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Generating functions for some families of the generalized Al-Salam–Carlitz q-polynomials

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Abstract

In this paper, by making use of the familiar q-difference operators D_q and $D_{q^{-1}}$, we first introduce two homogeneous q-difference operators $\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)$ and $\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})$, which turn out to be suitable for dealing with the families of the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$. We then apply each of these two homogeneous q-difference operators in order to derive generating functions, Rogers type formulas, the extended Rogers type formulas, and the Srivastava–Agarwal type linear as well as bilinear generating functions involving each of these families of the generalized Al-Salam–Carlitz q-polynomials. We also show how the various results presented here are related to those in many earlier works on the topics which we study in this paper.

MSC: Primary 05A30; 33D15; 33D45; secondary 05A40; 11B65

Keywords: Basic (or q-) hypergeometric series; Homogeneous q-difference operator; q-Binomial theorem; Cauchy polynomials; Al-Salam–Carlitz q-polynomials; Rogers type formulas; Extended Rogers type formulas; Srivastava–Agarwal type generating functions

1 Introduction, definitions, and preliminaries

The quantum (or q-) polynomials constitute a very interesting set of special functions and orthogonal polynomials. Their generating functions appear in several branches of mathematics and physics (see, for details, [1–5]) such as (for example) continued fractions, Eulerian series, theta functions, elliptic functions, quantum groups and algebras, discrete mathematics (including combinatorics and graph theory), coding theory, and so on.

Recently, new classes of special functions including (for example) q-hybrid special polynomials, q-Sheffer–Appell polynomials, twice-iterated 2D q-Appell polynomials, and a unified class of Apostol type q-polynomials were introduced in [6–8] and [9] in which some properties of the introduced polynomials were derived. For more information, the interested reader should refer to [6–8] and [9].

In the year 1997, Chen and Liu [10] developed a method of deriving basic (or q-) hypergeometric identities by parameter augmentation, which may be viewed as being analogous to the method used rather extensively in the theory of *ordinary* hypergeometric functions and hypergeometric generating functions (see, for details, [5]). Subsequent investigations along the lines developed in [10] can be found in [11–15] and [16]. The main objective of this paper is to investigate two families of the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ by first representing them by the homogeneous q-difference operators $\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)$ and $\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})$, which we have introduced here. We then derive a number of q-identities such as (among other results) generating functions, Rogers type formulas, two kind of the extended Rogers type formulas, and Srivastava–Agarwal type generating functions for each of the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

Here, in this paper, we adopt the common conventions and notations on q-series and q-hypergeometric functions. For the convenience of the reader, we provide a summary of mathematical notations and definitions, basic properties, and other relations to be used in the sequel. We refer, for details, to the general references (see [2, 17, 18]) for the definitions and notations. Throughout this paper, we assume that $|q|<1$.

For complex numbers a, the q-shifted factorials are defined by

$$(a;q)_0 := 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\bigl(1-aq^k\bigr) \quad \text{and} \quad (a;q)_\infty := \prod_{k=0}^{\infty}\bigl(1-aq^k\bigr), \tag{1.1}$$

where (see, for example, [2] and [19])

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad (a;q)_{n+m} = (a;q)_n\bigl(aq^n;q\bigr)_m$$

and

$$\bigl(aq^{-n};q\bigr)_n = (q/a;q)_n(-a)^n q^{-n-\binom{n}{2}}.$$

We adopt the following notation:

$$(a_1,a_2,\ldots,a_r;q)_m = (a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m \quad \bigl(m\in\mathbb{N}:=\{1,2,3,\ldots\}\bigr).$$

Also, for m large, we have

$$(a_1,a_2,\ldots,a_r;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_r;q)_\infty.$$

The q-numbers and the q-factorials are defined as follows:

$$[n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := \prod_{k=1}^{n}\left(\frac{1-q^k}{1-q}\right) \quad \text{and} \quad [0]_q! := 1. \tag{1.2}$$

The q-binomial coefficient is defined as follows (see, for example, [2]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q^{-n};q)_k}{(q;q)_k}(-1)^k q^{nk-\binom{k}{2}} \quad (0\leqq k\leqq n). \tag{1.3}$$

The basic (or q-) hypergeometric function of the variable z and with $\mathfrak{r}$ numerator and $\mathfrak{s}$ denominator parameters is defined as follows (see, for details, the monographs by Slater

[19, Chap. 3] and by Srivastava and Karlsson [18, p. 347, Eq. (272)]; see also [20, 21] and [17]):

$$ {}_{\mathfrak{r}}\Phi_{\mathfrak{s}}\left[\begin{matrix} a_1, a_2, \ldots, a_{\mathfrak{r}}; \\ b_1, b_2, \ldots, b_{\mathfrak{s}}; \end{matrix} q; z\right] := \sum_{n=0}^{\infty}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+\mathfrak{s}-\mathfrak{r}} \frac{(a_1, a_2, \ldots, a_{\mathfrak{r}}; q)_n}{(b_1, b_2, \ldots, b_{\mathfrak{s}}; q)_n} \frac{z^n}{(q;q)_n}, $$

where $q \neq 0$ when $\mathfrak{r} > \mathfrak{s} + 1$. We also note that

$$ {}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}}\left[\begin{matrix} a_1, a_2, \ldots, a_{\mathfrak{r}+1} \\ b_1, b_2, \ldots, b_{\mathfrak{r}}; \end{matrix} q; z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{\mathfrak{r}+1}; q)_n}{(b_1, b_2, \ldots, b_{\mathfrak{r}}; q)_n} \frac{z^n}{(q;q)_n}. $$

Here, in our present investigation, we are mainly concerned with the Cauchy polynomials $p_n(x, y)$ as given below (see [22] and [2]):

$$ p_n(x,y) := (x-y)(x-qy)\cdots\left(x-q^{n-1}y\right) = \left(\frac{y}{x}; q\right)_n x^n \tag{1.4} $$

together with the following Srivastava–Agarwal type generating function (see also [23]):

$$ \sum_{n=0}^{\infty} p_n(x,y) \frac{(\lambda;q)_n t^n}{(q;q)_n} = {}_2\Phi_1\left[\begin{matrix} \lambda, y/x; \\ 0; \end{matrix} q; xt\right]. \tag{1.5} $$

For $\lambda = 0$ in (1.5), we get the following simpler generating function [22]:

$$ \sum_{n=0}^{\infty} p_n(x,y) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{1.6} $$

The generating function (1.6) is also a homogeneous version of the Cauchy identity or the following q-binomial theorem (see, for example, [2, 19] and [18]):

$$ \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = {}_1\Phi_0\left[\begin{matrix} a; \\ —; \end{matrix} q; z\right] = \frac{(az;q)_\infty}{(z;q)_\infty} \quad \left(|z| < 1\right). \tag{1.7} $$

Upon further setting $a = 0$, this last relation (1.7) becomes Euler's identity (see, for example, [2])

$$ \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \quad \left(|z| < 1\right) \tag{1.8} $$

and its inverse relation is given as follows [2]:

$$ \sum_{k=0}^{\infty} \frac{(-1)^k}{(q;q)_k} q^{\binom{k}{2}} z^k = (z;q)_\infty. \tag{1.9} $$

The Jackson's q-difference or q-derivative operators D_q and $D_{q^{-1}}$ are defined as follows (see, for example, [2, 24, 25]):

$$ D_q\left\{f(x)\right\} := \frac{f(x) - f(qx)}{(1-q)x} \quad \text{and} \quad D_{q^{-1}}\left\{f(x)\right\} := \frac{f(q^{-1}x) - f(x)}{(q^{-1}-1)x}. \tag{1.10} $$

Evidently, in the limit when $q \to 1-$, we have

$$\lim_{q\to 1-}\{D_q\{f(x)\}\} = f'(x) \quad \text{and} \quad \lim_{q\to 1-}\{D_{q^{-1}}\{f(x)\}\} = f'(x),$$

provided that the derivative $f'(x)$ exists.

Suppose that D_q acts on the variable a. Then we have the q-identities asserted by Lemma 1.

Lemma 1 *Each of the following q-identities holds true for the q-derivative operator D_q acting on the variable a*:

$$D_q^k\left\{\frac{1}{(as;q)_\infty}\right\} = \frac{[(1-q)^{-1}s]^k}{(as;q)_\infty}, \tag{1.11}$$

$$(D_{q^{-1}})^k\left\{\frac{1}{(as;q)_\infty}\right\} = \frac{q^{-\binom{k}{2}}}{(asq^{-k};q)_\infty}\left(\frac{s}{1-q}\right)^k, \tag{1.12}$$

$$D_q^k\{(as;q)_\infty\} = (-1)^k q^{\binom{k}{2}}\left(asq^k;q\right)_\infty\left(\frac{s}{1-q}\right)^k, \tag{1.13}$$

and

$$(D_{q^{-1}})^k\{(as;q)_\infty\} = (as;q)_\infty\left(-\frac{s}{1-q}\right)^k. \tag{1.14}$$

The Leibniz rules for the q-derivative operators D_q and $D_{q^{-1}}$ are given by (see, for example, [10] and [12])

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^n q^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k\{f(x)\}D_q^{n-k}\{g(q^k x)\} \tag{1.15}$$

and

$$D_{q^{-1}}^n\{f(x)g(x)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_{q^{-1}}^k\{f(x)\}D_{q^{-1}}^{n-k}\{g(q^{-k} x)\}, \tag{1.16}$$

where D_q^0 and $D_{q^{-1}}^0$ are understood to be the identity operators.

Lemma 2 *Suppose that q-difference operator D_q acts on the variable a. Then*

$$D_q^n\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} = \left(\frac{\omega}{1-q}\right)^n\frac{(s/\omega;q)_n}{(as;q)_n}\frac{(as;q)_\infty}{(a\omega;q)_\infty} \tag{1.17}$$

and

$$D_{q^{-1}}^n\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} = \left(-\frac{q}{(1-q)a}\right)^n\frac{(s/\omega;q)_n}{(q/(a\omega);q)_n}\frac{(as;q)_\infty}{(a\omega;q)_\infty}. \tag{1.18}$$

Proof Suppose that the operator D_q acts upon the variable a. In light of (1.15), we then find that

$$\begin{aligned}
D_q^n\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} &= \sum_{k=0}^{n} q^{k(k-n)}\begin{bmatrix} n\\ k\end{bmatrix}_q D_q^k\{(as;q)_\infty\}D_q^{n-k}\left\{\frac{1}{(a\omega q^k;q)_\infty}\right\}\\
&= \sum_{k=0}^{n} q^{k(k-n)+\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q (asq^k;q)_\infty\left(-\frac{s}{1-q}\right)^k\frac{[(1-q)^{-1}\omega q^k]^{n-k}}{(a\omega q^k;q)_\infty}\\
&= \frac{(as;q)_\infty}{(a\omega;q)_\infty}\left(\frac{\omega}{1-q}\right)^n\sum_{k=0}^{n}\frac{(q^{-n},a\omega;q)_k}{(as,q;q)_k}\left(\frac{sq^n}{\omega}\right)^k\\
&= \left(\frac{\omega}{1-q}\right)^n\frac{(as;q)_\infty}{(a\omega;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} q^{-n},a\omega;\\ as;\end{matrix}\,q;\,\frac{sq^n}{\omega}\right],
\end{aligned}\tag{1.19}$$

where we have appropriately applied formulas (1.11), (1.13), and (1.3). The proof of the first assertion (1.17) of Lemma 2 is completed by using relation (1.21) in (1.19).

Similarly, by using relation (1.16), we have

$$\begin{aligned}
D_{q^{-1}}^n\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} &= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q D_{q^{-1}}^k\left\{\frac{1}{(a\omega;q)_\infty}\right\}D_{q^{-1}}^{n-k}\left\{\left(asq^{-k};q\right)_\infty\right\}\\
&= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{q^{-\binom{k}{2}}(\frac{\omega}{1-q})^k}{(a\omega q^{-k};q)_\infty}(-1)^{n-k}\left(\frac{sq^{-k}}{1-q}\right)^{n-k}\left(asq^{-k};q\right)_\infty\\
&= \frac{(as;q)_\infty}{(a\omega;q)_\infty}\left(-\frac{s}{1-q}\right)^n\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{(-1)^k q^{-\binom{k}{2}+k(k-n)}(asq^{-k};q)_k}{(a\omega q^{-k};q)_k}\left(\frac{\omega}{s}\right)^k\\
&= \frac{(as;q)_\infty}{(a\omega;q)_\infty}\left(-\frac{s}{1-q}\right)^n\sum_{k=0}^{n}\frac{(q^{-n},q/(as);q)_k q^k}{(q,q/(a\omega);q)_k}\\
&= \left(-\frac{s}{1-q}\right)^n\frac{(as;q)_\infty}{(a\omega;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix} q^{-n},q/(as);\\ q/(a\omega);\end{matrix}\,q;q\right],
\end{aligned}\tag{1.20}$$

where we have appropriately used relation (1.3).

Finally, by using relation (1.22) in (1.20), we are led to the second assertion (1.18) of Lemma 2. We thus have completed the proof of Lemma 2. □

Remark 1 For $s = 0$ and $\omega = s$, assertions (1.17) and (1.18) of Lemma 2 reduce to assertions (1.11) and (1.12) of Lemma 1. Moreover, for $\omega = 0$, assertions (1.17) and (1.18) of Lemma 2 reduce to assertions (1.13) and (1.14) of Lemma 1.

Lemma 3 (see, for example, [2, Eq. (0.58) and Eq. (II.6)]) *The q-Chu–Vandermonde formulas are given by*

$${}_2\Phi_1\left[\begin{matrix} q^{-n},a;\\ c;\end{matrix}\,q;\,\frac{cq^n}{a}\right] = \frac{(c/a;q)_n}{(c;q)_n}\quad\left(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right)\tag{1.21}$$

and

$$ {}_2\Phi_1\left[\begin{matrix} q^{-n}, a; \\ c; \end{matrix} q; q\right] = \frac{(c/a;q)_n}{(c;q)_n} a^n \quad \big(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\big). \tag{1.22} $$

We now state and prove the q-difference formulas asserted by Theorem 1.

Theorem 1 *Suppose that the q-difference operators D_q and $D_{q^{-1}}$ act upon the variable a. Then*

$$ D_q^n\left\{\frac{(as;q)_\infty}{(at,a\omega;q)_\infty}\right\} = \left(\frac{t}{1-q}\right)^n \frac{(as;q)_\infty}{(at,a\omega;q)_\infty} {}_3\Phi_1\left[\begin{matrix} q^{-n}, s/\omega, at; \\ as; \end{matrix} q; \frac{\omega q^n}{t}\right] \tag{1.23} $$

and

$$ D_{q^{-1}}^n\left\{\frac{(as,at;q)_\infty}{(a\omega;q)_\infty}\right\} = \left(-\frac{t}{1-q}\right)^n \frac{(at,as;q)_\infty}{(a\omega;q)_\infty} {}_3\Phi_2\left[\begin{matrix} q^{-n}, q/(at), s/\omega; \\ q/(a\omega), 0; \end{matrix} q; q\right]. \tag{1.24} $$

Proof Suppose first that the q-difference operator D_q acts upon the variable a. Then, in light of (1.15), and by using relations (1.17) and (1.11), it is easily seen that

$$ \begin{aligned} D_q^n\left\{\frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\right\} &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} D_q^k\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} D_q^{n-k}\left\{\frac{1}{(atq^k;q)_\infty}\right\} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} \frac{(s/\omega;q)_k}{(as;q)_k} \frac{(as;q)_\infty}{(a\omega;q)_\infty} \frac{[(1-q)^{-1}tq^k]^{n-k}}{(atq^k;q)_\infty} \left(\frac{\omega}{1-q}\right)^k \\ &= \frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\left(\frac{t}{1-q}\right)^n \sum_{k=0}^n \frac{(-1)^k q^{-\binom{k}{2}} (q^{-n},s/\omega,at;q)_k}{(as,q;q)_k}\left(\frac{\omega q^n}{t}\right)^k \\ &= \left(\frac{t}{1-q}\right)^n \frac{(as;q)_\infty}{(a\omega,at;q)_\infty} {}_3\Phi_1\left[\begin{matrix} q^{-n}, s/\omega, at; \\ as; \end{matrix} q; \frac{\omega q^n}{t}\right]. \end{aligned} \tag{1.25} $$

Similarly, by using (1.16), we find for the q-difference operator $D_{q^{-1}}$ acting on the variable a that

$$ \begin{aligned} &D_{q^{-1}}^n\left\{\frac{(at,as;q)_\infty}{(a\omega;q)_\infty}\right\} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_{q^{-1}}^k\left\{\frac{(as;q)_\infty}{(a\omega;q)_\infty}\right\} D_{q^{-1}}^{n-k}\left\{\left(atq^{-k};q\right)_\infty\right\} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[-qa^{-1}(1-q)^{-1}]^k (s/\omega;q)_k (as;q)_\infty}{(q/(a\omega);q)_k (a\omega;q)_\infty} \left(atq^{-k};q\right)_\infty \left(-\frac{tq^{-k}}{1-q}\right)^{n-k} \\ &= \left(-\frac{t}{1-q}\right)^n \frac{(as,at;q)_\infty}{(a\omega;q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(1+k-n)} (ta)^{-k} \frac{(s/\omega, atq^{-k};q)_k}{(q/(a\omega);q)_k} \\ &= \left(-\frac{t}{1-q}\right)^n \frac{(as,at;q)_\infty}{(a\omega;q)_\infty} \sum_{k=0}^n \frac{(q^{-n}, s/\omega, q/(at);q)_k}{(q,q/(a\omega);q)_k} q^k, \end{aligned} \tag{1.26} $$

where we have appropriately used relation (1.3) as well.

Equations (1.25) and (1.26) together complete the proof of Theorem 1. □

Remark 2 Upon first setting $\omega = 0$, we put $t = \omega$ in assertion (1.23) of Theorem 1. Then, if we make use of identity (1.21), we get (1.17). Furthermore, upon setting $s = 0$ in assertion (1.24) of Theorem 1, if we make use of the q-Chu–Vandermonde formula (1.22), we get (1.18).

This paper is organized as follows. In Sect. 2, we introduce two homogeneous q-difference operators $\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)$ and $\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})$. In addition, we define two families of the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and represent each of the families in terms of the homogeneous q-difference operators $\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)$ and $\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})$. We also derive generating functions for these families of the generalized Al-Salam–Carlitz q-polynomials. In Sect. 3, we first give the Rogers type formulas and the extended Rogers type formulas. The Srivastava–Agarwal type generating functions involving the generalized Al-Salam–Carlitz q-polynomials are derived in Sect. 4. Finally, in our last section (Sect. 5), we present the concluding remarks and observations concerning our present investigation.

2 Generalized Al-Salam–Carlitz q-polynomials

In this section, we first introduce two homogeneous q-difference operators $\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)$ and $\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})$ which are defined by

$$\mathbb{T}(\mathbf{a},\mathbf{b},cD_q) := \sum_{k=0}^{\infty} \frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_k}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_k}(cD_q)^k \tag{2.1}$$

and

$$\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}}) := \sum_{k=0}^{\infty} \frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_k}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_k}(cD_{q^{-1}})^k, \tag{2.2}$$

where, for convenience,

$$\mathbf{a} = (a_1,a_2,\ldots,a_{\mathfrak{r}+1}) \quad \text{and} \quad \mathbf{b} = (b_1,b_2,\ldots,b_{\mathfrak{r}}).$$

We now derive identities (2.3) and (2.4), which will be used later in order to derive the generating functions, the Rogers type formulas, the extended Rogers type formulas, and the Srivastava–Agarwal type generating functions involving the families of the generalized Al-Salam–Carlitz q-polynomials.

Theorem 2 *Suppose that the q-difference operator D_q acts on the variable a. Then*

$$\begin{aligned}&\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)\left\{\frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\right\}\\ &= \frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_n}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_n}\left(\frac{ct}{1-q}\right)^n\\ &\quad\cdot {}_3\Phi_1\left[\begin{matrix}q^{-n},s/\omega,at;\\ as;\end{matrix}\,q;\frac{\omega q^n}{t}\right] \quad \left(\max\left\{|a\omega|,|at|\right\}<1\right)\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})\left\{\frac{(at,as;q)_\infty}{(a\omega;q)_\infty}\right\}\\
&=\frac{(at,as;q)_\infty}{(a\omega;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_n}{(q,b_1,b_2,\dots,b_{\mathfrak{r}};q)_n}\left(-\frac{ct}{1-q}\right)^n\\
&\quad\cdot{}_3\Phi_2\left[\begin{matrix}q^{-n},q/(at),s/\omega;\\ q/(a\omega),0;\end{matrix}q;q\right]\quad\bigl(|a\omega|<1\bigr).
\end{aligned}\tag{2.4}$$

Proof Suppose that the operators D_q and $D_{q^{-1}}$ act on the variable a. We observe by applying (1.23) that

$$\begin{aligned}
&\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)\left\{\frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_nc^n}{(q,b_1,b_2,\dots,b_{\mathfrak{r}};q)_n}D_q^n\left\{\frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_nc^n}{(q,b_1,b_2,\dots,b_{\mathfrak{r}};q)_n}\frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\left(\frac{t}{1-q}\right)^n{}_3\Phi_1\left[\begin{matrix}q^{-n},s/\omega,at;\\ as;\end{matrix}q;\frac{\omega q^n}{t}\right]\\
&=\frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_n}{(q,b_1,b_2,\dots,b_{\mathfrak{r}};q)_n}{}_3\Phi_1\left[\begin{matrix}q^{-n},s/\omega,at;\\ as;\end{matrix}q;\frac{\omega q^n}{t}\right]\left(\frac{ct}{1-q}\right)^n.
\end{aligned}\tag{2.5}$$

Similarly, by applying (1.24), we find that

$$\begin{aligned}
&\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})\left\{\frac{(at,as;q)_\infty}{(a\omega;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_n(c)^n}{(q,b_1,b_2,\dots,b_{\mathfrak{r}};q)_n}D_{q^{-1}}^n\left\{\frac{(at,as;q)_\infty}{(a\omega;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_n(c)^n}{(q,b_1,b_2,\dots,b_{\mathfrak{r}};q)_n}\frac{(at,as;q)_\infty}{(a\omega;q)_\infty}\left(\frac{t}{q-1}\right)^n{}_3\Phi_2\left[\begin{matrix}q^{-n},q/(at),s/\omega;\\ q/(a\omega),0;\end{matrix}q;q\right]\\
&=\frac{(at,as;q)_\infty}{(a\omega;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_n}{(q,b_1,b_2,\dots,b_{\mathfrak{r}};q)_n}\left(\frac{ct}{q-1}\right)^n{}_3\Phi_2\left[\begin{matrix}q^{-n},q/(at),s/\omega;\\ q/(a\omega),0;\end{matrix}q;q\right],
\end{aligned}\tag{2.6}$$

as asserted by Theorem 2. □

Definition In terms of the q-binomial coefficient, the families of the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ are defined by

$$\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_k}{(b_1,b_2,\dots,b_{\mathfrak{r}};q)_k}x^ky^{n-k}\tag{2.7}$$

and

$$\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(a_1,a_2,\dots,a_{\mathfrak{r}+1};q)_k}{(b_1,b_2,\dots,b_{\mathfrak{r}};q)_k}q^{\binom{k+1}{2}-nk}x^ky^{n-k}.\tag{2.8}$$

Proposition *Suppose that the operators D_q and $D_{q^{-1}}$ act on the variable y. Then*

$$\begin{aligned}&\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)=\mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\big\{y^n\big\}\quad\text{and}\\&\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)=\mathbb{E}\big(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\big)\big\{y^n\big\}\end{aligned}\tag{2.9}$$

in terms of operators (2.1) *and* (2.2).

Theorem 3 (Generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)$) *Each of the following generating functions holds true*:

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x,y,z|q)\frac{t^n}{(q;q)_n}\\&\quad=\frac{1}{(yt;q)_\infty}{}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1};\\b_1,b_2,\ldots,b_{\mathfrak{r}};\end{matrix}q;xt\right]\quad\big(\max\big\{|xt|,|yt|\big\}<1\big)\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\\&\quad=(yt;q)_\infty{}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1};\\b_1,b_2,\ldots,b_{\mathfrak{r}};\end{matrix}q;xt\right]\quad\big(|xt|<1\big).\end{aligned}\tag{2.11}$$

In our proof of Theorem 3, the following easily derivable lemma will be needed.

Lemma 4 *Suppose that the operators D_q and $D_{q^{-1}}$ act on the variable a. Then*

$$\begin{aligned}&\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)\left\{\frac{1}{(as;q)_\infty}\right\}\\&\quad=\frac{1}{(as;q)_\infty}{}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}+1}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1};\\b_1,b_2,\ldots,b_{\mathfrak{r}};\end{matrix}q;\frac{cs}{1-q}\right]\quad\left(\max\left\{|as|,\left|\frac{cs}{1-q}\right|\right\}<1\right)\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}&\mathbb{E}(\mathbf{a},\mathbf{b},-cD_{q^{-1}})\big\{(as;q)_\infty\big\}\\&\quad=(as;q)_\infty{}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1};\\b_1,b_2,\ldots,b_{\mathfrak{r}};\end{matrix}q;\frac{cs}{1-q}\right]\quad\left(\left|\frac{cs}{1-q}\right|<1\right).\end{aligned}\tag{2.13}$$

Proof of Theorem 3 We suppose that the q-difference operator D_q acts upon the variable y. In light of the formulas in (2.9), and by applying (2.12), it is readily seen that

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}&=\sum_{n=0}^{\infty}\mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\big\{y^n\big\}\frac{t^n}{(q;q)_n}\\&=\mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\left\{\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\left\{\frac{1}{(yt;q)_\infty}\right\} \\
&= \frac{1}{(yt;q)_\infty}\,{}_{\mathfrak{r}+1}\Phi_{\mathfrak{r}}\left[\begin{matrix} a_1,a_2,\ldots,a_{\mathfrak{r}+1}; \\ b_1,b_2,\ldots,b_{\mathfrak{r}}; \end{matrix} q;xt\right]. \qquad (2.14)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n} \\
&\quad = \sum_{n=0}^{\infty}\mathbb{E}\big(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\big)\big\{y^n\big\}(-1)^n q^{\binom{n}{2}}\frac{t^n}{(q;q)_n} \\
&\quad = \mathbb{E}\big(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\big)\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(yt)^n}{(q;q)_n}\right\} \\
&\quad = \mathbb{E}\big(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\big)\big\{(yt;q)_\infty\big\}. \qquad (2.15)
\end{aligned}
$$

The proof of Theorem 3 can now be completed by making use of relation (2.13). □

3 The Rogers type formulas and the extended Rogers type formulas

In this section, we use the assertions in (2.9) to derive several q-identities such as the Rogers type formulas and the extended Rogers type formulas for the families of the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

Theorem 4 (Rogers type formula for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$) *The following Rogers type formula holds true for* $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n+m}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} \\
&\quad = \frac{1}{(yt,ys;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_n}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_n}(xt)^n\,{}_2\Phi_0\left[\begin{matrix} q^{-n},yt; \\ -; \end{matrix} q;\frac{sq^n}{t}\right] \\
&\quad \big(\max\big\{|yt|,|ys|\big\}<1\big). \qquad (3.1)
\end{aligned}
$$

Theorem 5 (Rogers type formula for $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$) *The following Rogers type formula holds true for* $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n}{2}+\binom{m}{2}}\psi_{n+m}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} \\
&\quad = (yt,ys;q)_\infty\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_n}{(b_1,b_2,\ldots,b_{\mathfrak{r}};q)_n}(xt)^n\,{}_2\Phi_1\left[\begin{matrix} q^{-n},q/(yt); \\ 0; \end{matrix} q;ys\right]. \qquad (3.2)
\end{aligned}
$$

In order to prove Theorems 4 and 5, we need Lemma 5.

Lemma 5 *It is asserted that*

$$\begin{aligned}&\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)\left\{\frac{1}{(a\omega,at;q)_\infty}\right\}\\&=\frac{(as;q)_\infty}{(a\omega,at;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_n}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_n}{}_2\Phi_0\left[\begin{matrix}q^{-n},at;\\ -;\end{matrix}q;\frac{\omega q^n}{t}\right]\left(\frac{ct}{1-q}\right)^n\\&\left(\max\bigl\{|a\omega|,|at|\bigr\}<1\right)\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\mathbb{E}(\mathbf{a},\mathbf{b},-cD_{q^{-1}})\bigl\{(at,as;q)_\infty\bigr\}\\&=(at,as;q)_\infty\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_n}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_n}{}_2\Phi_1\left[\begin{matrix}q^{-n},q/(at);\\ 0;\end{matrix}q;as\right]\left(\frac{ct}{1-q}\right)^n.\end{aligned}\tag{3.4}$$

Proof The first assertion (3.3) of Lemma 5 follows from (2.3) when $s = 0$. On the other hand, the second assertion (3.4) of Lemma 5 can be deduced from (2.4) by setting $\omega = 0$. □

Proof of Theorems 4 and 5 We suppose that the operator D_q acts upon the variable y. Then, in view of the formulas in (2.9), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\phi_{n+m}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{T}\bigl(\mathbf{a},\mathbf{b},(1-q)xD_q\bigr)\bigl\{y^{n+m}\bigr\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\&=\mathbb{T}\bigl(\mathbf{a},\mathbf{b},(1-q)xD_q\bigr)\left\{\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\right\}\\&=\mathbb{T}\bigl(\mathbf{a},\mathbf{b},(1-q)xD_q\bigr)\left\{\frac{1}{(yt,ys;q)_\infty}\right\}.\end{aligned}\tag{3.5}$$

The proof of assertion (3.1) of Theorem 4 can now be completed by using relation (3.3) in (3.5).

Similarly, we observe that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n}{2}+\binom{m}{2}}\psi_{n+m}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n}{2}+\binom{m}{2}}\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\bigl\{y^{n+m}\bigr\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\&=\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\left\{\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}(-1)^mq^{\binom{m}{2}}\frac{(ys)^m}{(q;q)_m}\right\}\\&=\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\bigl\{(yt,ys;q)_\infty\bigr\},\end{aligned}\tag{3.6}$$

which evidently completes the proof of assertion (3.2) of Theorem 5. □

We next derive another Rogers type formula for the family of the generalized Al-Salam–Carlitz q-polynomials $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ as follows.

Theorem 6 (Another Rogers type formula for $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$) *It is asserted that*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n q^{\binom{n}{2}}\psi_{n+m}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(yt;q)_\infty}{(ys;q)_\infty}\,{}_{\mathfrak{r}+2}\Phi_{\mathfrak{r}+1}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1},t/s;\\ b_1,b_2,\ldots,b_{\mathfrak{r}},q/(ys);\end{matrix}\,q;\frac{xq}{y}\right] \quad \bigl(|ys|<1\bigr). \tag{3.7}$$

Proof We suppose that the q-difference operator D_q acts upon the variable y. We then obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n q^{\binom{n}{2}}\psi_{n+m}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^n q^{\binom{n}{2}}\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\bigl\{y^{n+m}\bigr\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&\quad=\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\right\}\\
&\quad=\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\left\{\frac{(yt;q)_\infty}{(ys;q)_\infty}\right\}.
\end{aligned}\tag{3.8}$$

The proof of assertion (3.7) of Theorem 6 can now be completed by applying formula (2.4) with $s=0$ and $\omega=s$ in (3.8). □

Another extended Rogers type formula for the family of the generalized Al-Salam–Carlitz q-polynomials $\Psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ is given by Theorem 7.

Theorem 7 (Another extended Rogers type formula for $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$) *It is asserted that*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+m}q^{\binom{n}{2}+\binom{m}{2}}\psi_{n+m+k}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{\omega^k}{(q;q)_k}\\
&\quad=\frac{(yt,ys;q)_\infty}{(y\omega;q)_\infty}\sum_{j=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_j}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_j}(xt)^j\,{}_3\Phi_2\left[\begin{matrix}q^{-j},q/(yt),s/\omega;\\ q/(y\omega),0;\end{matrix}\,q;q\right]\\
&\quad\bigl(|y\omega|<1\bigr).
\end{aligned}\tag{3.9}$$

Proof We suppose that the operator D_q acts upon the variable y. By using formulas (2.9), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n}{2}+\binom{m}{2}}\sum_{k=0}^{\infty}\psi_{n+m+k}^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{\omega^k}{(q;q)_k}\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n}{2}+\binom{m}{2}}\sum_{k=0}^{\infty}\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\bigl\{y^{n+m+k}\bigr\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\frac{\omega^k}{(q;q)_k}
\end{aligned}$$

$$= \mathbb{E}\big(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\big)\left\{\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}(-1)^m q^{\binom{m}{2}}\frac{(ys)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(y\omega)^k}{(q;q)_k}\right\}$$

$$= \mathbb{E}\big(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\big)\left\{\frac{(yt,ys;q)_\infty}{(y\omega;q)_\infty}\right\}. \tag{3.10}$$

Thus, in light of (2.4), the proof of assertion (3.9) of Theorem 7 is completed. □

4 Srivastava–Agarwal type generating functions for the families of the Al-Salam–Carlitz q-polynomials

In this section, we use the formulas in (2.9) to derive the Srivastava–Agarwal type generating functions involving the families of the Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

The Hahn polynomials (see [26, 27], and [28]) (or, equivalently, the Al-Salam–Carlitz q-polynomials [29]) are defined as follows:

$$\phi_n^{(a)}(x|q)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q (a;q)_k x^k \quad \text{and} \quad \psi_n^{(a)}(x|q)=\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix} n\\ k\end{bmatrix}_q \big(aq^{1-k};q\big)_k x^k. \tag{4.1}$$

Recently, Srivastava and Agarwal [30] gave a generating function which we recall here as Lemma 6.

Lemma 6 (see [30, Eq. (3.20)]) *The following generating function holds true*:

$$\sum_{n=0}^{\infty}(\lambda;q)_n\phi_n^{(\alpha)}(x|q)\frac{t^n}{(q;q)_n}=\frac{(\lambda t;q)_\infty}{(t;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}\lambda,\alpha;\\ \lambda t;\end{matrix}\,q;xt\right] \quad \big(\max\big\{|t|,|xt|\big\}<1\big). \tag{4.2}$$

The generating function (4.2) is known as a Srivastava–Agarwal type generating function (see, for example, [23]).

In this section, we give the Srivastava–Agarwal type generating functions for the families of the Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$.

Theorem 8 (Srivastava–Agarwal type generating functions for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$) *The following Srivastava–Agarwal type generating functions hold true for the families of the Al-Salam–Carlitz q-polynomials* $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ *and* $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$:

$$\sum_{n=0}^{\infty}(\lambda;q)_n\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}$$

$$=\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\,{}_{\mathfrak{r}+2}\Phi_{\mathfrak{r}+1}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1},\lambda;\\ b_1,b_2,\ldots,b_{\mathfrak{r}},\lambda yt;\end{matrix}\,q;xt\right] \quad \big(|yt|<1\big) \tag{4.3}$$

and

$$\sum_{n=0}^{\infty}(\lambda;q)_n\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n}$$

$$=\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\,{}_{\mathfrak{r}+2}\Phi_{\mathfrak{r}+1}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1},\lambda;\\ b_1,b_2,\ldots,b_{\mathfrak{r}},q/(yt);\end{matrix}\,q;\frac{xq}{y}\right] \quad \big(|yt|<1\big). \tag{4.4}$$

Proof We suppose that the operator D_q acts upon the variable y. According to the formulas in (2.9), we then obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(\lambda;q)_n\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty}(\lambda;q)_n\mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\{y^n\}\frac{t^n}{(q;q)_n}\\ &= \mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\left\{\sum_{n=0}^{\infty}(\lambda;q)_n\frac{(yt)^n}{(q;q)_n}\right\}\\ &= \mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\left\{\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\right\}.\end{aligned}\tag{4.5}$$

Now, setting $\omega = 0$ in (2.3), we have

$$\mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\left\{\frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\right\} = \frac{(\lambda yt;q)_\infty}{(yt;q)_\infty}\,{}_{\mathfrak{r}+2}\Phi_{\mathfrak{r}+1}\left[\begin{matrix}a_1,a_2,\ldots,a_{\mathfrak{r}+1},\lambda;\\ b_1,b_2,\ldots,b_{\mathfrak{r}},y\lambda t;\end{matrix}q;xt\right],\tag{4.6}$$

which, in conjunction with (4.5), completes the proof of the first assertion (4.3) of Theorem 8.

The proof of the second assertion (4.4) of Theorem 8 is much akin to that of the first assertion (4.3). The details involved are, therefore, omitted here. □

Remark 3 Upon replacing t by λt, if we set $s = t$ in assertion (3.7) of Theorem 6, we get (4.4).

Theorem 9 (Srivastava–Agarwal type bilinear generating function for $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$) *The following Srivastava–Agarwal type bilinear generating function holds true for the family of the generalized Al-Salam–Carlitz q-polynomials* $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$:

$$\begin{aligned}&\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\phi_n^{(\alpha)}(\mu|q)\frac{t^n}{(q;q)_n}\\ &\quad=\frac{(\alpha\mu yt;q)_\infty}{(\mu yt,yt;q)_\infty}\sum_{j=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_j}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_j}(xt)^j\,{}_3\Phi_1\left[\begin{matrix}q^{-j},\alpha,yt;\\ \alpha\mu yt;\end{matrix}q;\mu q^j\right]\\ &\quad\big(\max\big\{|yt|,|\mu yt|\big\}<1\big).\end{aligned}\tag{4.7}$$

Proof We suppose that the q-difference operator D_q acts upon the variable y. We then find that

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\phi_n^{(\alpha)}(\mu|q)\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty}\mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\{y^n\}\phi_n^{(\alpha)}(\mu|q)\frac{t^n}{(q;q)_n}\\ &= \mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\left\{\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(\mu|q)\frac{(yt)^n}{(q;q)_n}\right\}\\ &= \mathbb{T}\big(\mathbf{a},\mathbf{b},(1-q)xD_q\big)\left\{\frac{(\alpha\mu yt;q)_\infty}{(yt,\mu yt;q)_\infty}\right\}.\end{aligned}\tag{4.8}$$

The proof of assertion (4.7) of Theorem 9 is now completed by making use of relation (2.3) in (4.8). □

Theorem 10 (Srivastava–Agarwal type bilinear generating function for $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$) *The following Srivastava–Agarwal type bilinear generating function holds true for the family of the generalized Al-Salam–Carlitz q-polynomials* $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\psi_n^{(\alpha)}(\mu|q)(-1)^n q^{\binom{n}{2}}\frac{t^n}{(q;q)_n}\\
&\quad=\frac{(\mu yt,yt;q)_\infty}{(\alpha\mu yt;q)_\infty}\sum_{j=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{\mathfrak{r}+1};q)_j}{(q,b_1,b_2,\ldots,b_{\mathfrak{r}};q)_j}(xt)^j{}_3\Phi_2\left[\begin{matrix}q^{-j},q/(yt),1/\alpha;\\ q/(\alpha\mu yt),0;\end{matrix}q;q\right]\\
&\quad\bigl(|\alpha\mu yt|<1\bigr).
\end{aligned}\tag{4.9}$$

Proof We suppose that the q-difference operator D_q acts upon the variable y. We then obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)\psi_n^{(\alpha)}(\mu|q)\frac{q^{\binom{n}{2}}(-t)^n}{(q;q)_n}\\
&\quad=\sum_{n=0}^{\infty}\mathbb{E}(\mathbf{a},\mathbf{b},-xD_{q^{-1}})\bigl\{y^n\bigr\}\psi_n^{(\alpha)}(\mu|q)\frac{q^{\binom{n}{2}}(-t)^n}{(q;q)_n}\\
&\quad=\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\left\{\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(\mu|q)\frac{(-1)^n q^{\frac{n}{2}}(yt)^n}{(q;q)_n}\right\}\\
&\quad=\mathbb{E}\bigl(\mathbf{a},\mathbf{b},-(1-q)xD_{q^{-1}}\bigr)\left\{\frac{(yt,\mu yt;q)_\infty}{(\alpha\mu yt;q)_\infty}\right\}.
\end{aligned}\tag{4.10}$$

The proof of assertion (4.9) of Theorem 10 can now be completed by making use of relation (2.4) in (4.10). □

5 Concluding remarks and observations

Our present investigation is motivated essentially by several recent studies of generating functions and other results for various families of basic (or q-) polynomials stemming mostly from the works by Hahn (see, for example, [26, 27] and [28]; see also Al-Salam and Carlitz [29], Srivastava and Agarwal [30], Cao and Srivastava [23], and other researchers cited herein).

In terms of the familiar q-difference operators D_q and $D_{q^{-1}}$, we have first introduced two homogeneous q-difference operators $\mathbb{T}(\mathbf{a},\mathbf{b},cD_q)$ and $\mathbb{E}(\mathbf{a},\mathbf{b},cD_{q^{-1}})$, which turn out to be suitable for dealing with the generalized Al-Salam–Carlitz q-polynomials $\phi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$ and $\psi_n^{(\mathbf{a},\mathbf{b})}(x,y|q)$. We have then applied each of these two homogeneous q-difference operators in order to derive generating functions, Rogers type formulas, the extended Rogers type formulas, and the Srivastava–Agarwal type linear and bilinear generating functions for each of these families of the generalized Al-Salam–Carlitz q-polynomials.

The various results, which we have presented in this paper, together with the citations of many related earlier works are believed to motivate and encourage interesting further research on the topics of study here.

In conclusion, it should be remarked that in a recently-published survey-cum-expository article Srivastava [31] presented an expository overview of the classical q-analysis versus the so-called (p,q)-analysis with an obviously redundant additional parameter p (see, for details, [31, p. 340]).

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Authors' contributions
The main idea of this paper was proposed by SA. SA prepared the manuscript initially and HMS performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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