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Existence of solutions for nonlinear fractional integro-differential equations

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Abstract

In this paper, by means of the Krasnoselskii fixed point theorem, the existence of solutions for a boundary value problem of nonlinear sequential fractional integro-differential equations are investigated. Two examples are given to illustrate our results.

MSC: Primary 34B40; secondary 34B15

Keywords: Fractional differential equation; Krasnoselskii fixed point theorem; Nonlocal boundary conditions

1 Introduction

Fractional differential equations have attracted much attention and have been the focus of many studies due mainly to their varied applications in many fields of science and engineering. In other words, fractional differential equations are widely used to describe many important phenomena in various fields such as physics, biophysics, chemistry, biology, control theory, economy and so on; see [14, 19, 23, 29, 33]. For an extensive literature in the study of fractional differential equations, we refer the reader to [2, 11, 15, 16, 18, 20, 21, 24, 26, 30, 32]. However, it should be noted that in recent years, there have been many works related to fractional integro-differential equations, see [1, 3, 4, 6, 8, 12, 17, 22, 28, 29] and the references therein. For some interesting and considerable applied works, we refer to [5, 7, 9, 10].

In [13], Baleanu et al. studied the existence and uniqueness of solutions for the multiterm nonlinear fractional integro-differential equation

$$\begin{cases} {}^cD^{\alpha}u(t) = f(t, u(t), \varphi u(t), \psi u(t), {}^cD^{\beta_1}u(t), \cdot, {}^cD^{\beta_n}u(t)) \quad (0 < t < 1), \\ u(0) + au(1) = 0 \quad \text{and} \quad u'(0) - bu'(1) = 0. \end{cases}$$

where $1 < \alpha < 2$, $0 < \beta_i < 1$, $\alpha - \beta_i \geq 1$, $a, b \neq -1$, $f : [0,1] \times \mathbb{R}^{n+3} \to \mathbb{R}$ is continuous, and for the mappings $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ with the property

$$\sup_{t\in[0,1]} \left(\int_0^t \gamma(t,s)\,ds \right) < \infty \quad \text{and} \quad \sup_{t\in[0,1]} \left(\int_0^t \lambda(t,s)\,ds \right) < \infty,$$

the maps φ and ψ being defined by $(\varphi u)(t) = \int_0^t \gamma(t,s)\,ds$ and $(\psi u)(t) = \int_0^t \lambda(t,s)\,ds$.

In [31], Wang et al. proved the existence and uniqueness of positive solutions for the following fractional integro-differential equation:

$$\begin{cases} {}^cD^\alpha u(t) + f(t, u(t), \varphi u(t), \psi u(t)) = 0 \quad (0 < t < 1), \\ u(0) = b_0, \qquad u'(0) = b_1, \dots, u^{(n-3)}(0) = b_{n-3}, \\ u^{(n-1)}(0) = b_{n-1}, \qquad u(1) = \mu \int_0^1 u(s)\, ds, \end{cases}$$

where $n-1 < \alpha \leq n, -1, n \geq 3, b_i \geq 0$ $(i = 1, 2, \dots, n-3, n-1)$, ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $(\varphi u)(t) = \int_0^t K(t,s)u(s)\, ds$, $(\psi u)(t) = \int_0^t H(t,s)u(s)\, ds$.

Motivated by the previous results, we discuss in this paper the existence of solutions for the following nonlinear sequential fractional boundary value problem:

$$\begin{cases} {}^cD^\alpha ({}^cD^\beta u)(t) = f(t, u(t), \varphi u(t), \psi u(t)) \quad (0 < t < 1), \\ u(1) = u(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and

$$\varphi u(t) = \int_0^t \gamma(t,s)u(s)\, ds, \qquad \psi u(t) = \int_0^t \lambda(t,s)u(s)\, ds.$$

where $\gamma, \lambda : [0,1] \times [0,1] \to [0, +\infty)$ are such that $\sup_{t\in[0,1]}(\int_0^1 \lambda(t,s)\, ds) < \infty$ and $\sup_{t\in[0,1]}(\int_0^1 \gamma(t,s)\, ds) < \infty$.

2 Preliminaries

For convenience, in this section we recall some basic definitions and properties of the fractional calculus theory and auxiliary lemmas which will be used throughout this paper, see [23, 25, 27].

Definition 2.1 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined by

$${}^cD^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined by

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 *If $\alpha > 0$, then the differential equation ${}^cD^\alpha u(t) = 0$ has a unique solution given by*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$ (n is the smallest integer such that $n \geq \alpha$).

Lemma 2.2 *For $\alpha > 0$, let $u \in C^n[0,1]$. Then*

$$I^\alpha\big({}^cDu\big)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}.$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ (n is the smallest integer such that $n \geq \alpha$).

Lemma 2.3 *If $y \in C[0,1]$, then the boundary value problem*

$$\begin{cases} {}^cD^\alpha({}^cD^\beta u)(t) = y(t), & 0 < t < 1, 1 < \alpha \leq 2 \text{ and } 0 < \beta \leq 1, \\ u(1) = u(0) = u'(1) = 0 \end{cases} \tag{2.1}$$

has the unique solution given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\beta+\alpha-1} y(s)\,ds \\ &\quad + \frac{t^\beta}{\Gamma(\alpha+\beta)}(-\beta - 1 + \beta t) \int_0^1 (1-s)^{\beta+\alpha-1} y(s)\,ds \\ &\quad + \frac{t^\beta}{\Gamma(\alpha+\beta-1)}(1-t) \int_0^1 (1-s)^{\alpha+\beta-2} y(s)\,ds \end{aligned} \tag{2.2}$$

Proof In view of Lemma 2.2, FBVP (2.1) is equivalent to the following integral equation:

$$u(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\beta+\alpha-1} y(s)\,ds + \frac{t^\beta}{\beta\Gamma(\beta)} c_0 + \frac{t^{\beta+1}}{\beta(\beta+1)\Gamma(\beta)} c_1 + c_2. \tag{2.3}$$

Differentiating both sides of (2.3), we get

$$u'(t) = \frac{1}{\Gamma(\alpha+\beta-1)} \int_0^t (t-s)^{\alpha+\beta-2} y(s)\,ds + \frac{t^{\beta-1}}{\Gamma(\beta)} c_0 + \frac{t^\beta}{\beta\Gamma(\beta)} c_1.$$

Using the boundary conditions $u(1) = u(0) = u'(1) = 0$, we obtain

$$\begin{aligned} c_0 &= -\left(\frac{\beta(\beta+1)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right) \int_0^1 (1-s)^{\beta+\alpha-1} y(s)\,ds + \frac{\beta\Gamma(\beta)}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} y(s)\,ds, \\ c_1 &= \frac{\beta^2(\beta+1)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\beta+\alpha-1} y(s)\,ds - \frac{\beta(\beta+1)\Gamma(\beta)}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} y(s)\,ds, \\ c_2 &= 0. \end{aligned}$$

Substituting the values of c_0, c_1, c_2 in (2.3) we obtain (2.2). This completes the proof. □

3 Main results

Theorem 3.1 (Krasnoselskii fixed point theorem) *Let X be a closed convex and nonempty subset of a Banach space E. Let A and B be two operators such that*

1. *$Ax + By \in X$, whenever $x, y \in X$;*
2. *A is compact and continuous;*
3. *B is a contraction.*

Then there exists $z \in X$ such that $z = Az + Bz$.

Let $X = C(I)$ be the space of all continuous real-valued functions on $I = [0,1]$ endowed with the norm $\|u\| = \max_{t\in I} |u(t)|$.

Theorem 3.2 *Assume that* $\alpha + \beta - 2 \geq 0$ *and there exists a nonnegative function* $\theta(t) \in L^1(0,1)$ *such that*

$$\left|f(t,x,y,z) - f\left(t,x',y',z'\right)\right| \leq \theta(t)\left(\left|x - x'\right| + \left|y - y'\right| + \left|z - z'\right|\right) \tag{3.1}$$

for all $t \in [0,1]$ *and* $t, x, y, z, t', y', z' \in \mathbb{R}$. *Then problem* (1.1) *has at least one solution on* X *whenever*

$$\frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2\beta + 1)\theta^*}{\Gamma(\alpha + \beta)} < 1, \tag{3.2}$$

where $\gamma_0 = \sup_{t\in I} |\int_0^t \gamma(t,s)\,ds|$, $\lambda_0 = \sup_{t\in I} |\int_0^t \lambda(t,s)\,ds|$, *and* $\theta^* = \int_0^1 \theta(s)\,ds$.

Proof Choose

$$R \geq \frac{\varpi(\alpha + 2\beta + 1)}{1 - \theta^*(1 + \lambda_0 + \gamma_0)(\alpha + 2\beta + 1)}$$

and set $\varpi = \max\{f(t,0,0,0) : t \in I\}$. Consider the set $B_R = \{u \in X : \|u\| \leq R\}$, then B_R is a closed, bounded, and convex set of X. We define the operators A and B on X as

$$\begin{aligned}
Au(t) &= \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\beta+\alpha-1} f\left(s, u(s), \varphi u(s), \psi u(s)\right) ds, \\
Bu(t) &= \frac{t^\beta}{\Gamma(\alpha + \beta)}(-\beta - 1 + \beta t) \int_0^1 (1 - s)^{\beta+\alpha-1} f\left(s, u(s), \varphi u(s), \psi u(s)\right) ds \\
&\quad + \frac{t^\beta(\alpha + \beta - 1)}{\Gamma(\alpha + \beta)}(1 - t) \int_0^1 (1 - s)^{\alpha+\beta-2} f\left(s, u(s), \varphi u(s), \psi u(s)\right) ds.
\end{aligned}$$

For any $u \in B_R$ and $t \in I$, we get with the help of inequality (3.1)

$$\begin{aligned}
\left|Au(t)\right| &= \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\beta+\alpha-1} \left|f\left(s, u(s), \varphi u(s), \psi u(s)\right)\right| ds \\
&\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\beta+\alpha-1} \left|f\left(s, u(s), \varphi u(s), \psi u(s)\right) - f(s,0,0,0)\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\beta+\alpha-1} \left|f(s,0,0,0)\right| ds \\
&\leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\beta+\alpha-1} \theta(s)\left(\left|u(s)\right| + \left|\varphi u(s)\right| + \left|\psi u(s)\right|\right) ds \\
&\quad + \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\beta+\alpha-1} \left|f(s,0,0,0)\right| ds \\
&\leq \frac{(1 + \lambda_0 + \gamma_0)\|u\|}{\Gamma(\alpha + \beta)} \int_0^1 \theta(s)\,ds + \frac{\varpi}{\Gamma(\alpha + \beta)} \int_0^t (t - s)^{\beta+\alpha-1}\,ds \\
&\leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)}\|u\| + \frac{\varpi}{\Gamma(\alpha + \beta)}.
\end{aligned}$$

Hence, we get

$$\|Au\| \leq \frac{\theta^*(1+\lambda_0+\gamma_0)}{\Gamma(\alpha+\beta)}\|u\| + \frac{\varpi}{\Gamma(\alpha+\beta)}. \tag{3.3}$$

Similarly, we estimate $\|Bv\|$. Let $v \in B_R$ and $t \in I$, then

$$\begin{aligned}
\bigl|Bv(t)\bigr| &\leq \frac{t^\beta}{\Gamma(\alpha+\beta)}(\beta+1-\beta t)\int_0^1 (1-s)^{\beta+\alpha-1}\bigl|f\bigl(s,v(s),\varphi v(s),\psi v(s)\bigr)\bigr|\,ds \\
&\quad + \frac{t^\beta(\alpha+\beta-1)}{\Gamma(\alpha+\beta)}(1-t)\int_0^1 (1-s)^{\alpha+\beta-2}\bigl|f\bigl(s,v(s),\varphi v(s),\psi v(s)\bigr)\bigr|\,ds \\
&\leq \frac{\beta+1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\beta+\alpha-1}\theta(s)\bigl(\bigl|v(s)\bigr|+\bigl|\varphi v(s)\bigr|+\bigl|\psi v(s)\bigr|\bigr)\,ds \\
&\quad + \frac{\beta+1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\beta+\alpha-1}\bigl|f(t,0,0,0)\bigr|\,ds \\
&\quad + \frac{(\alpha+\beta-1)}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-2}\theta(s)\bigl(\bigl|v(s)\bigr|+\bigl|\varphi v(s)\bigr|+\bigl|\psi v(s)\bigr|\bigr)\,ds \\
&\quad + \frac{(\alpha+\beta-1)}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-2}\bigl|f(t,0,0,0)\bigr|\,ds \\
&\leq \theta^*(1+\lambda_0+\gamma_0)\frac{(\beta+1)}{\Gamma(\alpha+\beta)}\|v\| + \frac{\varpi(\beta+1)}{\Gamma(\alpha+\beta)} \\
&\quad + \frac{(\alpha+\beta-1)}{\Gamma(\alpha+\beta)}\theta^*(1+\lambda_0+\gamma_0)\|v\| + \frac{\varpi(\alpha+\beta-1)}{\Gamma(\alpha+\beta)} \\
&= \frac{\theta^*(1+\lambda_0+\gamma_0)(\alpha+2\beta)}{\Gamma(\alpha+\beta)}\|v\| + \frac{\varpi(\alpha+2\beta)}{\Gamma(\alpha+\beta)} \\
&= \frac{(\alpha+2\beta)}{\Gamma(\alpha+\beta)}\bigl(\theta^*(1+\lambda_0+\gamma_0)\|v\|+\varpi\bigr).
\end{aligned}$$

Hence, we get

$$\|Bv\| \leq \frac{(\alpha+2\beta)}{\Gamma(\alpha+\beta)}\bigl(\theta^*(1+\lambda_0+\gamma_0)\|v\|+\varpi\bigr). \tag{3.4}$$

Taking estimates (3.3) and (3.4) into account, we get for any $u, v \in B_R$ and $t \in I$,

$$\begin{aligned}
\|Au+Bv\| &\leq \|Au\|+\|Bv\| \\
&\leq \frac{\theta^*(1+\lambda_0+\gamma_0)}{\Gamma(\alpha+\beta)}\|u\| + \frac{\varpi}{\Gamma(\alpha+\beta)} \\
&\quad + \frac{(\alpha+2\beta)}{\Gamma(\alpha+\beta)}\bigl(\theta^*(1+\lambda_0+\gamma_0)\|v\|+\varpi\bigr) \\
&\leq R\frac{\theta^*(1+\lambda_0+\gamma_0)(\alpha+2\beta+1)}{\Gamma(\alpha+\beta)} \\
&\quad + \frac{\varpi(\alpha+2\beta+1)}{\Gamma(\alpha+\beta)},
\end{aligned}$$

since if

$$R \geq \frac{\varpi(\alpha + 2\beta + 1)}{1 - \theta^*(1 + \lambda_0 + \gamma_0)(\alpha + 2\beta + 1)}$$

then $\|Au + Bv\| \leq R$.

Now, we prove that B is a contraction. Let $v, u \in B_R$ and $t \in I$. Then, thanks to (3.1), it yields

$$\begin{aligned}
\left|Bu(t) - Bv(t)\right| \leq{}& \frac{t^\beta(\beta + 1 - \beta t)}{\Gamma(\alpha + \beta)} \times \int_0^1 (1 - s)^{\beta+\alpha-1} \left|f\left(s, u(s), \varphi u(s), \psi u(s)\right)\right. \\
&\left. - f\left(s, v(s), \varphi v(s), \psi v(s)\right)\right| ds \\
&+ \frac{t^\beta(1 - t)}{\Gamma(\alpha + \beta - 1)} \times \int_0^1 (1 - s)^{\alpha+\beta-2} \left|f\left(s, u(s), \varphi u(s), \psi u(s)\right)\right. \\
&\left. - f\left(s, v(s), \varphi v(s), \psi v(s)\right)\right| ds \\
\leq{}& \frac{\beta + 1}{\Gamma(\alpha + \beta)} \int_0^1 (1 - s)^{\beta+\alpha-1} \theta(s)\left(\left|u(s) - v(s)\right|\right. \\
&\left. + \left|\varphi u(s) - \varphi v(s)\right| + \left|\psi u(s) - \psi v(s)\right|\right) ds \\
&+ \frac{1}{\Gamma(\alpha + \beta - 1)} \int_0^1 (1 - s)^{\alpha+\beta-2} \theta(s)\left(\left|u(s) - v(s)\right|\right. \\
&\left. + \left|\varphi u(s) - \varphi v(s)\right| + \left|\psi u(s) - \psi v(s)\right|\right) ds \\
\leq{}& \frac{(\beta + 1)(1 + \gamma_0 + \lambda_0)\|u - v\|}{\Gamma(\alpha + \beta)} \int_0^1 \theta(s)\, ds \\
&+ \frac{(1 + \gamma_0 + \lambda_0)(\alpha + \beta - 1)\|u - v\|}{\Gamma(\alpha + \beta)} \int_0^1 \theta(s)\, ds \\
\leq{}& \frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2\beta)\theta^*}{\Gamma(\alpha + \beta)} \|u - v\|,
\end{aligned}$$

thus

$$\|Bu - Bv\| \leq \frac{(1 + \gamma_0 + \lambda_0)(\alpha + 2\beta)\theta^*}{\Gamma(\alpha + \beta)} \|u - v\|,$$

so by (3.2) we conclude that B is a contraction.

Let us prove that A is compact and continuous. The continuity of f implies that A is continuous. Also A is uniformly bounded on B_R, indeed, from (3.3) we have

$$\|Au\| \leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} \|u\| + \frac{\varpi}{\Gamma(\alpha + \beta)} \leq \frac{\theta^*(1 + \lambda_0 + \gamma_0)}{\Gamma(\alpha + \beta)} R + \frac{\varpi}{\Gamma(\alpha + \beta)}.$$

Set $L = \max_{0 \leq s \leq 1}\{|f(s, u(s), \varphi u(s), \psi u(s))|, u \in B_R\}$. Let $u \in B_R$, $t_1, t_2 \in I$, with $t_1 \leq t_2$. We have

$$\begin{aligned}
&\left|Au(t_2) - Au(t_1)\right| \\
&\quad = \frac{1}{\Gamma(\alpha + \beta)} \left| \int_0^{t_2} (t_2 - s)^{\beta+\alpha-1} f\left(s, u(s), \varphi u(s), \psi u(s)\right) ds\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^{t_1} (t_1 - s)^{\beta+\alpha-1} f\bigl(s, u(s), \varphi u(s), \psi u(s)\bigr)\, ds \Big| \\
&\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_1} \bigl((t_2 - s)^{\beta+\alpha-1} - (t_1 - s)^{\beta+\alpha-1}\bigr) \bigl| f\bigl(s, u(s), \varphi u(s), \psi u(s)\bigr)\bigr|\, ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta+\alpha-1} \bigl| f\bigl(s, u(s), \varphi u(s), \psi u(s)\bigr)\bigr|\, ds \\
&\leq \frac{L}{\Gamma(\alpha+\beta)} \int_0^{t_1} \bigl((t_2 - s)^{\beta+\alpha-1} - (t_1 - s)^{\beta+\alpha-1}\bigr)\, ds + \frac{L}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta+\alpha-1}\, ds \\
&= \frac{L}{\Gamma(\alpha+\beta+1)} \bigl(t_2^{\beta+\alpha} - t_1^{\beta+\alpha}\bigr).
\end{aligned}
$$

Hence, if $t_2 \to t_1$, then $|Au(t_2) - Au(t_1)| \to 0$. Then A is equicontinuous and so, by Arzela–Ascoli theorem, we deduce that A is compact on B_R. So the operator A is completely continuous. Thus, by Theorem 3.1, problem (1.1) has at least one solution in X. The proof is complete. □

Example 3.1 We consider the boundary value problem (1.1) with $f(t, x_1, x_2, x_3) = \frac{t^{\frac{1}{2}} e^{-t}}{4} \times \sum_{i=1}^{3} \frac{1}{1+x_i^2(t)}$, $\alpha = \frac{9}{5}$, $\beta = \frac{3}{5}$. Also we have $f(t,0,0,0) = \frac{3t^{\frac{1}{2}} e^{-t}}{4}$ thus $\varpi = 0.31$. Let $\lambda(t,s) = \gamma(t,s) = ts$, so that $\gamma_0 = \lambda_0 = \frac{1}{2}$.

Moreover, we can verify that condition (3.1) is satisfied

$$
\begin{aligned}
&\bigl| f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\bigr| \\
&\quad \leq \frac{t^{\frac{1}{2}} e^{-t}}{4} \sum_{i=1}^{3} \left(\left| \frac{1}{1+x_i^2} - \frac{1}{1+y_i^2} \right| \right) \\
&\quad \leq \frac{t^{\frac{1}{2}} e^{-t}}{4} \sum_{i=1}^{3} \frac{|x_i - y_i||x_i + y_i|}{(1+x_i^2)(1+y_i^2)} \\
&\quad \leq \frac{t^{\frac{1}{2}} e^{-t}}{4} \sum_{i=1}^{3} |x_i - y_i|,
\end{aligned}
$$

so $\theta(t) = \frac{t^{\frac{1}{2}} e^{-t}}{4}$ and $\theta^* = \frac{0.37894}{4}$. Also, condition (3.2) holds:

$$
\frac{(1+\gamma_0+\lambda_0)(\alpha+2\beta+1)\theta^*}{\Gamma(\alpha+\beta)} = 0.61013 < 1.
$$

Therefore, by Theorem 3.2, the problem has at least one solution in B_R with

$$
R \geq \frac{\varpi(\alpha+2\beta+1)}{1-\theta^*(1+\lambda_0+\gamma_0)(\alpha+2\beta+1)} = 5.1214
$$

Example 3.2 Consider the boundary value problem (1.1) with

$$
f(t,x,y,z) = 10^{-2}\left(t\sin x + e^t \sin 2y + \frac{1+t^2}{1+z^2} \right),
$$

$\alpha = 1.3$, $\beta = 0.4$. Then $f(t,0,0,0) = 10^{-2}(1+t^2)$, thus $\varpi = 0.02$. Let $\lambda(t,s) = e^{t-s}$, $\gamma(t,s) = (t-s)^{\beta}$, thus $\gamma_0 = 1.7183$, $\lambda_0 = 0.71429$. Condition (3.1) is satisfied, in fact,

$$\bigl|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\bigr| \le 0.02e^t \sum_{i=1}^{3} |x_i - y_i|.$$

We choose $\theta(t) = 0.02e^t$ then $\theta^* = 3.4366 \times 10^{-2}$. We check condition (3.2):

$$\frac{(1+\gamma_0+\lambda_0)(\alpha+2\beta+1)\theta^*}{\Gamma(\alpha+\beta)} = 0.40246 < 1.$$

We conclude, by Theorem 3.2, that the problem has at least one solution in B_R with

$$R \ge \frac{\varpi(\alpha+2\beta+1)}{1-\theta^*(1+\lambda_0+\gamma_0)(\alpha+2\beta+1)} = 9.7744 \times 10^{-2}.$$

Conclusion. In the present work, we have studied the existence of solutions for a fractional sequential boundary value problem. To demonstrate the existence results, we transformed the posed problem into a sum of a contraction and a compact operator, then we applied the Krasnoselskii's fixed point theorem. We ended the article with some numerical examples illustrating the obtain results.

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References

1. Ahmad, B., Ntouyas, S.K.: Integro-differential equations of fractional order with nonlocal fractional boundary conditions associated with financial asset model. Electron. J. Differ. Equ. **2013**(60), 1 (2013)
2. Ahmad, B., Ntouyas, S.K.: Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions. Appl. Math. Comput. **266**, 615–622 (2015)
3. Ahmad, B., Ntouyas, S.K., Agarwal, R., Alsaedi, A.: Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions. Bound. Value Probl. **2016**, 205 (2016)
4. Alsaedi, A., Ahmad, B.: Existence of solutions for nonlinear fractional integro-differential equations with three-point nonlocal fractional boundary conditions. Adv. Differ. Equ. **2010**, 691721 (2010)
5. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020)

6. Baleanu, D., Ghafarnezhad, K., Rezapour, S.: On a three step crisis integro-differential equation. Adv. Differ. Equ. **2019**(1), 153 (2019)
7. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020)
8. Baleanu, D., Khadijeh, G., Shahram, R., Mehdi, S.: On the existence of solutions of a three steps crisis integro-differential, equation. Adv. Differ. Equ. **2018**(1), 135 (2018)
9. Baleanu, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of CD4+ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. **2020**, 71 (2020)
10. Baleanu, D., Mohammadi, H., Rezapour, S.: A mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model. Adv. Differ. Equ. **2020**, 184 (2020)
11. Baleanu, D., Rezapour, S., Mohammadi, H.: Some existence results on nonlinear fractional differential equation. Philos. Trans. R. Soc. A **371**, 20120144 (1990)
12. Baleanu, D., Rezapour, S., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional, Caputo–Fabrizio derivation. Bound. Value Probl. **2019**, 79 (2019)
13. Baleanu, D., Zahra Nazemi, S., Rezapour, S.: Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations. Adv. Differ. Equ. **2013**, 368 (2013)
14. Debnath, L.: Recent applications of fractional calculus to science and engineering. Int. J. Math. Math. Sci. **54**, 3413–3442 (2003)
15. Ferreira, R.A.C., Pinto, G.: Lyapunov-type inequalities for some sequential fractional boundary value problems. Adv. Dyn. Syst. Appl. **11**(1), 33–43 (2016)
16. Guezane Lakoud, A., Khaldi, R., Kılıçman, A.: Solvability of a boundary value problem at resonance. SpringerPlus **5**, 1504 (2016)
17. Guezane-Lakoud, A., Khaldi, R.: Solutions for a nonlinear fractional Euler–Lagrange type equation. SeMA **76**, 195 (2019). https://doi.org/10.1007/s40324-018-0170-4
18. Guezane-Lakoud, A., Rodríguez-López, R.: On a fractional boundary value problem in a weighted space. SeMA **75**(3), 435–443 (2018)
19. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
20. Khaldi, R., Guezane-Lakoud, A.: Upper and lower solutions, method for fractional oscillation equations. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **43**(2), 214–220 (2017)
21. Khaldi, R., Guezane-Lakoud, A.: On generalized nonlinear Euler–Bernoulli beam type equations. Acta Univ. Sapientiae Math. **10**(1), 90–100 (2018)
22. Khaldi, R., Guezane-Lakoud, A.: On a generalized Lyapunov inequality for a mixed fractional boundary value problem. AIMS Math. **4**(3), 506–515 (2019). https://doi.org/10.3934/math.2018.3.506
23. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Boston (2006)
24. Liu, L., Zhang, X., Jiang, J., Wu, Y.: The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems. J. Nonlinear Sci. Appl. **9**, 2943–2958 (2016)
25. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
26. Momani, S.M., Hadid, S.B.: Some comparison results for integro-fractional differential inequalities. J. Fract. Calc. **24**, 379–387 (2003)
27. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering. Academic Press, San Diego (1999)
28. Rezapour, S., Samei, M.E.: On the existence of solutions for a multi-singular point-wise defined fractional q-integro-differential equation. Bound. Value Probl. **2020**, 38 (2020)
29. Shahram, A., Baleanu, D., Shahram, R.: Analyzing transient response of the parallel RCL circuit by using the Caputo–Fabrizio fractional derivative. Adv. Differ. Equ. **2020**, 55 (2020)
30. Su, X., Zhang, S.: Solutions to boundary value problems for nonlinear differential equations of fractional order. Electron. J. Differ. Equ. **2009**, 26 (2009)
31. Wang, Y., Liu, L.: Uniqueness and existence of positive solutions for the fractional integro-differential equation. Bound. Value Probl. **2017**, 12 (2017)
32. Xinwei, S., Landong, L.: Existence of solution for boundary value problem of nonlinear fractional differential equation. Appl. Math. J. Chin. Univ. Ser. B **22**(3), 291–298 (2007)
33. Zhou, H., Alzabut, J., Rezapour, S., Samei, M.E.: Uniform persistence and almost periodic solutions of a nonautonomous patch occupancy model. Adv. Differ. Equ. **2020**, 143 (2020)