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Numerical solution of certain Cauchy singular integral equations using a collocation scheme

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Abstract

The present study is devoted to developing a computational collocation technique for solving the Cauchy singular integral equation of the second kind (CSIE-2). Although, several studies have investigated the numerical approximation solution of CSIEs, the strong singularity and accuracy of the numerical methods are still two important challenges for these integral equations. In this paper, we focus on the smooth transformation and implementation of Bessel basis polynomials (BBP). The reduction of the CSIEs-2 into a system of algebraic equations with the Gauss–Legendre collocation points simplifies this technique. The technique of performing numerical approximation of the solution is well presented and illustrated in the matrix form. Also, the convergence and error bound associated with the scheme are established. Finally, several experiments show the reliability and numerical efficiency of the proposed scheme in comparison with other methods.

Keywords: Collocation scheme; Cauchy singular integral equation; Bessel polynomial

1 Introduction

The theory of integral equations is one of the most important topics in applied mathematics and numerical analysis. Also, the singularity of the kernel is an important issue in the classification of integral equations. Applying the nonsingular kernel derivatives, Riemann–Liouville fractional integrals, Riemann–Liouville and Caputo fractional derivatives, and AB derivatives in the study of the behavior of fractional differential models leads to the singular integral equations [1–5]. Singular integral equations with Cauchy kernels have many applications in a wide variety of physics and engineering fields like airfoils, contact radiations, fracture mechanics, molecular conductions, and elastodynamics [6–8]. Since it is very difficult to find analytical solutions of integral equations with weak or strong singularity, many researchers have been developing several numerical methods with significant accuracy to solve these equations [9–11].

Cauchy-type kernel of the singular integral equation is defined by

$$\lambda x(s) + \mu \int_{-1}^1 \frac{x(t)}{t-s} dt = \xi(s), \quad s \in (-1, 1), \quad (1)$$

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where $\xi(s)$ is a given function, λ, μ are constants, and $x(s)$ is an unknown function. The CSIEs have been solved via various numerical techniques such as using orthogonal Legendre polynomial [6], Lagrangian interpolation with Gauss–Jacobi mechanical quadrature [8], spline method [12, 13], Galerkin technique [14], collocation method [15–17], application of Jacobi polynomials [18], using Chebyshev polynomials of the second kind [19], quadrature formula [20–22], reproducing kernel Hilbert space method [23, 24], and other schemes [25–27]. Recently, several types of operational matrix methods with truncated series have been proposed for solving the integral and integro-differential equations (see [16, 28]).

To obtain an approximate solution for the CSIEs-2, it is necessary to eliminate or weaken singularity by applying a smooth transformation. Hence, we use a smooth transformation in this article as well. The main part of this work deals with the use of Legendre polynomial roots as collocation points and the operational matrix approach. The essential focus of the scheme is the use of the numerical method designed based on Bessel basis functions for the CSIEs-2. Determining the theoretical error bound for the two functions of the exact and approximate solutions showed the convergence of the proposed method. The tested functions as the exact and approximate solutions in the numerical examples confirm the accuracy and efficiency of this method.

The rest of this work is outlined as follows: Sect. 2 discusses the Bessel basis polynomials and removing singularity of Eq. (1) by a suitable smooth transformation. In Sect. 3, we present the computational matrix approach for solving CSIEs-2. Error estimation and convergence analysis are given in Sect. 4. Some numerical experiments with graphical results are provided in Sect. 5. Finally, a brief conclusion is given in Sect. 6.

2 Preliminaries

2.1 The Bessel polynomials

The Bessel polynomials of the first kind and order i are defined by the truncated series

$$J_i(s) = \left(\frac{s}{2}\right)^i \sum_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} \frac{(-1)^j s^{2j}}{j!(j+i)!4^j}, \quad i = 0, 1, 2, \dots, n, \tag{2}$$

which is convergent absolutely and uniformly in $[-1, 1]$.

2.2 Construction approximation of Bessel basis function

Let $s_k; k = 0, 1, \dots, n$, for any positive integer n , denote the $n + 1$ roots of $p_n(s)$ where $p_n(s)$ is the Legendre polynomial of order n . On the other hand, assume that $J_i(s)$ can be expressed by BBF as follows:

$$f(s) = \sum_{i=0}^{\infty} \psi(s) J_i(s). \tag{3}$$

If the infinite series (3) is truncated for $i = n$, then the approximation of (3) can be represented in the following form:

$$f(s) = \sum_{i=0}^{\infty} \psi(s) J_i(s) \simeq \sum_{i=0}^n \psi(s) J_i(s). \tag{4}$$

By the operational matrices, we have

$$f(s) \simeq \Psi^T J(s), \tag{5}$$

where

$$\Psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_n \end{bmatrix} \quad \text{and} \quad J(s) = \begin{bmatrix} J_0(s) \\ J_1(s) \\ \vdots \\ J_n(s) \end{bmatrix}.$$

2.3 Removing singularity of Eq. (1)

For solving Eq. (1), it is clear that we need to use an equivalent presentation of Eq. (1). We suggest to weaken the singularity of the integral part by the following technique:

$$\int_{-1}^1 \frac{x(t)}{t-s} dt = \int_{-1}^1 \frac{x(t) - x(s)}{t-s} dt + \int_{-1}^1 \frac{x(s)}{t-s} dt, \tag{6}$$

where $\int_{-1}^1 \frac{x(s)}{t-s} dt = x(s) \ln\left(\frac{1-s}{1+s}\right)$ [23]. By applying this separation to the right part of the integral term of Eq. (1), the equivalent smooth form of Eq. (1) is as follows:

$$\lambda x(s) + \mu \int_{-1}^1 \frac{x(t) - x(s)}{t-s} dt + \mu x(s) \ln\left(\frac{1-s}{1+s}\right) = \xi(s), \quad s \in (-1, 1), \tag{7}$$

where $\frac{x(t)-x(s)}{t-s} = x'(s)$, when $t \rightarrow s$. Note that

$$\lim_{s \rightarrow -1,1} \ln\left(\frac{1-s}{1+s}\right) = \infty,$$

therefore, for defining Eq. (7) at two endpoints of $[-1, 1]$, we consider $x(-1) = x(1) = 0$ and

$$\lim_{s \rightarrow -1,1} x(s) \ln\left(\frac{1-s}{1+s}\right) = 0.$$

3 Description of the method

3.1 Construction of the residual operator

In this section, we extend the collocation scheme for solving the CSIE-2. First, consider the operation form of Eq. (1)

$$\chi(x) = \zeta, \tag{8}$$

where χ is in the Banach space S . Assume that a sequence of finite dimensional subspace $S_n \subset S$, $n \geq 1$, has dimension $n + 1$. Choose S_n that has a basis $J = \{J_0(s), J_1(s), \dots, J_n(s)\}$ in S . Assume $x_n(s) \in S_n$, which is the best approximate of $x(s)$ such that

$$x_n(s) = \sum_{i=0}^n x_i J_i(s), \quad s \in [-1, 1]. \tag{9}$$

So, we can approximate the unknown function $x(s)$ in terms of BBF $J_i(s)$ as follows:

$$x(s) \simeq \sum_{i=0}^n x_i J_i(s) = X^T J(s), \tag{10}$$

where $X = [x_0, x_1, \dots, x_n]^T$ is the unknown coefficients matrix. After applying the smooth transformation, we substitute the main equation Eq. (7) into (1). So, by placing Eq. (7) into (1), the unknown Bessel coefficients $\{x_i \mid i = 0, 1, \dots, n\}$ are determined by forcing the equation to be accurate in some sense. However, we conduct it by the operational matrix. The residual function $\tau_n(s)$ is obtained in approximating Eq. (7) by substituting $x(s)$ with $x_n(s)$

$$\tau_n(s) = \lambda \sum_{i=0}^n x_i J_i(s) + \mu \int_{-1}^1 \sum_{i=0}^n x_i \frac{J_i(t) - J_i(s)}{t - s} dt + \mu \sum_{i=0}^n x_i J_i(s) \ln\left(\frac{1-s}{1+s}\right) - \xi(s) \tag{11}$$

and it can be demonstrated by the operational matrices

$$\tau_n(s) = \lambda X^T J(s) + \mu X^T \tilde{J}(t, s) + \mu X^T J(s) \ln\left(\frac{1-s}{1+s}\right) - \xi(s), \tag{12}$$

where

$$\tilde{J}(t, s) = \begin{bmatrix} \int_{-1}^1 \frac{J_0(t) - J_0(s)}{t - s} dt \\ \int_{-1}^1 \frac{J_1(t) - J_1(s)}{t - s} dt \\ \vdots \\ \int_{-1}^1 \frac{J_n(t) - J_n(s)}{t - s} dt \end{bmatrix}.$$

The symbolic presentation of (11) is

$$\tau_n(s) = \chi(x_n(s)) - \zeta(s), \tag{13}$$

or with the Bessel series

$$\tau_n(s) = \chi\left(\sum_{i=0}^n x_i J_i(s)\right) - \zeta(s). \tag{14}$$

3.2 Construction of the operational fundamental matrices

Let us set $\tau_n(s)$ to be zero approximately, so the unknown Bessel coefficients can be approximated. We peek distinct $n + 1$ roots of the Gauss–Legendre $s_0, s_1, \dots, s_n \in [-1, 1]$, and by using the collocation scheme, we consider the zero value of the residual function for these separate points

$$\tau_n(s_j) = 0, \quad j = 0, 1, \dots, n, \tag{15}$$

therefore

$$\chi(x_n(s_j)) = \zeta(s_j) \tag{16}$$

and

$$\chi \left(\sum_{i=0}^n x_i J_i(s_j) \right) = \zeta(s_j). \tag{17}$$

For determining $\{x_i | i = 0, 1, \dots, n\}$, we have to solve the system of linear Fredholm integral equations

$$\lambda \sum_{i=0}^n x_i J_i(s_j) + \mu \int_{-1}^1 \sum_{i=0}^n x_i \frac{J_i(t) - J_i(s_j)}{t - s_j} dt + \mu \sum_{i=0}^n x_i J_i(s_j) \ln \left(\frac{1 - s_j}{1 + s_j} \right) = \xi(s_j), \tag{18}$$

$$s_j \in [-1, 1], \quad j = 0, \dots, n,$$

or we have to solve the operational matrix equation

$$\lambda X^T \bar{J} + \mu X^T \bar{\bar{J}} + \mu X^T \bar{J} L = Z, \tag{19}$$

where

$$\bar{J} = \begin{bmatrix} J_0(s_0) & J_0(s_1) & \dots & J_0(s_n) \\ J_1(s_0) & J_1(s_1) & \dots & J_1(s_n) \\ \vdots & \vdots & \ddots & \vdots \\ J_n(s_0) & J_n(s_1) & \dots & J_n(s_n) \end{bmatrix}, \tag{20}$$

$$\bar{\bar{J}} = \begin{bmatrix} \int_{-1}^1 \frac{J_0(t) - J_0(s_0)}{t - s_0} dt & \int_{-1}^1 \frac{J_0(t) - J_0(s_1)}{t - s_1} dt & \dots & \int_{-1}^1 \frac{J_0(t) - J_0(s_n)}{t - s_n} dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_{-1}^1 \frac{J_n(t) - J_n(s_0)}{t - s_0} dt & \int_{-1}^1 \frac{J_n(t) - J_n(s_1)}{t - s_1} dt & \dots & \int_{-1}^1 \frac{J_n(t) - J_n(s_n)}{t - s_n} dt \end{bmatrix},$$

and

$$L = \begin{bmatrix} \ln \left(\frac{1 - s_0}{1 + s_0} \right) & 0 & \dots & 0 \\ 0 & \ln \left(\frac{1 - s_1}{1 + s_1} \right) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \ln \left(\frac{1 - s_n}{1 + s_n} \right) \end{bmatrix}, \quad Z = [\xi(s_0) \quad \xi(s_1) \quad \dots \quad \xi(s_n)]. \tag{21}$$

Since the integral term of Eq. (18) cannot be solved analytically, we suggest using one of the quadrature rules such as the Gaussian–Legendre formula for approximating the value integrals or the entries of the matrix $\bar{\bar{J}}$ of relation (20). By applying the quadrature rule, Eq. (18) is converted into

$$\sum_{i=0}^n x_i \left(\lambda J_i(s_j) + \mu \left(\sum_{k=1}^m \omega_k \frac{J_i(t_k) - J_i(s_j)}{t_k - s_j} \right) + \mu J_i(s_j) \ln \left(\frac{1 - s_j}{1 + s_j} \right) \right) = \xi(s_j), \tag{22}$$

$$s_j \in [-1, 1], \quad j = 0, \dots, n,$$

with m Gauss nodes such that t_k is the k th root of Legendre polynomial $P_{n+1}(s)$ and the weight functions are given by $\omega_k = \frac{2}{(1 - t_k)^2 (P'_n(t_k))^2}$. Using this quadrature rule, the matrix

form of Eq. (19) does not change, while \bar{J} could be written as

$$\bar{J} = \begin{bmatrix} \sum_{k=1}^m \omega_k \frac{J_0(t_k) - J_0(s_0)}{t_k - s_0} & \sum_{k=1}^m \omega_k \frac{J_0(t_k) - J_0(s_1)}{t_k - s_1} & \dots & \sum_{k=1}^m \omega_k \frac{J_0(t_k) - J_0(s_n)}{t_k - s_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^m \omega_k \frac{J_n(t_k) - J_n(s_0)}{t_k - s_0} & \sum_{k=1}^m \omega_k \frac{J_n(t_k) - J_n(s_1)}{t_k - s_1} & \dots & \sum_{k=1}^m \omega_k \frac{J_n(t_k) - J_n(s_n)}{t_k - s_n} \end{bmatrix}. \tag{23}$$

However, the linear equations in relation (22) and matrix equation (19) are the same and are generally written as the following

$$X^T(\lambda \bar{J} + \mu \bar{J} + \mu \bar{J}L) = Z. \tag{24}$$

Note that, by considering $\lambda \bar{J} + \mu \bar{J} + \mu \bar{J}L = Y$, we have

$$X^T Y = Z. \tag{25}$$

Using the previously described process, Eq. (7) is changed to matrix equation (25). If $\det(Y) \neq 0$, then Eq. (25) has a unique solution, and we can obtain the coefficient matrix by

$$X^T = ZY^{-1}. \tag{26}$$

After determining the Bessel coefficients $\{x_i \mid i = 0, 1, \dots, n\}$ and by substituting $x_i, i = 0, 1, \dots, n$, in Eq. (9), the solution function $x(s)$ can be efficiently approximated.

4 Error estimation analysis

Here, we explain the convergence analysis and error bound of the approximation solutions of the presented scheme in Sect. 3 for solving Eq. (1).

Lemma 1 ([29]) *Suppose that s_0, s_1, \dots, s_n are $n + 1$ distinct Legendre roots in $[-1, 1]$, and n is a positive integer. If we approximate the function $x(s) \in C^{n+1}[-1, 1]$ by BBF $J(s)$ and $x(s)$ is a sufficiently smooth function on $[-1, 1]$, then there exists $\eta \in (s_0, s_n)$ such that*

$$x(s) - x_n(s) = \frac{x^{(n+1)}(s)}{2^{2n+1}(n+1)!} (s - s_0)(s - s_1) \dots (s - s_n), \quad s \in [-1, 1], \tag{27}$$

where $x_n(s)$ is the approximation of $x(s)$ by BBF. Now consider

$$M_1 = \max_{-1 \leq \eta \leq 1} |x^{(n+1)}(\eta)| \quad \text{and} \quad M = \max_{-1 \leq s \leq 1} \left| \prod_{i=0}^n (s - s_i) \right|,$$

then

$$|x(s) - x_n(s)| \leq \frac{M_1 M}{2^{2n+1}(n+1)!}. \tag{28}$$

Lemma 2 *Suppose that $x(s)$ is a sufficiently smooth function. Let χ_n, χ and A be linear operators on $L^2[-1, 1]$ and defined for $s \in [-1, 1]$ as follows:*

$$\chi_n(x(s)) = \lambda x(s) + \mu \sum_{k=1}^m \frac{\omega_k x(t_k) - \omega_k x(s)}{t_k - s} + \mu x(s) \ln \left(\frac{1-s}{1+s} \right), \tag{29}$$

where ω_k ; $k = 0, 1, \dots, m$, are the weight functions and t_k , $k = 0, 1, \dots, m$, are the abscissas numbers of the Gauss–quadrature. If the Gauss–quadrature rule is changed into the real–valued integral term, then we have

$$\chi(x(s)) = \lambda x(s) + \mu \int_{-1}^1 \frac{x(t) - x(s)}{t - s} dt + \mu x(s) \ln\left(\frac{1 - s}{1 + s}\right), \tag{30}$$

where $\chi(x(s)) = \zeta(s)$. On the other hand, we can show Eq. (30) by the equivalent integral equation

$$\chi(x(s)) = \lambda x(s) + \mu \int_{-1}^1 \frac{x(t) - x(s)}{t - s} dt + \mu \int_{-1}^1 \frac{x(s)}{t - s} dt, \tag{31}$$

or

$$\chi(x(s)) = \lambda x(s) + \mu \int_{-1}^1 \frac{x(t)}{t - s} dt, \tag{32}$$

where $\chi(x) = \zeta$, and by using linear operator A , we have

$$\chi(x(s)) = \lambda x(s) + \mu A(x(s)). \tag{33}$$

Theorem 3 Suppose that the unknown function $x(s)$ is $n + 1$ -times continuously differentiable on the interval $[-1, 1]$ and $\bar{x}_n(s) = \sum_{i=0}^n \bar{x}_i J_i(s)$ is the expansion of the exact solution $x(s)$ with respect to the basis functions in J . Let $x_n(s) = \sum_{i=0}^n x_i J_i(s)$ be the approximate solution obtained by the purposed scheme in Sect. 3 and $M_1 = \max_{s \in [-1, 1]} |x^{(n+1)}(s)|$, $M = \max_{-1 \leq s \leq 1} |\prod_{i=0}^n (s - s_i)|$, then

$$\|E_n(s)\|_2 \leq (\Omega_1 + \Omega_2) \left(\epsilon_1 \frac{M_1 M}{2^{2n+1}(n+1)!} + \epsilon_2 \|x - \bar{x}\| \right). \tag{34}$$

Proof From the notation mentioned in Eqs. (32) and (33), the exact and approximate solutions of Eq. (1) can be written as follows:

$$\begin{aligned} \chi(x(s)) &= \lambda x(s) + \mu \int_{-1}^1 \frac{x(t)}{t - s} dt, \\ \chi(x_n(s)) &= \lambda x_n(s) + \mu \int_{-1}^1 \frac{x_n(t)}{t - s} dt, \\ \chi(x(s)) &= \lambda x(s) + \mu A(x(s)), \\ \chi(x_n(s)) &= \lambda x_n(s) + \mu A(x_n(s)), \end{aligned} \tag{35}$$

so

$$\begin{aligned} \frac{1}{\lambda} \chi(x(s)) &= x(s) + \frac{\mu}{\lambda} A(x(s)), \\ \frac{1}{\lambda} \chi(x_n(s)) &= x_n(s) + \frac{\mu}{\lambda} A(x_n(s)). \end{aligned} \tag{36}$$

From Eq. (36), we get

$$E_n(s) = \frac{1}{\lambda} \chi(x_n(s) - x(s)) - \frac{\mu}{\lambda} A(x_n(s) - x(s)). \tag{37}$$

Since χ and A are bounded, there exist two constants Ω_1 and Ω_2 such that

$$\|E_n(s)\|_2 = (\Omega_1 + \Omega_2)\|x_n(s) - x(s)\|_2.$$

Therefore

$$\begin{aligned} \|x(s) - x_n(s)\|_2 &= \|x(s) - \bar{x}_n(s) + \bar{x}_n(s) - x_n(s)\|_2 \\ &\leq \|x(s) - \bar{x}_n(s)\|_2 + \|\bar{x}_n(s) - x_n(s)\|_2 \\ &\leq \left(\int_{-1}^1 \left(\frac{M_1 M}{2^{2n+1}(n+1)!} \right)^2 ds \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{-1}^1 \left(\sum_{i=0}^n (\bar{x}_i - x_i) J_i(s) \right)^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{M_1 M}{2^{2n+1}(n+1)!} \left(\int_{-1}^1 ds \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{-1}^1 \left(\sum_{i=0}^n |\bar{x}_i - x_i|^2 \right) ds \right)^{\frac{1}{2}} \left(\sum_{i=0}^n \int_{-1}^1 |J_i(s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2} M_1 M}{2^{2n+1}(n+1)!} + \epsilon_2 \|\bar{x} - x\|. \end{aligned} \tag{38}$$

Now, we have

$$\|E_n(s)\|_2 \leq (\Omega_1 + \Omega_2) \left(\frac{\epsilon_1 M_1 M}{2^{2n+1}(n+1)!} + \epsilon_2 \|\bar{x} - x\| \right). \tag{39}$$

5 Numerical experiments

In order to illustrate the efficiency and accuracy of the scheme presented in Sect. 3, four numerical experiments are presented by testing the functional solutions. By considering the smooth transform and nonuniform collocation points, the following experiments are given for solving the CSIEs-2. These numerical results are computed, compared, and plotted using a program written in Mathematica 12.0. The absolute error $|E_n(s)|$ can be defined by $|E_n(s)| = |x(s) - x_n(s)|$. Graphs and tables of absolute errors for different values of n are plotted in various examples. By changing the values of m , the error also changes, which is evident in the examples. The absolute errors of two numerical examples can be compared with the results of the well-known method [17].

Example 1 ([17]) Consider the Cauchy singular integral equation Eq. (1) with

$$\int_{-1}^1 \frac{x(t)}{t-s} dt = \frac{4}{3} - 2s^2 + (s-s^3) \ln((1-s)/(1+s)).$$

The exact solution is $x(s) = -s^3 + s$.

This example is solved for values of $n = 3, 5, 7$ and $m = 16$. Since the exact solution is a polynomial of degree 3, then the best value for n is using a Bessel basis polynomial of degree $n \geq 3$. The results well indicate the high accuracy of the method based on this

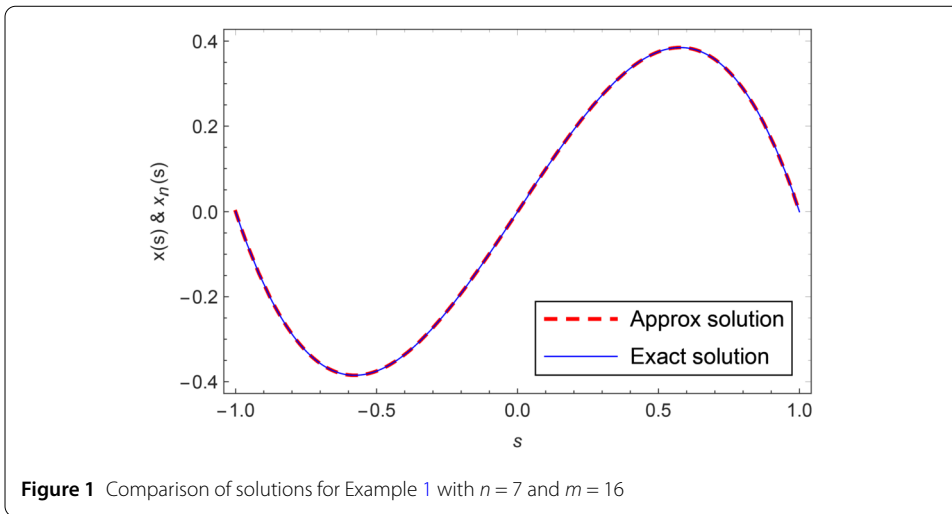


Figure 1 Comparison of solutions for Example 1 with $n = 7$ and $m = 16$

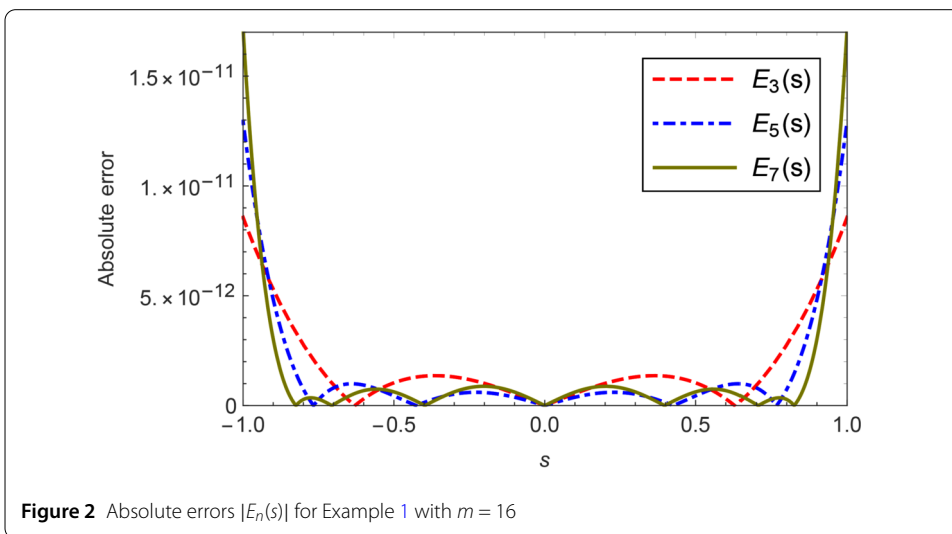


Figure 2 Absolute errors $|E_n(s)|$ for Example 1 with $m = 16$

important polynomial. The graph of the solution functions in Fig. 1, the figures of absolute errors for $n = 3, 5, 7$ and $m = 16$ in Fig. 2, and Table 1 show the efficiency of the current method BBF. In Fig. 2, we can say that by increasing the number of roots of the Legendre polynomials as the collocation points, the maximum absolute error is increased.

Example 2 Consider the CSIE-2

$$\lambda x(s) + \mu \int_{-1}^1 \frac{x(t)}{t-s} dt = \xi(s), \quad s \in (-1, 1),$$

with $\lambda = \mu = 1$, $\xi(s) = \sinh(s) - \text{Chi}(-s - 1) \sinh(s) + \text{Chi}(1 - s) \sinh(s) + \ln(-1 - s) \sinh(s) - \ln(1 - s) \sinh(s) + \ln\left(\frac{-(-1+s)}{(1+s)}\right) \sinh(s) + \cosh(s) \text{Shi}(1 - s) + \cosh(s) \text{Shi}(s + 1)$, and the exact solution is $x(s) = \sinh(s)$.

Here, we test the performance of the discussed method in Sect. 3 on an example with a non-polynomial exact solution. In this example, we obtained absolute errors for $n = 3, 5, 7$ and $m = 16$, where we see the difference and improvement of the approximate solutions

Table 1 Absolute error for Example 1

Node	Exact solution	Absolute error		
		$n = 3$ and $m = 16$	$n = 5$ and $m = 16$	$n = 7$ and $m = 16$
-1.0	0.0	8.57103×10^{-12}	1.29419×10^{-11}	1.71944×10^{-11}
-0.9	-0.171	5.28733×10^{-12}	4.82814×10^{-12}	3.02805×10^{-12}
-0.8	-0.288	2.76978×10^{-12}	7.42573×10^{-13}	2.76779×10^{-13}
-0.7	-0.357	9.33587×10^{-13}	8.03413×10^{-13}	4.52971×10^{-14}
-0.6	-0.384	3.06755×10^{-13}	9.32199×10^{-13}	6.80345×10^{-13}
-0.5	-0.375	1.03606×10^{-12}	4.42868×10^{-13}	6.23890×10^{-13}
-0.4	-0.336	1.33976×10^{-12}	1.39888×10^{-13}	1.88183×10^{-14}
-0.3	-0.273	1.30285×10^{-12}	5.21860×10^{-13}	6.22891×10^{-13}
-0.2	-0.192	1.01039×10^{-12}	5.92443×10^{-13}	8.7802×10^{-13}
-0.1	-0.099	5.47729×10^{-13}	3.78711×10^{-13}	6.14328×10^{-13}
0.0	0.0	7.83203×10^{-17}	3.64086×10^{-17}	9.54481×10^{-17}
0.1	0.099	5.47895×10^{-13}	3.78794×10^{-13}	6.14148×10^{-13}
0.2	0.192	1.01058×10^{-12}	5.92471×10^{-13}	8.77770×10^{-13}
0.3	0.273	1.30296×10^{-12}	5.21916×10^{-13}	6.22558×10^{-13}
0.4	0.336	1.33993×10^{-12}	1.39944×10^{-13}	1.91513×10^{-14}
0.5	0.375	1.03628×10^{-12}	4.42868×10^{-13}	6.24223×10^{-13}
0.6	0.384	3.06977×10^{-13}	9.32199×10^{-13}	6.80678×10^{-13}
0.7	0.357	9.33309×10^{-13}	8.03468×10^{-13}	4.56302×10^{-14}
0.8	0.288	2.76945×10^{-12}	7.42573×10^{-13}	2.76501×10^{-13}
0.9	0.171	5.28688×10^{-12}	4.82803×10^{-12}	3.02883×10^{-12}
1.0	0.0	8.57059×10^{-12}	1.29418×10^{-11}	1.71959×10^{-11}

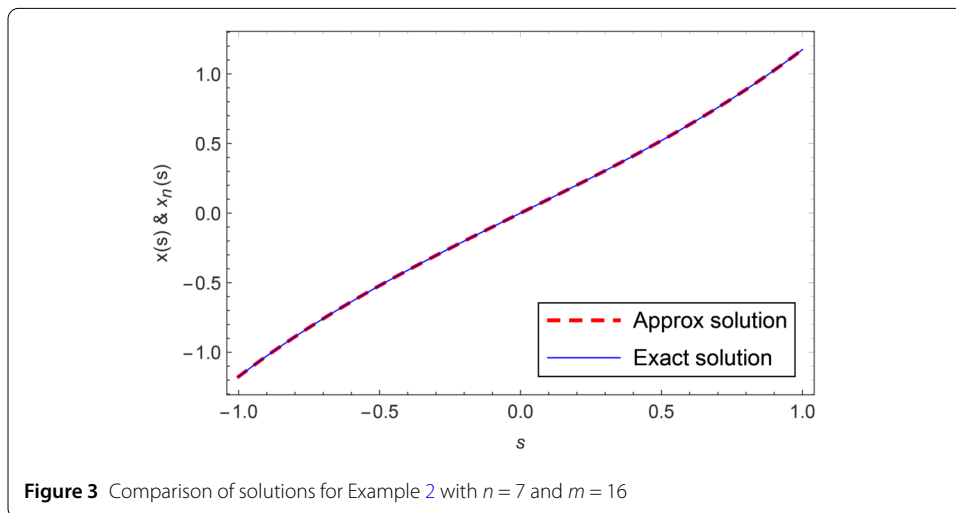


Figure 3 Comparison of solutions for Example 2 with $n = 7$ and $m = 16$

by increasing n . The exact and approximate solution are plotted in Fig. 3. The obtained absolute errors in this scheme are described in Fig. 4 and Table 2. Here, it is well seen that by increasing n , the obtained absolute error decreases and the approximate solution becomes more accurate.

Example 3 Consider the CSIE-2 Eq. (1) with

$$\begin{aligned} \xi(s) = & -\text{Ci}(-s-1)\cos(s) + \text{Ci}(1-s)\cos(s) - \text{Si}(1-s)\sin(s) - \text{Si}(s+1)\sin(s) \\ & + \cos(s) + \ln(-s-1)\cos(s) - \ln(1-s)\cos(s) + \ln\left(-\frac{s-1}{s+1}\right)\cos(s), \end{aligned}$$

with $\lambda = \mu = 1$, and the exact solution $x(s) = \cos(s)$.

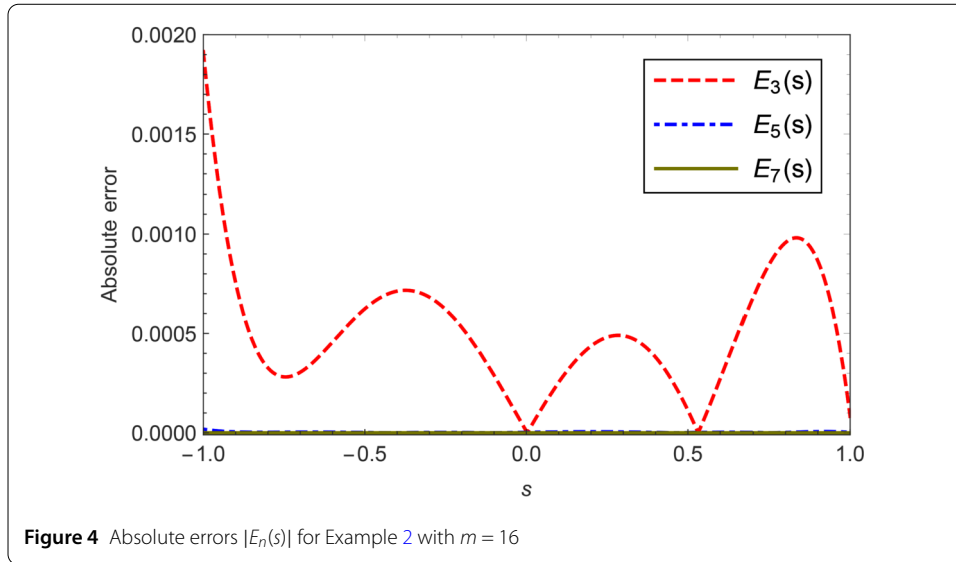


Table 2 Absolute error for Example 2

Node	Exact solution	Absolute error	Absolute error	Absolute error
		$n = 3$ and $m = 16$	$n = 5$ and $m = 16$	$n = 7$ and $m = 16$
-1.0	-1.17520	1.91441×10^{-3}	1.81254×10^{-5}	8.91703×10^{-8}
-0.9	-1.02652	7.55784×10^{-4}	4.56592×10^{-6}	1.21329×10^{-8}
-0.8	-0.888106	3.26831×10^{-4}	2.70516×10^{-6}	1.01281×10^{-8}
-0.7	-0.758584	3.04940×10^{-4}	3.37668×10^{-6}	7.32927×10^{-9}
-0.6	-0.636654	4.56464×10^{-4}	3.09250×10^{-6}	2.74983×10^{-10}
-0.5	-0.521095	6.23736×10^{-4}	1.48584×10^{-6}	1.38631×10^{-9}
-0.4	-0.410752	7.12864×10^{-4}	4.85742×10^{-7}	4.69581×10^{-9}
-0.3	-0.30452	6.82151×10^{-4}	1.69887×10^{-6}	1.28989×10^{-8}
-0.2	-0.201336	5.31044×10^{-4}	1.50059×10^{-6}	1.59589×10^{-8}
-0.1	-0.100167	2.89494×10^{-4}	6.72324×10^{-8}	1.08082×10^{-8}
0.0	0.0	7.61784×10^{-6}	2.34882×10^{-6}	5.43884×10^{-10}
0.1	0.100167	2.54435×10^{-4}	4.35718×10^{-6}	7.79669×10^{-9}
0.2	0.201336	4.36513×10^{-4}	5.15723×10^{-6}	7.95496×10^{-9}
0.3	0.30452	4.88499×10^{-4}	4.24896×10^{-6}	1.03862×10^{-9}
0.4	0.410752	3.80444×10^{-4}	1.85014×10^{-6}	1.32656×10^{-8}
0.5	0.521095	1.12900×10^{-4}	1.02282×10^{-6}	1.89303×10^{-8}
0.6	0.636654	2.72436×10^{-4}	2.77923×10^{-6}	1.16573×10^{-8}
0.7	0.758584	6.81671×10^{-4}	1.89035×10^{-6}	3.30044×10^{-9}
0.8	0.888106	9.57141×10^{-4}	1.95203×10^{-6}	7.44314×10^{-9}
0.9	1.02652	8.65196×10^{-4}	6.03881×10^{-6}	1.32789×10^{-8}
1.0	1.17520	8.32275×10^{-5}	2.08619×10^{-6}	1.23370×10^{-8}

The behaviors of the exact and numerical solution with $n = 10$ and $m = 10$ are being compared in Fig. 5. Table 3 and Fig. 6 report the obtained errors for $n = 8, 10$ and $m = 10$. Note that the computational numerical results improve by adding the value of n .

Example 4 ([17]) Consider the Cauchy singular integral equation Eq. (1) with

$$\int_{-1}^1 \frac{x(t)}{t-s} dt = \frac{1}{3} + \frac{s}{2} + s^2 + s^3 \ln((1-s)/s).$$

The exact solution is $x(s) = s^3$.

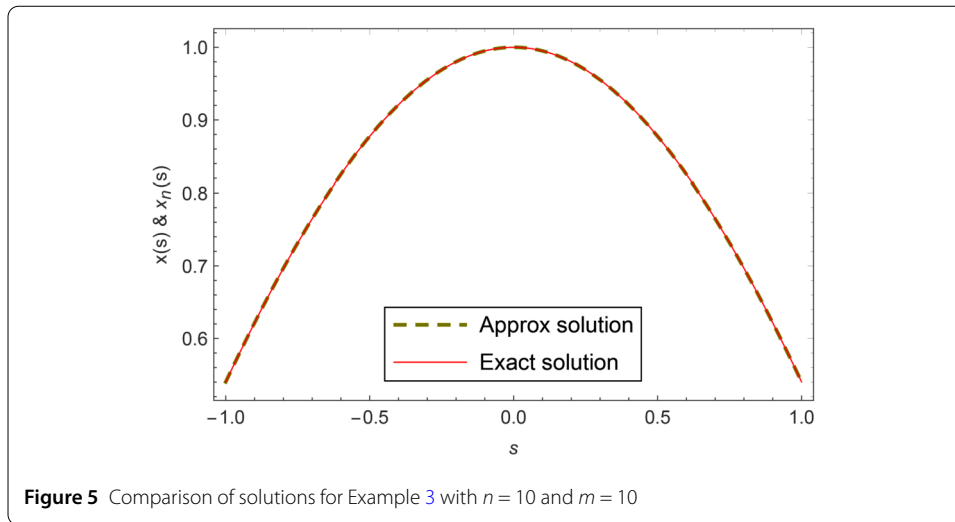


Figure 5 Comparison of solutions for Example 3 with $n = 10$ and $m = 10$

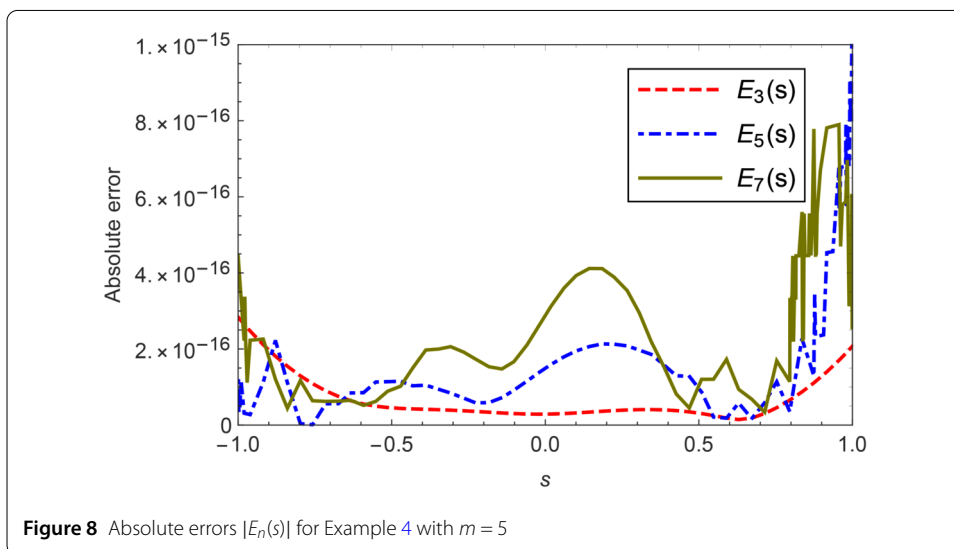
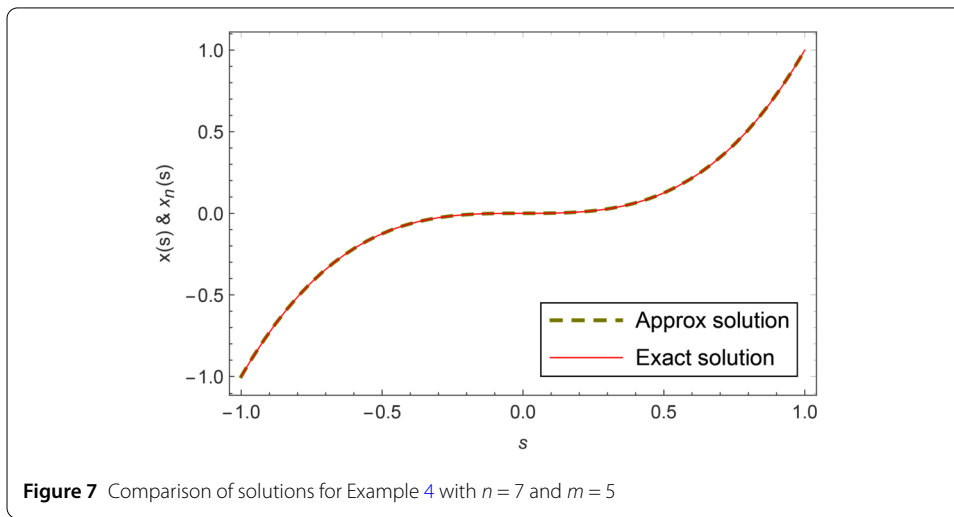
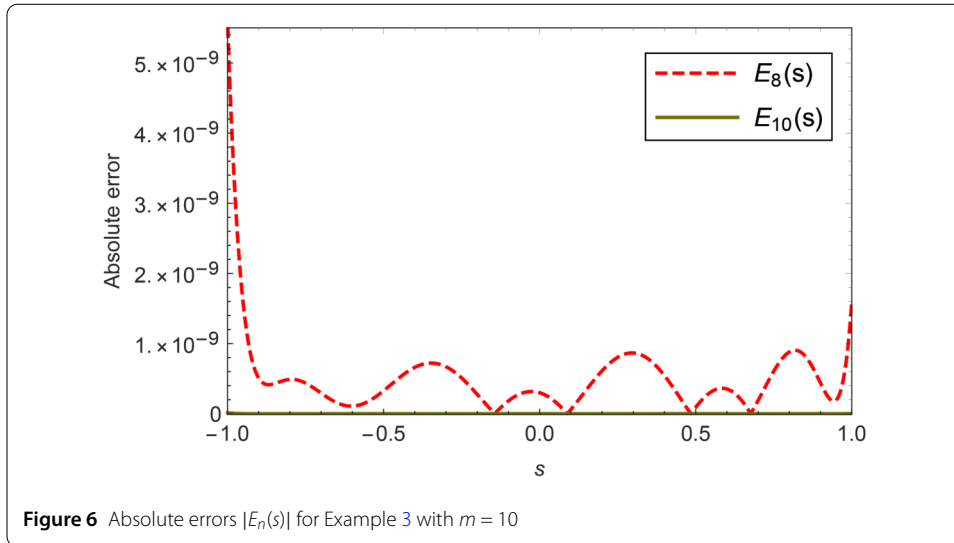
Table 3 Absolute error for Example 3

Node	Exact solution	Absolute error	
		$n = 8$ and $m = 10$	$n = 10$ and $m = 10$
-1.0	0.540302	5.63786×10^{-09}	1.3509×10^{-11}
-0.9	0.62161	5.08218×10^{-10}	6.73017×10^{-13}
-0.8	0.696707	4.88729×10^{-10}	9.73333×10^{-13}
-0.7	0.764842	2.94039×10^{-10}	5.64326×10^{-13}
-0.6	0.825336	1.09171×10^{-10}	1.02507×10^{-12}
-0.5	0.877583	3.27639×10^{-10}	1.17395×10^{-12}
-0.4	0.921061	6.6095×10^{-10}	3.34288×10^{-13}
-0.3	0.955336	6.70017×10^{-10}	3.69038×10^{-13}
-0.2	0.980067	2.78499×10^{-10}	8.53762×10^{-14}
-0.1	0.995004	1.83289×10^{-10}	1.17184×10^{-12}
0.0	1.0	2.9988×10^{-10}	1.51223×10^{-12}
0.1	0.995004	6.01889×10^{-11}	5.87641×10^{-13}
0.2	0.980067	6.12546×10^{-10}	5.58442×10^{-13}
0.3	0.955336	8.65893×10^{-10}	4.71290×10^{-13}
0.4	0.921061	5.43585×10^{-10}	8.68305×10^{-13}
0.5	0.877583	9.17413×10^{-11}	1.70097×10^{-12}
0.6	0.825336	3.52923×10^{-10}	6.77236×10^{-13}
0.7	0.764842	1.88950×10^{-10}	5.97855×10^{-13}
0.8	0.696707	8.80399×10^{-10}	6.23279×10^{-13}
0.9	0.621610	4.35435×10^{-10}	1.58096×10^{-12}
1.0	0.540302	1.58424×10^{-09}	3.40161×10^{-12}

In this example, the exact solution function and the approximate solution function for $n = 7$ and $m = 5$ are plotted in Fig. 7. In this numerical experiment, the exact solution is given by an example of a degree 3 polynomial, therefore the best degree of BBF which can be used is a Bessel polynomial of degree 3 where the corresponding absolute error is plotted in Fig. 8. Clearly, all of the above points indicate the applicability and accuracy of the proposed method.

6 Conclusion

This paper deals with a matrix scheme with truncated series to find the numerical solution for the second-type singular integral equation with Cauchy kernel based on the Bessel basis function of the first kind. Using the smooth transform, the proposed scheme leads to converting the CSIE-2 into a system of linear equations and matrix equations. Another



part of this work was dedicated to using the roots of Legendre polynomials as the collocation points and applying the Gaussian–Legendre quadrature rule. By increasing the number of collocation points, we obtain the accurate numerical results for CSIEs-2, but we cannot repeat the statement about increasing the value of m because the error of the quadrature rule may reduce the accuracy of the approximate solution. In numerical experiments with an exact solution in a polynomial form, the best number of n for approximate solution is the degree of the corresponding polynomial. Finally, we provided the efficiency and accuracy of the presented scheme by the computational examples. We suggest this method for the CSFIEs by using different smooth transformations and quadrature rules.

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Ethics approval and consent to participate

Author read and approved the final manuscript.

Competing interests

The author declares that he has no competing interests.

Authors' contributions

The author read and approved the final version of the current paper.

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