,ds], \end{cases} \tag{3.14}$$

where $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, $\rho = \frac{1}{4}$, $\alpha = \beta = \frac{1}{2}$, $\delta_1 = 1$, $\delta_2 = 2$, $\eta_1 = \frac{1}{8}$, $\eta_2 = \frac{1}{4}$, $\xi_1 = \frac{1}{6}$, $\xi_2 = \frac{1}{5}$. Clearly, $|f(t,u)| \le \frac{e^t}{\sqrt{63+t}} + \frac{\pi}{2(2t^5+40)}$, $|f(t,u) - f(t,v)| \le L|u-v|$, with $L = \frac{1}{42}$. Using the given values, we find that $\Omega = 0.02217983$, $\Phi = 0.0059975$, and $K = 25.4438$. Also we have

$$L\hat{K} \approx 0.31923 \le 1.$$

Thus, all the conditions of Theorem 9 are satisfied and problem (3.14) has at least one solution on $[0,1]$.

Example 2 Consider the following problem:

$$\begin{cases} {}^CD_{0^+}^{\frac{5}{2},\frac{1}{4}}u(t) + 3\,{}^CD_{0^+}^{\frac{3}{2},\frac{1}{4}}u(t) + 2\,{}^CD_{0^+}^{\frac{1}{2},\frac{1}{4}}u(t) = \frac{t}{5\sqrt{t^2+35}}(e^{u(t)} + \frac{1}{2}), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ \sum_{i=1}^2 \delta_i u(\xi_i) = \frac{1}{\rho^\beta\Gamma(\beta)}[a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}u(s)\,ds \\ \qquad + a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}u(s)\,ds], \end{cases} \tag{3.15}$$

supplemented with the boundary conditions of problem (3.14). Evidently, $|f(t,u)| \le q(t)g(\|u\|)$ with $g(\|u\|) = \|u\| + \frac{1}{2}$ (where g is defined by $g(M) = M + \frac{1}{2}$) and $q(t) = \frac{t}{5\sqrt{t^2+35}}$ and $\|q\| = \frac{1}{30}$. By condition (H_4), that is, $\frac{M}{\|q\|g(M)K} > 1$, we find that $M > 2.79221$. Thus, we deduce by Theorem 10 that problem (3.15) has at least one solution on $[0,1]$.

4 Inclusion problem

In this section, we extend our study to the multivalued analogue of problem (1.2). We recall some basic notions needed throughout this section.

For a normed space $(X, \|\cdot\|)$, let

$$\mathcal{P}_{cp,c}(X) = \{\mathcal{Y} \in \mathcal{P}(X) : \mathcal{Y} \text{ is compact and convex}\}.$$

A multivalued map $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is called Carathéodory if

(i) $t\mapsto F(t,u)$ is measurable for each $u\in\mathbb{R}$;

(ii) $u\mapsto F(t,u)$ is upper semicontinuous for almost all $t\in[0,1]$;

Further, a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $a>0$, there exists $\varphi_a\in L^1([0,1],\mathbb{R}^+)$ such that

$$\big\|F(t,u)\big\|=\sup\big\{|x|:x\in F(t,u)\big\}\le\varphi_a(t)$$

for all $\|u\|<a$ and for a.e. $t\in[0,1]$.

Definition 15 A function $u\in\mathcal{C}$ is called a solution of problem (1.2) if we can find a function $f\in L^1([0,1],\mathbb{R})$ with $f(t)\in F(t,u)$ for a.e. $t\in[0,1]$ and

$$\begin{aligned} u(t)={}&\frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\left\{\int_0^t w(t)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds\right.\\ &+\frac{\psi(t)}{\Omega}\left[\frac{1}{\rho^\beta\Gamma(\beta)}\left(a_1\int_0^{\eta_1}e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s)\right.\right.\\ &\cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds\\ &+a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s)\\ &\cdot\left.\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds\right)\\ &\left.\left.-\sum_{i=1}^m\delta_i\int_0^{\xi_i}w(\xi_i)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds\right]\right\}, \end{aligned}$$

where $\psi(t)$, Ω, Ω_1, and Ω_2 are given by (2.4).

We define the set of selections of F by $S_{F,u}:=\{x\in L^1([0,1],\mathbb{R}):x(t)\in F(t,u(t))\text{ on }[0,1]\}$ for each $u\in\mathcal{C}$. The following lemma is helpful in the sequel.

Lemma 16 ([30]) *Let X be a Banach space. Let $F:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1-Carathéodory multivalued map, and let $\mathcal{T}$ be a linear continuous mapping from $L^1([0,1],\mathbb{R})$ to $C([0,1],\mathbb{R})$. Then the operator*

$$\mathcal{T}\circ S_F:C\big([0,1],\mathbb{R}\big)\to\mathcal{P}_{cp,c}\big(C\big([0,1],\mathbb{R}\big)\big),\qquad u\mapsto(\mathcal{T}\circ S_F)(u)=\mathcal{T}(S_{F,u})$$

is a closed graph operator in $C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$.

4.1 The upper semicontinuous case

In the following result, we assume that the multivalued map F is convex-valued and apply the Leray–Schauder nonlinear alternative for multivalued maps to prove the existence of solutions for the problem at hand.

Theorem 17 *Assume that*:

(A_1) $F:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory.*

(A_2) *There exist a function* $q\in C([0,1],\mathbb{R}^+)$ *and a continuous nondecreasing function* $g:[0,\infty)\to(0,\infty)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}}:=\sup\{|x|:x\in F(t,u)\}\le q(t)g(\|u\|)\quad \textit{for each } (t,u)\in[0,1]\times\mathbb{R}.$$

(A_3) *There exists a constant* $M>0$ *such that*

$$\frac{M}{\|q\|g(M)K}>1.$$

Then problem (1.2) *with* $A_2^2-4A_1A_3>0$ *has at least one solution on* $[0,1]$.

Proof Define an operator $T:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ by

$$\begin{aligned} T(u)=\Bigg\{&h(t)\in\mathcal{C}:h(t)=\frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\Bigg\{\int_0^t w(t)\\ &\cdot\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds\\ &+\frac{\psi(t)}{\Omega}\Bigg[\frac{1}{\rho^\beta\Gamma(\beta)}\Bigg(a_1\int_0^{\eta_1}e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s)\\ &\cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds\\ &\cdot+a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\cdot\int_0^s w(s)\\ &\cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds\Bigg)\\ &-\sum_{i=1}^m\delta_i\int_0^{\xi_i}w(\xi_i)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds\Bigg],f\in S_{F,u}\Bigg\}. \end{aligned}\tag{4.1}$$

It is clear that fixed points of T are solutions of problem (1.2). So, we need to verify that the operator T satisfies all the conditions of Leray–Schauder nonlinear alternative. This will be done in several steps.

Step 1. $T(u)$ is convex for each $u\in\mathcal{C}$. Indeed, if h_1, h_2 belongs to $T(u)$, then there exist $f_1,f_2\in S_{F,u}$ such that, for each $t\in[0,1]$, we get

$$\begin{aligned} h_i(t)={}&\frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\Bigg\{\int_0^t w(t)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f_i(\tau)\,d\tau\right)ds\\ &+\frac{\psi(t)}{\Omega}\Bigg[\frac{1}{\rho^\beta\Gamma(\beta)}\Bigg(a_1\int_0^{\eta_1}e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s)\\ &\cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f_i(y)\,dy\right)d\tau\,ds\\ &+a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f_i(y)\,dy\right)d\tau\,ds\Bigg) \\
&-\sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f_i(\tau)\,d\tau\right)ds\Bigg]\Bigg\}, \quad i=1,2.
\end{aligned}
$$

For $0 \leq \sigma \leq 1$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
\big[\sigma h_1 + (1-\sigma)h_2\big](t) = {}& \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\Bigg\{\int_0^t w(t) \\
&\cdot \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\big[\sigma f_1(\tau)+(1-\sigma)f_2(\tau)\big]\,d\tau\right)ds \\
&+\frac{\psi(t)}{\Omega}\Bigg[\frac{1}{\rho^\beta\Gamma(\beta)}\Bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\big[\sigma f_1(y)+(1-\sigma)f_2(y)\big]\,dy\right)d\tau\,ds \\
&+a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}\big[\sigma f_1(y)+(1-\sigma)f_2(y)\big]\,dy\right)d\tau\,ds\Bigg) \\
&-\sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \\
&\cdot \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}\big[\sigma f_1(\tau)+(1-\sigma)f_2(\tau)\big]\,d\tau\right)ds\Bigg]\Bigg\}.
\end{aligned}
$$

Hence, by the convexity of $S_{F,u}$, it follows that $\sigma h_1 + (1-\sigma)h_2 \in T(u)$.

Step 2. $T(u)$ maps bounded sets into bounded sets in $\mathcal{C}$. Let $B_r = \{u \in \mathcal{C} : \|u\| \leq r\}$ be a bounded ball in $\mathcal{C}$, where r is a positive number. Thus, for each $h \in T(u)$, $u \in B_r$, there exists $f \in S_{F,u}$ such that

$$
\begin{aligned}
h(t) = {}& \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\Bigg\{\int_0^t w(t)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds \\
&+\frac{\psi(t)}{\Omega}\Bigg[\frac{1}{\rho^\beta\Gamma(\beta)}\Bigg(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds \\
&+a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds\Bigg) \\
&-\sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds\Bigg]\Bigg\}.
\end{aligned}
$$

In view of (A_2), for each $t \in [0,1]$, we find that

$$\begin{aligned}
\bigl|h(t)\bigr| \leq{}& \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\Biggl\{\int_0^t \bigl|w(t)\bigr| \\
&\cdot\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}q(\tau)g\bigl(\|u\|\bigr)\,d\tau\biggr)\,ds \\
&+\frac{|\psi(t)|}{|\Omega|}\Biggl[\frac{1}{\rho^\beta\Gamma(\beta)}\cdot\biggl(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s \bigl|w(s)\bigr| \\
&\cdot\biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}q(y)g\bigl(\|u\|\bigr)\,dy\biggr)\,d\tau\,ds \\
&+a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s \bigl|w(s)\bigr| \\
&\cdot\biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}q(y)g\bigl(\|u\|\bigr)\,dy\biggr)\,d\tau\,ds\biggr) \\
&-\sum_{i=1}^m \delta_i\int_0^{\xi_i}\bigl|w(\xi_i)\bigr|\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}q(\tau)g\bigl(\|u\|\bigr)\,d\tau\biggr)\,ds\Biggr]\Biggr\} \\
\leq{}& \|q\|g\bigl(\|u\|\bigr)K,
\end{aligned}$$

which leads to $\|h\| \leq \|q\|g(r)K$, where K is given by (3.2).

Step 3. $T(u)$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and $u \in B_r$. Then we have

$$\begin{aligned}
\bigl|h(t_2)-h(t_1)\bigr| \leq{}& \frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\Biggl\{\biggl|\int_0^{t_1}\bigl[w(t_2)-w(t_1)\bigr] \\
&\cdot\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\biggr)\,ds \\
&+\int_{t_1}^{t_2} w(t_2)\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\biggr)\,ds\biggr| \\
&+\biggl|\frac{\psi(t_2)-\psi(t_1)}{\Omega}\biggl[\frac{1}{\rho^\beta\Gamma(\beta)}\cdot\biggl(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1} \\
&\cdot\int_0^s w(s)\biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\biggr)\,d\tau\,ds \\
&+a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\
&\cdot\biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\biggr)\,d\tau\,ds\biggr) \\
&-\sum_{i=1}^m \delta_i\int_0^{\xi_i} w(\xi_i)\biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\biggr)\,ds\biggr]\biggr|\Biggr\} \\
\leq{}& \frac{2\|q\|g(r)}{A_1(\chi_2-\chi_1)\rho^\alpha}\Biggl[\frac{|t_2^{\alpha+1}-t_1^{\alpha+1}|}{\Gamma(\alpha+2)}+\frac{(\psi(t_2)-\psi(t_1))}{|\Omega|} \\
&\cdot\Biggl(\frac{(a_1\eta_1^{\beta+\alpha+1}+a_2)}{\rho^\beta\Gamma(\alpha+\beta+2)}-\sum_{i=1}^m\frac{\delta_i\xi_i^{\alpha+1}}{\Gamma(\alpha+2)}\Biggr)\Biggr],
\end{aligned}$$

which tends to zero independent of $u \in B_r$ as $(t_2 - t_1) \to 0$. Combining the outcome of Steps 1–3 with the Arzelá–Ascoli theorem leads to the conclusion that $T : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Step 4. T has a closed graph. Suppose that there exists $u_n \to u^*$, $h_n \in T(u_n)$ and $h_n \to h^*$. Then we have to establish that $h^* \in T(u^*)$. Since $h_n \in T(u_n)$, there exists $f_n \in S_{F,u_n}$. In consequence, for each $t \in [0,1]$, we get

$$\begin{aligned} h_n(t) = {} & \frac{1}{A_1(\chi_2 - \chi_1)\rho^\alpha \Gamma(\alpha)} \Biggl\{ \int_0^t w(t) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f_n(\tau)\,d\tau \biggr) ds \\ & + \frac{\psi(t)}{\Omega} \Biggl[\frac{1}{\rho^\beta \Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)} (\eta_1 - s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f_n(y)\,dy \biggr) d\tau\,ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f_n(y)\,dy \biggr) d\tau\,ds \biggr) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f_n(\tau)\,d\tau \biggr) ds \Biggr] \Biggr\}. \end{aligned}$$

Next, we have to show that there exists $f^* \in S_{F,u^*}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h^*(t) = {} & \frac{1}{A_1(\chi_2 - \chi_1)\rho^\alpha \Gamma(\alpha)} \Biggl\{ \int_0^t w(t) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f^*(\tau)\,d\tau \biggr) ds \\ & + \frac{\psi(t)}{\Omega} \Biggl[\frac{1}{\rho^\beta \Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)} (\eta_1 - s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f^*(y)\,dy \biggr) d\tau\,ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f^*(y)\,dy \biggr) d\tau\,ds \biggr) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f^*(\tau)\,d\tau \biggr) ds \Biggr] \Biggr\}. \end{aligned}$$

Considering the continuous linear operator $\mathcal{T} : L^1([0,1],\mathbb{R}) \to \mathcal{C}$, we get

$$\begin{aligned} f \mapsto \mathcal{T}(f)(t) = {} & \frac{1}{A_1(\chi_2 - \chi_1)\rho^\alpha \Gamma(\alpha)} \Biggl\{ \int_0^t w(t) \\ & \cdot \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f(\tau)\,d\tau \biggr) ds \\ & + \frac{\psi(t)}{\Omega} \Biggl[\frac{1}{\rho^\beta \Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)} (\eta_1 - s)^{\beta-1} \int_0^s w(s) \end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f(y)\,dy \right) d\tau\,ds \\
&+ a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f(y)\,dy \right) d\tau\,ds \Bigg) \\
&- \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f(\tau)\,d\tau \right) ds \Bigg] \Bigg\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\|h_n(t)-h^*(t)\| = \sup_{t\in[0,1]} \Bigg| &\frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)} \Bigg\{ \int_0^t w(t) \\
&\cdot \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} \big[f_n(\tau)-f^*(\tau)\big]\,d\tau \right) ds \\
&+ \frac{\psi(t)}{\phi} \Bigg[\frac{1}{\rho^\beta\Gamma(\beta)} \Bigg(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)} (\eta_1-s)^{\beta-1} \int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} \big[f_n(y)-f^*(y)\big]\,dy \right) d\tau\,ds \\
&+ a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} \big[f_n(y)-f^*(y)\big]\,dy \right) d\tau\,ds \Bigg) \\
&- \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} \big[f_n(\tau)-f^*(\tau)\big]\,d\tau \right) ds \Bigg] \Bigg\} \Bigg|
\end{aligned}
$$

tends to 0 as $n\to\infty$. Thus, it follows by Lemma 16 that $\mathcal{T}\circ S_{F,u}$ is a closed graph operator. Furthermore, $h_n(t)\in\mathcal{T}(S_{F,u_n})$. Since $u_n\to u^*$, we have

$$
\begin{aligned}
h^*(t) = &\frac{1}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)} \Bigg\{ \int_0^t w(t) \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f^*(\tau)\,d\tau \right) ds \\
&+ \frac{\psi(t)}{\Omega} \Bigg[\frac{1}{\rho^\beta\Gamma(\beta)} \Bigg(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)} (\eta_1-s)^{\beta-1} \int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f^*(y)\,dy \right) d\tau\,ds \\
&+ a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\
&\cdot \left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f^*(y)\,dy \right) d\tau\,ds \Bigg) \\
&- \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f^*(\tau)\,d\tau \right) ds \Bigg] \Bigg\}
\end{aligned}
$$

for some $f^*\in S_{F,u^*}$.

Step 5. There exists an open set $V \subseteq \mathcal{C}$ with $u \notin \nu T(u)$ for any $\nu \in (0,1)$ and all $u \in \partial V$. Let u be a solution of (1.2). Then there exists $f \in L^1([0,1],\mathbb{R})$ with $f \in S_{F,u}$ such that, for $t \in [0,1]$, we have

$$\begin{aligned}
u(t) = {} & \frac{\lambda}{A_1(\chi_2-\chi_1)\rho^\alpha\Gamma(\alpha)}\left\{\int_0^t w(t)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds\right. \\
& + \frac{\psi(t)}{\Omega}\left[\frac{1}{\rho^\beta\Gamma(\beta)}\left(a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1}\int_0^s w(s)\right.\right. \\
& \cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds \\
& + a_2\int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}\int_0^s w(s) \\
& \left.\cdot\left(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1}f(y)\,dy\right)d\tau\,ds\right) \\
& \left.\left.-\sum_{i=1}^m \delta_i\int_0^{\xi_i} w(\xi_i)\left(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1}f(\tau)\,d\tau\right)ds\right]\right\}.
\end{aligned}$$

Using the computations done in Step 2, for each $t \in [0,1]$, we get

$$\begin{aligned}
\left|u(t)\right| &= \left|\lambda\mathcal{L}u(t)\right| \\
&\leq \left|\mathcal{L}u(t)\right|,
\end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|u\| \leq \|q\|g(r)K,$$

and then

$$\frac{\|u\|}{\|q\|g(r)K} \leq 1,$$

By (A_3), there exists M such that $\|u\| \neq M$. Let us set

$$V = \left\{u \in \mathcal{C} : \|u\| < M\right\}.$$

Note that the operator $T : \overline{V} \to \mathcal{P}(\mathcal{C})$ is a compact multivalued map, u.s.c. with convex closed values. With the given choice of V, it is not possible to find $u \in \partial V$ satisfying $u \in \nu T(u)$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type, the operator T has a fixed point $u \in \overline{V}$, which corresponds to a solution of problem (1.2) with $A_2^2 - 4A_1A_3 > 0$. □

4.2 The Lipschitz case

This subsection concerns the existence of solutions for problem (1.2) with a nonconvex-valued right-hand side by applying a fixed point theorem for multivalued maps due to

Covitz and Nadler [31]: "If $T : \mathcal{Z} \to \mathcal{P}_{cl}(\mathcal{Z})$ is a contraction, then $FixT \neq \phi$, where $\mathcal{P}_{cl}(\mathcal{Z}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{Z}) : \mathcal{Y} \text{ is closed}\}$."

Let $(\mathcal{Z}, d)$ be a metric space induced from the normed space $(\mathcal{Z}, \|\cdot\|)$. Consider $H_d : \mathcal{P}(\mathcal{Z}) \times \mathcal{P}(\mathcal{Z}) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(\mathcal{Z}), H_d)$ is a metric space, where $\mathcal{P}_{b,cl}(\mathcal{Z}) = \{\mathcal{Y} \in \mathcal{P}(\mathcal{Z}) : \mathcal{Y} \text{ is bounded and closed}\}$.

Theorem 18 *Assume that*

(A_4) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,u) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $u \in \mathbb{R}$, where $\mathcal{P}_{cp}(\mathbb{R}) = \{\mathcal{Y} \in \mathcal{P}(\mathbb{R}) : \mathcal{Y} \text{ is compact}\}$;*

(A_5) *$H_d(F(t,u), F(t,\hat{u})) \leq \varpi(t)|u - \hat{u}|$ for almost all $t \in [0,1]$ and $u, \hat{u} \in \mathbb{R}$ with $\varpi \in C([0,1], \mathbb{R}^+)$ and $d(0, F(t,0)) \leq \varpi(t)$ for almost all $t \in [0,1]$.*

Then problem (1.2) with $A_2^2 - 4A_1A_3 > 0$ has at least one solution on $[0,1]$ if

$$\|\varpi\|K < 1. \tag{4.2}$$

Proof Let us verify that the operator $T : \mathcal{C} \to \mathcal{P}(\mathcal{C})$, defined in (4.1), satisfies the hypothesis of the Covitz and Nadler fixed point theorem. We establish it in two steps.

Step 1. $T(u)$ is nonempty and closed for every $f \in S_{F,u}$. Since the set-valued map $F(\cdot, u(\cdot))$ is measurable, it admits a measurable selection $f : [0,1] \to \mathbb{R}$ by the measurable selection theorem ([32], Theorem III.6). By (A_4), we have

$$\big|f(t)\big| \leq \varpi(t)\big(1 + \big|u(t)\big|\big),$$

that is, $f \in L^1([0,1], \mathbb{R})$. So, F is integrable bounded. Therefore, $S_{F,u} \neq \phi$.

Now, we establish that $T(u)$ is closed for each $u \in \mathcal{C}$. Let $\{m_n\}_{n\geq 0} \in T(u)$ be such that $m_n \to m$ as $n \to \infty$ in $\mathcal{C}$. Then, $m \in \mathcal{C}$ and we can find $f_n \in S_{F,u_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}
m_n(t) = {}& \frac{1}{A_1(\chi_2 - \chi_1)\rho^\alpha \Gamma(\alpha)} \Bigg\{ \int_0^t w(t) \bigg(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f_n(\tau)\,d\tau \bigg)\,ds \\
& + \frac{\psi(t)}{\Omega} \Bigg[\frac{1}{\rho^\beta \Gamma(\beta)} \bigg(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} \int_0^s w(s) \\
& \cdot \bigg(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f_n(y)\,dy \bigg)\,d\tau\,ds \\
& + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\
& \cdot \bigg(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f_n(y)\,dy \bigg)\,d\tau\,ds \bigg) \\
& - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \bigg(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f_n(\tau)\,d\tau \bigg)\,ds \Bigg] \Bigg\}.
\end{aligned}$$

As F has compact values, we can pass onto a subsequence (if necessary) to obtain that f_n converges to f in $L^1([0,1],\mathbb{R})$. So $f \in S_{F,u}$. Then, for each $t \in [0,1]$, we get

$$\begin{aligned} m_n(t) \to m(t) = {} & \frac{1}{A_1(\chi_2-\chi_1)\rho^{\alpha}\Gamma(\alpha)} \Biggl\{ \int_0^t w(t) \\ & \cdot \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1} f(\tau)\,d\tau \biggr)\,ds \\ & + \frac{\psi(t)}{\Omega} \biggl[\frac{1}{\rho^{\beta}\Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1} f(y)\,dy \biggr)\,d\tau\,ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1} f(y)\,dy \biggr)\,d\tau\,ds \biggr) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1} f(\tau)\,d\tau \biggr)\,ds \biggr] \Biggr\}, \end{aligned}$$

which implies that $m \in T(u)$.

Step 2. We show that there exists $0 < \Lambda < 1$ ($\Lambda = \|\varpi\|K$) satisfying

$$H_d\bigl(T(u),T(\hat{u})\bigr) \le \Lambda \|u-\hat{u}\| \quad \text{for each } u,\hat{u} \in \mathcal{C}.$$

Let us take $u,\hat{u} \in \mathcal{C}$ and $h_1 \in T(u)$. Then there exists $f_1(t) \in F(t,u(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) = {} & \frac{1}{A_1(\chi_2-\chi_1)\rho^{\alpha}\Gamma(\alpha)} \Biggl\{ \int_0^t w(t) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1} f_1(\tau)\,d\tau \biggr)\,ds \\ & + \frac{\psi(t)}{\Omega} \biggl[\frac{1}{\rho^{\beta}\Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1} f_1(y)\,dy \biggr)\,d\tau\,ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^{\tau} e^{\frac{\rho-1}{\rho}(\tau-y)}(\tau-y)^{\alpha-1} f_1(y)\,dy \biggr)\,d\tau\,ds \biggr) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\alpha-1} f_1(\tau)\,d\tau \biggr)\,ds \biggr] \Biggr\}. \end{aligned}$$

By (A_5), we have that $H_d(F(t,u),F(t,\hat{u})) \le \varpi(t)|u(t)-\hat{u}(t)|$. So, there exists $\nu(t) \in F(t,\hat{u}(t))$ satisfying $|f_1(t)-\nu| \le \varpi(t)|u(t)-\hat{u}(t)|$, $t \in [0,1]$.

Define $\mathcal{W} : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{W}(t) = \{ v \in \mathbb{R} : |f_1(t) - v| \le \varpi(t) |u(t) - \hat{u}(t)| \}.$$

As the multivalued operator $\mathcal{W}(t) \cap F(t, \hat{u}(t))$ is measurable by (Proposition III.4 [32]), we can find a function $f_2(t)$, which is a measurable selection for $\mathcal{W}$. So $f_2(t) \in F(t, \hat{u}(t))$, and for each $t \in [0,1]$, we have $|f_1(t) - f_2(t)| \le \varpi(t)|u(t) - \hat{u}(t)|$. For each $t \in [0,1]$, we define

$$\begin{aligned} h_2(t) = {} & \frac{1}{A_1(\chi_2 - \chi_1)\rho^\alpha \Gamma(\alpha)} \Biggl\{ \int_0^t w(t) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f_2(\tau)\, d\tau \biggr) ds \\ & + \frac{\psi(t)}{\Omega} \Biggl[\frac{1}{\rho^\beta \Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f_2(y)\, dy \biggr) d\tau\, ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} f_2(y)\, dy \biggr) d\tau\, ds \biggr) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} f_2(\tau)\, d\tau \biggr) ds \Biggr] \Biggr\}. \end{aligned}$$

As a result, we get

$$\begin{aligned} |h_1(t) - h_2(t)| = {} & \Biggl| \frac{1}{A_1(\chi_2 - \chi_1)\rho^\alpha \Gamma(\alpha)} \Biggl\{ \int_0^t w(t) \\ & \cdot \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} [f_1(\tau) - f_2(\tau)]\, d\tau \biggr) ds \\ & + \frac{\psi(t)}{\Omega} \Biggl[\frac{1}{\rho^\beta \Gamma(\beta)} \biggl(a_1 \int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1 - s)} (\eta_1 - s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} [f_1(y) - f_2(y)]\, dy \biggr) d\tau\, ds \\ & + a_2 \int_{\eta_2}^1 e^{\frac{\rho-1}{\rho}(1-s)} (1-s)^{\beta-1} \int_0^s w(s) \\ & \cdot \biggl(\int_0^\tau e^{\frac{\rho-1}{\rho}(\tau-y)} (\tau-y)^{\alpha-1} [f_1(y) - f_2(y)]\, dy \biggr) d\tau\, ds \biggr) \\ & - \sum_{i=1}^m \delta_i \int_0^{\xi_i} w(\xi_i) \biggl(\int_0^s e^{\frac{\rho-1}{\rho}(s-\tau)} (s-\tau)^{\alpha-1} [f_1(\tau) - f_2(\tau)]\, d\tau \biggr) ds \Biggr] \Biggr\} \Biggr| \\ \le {} & K \|\varpi\| \|u - \hat{u}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le K \|\varpi\| \|u - \hat{u}\|.$$

Analogously, we can interchange the roles of u and $\hat{u}$ to get

$$H_d\bigl(T(u), T(\hat{u})\bigr) \leq K\|\varpi\|\|u - \hat{u}\|,$$

which implies that T is a contraction by condition (4.2). Therefore, by the conclusion of the Covitz and Nadler fixed point theorem, T has a fixed point u, which corresponds to a solution of (1.2). □

Now, we will discuss the remaining two cases as follows.

Corollary 19 *Assume that* (A_1) *and* (A_2) *hold. In addition, we assume that*

(A_6) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{\|q\|g(M)Q} > 1.$$

Then problem (1.2) *with* $A_2^2 - 4A_1A_3 = 0$ *has at least one solution on* $[0,1]$.

Corollary 20 *Assume that* (A_4) *and* (A_5) *hold. Then problem* (1.2) *with* $A_2^2 - 4A_1A_3 = 0$ *has at least one solution on* $[0,1]$ *if*

$$\|\varpi\|Q < 1. \tag{4.3}$$

Corollary 21 *Assume that* (A_1) *and* (A_2) *hold. In addition, we assume that*

(A_7) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{\|q\|g(M)H} > 1.$$

Then problem (1.2) *with* $A_2^2 - 4A_1A_3 < 0$ *has at least one solution on* $[0,1]$.

Corollary 22 *Assume that* (A_4) *and* (A_5) *hold. Then problem* (1.2) *with* $A_2^2 - 4A_1A_3 < 0$ *has at least one solution on* $[0,1]$ *if*

$$\|\varpi\|H < 1. \tag{4.4}$$

The following examples show the applicability of inclusion results.

Example 3 Consider the following problem:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{5}{2},\frac{1}{4}}u(t) + 3\,{}^{C}D_{0^+}^{\frac{3}{2},\frac{1}{4}}u(t) + 2\,{}^{C}D_{0^+}^{\frac{1}{2},\frac{1}{4}}u(t) \in F(t,u(t)), \quad t \in [0,1], \\ u(0) = u'(0) = 0, \\ \sum_{i=1}^{2}\delta_i u(\xi_i) = \frac{1}{\rho^\beta\Gamma(\beta)}[a_1\int_0^{\eta_1} e^{\frac{\rho-1}{\rho}(\eta_1-s)}(\eta_1 - s)^{\beta-1}u(s)\,ds \\ \qquad + a_2\int_{\eta_2}^{1} e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\beta-1}u(s)\,ds], \end{cases} \tag{4.5}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$u \to F(t,u) = \left[\frac{u^4}{u^4+2} + t^2 + 5, \frac{u}{u+1} + t + 1\right]. \tag{4.6}$$

For $f \in F$, we have

$$|f| \leq \max\left[\frac{u^4}{u^4+2} + t^2 + 5, \frac{u}{u+1} + t + 1\right] \leq 7.$$

Here, $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, $\rho = \frac{1}{4}$, $\alpha = \beta = \frac{1}{2}$, $\delta_1 = 1$, $\delta_2 = 2$, $\eta_1 = \frac{1}{8}$, $\eta_2 = \frac{1}{4}$, $\xi_1 = \frac{1}{6}$, $\xi_2 = \frac{1}{5}$. Clearly,

$$\left\|F(t,u)\right\|_{\mathcal{P}} := \sup\left\{|x| : x \in F(t,u)\right\} \leq q(t)g\left(\|u\|\right), \quad u \in \mathbb{R},$$

with $q(t) = 1$, $g(\|u\|) = 7$. Using the given values, we find that $\Omega = 0.02217983$, $\Phi = 0.0059975$, and$K = 25.4438$. Thus,

$$M > \|q\|g(M)K \approx 178.1066.$$

Clearly, all the conditions of Theorem 17 are satisfied. So, there exists at least one solution of problem (4.5) on $[0,1]$.

Example 4 Consider the fractional inclusion boundary value problem (4.5) with $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$u \to F(t,u) = \left[0, \frac{1}{(t+6)^2}\left(\sin u + \frac{1}{15}\right)\right]. \tag{4.7}$$

Clearly, $H_d(F(t,u),F(t,\hat{u})) \leq \varpi(t)|u-\hat{u}|$, where $\varpi(t) = \frac{1}{(t+6)^2}$. Also $d(0,F(t,0)) \leq \varpi(t)$ for almost all $t \in [0,1]$ and

$$\|\varpi\|K \approx 0.706772222 < 1.$$

Since all the conditions of Theorem 18 hold, problem (4.5) with F given by (4.7) has at least one solution on $[0,1]$.

Acknowledgements
Not applicable.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
WS designed the research problem. WS, HZA, and ZA contributed to the implementation of the research, to the analysis of the results, and to the writing of the manuscript. All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 27 April 2020 Accepted: 5 November 2020 Published online: 18 November 2020

References

1. Podlubny, L.: Fractional Differential Equations. Academic Press, San Diego (1999)
2. Torvik, P.J., Bagley, R.L.: On the appearance of the fractional derivative in the behavior of real materials. J. Appl. Mech. **51**, 294–298 (1984)
3. Mainardi, F.: Some basic problems in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics, pp. 291–348. Springer, Berlin (1997)
4. Pedas, A., Tamme, E.: Piecewise polynomial collocation for linear boundary value problems of fractional differential equations. J. Comput. Appl. Math. **236**, 3349–3359 (2012)
5. ur Rehman, M., Khan, R.A.: A numerical method for solving boundary value problems for fractional differential equations. Appl. Math. Model. **36**, 894–907 (2012)
6. ur Rehman, M., Khan, R.A.: The Legendre wavelet method for solving fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **16**, 4163–4173 (2011)
7. Jafari, H., Yousefi, S.A., Firoozjaee, M.A., Momani, S., Khalique, C.M.: Application of Legendre wavelets for solving fractional differential equations. Comput. Math. Appl. **62**, 1038–1045 (2011)
8. Wang, Y., Fan, Q.: The second kind Chebyshev wavelet method for solving fractional differential equations. Appl. Math. Comput. **218**, 8592–8601 (2012)
9. Bhrawy, A.H., Alofi, A.S., Ezz-Eldien, S.S.: A quadrature tau method for fractional differential equations with variable coefficients. Appl. Math. Lett. **24**, 2146–2152 (2011)
10. Ahmad, B., Alghamdi, N., Alsaedi, A., Ntouyas, S.K.: On multi-term fractional differential equations with multi-point boundary conditions. Eur. Phys. J. Spec. Top. **226**, 3369–3390 (2017)
11. Ahmad, B., Alghamdi, N., Alsaedi, A., Ntouyas, S.K.: A system of coupled multi-term fractional differential equations with three-point coupled boundary conditions. Fract. Calc. Appl. Anal. **22**, 601–618 (2019)
12. Ahmad, B., Alsaedi, A., Alghamdi, N., Ntouyas, S.K., Alruwaily, Y., Salem, S.: Existence theorems for a coupled system of nonlinear multi-term fractional differential equations with nonlocal boundary conditions. Kragujev. J. Math. **46**, 317–331 (2019)
13. Alsaedi, A., Alghamdi, N., Ahmad, B., Ntouyas, S.K.: Nonlocal three-point multi-term multivalued fractional-order boundary value problems. Mat. Vesn. **72**, 66–80 (2020)
14. Ahmad, B., Agarwal, R.P.: Some new versions of fractional boundary value problems with slit-strips conditions. Bound. Value Probl. **2014**, 175 (2014)
15. Ahmad, B., Karthikeyan, P., Buvaneswari, K.: Fractional differential equations with coupled slit-strips type integral boundary conditions. AIMS Math. **4**(6), 1596–1609 (2019)
16. Lundqvist, M.: Silicon strip detectors for scanned multi-slit x-ray imaging. Kungl Tekniska Hogskolan, Stockholm (2003)
17. Mellow, T., Kärkkäinen, L.: On the sound fields of infinitely long strips. J. Acoust. Soc. Am. **130**, 153–167 (2011)
18. Akyildiz, F.T., Bellout, H., Vajravelu, K., Van Gorder, R.A.: Existence results for third order nonlinear boundary value problems arising in nano boundary layer fluid flows over stretching surfaces. Nonlinear Anal., Real World Appl. **12**, 2919–2930 (2011)
19. Agarwal, R.P., Alsaedi, A., Alghamdi, N., Ntouyas, S.K., Ahmad, B.: Existence results for multi-term fractional differential equations with nonlocal multi-point and multi-strip boundary conditions. Adv. Differ. Equ. **2018**, 342 (2018)
20. Shammakh, W., Alzumi, H.Z.: Existence results for nonlinear fractional boundary value problem involving generalized proportional derivative. Adv. Differ. Equ. **2019**, 94 (2019)
21. Wang, G., Qin, J., Zhang, L., Baleanu, D.: Explicit iteration to a nonlinear fractional Langevin equation with non-separated integro-differential strip-multi-point boundary conditions. CSF. 131 (2020)
22. Anderson, D.R., Ulness, D.J.: Newly defined conformable derivatives. Adv. Dyn. Syst. Appl. **10**, 109–137 (2015)
23. Anderson, D.R.: Second-order self-adjoint differential equations using a proportional-derivative controller. Commun. Appl. Nonlinear Anal. **24**, 17–48 (2017)
24. Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. Prog. Fract. Differ. Appl. **1**, 1–13 (2015)
25. Abdeljawad, T., Baleanu, D.: On fractional derivatives with exponential kernel and their discrete versions. Rep. Math. Phys. **80**, 11–27 (2017)
26. Atangana, A., Baleanu, D.: New fractional derivative with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm. Sci. **18**, 1–10 (2016)
27. Jarad, F., Abdeliawad, T., Alzabut, J.: Generalized fractional derivatives generated by a class of local proportional derivatives. Eur. Phys. J. Spec. Top. **226**, 3457–3471 (2017)
28. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
29. Granas, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003)
30. Lasota, A., Opial, Z.: An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13**, 781–786 (1965)
31. Covitz, H., Nadler, S.B.: Multi-valued contraction mappings in generalized metric spaces. Isr. J. Math. **8**, 5–11 (1970)
32. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, vol. 580. Springer, Berlin (1977)