Abdeljawad et al. *Advances in Difference Equations* (2020) 2020:330
https://doi.org/10.1186/s13662-020-02782-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On new fractional integral inequalities for p-convexity within interval-valued functions

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Abstract

This work mainly investigates a class of convex interval-valued functions via the Katugampola fractional integral operator. By considering the p-convexity of the interval-valued functions, we establish some integral inequalities of the Hermite–Hadamard type and Hermite–Hadamard–Fejér type as well as some product inequalities via the Katugampola fractional integral operator. In addition, we compare our results with the results given in the literature. Applications of the main results are illustrated by using examples. These results may open a new avenue for modeling, optimization problems, and fuzzy interval-valued functions that involve both discrete and continuous variables at the same time.

MSC: 26A51; 26D15; 05A30

Keywords: p-convexity; Katugampola fractional integral operator; Interval-valued function; Hermite–Hadamard inequality

1 Introduction

Fractional calculus [1–21] is invariably important in almost all areas of mathematics and other natural sciences. Indeed, we can clearly realize that fractional operators have appeared in all fields of natural science and in fractional differential equations [22–35]. In particular, it has been used in the study of waves in liquids, propagation of sound, gravitational attraction, and vibrations of strings. Numerous significant definitions and concepts have been established for the investigation of the fractional operators, for instance, Riemann, Liouville, Caputo, Hadamard, Katugampola, Atangana–Baleanu operators, and so on. Some well-known operators have been utilized for finding the existence of solutions to the boundary value problems, fractional integrodifferential equations or inclusions were elaborated [36–40].

In the present scenario, numerous significant fractional derivative and integral operators are systematically and successfully analyzed with the assistance of fractional integral inequalities [41–50]. It is known that variants have many important applications in all parts of mathematics as well as in different areas of natural science. Among others, numerous sorts of variants, those conveying the names of Jensen, Hermite–Hadamard, Hardy, Ostrowski, Minkowski, and Opial et al., have a profound noteworthiness; also, they have an extraordinary effect in significant fields of research. Convexity [51–60] has received renewed attention in mathematical sciences, statistical theory, optimization theory, fixed point theory, and several other areas of science and technology. Over the years, convex sets and convex functions have been modified to a remarkable variety of convexities such as $H_{p,q}$-convexity [61–64], harmonic convexity [65], strong convexity [66, 67], Schur convexity [68, 69], quasi-convexity [70], generalized convexity [71], and so on. In particular, many inequalities can be found in the literature [72–93] via the convexity theory.

Recently, the following Hermite–Hadamard inequality [94], one of the famous distinguished classical inequalities, has gained much consideration.

Let $\mathcal{Q} : \mathcal{I} \to \mathbb{R}$ be a convex function. Then the double inequality

$$(f-e)\mathcal{Q}\left(\frac{e+f}{2}\right) \le \int_e^f \mathcal{Q}(z)\,dz \le (f-e)\frac{\mathcal{Q}(e)+\mathcal{Q}(f)}{2} \tag{1.1}$$

holds for all $e,f \in \mathcal{I}$ with $f \neq e$. If $\mathcal{Q}$ is concave, then both the inequalities in (1.1) hold in the reverse direction. Many generalizations, modifications, applications, refinements, and variants for Hermite–Hadamard inequality (1.1) can be found in the literature [95, 96].

The following weighted generalization of Hermite–Hadamard inequality (1.1) was derived by Fejér:

$$\mathcal{Q}\left(\frac{e+f}{2}\right)\int_e^f \mathcal{W}(z)\,dz \le \int_e^f \mathcal{Q}(z)\mathcal{W}(z)\,dz \le \frac{\mathcal{Q}(e)+\mathcal{Q}(f)}{2}\int_e^f \mathcal{W}(z)\,dz. \tag{1.2}$$

Due to the modification among the ideas of convexity, the refinements for double inequality (1.2) have been widely investigated by many researchers. To meet the development trend of this research field, we delineate a new scheme and future plan in the present framework. We consider the p-convex function which assumes a dynamic job in portraying the idea of the interval-valued function just as establishing several generalizations by employing the Katugampola fractional integral operator.

On the other hand, a long history that can be followed back to Archimede's computation of the circumference of a circle has based on the theory of interval analysis. It fell into obscurity for a long time because of the dearth of utilities to different sciences. To the preeminence of our understanding, the substantial effort did not seem to this extent until the 1950s. In 1966, the first celebrated monograph concerned with interval analysis was written by Moore, who is famous as the founder of intervals, in order to compute the error bounds of the numerical solutions of a finite state machine. After his exploration, several researchers focused on studying the literature and applications of interval analysis in automatic error analysis, computer graphics, neural network output optimization, robotics, computational physics, and several other well-known areas in science and technology. Since then, several analysts have been broadly concentrated on and investigated the interval analysis and interval-valued functions in both mathematics and its applications.

The principal objective of this article is that we propose the notion of p-convex function for the interval-valued function. We also present the results concerning Hermite–Hadamard inequality, Fejér type inequality, and certain other related variants by employing p-convexity, which correlates with the Katugampola fractional integral operator. Finally, the repercussions of the employed technique depict the presentations for various existing outcomes. Results obtained by the novel approach disclose that the suggested scheme is very accurate, flexible, effective, and simple to use.

2 Preliminaries

For the basic notions and definitions on interval analysis, we use the literature [97].

Let $\mathcal{M}$ be the space of all intervals of $\mathbb{R}$ and $\mathcal{D} \in \mathcal{M}$ be defined by

$$\mathcal{D} = [\underline{\mathfrak{d}}, \bar{\mathfrak{d}}] = \{z \in \mathbb{R} | \underline{\mathfrak{d}} \leq z \leq \bar{\mathfrak{d}}\} \quad (\underline{\mathfrak{d}}, \bar{\mathfrak{d}} \in \mathbb{R}).$$

Then $\mathcal{D}$ is said to be degenerate if $\underline{\mathfrak{d}} = \bar{\mathfrak{d}}$. If $\underline{\mathfrak{d}} > 0$, then $\mathcal{D}$ is said to be positive, and if $\bar{\mathfrak{d}} < 0$, then $\mathcal{D}$ is said to be negative. We use $\mathcal{M}^+$ and $\mathcal{M}^-$ to symbolize the sets of all positive and negative intervals.

Let $\eta \in \mathbb{R}$ and $\eta\mathcal{D}$ be defined by

$$\eta\mathcal{D} = \begin{cases} [\eta\underline{\mathfrak{d}}, \eta\bar{\mathfrak{d}}], & \eta \geq 0, \\ [\eta\bar{\mathfrak{d}}, \eta\underline{\mathfrak{d}}], & \eta < 0. \end{cases}$$

Then the addition $\mathcal{D}_1 + \mathcal{D}_2$ and Minkowski difference $\mathcal{D}_1 - \mathcal{D}_2$ for $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{M}$ are defined by

$$\mathcal{D}_1 + \mathcal{D}_2 = [\underline{\mathfrak{d}_1}, \bar{\mathfrak{d}_1}] + [\bar{\mathfrak{d}_2}, \underline{\mathfrak{d}_2}] = [\underline{\mathfrak{d}_1} + \underline{\mathfrak{d}_2}, \bar{\mathfrak{d}_1} + \bar{\mathfrak{d}_2}]$$

and

$$\mathcal{D}_1 - \mathcal{D}_2 = [\underline{\mathfrak{d}_1}, \bar{\mathfrak{d}_1}] - [\bar{\mathfrak{d}_2}, \underline{\mathfrak{d}_2}] = [\underline{\mathfrak{d}_1} - \underline{\mathfrak{d}_2}, \bar{\mathfrak{d}_1} - \bar{\mathfrak{d}_2}],$$

respectively.

The inclusion relation "$\supseteq$" means that

$$\mathcal{D}_2 \supseteq \mathcal{D}_1 \quad \Longleftrightarrow \quad [\underline{\mathfrak{d}_2}, \bar{\mathfrak{d}_2}] \supseteq [\underline{\mathfrak{d}_1}, \bar{\mathfrak{d}_1}] \quad \Longleftrightarrow \quad \underline{\mathfrak{d}_1} \geq \underline{\mathfrak{d}_2}, \bar{\mathfrak{d}_2} \geq \bar{\mathfrak{d}_1}.$$

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval and $\mathcal{Q}(z) = [\underline{\mathcal{Q}}(z), \bar{\mathcal{Q}}(z)]$ $(z \in \mathcal{I})$. Then $\mathcal{Q}(z)$ is said to be Lebesgue integrable if $\underline{\mathcal{Q}}(z)$ and $\bar{\mathcal{Q}}(z)$ are measurable and Lebesgue integrable on $\mathcal{I}$. Moreover, $\int_e^f \mathcal{Q}(z)\,dz$ is defined by

$$\int_e^f \mathcal{Q}(z)\,dz = \left[\int_e^f \underline{\mathcal{Q}}(z)\,dz + \int_e^f \bar{\mathcal{Q}}(z)\,dz\right]. \tag{2.1}$$

Now, we introduce the concept of Katugampola fractional integral operator for interval-valued function.

Let $q \geq 1$, $c \in \mathbb{R}$, and $\chi_c^q(e,f)$ be the set of all complex-valued Lebesgue integrable interval-valued functions $\mathcal{Q}$ on $[e,f]$ for which the norm $\|\mathcal{Q}\|\chi_c^q$ is defined by

$$\|\mathcal{Q}\|\chi_c^q = \left(\int_e^f \left|\eta^c \mathcal{Q}(\eta)\right|^q \frac{d\eta}{\eta}\right)^{\frac{1}{q}} < \infty \tag{2.2}$$

for $1 \leq q < \infty$ and

$$\|\mathcal{Q}\|\chi_c^\infty = \operatorname*{ess\,sup}_{e \leq \eta \leq f} \eta^c \left|\mathcal{Q}(\eta)\right|. \tag{2.3}$$

Katugampola [98] presented a new fractional integral to generalize the Riemann–Liouville and Hadamard fractional integrals under certain conditions.

Let $p, \delta > 0$ and $\mathfrak{IL}_{([e,f])}$ be the collection of all complex-valued Lebesgue integrable interval-valued functions on $[e,f]$. Then the interval left and right Katugampola fractional integrals of $\mathcal{Q} \in \mathfrak{IL}_{([e,f])}$ with order $\delta > 0$ are defined by

$$\mathcal{J}^{p,\delta}_{e^+}\mathcal{Q}(z) = \frac{p^{1-\delta}}{\Gamma(\delta)}\int_e^z \left(z^p - \zeta^p\right)^{\delta-1}\zeta^{p-1}\mathcal{Q}(\zeta)\,d\zeta \quad (z > e) \tag{2.4}$$

and

$$\mathcal{J}^{p,\delta}_{f^-}\mathcal{Q}(z) = \frac{p^{1-\delta}}{\Gamma(\delta)}\int_z^f \left(\zeta^p - z^p\right)^{\delta-1}\zeta^{p-1}\mathcal{Q}(\zeta)\,d\zeta \quad (z < f), \tag{2.5}$$

respectively, where $\Gamma(z) = \int_0^\infty \zeta^{z-1}e^{-\zeta}\,d\zeta$ is the Euler gamma function [99].

In [100], Zhang and Wan presented a definition of the p-convex function as follows.

Definition 2.1 ([100]) Let $p \in \mathbb{R}$ with $p \neq 0$. Then the interval $\mathcal{I}$ is said to be p-convex if

$$\left[\eta e^p + (1-\eta)f^p\right]^{\frac{1}{p}} \in \mathcal{I} \tag{2.6}$$

for all $e,f \in \mathcal{I}$ and $\eta \in [0,1]$.

Definition 2.2 ([100]) Let $p \in \mathbb{R}$ with $p \neq 0$ and $\mathcal{I} \subseteq \mathbb{R}$ be a p-convex interval. Then the function $\mathcal{Q} : \mathcal{I} \to \mathbb{R}$ is said to be a p-convex function if the inequality

$$\mathcal{Q}\left(\left[\eta e^p + (1-\eta)f^p\right]^{\frac{1}{p}}\right) \le \eta\mathcal{Q}(e) + (1-\eta)\mathcal{Q}(f) \tag{2.7}$$

holds for all $e,f \in \mathcal{I}$ and $\eta \in [0,1]$.

From Definition 2.2 we clearly see that the p-convexity reduces to classical convexity and harmonic convexity if $p = 1$ and $p = -1$, respectively.

Next, we introduce a novel concept of interval p-convexity.

Definition 2.3 Let $p \in \mathbb{R}$ with $p \neq 0$ and $\mathcal{I} \subseteq \mathbb{R}$ be a p-convex interval. Then the function $\mathcal{Q} : \mathcal{I} \to \mathcal{M}^+$ is said to be a p-convex interval-valued function if

$$\mathcal{Q}\left(\left[\eta e^p + (1-\eta)f^p\right]^{\frac{1}{p}}\right) \supseteq \eta\mathcal{Q}(e) + (1-\eta)\mathcal{Q}(f) \tag{2.8}$$

for all $e,f \in \mathcal{I}$ and $\eta \in [0,1]$. If the set inclusion (2.8) is reversed, then $\mathcal{Q}$ is said to be a p-concave interval-valued function.

Remark 2.4 From Definition 2.3 we clearly see that

(1) If $p = 1$, then we get the definition given in [101].

(2) If $p = -1$, then Definition 2.3 reduces to Definition 3.1 of [102].

3 Results and discussions

In this section, we establish several Hermite–Hadamard type inequalities for the p-convex interval-valued functions by employing the Katugampola fractional integral operator. In what follows, we denote by $\mathcal{QC}(\mathcal{I},\mathcal{M}^+)$ the family of interval p-convex functions of the interval $\mathcal{I}$.

Theorem 3.1 *Let $p,\delta > 0$, $e,f \in \mathcal{I}$ such that $f > e$, $\mathcal{Q} \in \mathfrak{IL}_{([e,f])}$. Then*

$$\begin{aligned} \mathcal{Q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right) &\supseteq \frac{p^\delta \Gamma(\delta+1)}{2(f^p-e^p)^\delta}\left[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}(e)\right] \\ &\supseteq \frac{\mathcal{Q}(e)+\mathcal{Q}(f)}{2} \end{aligned} \tag{3.1}$$

if $\mathcal{Q} \in \mathcal{QC}(\mathcal{I},\mathcal{M}^+)$.

Proof It follows from $\mathcal{Q} \in \mathcal{QC}(\mathcal{I},\mathcal{M}^+)$ that

$$\mathcal{Q}\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right) \supseteq \frac{\mathcal{Q}(x)+\mathcal{Q}(y)}{2} \tag{3.2}$$

for all $x,y \in [e,f]$.

Let $\eta \in [0,1]$, $x^p = \eta e^p + (1-\eta)f^p$ and $y^p = (1-\eta)e^p + \eta f^p$. Then (3.2) leads to

$$2\mathcal{Q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right) \supseteq \mathcal{Q}\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)+\mathcal{Q}\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right). \tag{3.3}$$

Multiplying both sides (3.3) by $\eta^{\delta-1}$ and integrating the obtained result with respect to η over $(0,1)$, we have

$$\begin{aligned} &2\int_0^1 \eta^{\delta-1}\mathcal{Q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta \\ &\quad\supseteq \int_0^1 \eta^{\delta-1}\left[\mathcal{Q}\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)+\mathcal{Q}\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\right]d\eta. \end{aligned} \tag{3.4}$$

From (2.1) and (3.4), we get

$$\begin{aligned} &2\int_0^1 \eta^{\delta-1}\mathcal{Q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta \\ &\quad= 2\left[\int_0^1 \eta^{\delta-1}\underline{q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta, \int_0^1 \eta^{\delta-1}\bar{q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta\right] \\ &\quad= 2\left[\frac{1}{\delta}\underline{q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right), \frac{1}{\delta}\bar{q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\right] \\ &\quad= 2\frac{1}{\delta}\mathcal{Q}\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\int_0^1 \eta^{\delta-1}\big[\mathcal{Q}\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big) + \mathcal{Q}\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\big]\,d\eta \\
&\quad = \biggl[\int_0^1 \eta^{\delta-1}\big[\underline{q}\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big), \underline{q}\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\big]\,d\eta \\
&\qquad + \int_0^1 \eta^{\delta-1}\big[\bar{q}\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big), \bar{q}\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\big]\,d\eta\biggr] \\
&\quad = \frac{p}{(f^p - e^p)^\delta}\biggl[\int_e^f \big(f^p - y^p\big)^{\delta-1}\frac{\underline{q}(y)}{y^{1-p}}\,dy + \int_e^f \big(x^p - e^p\big)^{\delta-1}\frac{\underline{q}(x)}{x^{1-p}}\,dx, \\
&\qquad \int_e^f \big(f^p - y^p\big)^{\delta-1}\frac{\bar{q}(y)}{y^{1-p}}\,dy + \int_e^f \big(x^p - e^p\big)^{\delta-1}\frac{\bar{q}(x)}{x^{1-p}}\,dx\biggr] \\
&\quad = \frac{p}{(f^p - e^p)^\delta}\biggl[\int_e^f \big(f^p - y^p\big)^{\delta-1}\frac{\mathcal{Q}(y)}{y^{1-p}}\,dy + \int_e^f \big(x^p - e^p\big)^{\delta-1}\frac{\mathcal{Q}(x)}{x^{1-p}}\,dx\biggr] \\
&\quad \supseteq \frac{p^\delta\Gamma(\delta)}{(f^p - e^p)^\delta}\big[\mathcal{J}^{p,\delta}_{e^+}\mathcal{Q}(f) + \mathcal{J}^{p,\delta}_{f^-}\mathcal{Q}(e)\big].
\end{aligned} \tag{3.6}$$

Since $\mathcal{Q} \in \mathcal{QC}(\mathcal{I}, \mathcal{M}^+)$, we get

$$\mathcal{Q}\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big) \supseteq \eta\mathcal{Q}(e) + (1-\eta)\mathcal{Q}(f) \tag{3.7}$$

and

$$\mathcal{Q}\big(\big[\eta f^p + (1-\eta)e^p\big]^{\frac{1}{p}}\big) \supseteq \eta\mathcal{Q}(f) + (1-\eta)\mathcal{Q}(e). \tag{3.8}$$

Adding (3.7) and (3.8), we obtain

$$\mathcal{Q}\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big) + \mathcal{Q}\big(\big[\eta f^p + (1-\eta)e^p\big]^{\frac{1}{p}}\big) \supseteq \mathcal{Q}(e) + \mathcal{Q}(f). \tag{3.9}$$

Multiplying both sides (3.9) by $\eta^{\delta-1}$ and integrating both sides of the obtained result with respect to η over $(0, 1)$, we get

$$\frac{p^\delta\Gamma(\delta)}{(f^p - e^p)^\delta}\big[\mathcal{J}^{p,\delta}_{e^+}\mathcal{Q}(f) + \mathcal{J}^{p,\delta}_{f^-}\mathcal{Q}(e)\big] \supseteq \frac{\mathcal{Q}(e) + \mathcal{Q}(f)}{\delta},$$

which completes the proof of Theorem 3.1. □

Remark 3.2 From Theorem 3.1 we clearly see that

(1) Let $\underline{q} = \bar{q}$. Then we get Theorem 2.1 [103].
(2) If $p = 1$ and $\underline{q} = \bar{q}$, then Theorem 3.1 reduces to the result given in [104].
(3) If $\delta = p = 1$ and $\underline{q} = \bar{q}$, then Theorem 3.1 becomes the result in [105].

Example 3.3 Let p be an odd number, $\delta = \frac{1}{2}$, $u \in [2, 3]$, and $\mathcal{Q}(u) = [2 - u^{\frac{p}{2}}, u^{\frac{p}{2}} + 2]$. Then we clearly see that $\mathcal{Q} \in \mathfrak{J}\mathfrak{L}_{([2,3])}$ and

$$\mathcal{Q}\biggl(\biggl[\frac{e^p + f^p}{2}\biggr]^{\frac{1}{p}}\biggr) = \mathcal{Q}(2.5) = \biggl[\frac{4 - \sqrt{10}}{2}, \frac{4 + \sqrt{10}}{2}\biggr],$$

$$\frac{[\mathcal{Q}(e) + \mathcal{Q}(f)]}{2} = \biggl[2 - \frac{\sqrt{2} + \sqrt{3}}{2}, 2 + \frac{\sqrt{2} + \sqrt{3}}{2}\biggr].$$

Note that

$$
\begin{aligned}
&\frac{p^\delta \Gamma(\delta+1)}{2(f^p-e^p)^\delta}\big[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}(e)\big] \\
&\quad=\frac{\Gamma(\frac{3}{2})}{2}\bigg[\frac{1}{\sqrt{\pi}}\int_2^3\big(3^p-u^p\big)^{\frac{-1}{2}}u^{p-1}\big[2-u^{\frac{p}{2}},u^{\frac{p}{2}}+2\big]\,du \\
&\qquad+\frac{1}{\sqrt{\pi}}\int_2^3\big(u^p-2^p\big)^{\frac{-1}{2}}u^{p-1}\big[2-u^{\frac{p}{2}},u^{\frac{p}{2}}+2\big]\,du\bigg] \\
&\quad=\frac{1}{4}\bigg[\bigg[\frac{7393}{10{,}000},\frac{7260}{1000}\bigg]+\bigg[\frac{9501}{10{,}000},\frac{7049}{1000}\bigg]\bigg] \\
&\quad=\bigg[\frac{8447}{20{,}000},\frac{14{,}309}{4000}\bigg].
\end{aligned}
$$

Therefore,

$$
\bigg[\frac{4-\sqrt{10}}{2},\frac{4+\sqrt{10}}{2}\bigg]\supseteq\bigg[\frac{8447}{20{,}000},\frac{14{,}309}{4000}\bigg]\supseteq\bigg[2-\frac{\sqrt{2}+\sqrt{3}}{2},2+\frac{\sqrt{2}+\sqrt{3}}{2}\bigg]
$$

and Theorem 3.1 is verified.

The next Theorem 3.4 gives the Hermite–Hadamard–Fejér type inequality for interval-valued p-convex functions.

Theorem 3.4 *Let $p,\delta>0$, $e,f\in\mathcal{I}$ with $f>e$, $\mathcal{Q}\in\mathfrak{IL}_{([e,f])}$, and $\mathcal{W}(x)=\mathcal{W}([e^p+f^p-x^p]^{\frac{1}{p}})\geq 0$ for $x\in\mathcal{I}$. Then we have the Hermite–Hadamard–Fejér type inequality for interval-valued p-convex functions as follows*:

$$
\begin{aligned}
&\mathcal{Q}\bigg(\bigg[\frac{e^p+f^p}{2}\bigg]^{\frac{1}{p}}\bigg)\big[\mathcal{J}_{e^+}^{p,\delta}\mathcal{W}(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{W}(e)\big] \\
&\quad\supseteq\big[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}\mathcal{W}(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}\mathcal{W}(e)\big] \\
&\quad\supseteq\frac{\mathcal{Q}(e)+\mathcal{Q}(f)}{2}\big[\mathcal{J}_{e^+}^{p,\delta}\mathcal{W}(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{W}(e)\big]
\end{aligned}
\tag{3.10}
$$

if $\mathcal{Q}\in\mathcal{QC}(\mathcal{I},\mathcal{M}^+)$.

Proof Since $\mathcal{W}$ is nonnegative, integrable, and p-symmetric with respect to $[\frac{e^p+f^p}{2}]^{\frac{1}{p}}$, we get

$$
\mathcal{W}\big(\big[\eta e^p+(1-\eta)f^p\big]^{\frac{1}{p}}\big)=\mathcal{W}\big(\big[\eta f^p+(1-\eta)e^p\big]^{\frac{1}{p}}\big).
$$

Multiplying both sides of (3.4) by $\eta^{\delta-1}\mathcal{W}([\eta f^p+(1-\eta)u^p]^{\frac{1}{p}})$, we have

$$
\begin{aligned}
&2\int_0^1\eta^{\delta-1}\mathcal{Q}\bigg(\bigg[\frac{e^p+f^p}{2}\bigg]^{\frac{1}{p}}\bigg)\mathcal{W}\big(\big[\eta f^p+(1-\eta)e^p\big]^{\frac{1}{p}}\big)\,d\eta \\
&\quad\supseteq\int_0^1\eta^{\delta-1}\mathcal{Q}\big(\big[\eta e^p+(1-\eta)f^p\big]^{\frac{1}{p}}\big)\mathcal{W}\big(\big[\eta f^p+(1-\eta)e^p\big]^{\frac{1}{p}}\big)\,d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1 \eta^{\delta-1}\mathcal{Q}\big(\big[(1-\eta)e^p+\eta f^p\big]^{\frac{1}{p}}\big)\mathcal{W}\big(\big[\eta f^p+(1-\eta)e^p\big]^{\frac{1}{p}}\big)\,d\eta \\
&=\int_0^1 \eta^{\delta-1}\big[\underline{q}\big(\big[\eta e^p+(1-\eta)f^p\big]^{\frac{1}{p}}\big)+\underline{q}\big(\big[(1-\eta)e^p+\eta f^p\big]^{\frac{1}{p}}\big)\big] \\
&\quad\times\mathcal{W}\big(\big[\eta f^p+(1-\eta)e^p\big]^{\frac{1}{p}}\big)\,d\eta \\
&\quad+\int_0^1 \eta^{\delta-1}\big[\bar{q}\big(\big[\eta e^p+(1-\eta)f^p\big]^{\frac{1}{p}}\big)+\bar{q}\big(\big[(1-\eta)e^p+\eta f^p\big]^{\frac{1}{p}}\big)\big] \\
&\quad\times\mathcal{W}\big(\big[\eta f^p+(1-\eta)e^p\big]^{\frac{1}{p}}\big)\,d\eta.
\end{aligned}
$$

Let $u^p=\eta f^p+(1-\eta)e^p$. Then one has

$$
\begin{aligned}
&\frac{2p}{(f^p-e^p)^\delta}\mathcal{Q}\bigg(\bigg[\frac{e^p+f^p}{2}\bigg]^{\frac{1}{p}}\bigg)\int_0^1\big(u^p-e^p\big)^{\delta-1}\mathcal{W}(u)\,du \\
&\supseteq\frac{p}{(f^p-e^p)^\delta}\bigg[\int_e^f\big(u^p-e^p\big)^{\delta-1}\underline{q}\big(\big[e^p+f^p-u^p\big]^{\frac{1}{p}}\big)\mathcal{W}(u)u^{p-1}\,du \\
&\quad+\int_e^f\big(u^p-e^p\big)^{\delta-1}\underline{q}(u)\mathcal{W}(u)u^{p-1}\,du, \\
&\quad\int_e^f\big(u^p-e^p\big)^{\delta-1}\bar{q}\big(\big[e^p+f^p-u^p\big]^{\frac{1}{p}}\big)\mathcal{W}(u)u^{p-1}\,du \\
&\quad+\int_e^f\big(u^p-e^p\big)^{\delta-1}\bar{q}(u)\mathcal{W}(u)u^{p-1}\,du\bigg] \\
&=\frac{p}{(f^p-e^p)^\delta}\bigg[\int_e^f\big(f^p-u^p\big)^{\delta-1}\underline{q}(u)\mathcal{W}\big(\big[e^p+f^p-u^p\big]^{\frac{1}{p}}\big)u^{p-1}\,du \\
&\quad+\int_e^f\big(u^p-e^p\big)^{\delta-1}\underline{q}(u)\mathcal{W}(u)u^{p-1}\,du, \\
&\quad\int_e^f\big(f^p-u^p\big)^{\delta-1}\bar{q}(u)\mathcal{W}\big(\big[e^p+f^p-u^p\big]^{\frac{1}{p}}\big)u^{p-1}\,du \\
&\quad+\int_e^f\big(u^p-e^p\big)^{\delta-1}\bar{q}(u)\mathcal{W}(u)u^{p-1}\,du\bigg] \\
&=\frac{p}{(f^p-e^p)^\delta}\bigg[\int_e^f\big(f^p-u^p\big)^{\delta-1}\underline{q}(u)\mathcal{W}(u)u^{p-1}\,du \\
&\quad+\int_e^f\big(u^p-e^p\big)^{\delta-1}\underline{q}(u)\mathcal{W}(u)u^{p-1}\,du, \\
&\quad\int_e^f\big(f^p-u^p\big)^{\delta-1}\bar{q}(u)\mathcal{W}(u)u^{p-1}\,du \\
&\quad+\int_e^f\big(u^p-e^p\big)^{\delta-1}\bar{q}(u)\mathcal{W}(u)u^{p-1}\,du\bigg] \\
&=\frac{p}{(f^p-e^p)^\delta}\bigg[\int_e^f\big(f^p-u^p\big)^{\delta-1}\mathcal{Q}(u)\mathcal{W}(u)u^{p-1}\,du \\
&\quad+\int_e^f\big(u^p-e^p\big)^{\delta-1}\mathcal{Q}(u)\mathcal{W}(u)u^{p-1}\,du\bigg].
\end{aligned}
\tag{3.11}
$$

Therefore,

$$\begin{aligned} &\frac{p^\delta \Gamma(\delta)}{(f^p - e^p)^\delta} \mathcal{Q}\left(\left[\frac{e^p + f^p}{2}\right]^{\frac{1}{p}}\right)\left[\mathcal{J}_{e^+}^{p,\delta}\mathcal{W}(f) + \mathcal{J}_{f^-}^{p,\delta}\mathcal{W}(e)\right] \\ &\quad\supseteq \frac{p^\delta \Gamma(\delta)}{(f^p - e^p)^\delta}\left[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}\mathcal{W}(f) + \mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}\mathcal{W}(e)\right]. \end{aligned} \tag{3.12}$$

Multiplying both sides of (3.9) by $\eta^{\delta-1}\mathcal{W}([\eta f^p + (1-\eta)u^p]^{\frac{1}{p}})$, we get

$$\begin{aligned} &\int_0^1 \eta^{\delta-1}\mathcal{W}\left(\left[\eta f^p + (1-\eta)u^p\right]^{\frac{1}{p}}\right)\mathcal{Q}\left(\left[\eta e^p + (1-\eta)f^p\right]^{\frac{1}{p}}\right) d\eta \\ &\qquad + \int_0^1 \eta^{\delta-1}\mathcal{W}\left(\left[\eta f^p + (1-\eta)u^p\right]^{\frac{1}{p}}\right)\mathcal{Q}\left(\left[\eta f^p + (1-\eta)e^p\right]^{\frac{1}{p}}\right) d\eta \\ &\quad\supseteq \left[\mathcal{Q}(e) + \mathcal{Q}(f)\right]\int_0^1 \eta^{\delta-1}\mathcal{W}\left(\left[\eta f^p + (1-\eta)u^p\right]^{\frac{1}{p}}\right) d\eta. \end{aligned} \tag{3.13}$$

□

Remark 3.5 Theorem 3.4 leads to the conclusion that

(1) Let $\mathcal{W}(x) = 1$. Then we get Theorem 3.1.
(2) If $\underline{q} = \bar{q}$ and $\delta = 1$, then we get Theorem 5 of [106].
(3) Let $\underline{q} = \bar{q}$ and $\mathcal{W}(x) = p = \delta = 1$. Then we get the classical Hermite–Hadamard inequality (1.1).
(4) If $\underline{q} = \bar{q}$ and $\delta = 1$, then we obtain the classical Hermite–Hadamard–Fejér type inequality (1.2).

Theorem 3.6 *Let $p, \delta > 0$, $e, f \in \mathcal{I}$ with $f > e$, and $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathfrak{J}\mathfrak{L}_{([e,f])}$. Then we have*

$$\begin{aligned} &\frac{p^\delta \Gamma(1+\delta)}{2(f^p - e^p)^\delta}\left[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}_1(f)\mathcal{Q}_2(f) + \mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}_1(e)\mathcal{Q}_2(e)\right] \\ &\quad\supseteq \left(\frac{1}{2} - \frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_1(e,f) + \left(\frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_2(e,f) \end{aligned} \tag{3.14}$$

if $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{QC}(\mathcal{I}, \mathcal{M}^+)$, where

$$\Upsilon_1(e,f) = \left[\mathcal{Q}_1(e)\mathcal{Q}_2(e) + \mathcal{Q}_1(f)\mathcal{Q}_2(f)\right] \tag{3.15}$$

and

$$\Upsilon_2(e,f) = \left[\mathcal{Q}_1(e)\mathcal{Q}_2(f) + \mathcal{Q}_1(f)\mathcal{Q}_2(e)\right]. \tag{3.16}$$

Proof Let $\eta \in [0,1]$. Then it follows from the assumption of Theorem 3.6 that

$$\mathcal{Q}_1\left(\left[\eta e^p + (1-\eta)f^p\right]^{\frac{1}{p}}\right) \supseteq \eta\mathcal{Q}_1(e) + (1-\eta)\mathcal{Q}_1(f) \tag{3.17}$$

and

$$\mathcal{Q}_2\left(\left[\eta e^p + (1-\eta)f^p\right]^{\frac{1}{p}}\right) \supseteq \eta\mathcal{Q}_2(e) + (1-\eta)\mathcal{Q}_2(f). \tag{3.18}$$

From (3.17) and (3.18) we get

$$\begin{aligned}
&\mathcal{Q}_1\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big) \\
&\quad\supseteq \eta^2\mathcal{Q}_1(e)\mathcal{Q}_2(e) + (1-\eta)^2\mathcal{Q}_1(f)\mathcal{Q}_2(f) \\
&\qquad + \eta(1-\eta)\big[\mathcal{Q}_1(f)\mathcal{Q}_2(e) + \mathcal{Q}_1(e)\mathcal{Q}_2(f)\big].
\end{aligned} \tag{3.19}$$

Analogously, we have

$$\begin{aligned}
&\mathcal{Q}_1\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big) \\
&\quad\supseteq \eta^2\mathcal{Q}_1(f)\mathcal{Q}_2(f) + (1-\eta)^2\mathcal{Q}_2(f)\mathcal{Q}_1(f) \\
&\qquad + \eta(1-\eta)\big[\mathcal{Q}_2(f)\mathcal{Q}_1(e) + \mathcal{Q}_2(e)\mathcal{Q}_1(f)\big].
\end{aligned} \tag{3.20}$$

Adding (3.19) and (3.20), we obtain

$$\begin{aligned}
&\mathcal{Q}_1\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big) \\
&\qquad + \mathcal{Q}_1\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big) \\
&\quad\supseteq \big[\eta^2 + (1-\eta)^2\big]\big[\mathcal{Q}_1(e)\mathcal{Q}_2(e) + \mathcal{Q}_1(f)\mathcal{Q}_2(f)\big] \\
&\qquad + 2\eta(1-\eta)\big[\mathcal{Q}_1(f)\mathcal{Q}_2(e) + \mathcal{Q}_1(e)\mathcal{Q}_2(f)\big].
\end{aligned} \tag{3.21}$$

Multiplying both sides of (3.21) by $\eta^{\delta-1}$ and integrating the obtained result with respect to η over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big)\,d\eta \\
&\qquad + \int_0^1 \eta^{\delta-1}\mathcal{Q}_1\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\,d\eta \\
&\quad\supseteq \Upsilon_1(e,f)\int_0^1 \eta^{\delta-1}\big[\eta^2 + (1-\eta)^2\big] + 2\Upsilon_2(e,f)\int_0^1 \eta^{\delta-1}\eta(1-\eta)\,d\eta.
\end{aligned} \tag{3.22}$$

From (2.1) and (3.22), we have

$$\begin{aligned}
&\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[\eta e^p + (1-\eta)f^p\big]^{\frac{1}{p}}\big)\,d\eta \\
&\qquad + \int_0^1 \eta^{\delta-1}\mathcal{Q}_1\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\mathcal{Q}_2\big(\big[(1-\eta)e^p + \eta f^p\big]^{\frac{1}{p}}\big)\,d\eta \\
&\quad= \frac{p^\delta\Gamma(\delta)}{(f^p - e^p)^\delta}\big[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}_1(f)\mathcal{Q}_2(f) + \mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}_1(e)\mathcal{Q}_2(e)\big]
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
&\Upsilon_1(e,f)\int_0^1 \eta^{\delta-1}\big[\eta^2 + (1-\eta)^2\big] + 2\Upsilon_2(e,f)\int_0^1 \eta^{\delta-1}\eta(1-\eta)\,d\eta \\
&\quad= \frac{2}{\delta}\left(\frac{1}{2} - \frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_1(e,f) + \frac{2}{\delta}\left(\frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_2(e,f).
\end{aligned} \tag{3.24}$$

Therefore, the desired result (3.14) follows from (3.22)–(3.24). □

Example 3.7 Let p be an odd number, $[e,f]=[0,2]$, $\delta=\frac{1}{2}$, $\mathcal{Q}_1(u)=[u^p,4-e^{u^p}]$, and $\mathcal{Q}_2(u)=[u^p,3-u^p]$. Then $\mathcal{Q}_1,\mathcal{Q}_2\in\mathfrak{IL}_{([0,2])}$ and

$$\begin{aligned}&\frac{p^\delta\Gamma(1+\delta)}{2(f^p-e^p)^\delta}\left[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}_1(f)\mathcal{Q}_2(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}_1(e)\mathcal{Q}_2(e)\right]\\&\quad=\frac{\Gamma(\frac{3}{2})}{2\sqrt{2}}\left[\frac{1}{\sqrt{\pi}}\int_0^2\left(2^p-u^p\right)^{-\frac{1}{2}}u^{p-1}\left[u^{2p},\left(4-e^{u^p}\right)\left(3-u^p\right)\right]du\right.\\&\qquad\left.+\frac{1}{\sqrt{\pi}}\int_0^2\left(u^p\right)^{-\frac{1}{2}}u^{p-1}\left[u^{2p},\left(4-e^{u^p}\right)\left(3-u^p\right)\right]du\right]\\&\quad\approx[1.4666,2.6446].\end{aligned}\tag{3.25}$$

Note that

$$\begin{aligned}\Upsilon_1(e,f)&=\left[\mathcal{Q}_1(e)\mathcal{Q}_2(e)+\mathcal{Q}_1(f)\mathcal{Q}_2(f)\right]=\left[4,13-e^2\right],\\\Upsilon_2(e,f)&=\left[\mathcal{Q}_1(e)\mathcal{Q}_2(f)+\mathcal{Q}_1(f)\mathcal{Q}_2(e)\right]=\left[0,15-3e^2\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left(\frac{1}{2}-\frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_1(e,f)+\left(\frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_2(e,f)\\&\quad=\frac{11}{30}\left[4,13-e^2\right]+\frac{2}{15}\left[0,15-3e^2\right]\approx[1.4666,1.1017].\end{aligned}\tag{3.26}$$

It follows that

$$[1.4666,2.6446]\supseteq[1.4666,1.1017]$$

and Theorem 3.6 is verified.

Theorem 3.8 *Let $p,\delta>0$, $e,f\in\mathcal{I}$ with $f>e$, and $\mathcal{Q}_1,\mathcal{Q}_2\in\mathfrak{IL}_{([e,f])}$. Then*

$$\begin{aligned}&\mathcal{Q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\\&\quad\supseteq\frac{p^\delta\Gamma(\delta+1)}{4(f^p-e^p)^\delta}\left[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}_1(f)\mathcal{Q}_2(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}_1(e)\mathcal{Q}_2(e)\right]\\&\qquad+\frac{1}{2}\left(\frac{1}{2}-\frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_2(e,f)+\frac{\delta}{2(\delta+1)(\delta+2)}\Upsilon_1(e,f)\end{aligned}\tag{3.27}$$

if $\mathcal{Q}_1,\mathcal{Q}_2\in\mathcal{QC}(\mathcal{I},\mathcal{M}^+)$, where $\Upsilon_1(a,b)$ and $\Upsilon_2(a,b)$ are given in (3.15) and (3.16), respectively.

Proof Let $\eta\in[0,1]$. Then we clearly see that

$$\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)=\frac{[(1-\eta)e^p+\eta f^p]^{\frac{1}{p}}}{2}+\frac{[\eta e^p+(1-\eta)f^p]^{\frac{1}{p}}}{2}.\tag{3.28}$$

Since $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{QC}([e,f], \mathcal{K}^+)$, we have

$$\begin{aligned}
&\mathcal{Q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\\
&= \mathcal{Q}_1\left[\frac{[(1-\eta)e^p+\eta f^p]^{\frac{1}{p}}}{2}+\frac{[\eta e^p+(1-\eta)f^p]^{\frac{1}{p}}}{2}\right]\\
&\quad + \mathcal{Q}_2\left[\frac{[(1-\eta)e^p+\eta f^p]^{\frac{1}{p}}}{2}+\frac{[\eta e^p+(1-\eta)f^p]^{\frac{1}{p}}}{2}\right]\\
&\supseteq \frac{1}{4}\left[\mathcal{Q}_1\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)+\mathcal{Q}_1\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\right]\\
&\quad \times \left[\mathcal{Q}_2\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)+\mathcal{Q}_2\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\right]\\
&= \frac{1}{4}\left[\mathcal{Q}_1\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\right.\\
&\quad + \mathcal{Q}_1\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\Big]\\
&\quad + \mathcal{Q}_2\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_1\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\\
&\quad + \mathcal{Q}_1\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\\
&\supseteq \frac{1}{4}\left[\mathcal{Q}_1\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\right.\\
&\quad + \mathcal{Q}_1\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\Big]\\
&\quad + \frac{1}{4}\left(2\eta^2-2\eta+1\right)\Upsilon_2(e,f)+\frac{1}{2}\eta(1-\eta)\Upsilon_1(e,f).
\end{aligned} \tag{3.29}$$

Multiplying both sides of (3.29) by $\eta^{\delta-1}$ and integrating the obtained result with respect to η over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta\\
&\supseteq \frac{1}{4}\left[\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)d\eta\right.\\
&\quad \left.+\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)d\eta\right]\\
&\quad + \frac{1}{4}\int_0^1 \eta^{\delta-1}\left(2\eta^2-2\eta+1\right)\Upsilon_2(e,f)\,d\eta+\frac{1}{2}\int_0^1 \eta^{\delta-1}\eta(1-\eta)\Upsilon_1(e,f)\,d\eta.
\end{aligned} \tag{3.30}$$

From (2.1) and (3.30), we get

$$\begin{aligned}
&\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta\\
&= \left[\int_0^1 \eta^{\delta-1}\underline{q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\underline{q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta,\right.\\
&\quad \left.\int_0^1 \eta^{\delta-1}\bar{q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\bar{q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)d\eta\right]
\end{aligned}$$

$$= \left[\frac{1}{\delta}\underline{q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\underline{q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right), \frac{1}{\delta}\bar{q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\bar{q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\right]$$
$$= \frac{1}{\delta}\mathcal{Q}_1\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\frac{e^p+f^p}{2}\right]^{\frac{1}{p}}\right). \tag{3.31}$$

On the other hand, making suitable substitution and applying (2.1), we obtain

$$\begin{aligned}
&\frac{1}{4}\left[\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[\eta e^p+(1-\eta)f^p\right]^{\frac{1}{p}}\right)d\eta\right.\\
&\quad\left.+\int_0^1 \eta^{\delta-1}\mathcal{Q}_1\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)\mathcal{Q}_2\left(\left[(1-\eta)e^p+\eta f^p\right]^{\frac{1}{p}}\right)d\eta\right]\\
&\quad+\frac{1}{4}\int_0^1 \eta^{\delta-1}\left(2\eta^2-2\eta+1\right)\Upsilon_2(e,f)\,d\eta+\frac{1}{2}\int_0^1 \eta^{\delta-1}\eta(1-\eta)\Upsilon_1(e,f)\,d\eta\\
&=\frac{p}{4(f^p-e^p)^\delta}\left[\int_e^f \left(f^p-x^p\right)^{\delta-1}\underline{q}_1(x)\underline{q}_2(x)x^{p-1}\,dx+\int_e^f \left(x^p-e^p\right)^{\delta-1}\underline{q}_1(y)\underline{q}_2(y)y^{p-1}\,dy,\right.\\
&\quad\left.\int_e^f \left(f^p-x^p\right)^{\delta-1}\bar{q}_1(x)\bar{q}_2(x)x^{p-1}\,dx+\int_e^f \left(y^p-e^p\right)^{\delta-1}\bar{q}_1(y)\bar{q}_2(y)y^{p-1}\,dy\right]\\
&\quad+\frac{1}{2\delta}\left(\frac{1}{2}-\frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_2(e,f)+\frac{\delta}{2\delta(\delta+1)(\delta+2)}\Upsilon_1(e,f)\\
&=\frac{p^\delta\Gamma(\delta)}{4(f^p-e^p)^\delta}\left[\mathcal{J}_{e^+}^{p,\delta}\mathcal{Q}_1(f)\mathcal{Q}_2(f)+\mathcal{J}_{f^-}^{p,\delta}\mathcal{Q}_1(e)\mathcal{Q}_2(e)\right]\\
&\quad+\frac{1}{2\delta}\left(\frac{1}{2}-\frac{\delta}{(\delta+1)(\delta+2)}\right)\Upsilon_2(e,f)+\frac{\delta}{2\delta(\delta+1)(\delta+2)}\Upsilon_1(e,f).
\end{aligned}\tag{3.32}$$

Combining (3.30)–(3.32) gives the desired result (3.27). □

4 Conclusion

We have proposed the concept of p-convexity for the interval-valued functions, established several novel Hermite–Hadamard type and Hermite–Hadamard–Fejér type inequalities for the p-convex interval-valued functions. Our results provided the interval-valued counterparts of the inequalities presented in [103, 106], and our ideas may lead to a lot of follow-up research.

Acknowledgements
The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding
The work was supported by the Natural Science Foundation of China (Grant Nos. 11971142, 61673169, 11871202, 11701176, 11626101, 11601485).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 23 April 2020 Accepted: 17 June 2020 Published online: 03 July 2020

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