Xin and Hu *Advances in Difference Equations* (2020) 2020:269
https://doi.org/10.1186/s13662-020-02716-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Periodic solution for prescribed mean curvature Rayleigh equation with a singularity

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Abstract

In this paper, we consider the existence of a periodic solution for a prescribed mean curvature Rayleigh equation with singularity (weak and strong singularities of attractive type or weak and strong singularities of repulsive type). Our proof is based on an extension of Mawhin's continuation theorem.

MSC: 34B16; 34B18; 34C25

Keywords: Periodic solution; Prescribed mean curvature; Weak and strong; Attractive and repulsive; Rayleigh equation

1 Introduction

In this paper, we consider the following p-Laplacian prescribed mean curvature Rayleigh equation:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + f\bigl(t,u'(t)\bigr) + g\bigl(u(t)\bigr) = e(t), \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s$, p is a positive constant and $p > 1$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is an ω-periodic function about t, and $f(t,0) \equiv 0$, $g \in C((0,+\infty), \mathbb{R})$ has a singularity at the origin, $e \in L^\sigma(\mathbb{R})$ is an ω-periodic function and $1 \leq \sigma < \infty$, ω is a positive constant.

During the past ten years, the problem of existence of periodic solutions to singular equations has been extensively studied by may researchers [1–13]. Among these results, some results on Liénard equations with singularity of attractive type (or a singularity of repulsive type) have been published (see [1, 5, 6, 9, 10]). For example, Wang and Ma [9] investigated in 2015 a special of equation (1.1), where $\frac{u'(t)}{\sqrt{1+(u'(t))^2}} = u'(t)$ and $p = 2$, $\sigma = 0$, g satisfied a semilinear condition and had a strong singularity of repulsive type, i.e.,

$$\lim_{u\to 0^+} g(u) = -\infty \quad \text{and} \quad \lim_{u\to 0^+} \int_u^1 g(v)\,dv = +\infty. \tag{1.2}$$

Applying the limit properties of time map, the authors obtained the existence of periodic solution for this equation. After that, Lu et al. [1] improved the results of [9], they required that $p > 1$. Their proof was based on the topological degree theory.

Compared with Rayleigh equations, only a few works focus on prescribed mean curvature Rayleigh equations, especially prescribed mean curvature equations with singularity. As far as we know, prescribed mean curvature $\frac{u'(t)}{\sqrt{1+(u'(t))^2}}$ of $u(t)$ appears in different geometry and physics [14–17]. Recently, Li and Ge's work [18] has been performed on the existence of a periodic solution of equation (1.1) with strong singularity of repulsive type by using Manásevich–Mawhin continuation theorem, where $p = 2$, g satisfied a semilinear condition and equation (1.2).

Inspired by the above paper [1, 9, 18], in this paper, we further consider the existence of a periodic solution for equation (1.1) by means of an extension of Mawhin's continuation theorem due to Ge and Ren [19]. It is worth mentioning that our results are more general than those in [1, 9, 18]. First, g of this paper satisfies weak and strong singularities of attractive type (or weak and strong singularities of repulsive type) at the origin. Second, g of this paper may satisfy sub-linearity, semi-linearity, and super-linearity conditions at infinity.

2 Periodic solution for equation (1.1) in the case that $p > 1$

In this section, we study the existence of a periodic solution to equation (1.1). Since $(\phi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}))'$ is a nonlinear term, coincidence degree theory does not apply directly. The traditional method of studying is to translate equation (1.1) into the following two-dimensional system:

$$\begin{cases} u_1'(t) = \frac{\phi_q(u_2(t))}{\sqrt{1-\phi_q^2(u_2(t))}}, \\ u_2'(t) = -f(t, u_1'(t)) - g(u_1(t)) + e(t), \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, for which coincidence degree theory can be applied. However, from the first equation of the above system, it is obvious that $\|u_2\| < 1$, here $\|u_2\| := \max_{t\in\mathbb{R}} |u'(t)|$. Therefore, estimating an upper bound of $u_2(t)$ is very complicated; to get around this difficulty, we find other methods to study equation (1.1). We first investigate the following second-order prescribe mean curvature equation:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' = \tilde{f}\big(t, u(t), u'(t)\big), \tag{2.1}$$

where $\tilde{f} : [0, \omega] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Applying the extension of Mawhin's continuation theorem due to Ge and Ren [19, Theorem 2.1], we get the following conclusion.

Lemma 2.1 *Assume that Ω is an open bounded set in $C^1_\omega := \{u \in C^1(\mathbb{R},\mathbb{R}) : u(t+\omega) \equiv u(t)$ and $u'(t+\omega) \equiv u'(t), \forall t \in \mathbb{R}\}$. Suppose that the following conditions hold*:

(i) *For each $\lambda \in (0, 1)$, the equation*

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' = \lambda\tilde{f}\big(t, u(t), u'(t)\big)$$

has no solution on $\partial\Omega$.

(ii) *The equation*

$$F(a) := \frac{1}{\omega}\int_0^{\omega} \tilde{f}(t,a,0)\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) *The Brouwer degree*

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then equation (2.1) *has at least one* ω*-periodic solution on* $\bar{\Omega}$.

Proof First, operators M and N_λ are defined by

$$M : \operatorname{dom} M \cap X \to Z, \qquad (Mu)(t) = \left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)', \quad t \in \mathbb{R},$$

$$N_\lambda : X \to Z, \qquad (N_\lambda u)(t) = \lambda \tilde{f}\bigl(t, u(t), u'(t)\bigr).$$

Obviously, equation (2.1) can be converted to

$$Mu = N_\lambda u, \quad \lambda \in (0,1).$$

By [20, Lemma 3.1 and Lemma 3.2], we know that M is a quasi-linear operator, N_λ is M-compact. From assumption (i), one finds

$$Mu \neq N_\lambda u, \quad \lambda \in (0,1) \text{ and } u \in \partial\Omega,$$

and assumptions (ii) and (iii) imply that $\deg\{JQN, \Omega \cap \ker M, \theta\}$ is valid and

$$\deg\{JQN, \Omega \cap \ker M, \theta\} \neq 0.$$

Therefore, applying the extension of Mawhin's continuation theorem, equation (2.1) has at least one T-periodic solution. □

In the following, applying Lemma 2.1, we prove the existence of a periodic solution for equation (1.1) with singularity of repulsive type.

Theorem 2.1 *Assume that equation* (1.2) *holds. Furthermore, suppose that the following conditions hold*:

(H_1) *There exist positive constants* α *and* $m > 1$ *such that*

$$f(t,v)v \geq \alpha |v|^m \quad \text{for } (t,v) \in [0,\omega] \times \mathbb{R}.$$

(H_2) *There exist two positive constants* d_1, d_2 *with* $d_1 < d_2$ *such that* $g(u) - e(t) < 0$ *for* $(t,u) \in [0,\omega] \times (0,d_1)$ *and* $g(u) - e(t) > 0$ *for* $(t,u) \in [0,\omega] \times (d_2,+\infty)$.

Then equation (1.1) *has at least one periodic solution.*

Proof We embed equation (1.1) into the following family equation:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + \lambda f\big(t,u'(t)\big) + \lambda g\big(u(t)\big) = \lambda e(t), \tag{2.2}$$

where $\lambda \in (0,1]$. Firstly, we claim that there exist two points $\tau, \xi \in (0,T)$ such that

$$u(\tau) \geq d_1 \quad \text{and} \quad u(\xi) \leq d_2. \tag{2.3}$$

In fact, since $\int_0^\omega u'(t)\,dt = 0$, it is easy to verify that there exist two points $t_1, t_2 \in (0,T)$ such that

$$u'(t_1) \leq 0 \quad \text{and} \quad u'(t_2) \geq 0.$$

Therefore, we get

$$\phi_p\left(\frac{u'(t_1)}{\sqrt{1+(u'(t_1))^2}}\right) \leq 0 \quad \text{and} \quad \phi_p\left(\frac{u'(t_2)}{\sqrt{1+(u'(t_2))^2}}\right) \geq 0.$$

Let $t_*, t^* \in (0,\omega)$ be minimum and maximum points of the prescribed mean curvature term $\phi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})$, the above equation implies

$$\phi_p\left(\frac{u'(t_*)}{\sqrt{1+(u'(t_*))^2}}\right) \leq 0 \quad \text{and} \quad \left(\phi_p\left(\frac{u'(t_*)}{\sqrt{1+(u'(t_*))^2}}\right)\right)' = 0; \tag{2.4}$$

and

$$\phi_p\left(\frac{u'(t^*)}{\sqrt{1+(u'(t^*))^2}}\right) \geq 0 \quad \text{and} \quad \left(\phi_p\left(\frac{u'(t^*)}{\sqrt{1+(u'(t^*))^2}}\right)\right)' = 0. \tag{2.5}$$

Besides, since

$$\phi_p\left(\frac{u'(t_*)}{\sqrt{1+(u'(t_*))^2}}\right) = \left|\frac{u'(t_*)}{\sqrt{1+(u'(t_*))^2}}\right|^{p-2} \frac{u'(t_*)}{\sqrt{1+(u'(t_*))^2}} \leq 0,$$

then it is clear that $u'(t_*) \leq 0$. By condition (H_1), we have

$$f\big(t_*, u'(t_*)\big) \leq 0. \tag{2.6}$$

Applying equations (2.4) and (2.6) into (2.2), we deduce

$$g\big(u(t_*)\big) - e(t_*) = -f\big(t_*, u'(t_*)\big) \geq 0.$$

By condition (H_2), we get that $u(t_*) \geq d_1$.

Similarly, by conditions (H_1), (H_2) and equation (2.4), we obtain that $u(t^*) \leq d_2$. Take $\tau = t_*$ and $\xi = t^*$, then (2.3) is proved.

Multiplying both sides of equation (2.2) by $u'(t)$ and integrating from 0 to ω, we have

$$\begin{aligned}&\int_0^\omega \biggl(\phi_p\biggl(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\biggr)\biggr)' u'(t)\,dt+\lambda\int_0^\omega f\bigl(t,u'(t)\bigr)u'(t)\,dt+\lambda\int_0^\omega g\bigl(u(t)\bigr)u'(t)\,dt\\&\quad=\lambda\int_0^\omega e(t)u'(t)\,dt.\end{aligned}\tag{2.7}$$

Substituting

$$\begin{aligned}&\int_0^\omega \biggl(\phi_p\biggl(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\biggr)\biggr)' u'(t)\,dt\\&\quad=\phi_p\biggl(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\biggr)u'(t)\Big|_0^\omega-\int_0^\omega\phi_p\biggl(\frac{u'}{\sqrt{1+(u')^2}}\biggr)\,du'=0\end{aligned}$$

and $\int_0^\omega g(u(t))u'(t)\,dt=0$ into (2.7), it is clear that

$$\biggl|\int_0^\omega f\bigl(t,u'(t)\bigr)u'(t)\,dt\biggr|=\biggl|\int_0^\omega e(t)u'(t)\,dt\biggr|.$$

By condition (H_1) and the Hölder inequality, the above equation implies

$$\begin{aligned}\alpha\int_0^\omega\bigl|u'(t)\bigr|^m\,dt&\le\biggl|\int_0^\omega f\bigl(t,u'(t)\bigr)u'(t)\,dt\biggr|\\&\le\int_0^\omega\bigl|e(t)\bigr|\bigl|u'(t)\bigr|\,dt\\&\le\biggl(\int_0^\omega\bigl|e(t)\bigr|^{\frac{m}{m-1}}\,dt\biggr)^{\frac{m-1}{m}}\biggl(\int_0^\omega\bigl|u'(t)\bigr|^m\,dt\biggr)^{\frac{1}{m}}.\end{aligned}$$

Since $\int_0^\omega|u'(t)|^m\,dt\ne0$ and $\alpha>0$, we arrive at

$$\biggl(\int_0^\omega\bigl|u'(t)\bigr|^m\,dt\biggr)^{\frac{m-1}{m}}\le\frac{\|e\|_{\frac{m-1}{m}}}{\alpha},\tag{2.8}$$

where $\|e\|_{\frac{m-1}{m}}:=(\int_0^T|e(t)|^{\frac{m}{m-1}}\,dt)^{\frac{m-1}{m}}$. From equations (2.3) and (2.8), using the Hölder inequality, we get

$$\begin{aligned}u(t)&\le d_2+\int_0^\omega\bigl|u'(t)\bigr|\,dt\\&\le d_2+\omega^{\frac{m-1}{m}}\biggl(\int_0^\omega\bigl|u'(t)\bigr|^m\,dt\biggr)^{\frac{1}{m}}\\&\le d_2+\frac{\omega^{\frac{m-1}{m}}(\|e\|_{\frac{m-1}{m}})^{\frac{1}{m-1}}}{\alpha^{\frac{1}{m-1}}}:=M_1.\end{aligned}\tag{2.9}$$

From equation (2.8) and the Hölder inequality, we deduce

$$\bigl\|u'\bigr\|=\frac{1}{\omega}\int_0^\omega\bigl\|u'\bigr\|\,dt\le\omega^{-\frac{1}{m}}\biggl(\int_0^\omega\bigl\|u'\bigr\|^m\,dt\biggr)^{\frac{1}{m}}\le\omega^{-\frac{1}{2}}\biggl(\frac{\|e\|_{\frac{m-1}{m}}}{\alpha}\biggr)^{\frac{1}{m-1}}:=M_2.\tag{2.10}$$

On the other hand, let $\tau \in (0,\omega)$ be as in equation (2.3). Multiplying both sides of equation (2.2) by $u'(t)$ and integrating over the interval $[\tau, t]$, here $t \in [\tau,\omega]$, we see that

$$\begin{aligned}\lambda \int_{u(\tau)}^{u(t)} g(u)\,du &= \lambda \int_{\tau}^{t} g\big(u(s)\big)u'(s)\,ds \\ &= -\int_{\tau}^{t}\left(\phi_p\left(\frac{u'(s)}{\sqrt{1+(u'(s))^2}}\right)\right)' u'(s)\,ds \\ &\quad - \lambda \int_{\tau}^{t} f\big(s,u'(s)\big)u'(s)\,ds + \lambda \int_{\tau}^{t} e(s)u'(s)\,ds.\end{aligned}$$

Furthermore, from equations (2.2), (2.9), and (2.10), applying the Hölder inequality, the above equation implies

$$\begin{aligned}\lambda\left|\int_{u(\tau)}^{u(t)} g(u)\,du\right| &\le \int_0^{\omega}\left|\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)'\right|\big|u'(t)\big|\,dt \\ &\quad + \lambda \int_0^{\omega}\big|f\big(t,u'(t)\big)\big|\big|u'(t)\big|\,dt + \lambda \int_0^{\omega}\big|e(t)\big|\big|u'(t)\big|\,dt \\ &\le \lambda M_2\left(\int_0^{\omega}\big|f\big(t,u'(t)\big)\big|\,dt + \int_0^{\omega}\big|g\big(u(t)\big)\big|\,dt + \int_0^{\omega}\big|e(t)\big|\,dt\right) \\ &\quad + \lambda M_2 \int_0^{\omega}\big|f\big(t,u'(t)\big)\big|\,dt + \lambda M_1 \omega^{\frac{1}{m}}\|e\|_{\frac{m-1}{m}} \\ &\le 2\lambda M_2\big(\omega\|f_{M_2}\| + \omega^{\frac{1}{m}}\|e\|_{\frac{m-1}{m}}\big) + \lambda M_2 \int_0^{\omega}\big|g\big(u(t)\big)\big|\,dt,\end{aligned} \tag{2.11}$$

where $\|f_{M_2}\| := \max_{(t,v)\in[0,\omega]\times[-M_2,M_2]} |f(t,v)|$.

Next, we consider $\int_0^{\omega} |g(u(t))|\,dt$. Integrating equation (2.2) over the interval $[0,\omega]$, we obtain

$$\int_0^{\omega}\big(f\big(t,u'(t)\big) + g\big(u(t)\big) - e(t)\big)\,dt = 0. \tag{2.12}$$

From equation (2.12), we see that

$$\begin{aligned}\int_0^{T}\big|g\big(u(t)\big)\big|\,dt &= \int_{g(u(t))\ge 0} g\big(u(t)\big)\,dt - \int_{g(u(t))\le 0} g\big(u(t)\big)\,dt \\ &= 2\int_{g(u(t))\ge 0} g^+\big(u(t)\big)\,dt + \int_0^{\omega} f\big(t,u'(t)\big)\,dt - \int_0^{\omega} e(t)\,dt \\ &\le 2\int_0^{\omega} g^+\big(u(t)\big)\,dt + \int_0^{\omega}\big|f\big(t,u'(t)\big)\big|\,dt + \int_0^{\omega}\big|e(t)\big|\,dt,\end{aligned} \tag{2.13}$$

where $g^+(u) := \max\{g(u),0\}$. Since $g^+(u(t)) \ge 0$, from condition (H_2) and equation (1.2), we know that there exists a positive constant d_2^* with $d_2^* > d_1$ such that $u(t) \ge d_2^*$. Therefore, from equations (2.9) and (2.10), equation (2.13) implies

$$\begin{aligned}\int_0^{\omega}\big|g\big(u(t)\big)\big|\,dt &\le 2\omega\big\|g_{M_1}^+\big\| + \int_0^{\omega}\big|f\big(t,u(t)\big)\big|\big|u'(t)\big| + \int_0^{\omega}\big|e(t)\big|\,dt \\ &\le 2\omega\big\|g_{M_1}^+\big\| + \omega\|f_{M_2}\| + \omega^{\frac{1}{m}}\|e\|_{\frac{m-1}{m}},\end{aligned} \tag{2.14}$$

where $\|g^+_{M_1}\| := \max_{d_2^* \le u \le M_1} g^+(u)$. Applying equations (2.14) into (2.11), we have

$$\lambda\left|\int_{u(\tau)}^{u(t)} g(u)\,du\right| \le 3\lambda M_2\big(\omega\|f_{M_2}\| + \omega^{\frac{1}{m}}\|e\|_{\frac{m-1}{m}}\big) + 2\lambda M_2\omega\big\|g^+_{M_1}\big\|.$$

According to equation (1.2), we see that there exists a positive constant M_3' such that

$$u(t) \ge M_3' \quad \text{for } t \in [\tau, \omega]. \tag{2.15}$$

For the case $t \in [0, \tau]$, we can handle it similarly.

From equations (2.9), (2.10), and (2.15), we obtain that periodic solution u of equation (2.2) satisfies

$$M_3 < u(t) < M_1, \qquad \big\|u'\big\| < M_2,$$

where $M_3 := \min\{d_1, M_3'\}$. Then condition (1) of Lemma 2.1 is satisfied. There exists a constant $C \in [M_3, M_1]$ such that

$$g(C) - \frac{1}{\omega}\int_0^\omega e(t)\,dt = 0,$$

since $f(t,0) \equiv 0$. Therefore, condition (2) of Lemma 2.1 holds. Finally, by condition (H_2), we arrive at

$$g(M_3) - \frac{1}{\omega}\int_0^\omega e(t)\,dt < 0 \quad \text{and} \quad g(M_1) - \frac{1}{\omega}\int_0^\omega e(t)\,dt > 0.$$

So condition (3) of Lemma 2.1 is also satisfied. By Theorem 2.1, equation (1.1) has at least one periodic solution. □

By condition (H_1), we know that $f(t,v)v$ may not change sign for $(t,v) \in [0,\omega] \times \mathbb{R}$. Similarly, we give the following condition:

(H_1') There exist positive constants β and $n > 1$ such that

$$f(t,v)v \le -\beta|v|^n \quad \text{for } (t,v) \in [0,\omega] \times \mathbb{R}.$$

In the following, applying Theorem 2.1, we get the following corollary.

Corollary 2.1 *Assume that conditions (H_1'), (H_2) and equation (1.2) hold. Then equation (1.1) has at least one periodic solution.*

In equation (1.2), the nonlinear term g requires a strong singularity of repulsive type (i.e., $\lim_{u\to 0^+}\int_u^1 g(v)\,dv = +\infty$). It is clear that the method of Theorem 2.1 is no longer applicable to estimating lower bound of periodic solution $u(t)$ of equation (1.1) in the case of a weak singularity of repulsive type (i.e., $\lim_{u\to 0^+}\int_u^1 g(v)\,dv < +\infty$). Therefore, we find another method to consider equation (1.1) in the case of a weak singularity of repulsive type.

Theorem 2.2 *Assume that conditions (H_1) and (H_2) hold. If $\|e\|_{\frac{m-1}{m}} < 2\alpha d_1^{m-1}\omega^m$, here d_1 is defined in Theorem 2.1, then equation (1.1) has at least one periodic solution.*

Proof We follow the same strategy and notation as in the proof of Theorem 2.1. Next, we consider the lower bound of periodic solution $u(t)$ of equation (1.1). From equations (2.3) and (2.8), applying the Hölder inequality, we get

$$\begin{aligned} u(t) &= \frac{1}{2}\big(u(t)+u(t-\omega)\big) \\ &= \frac{1}{2}\left(u(\tau)+\int_{\tau}^{t} u'(s)\,ds + u(\tau) - \int_{t-\omega}^{\tau} u'(s)\,ds\right) \\ &\geq u(\tau) - \frac{1}{2}\left(\int_{\tau}^{t} \left|u'(s)\right|ds + \int_{t-\omega}^{\tau}\left|u'(s)\right|ds\right) \\ &= u(\tau) - \frac{1}{2}\int_{t-\omega}^{t}\left|u'(s)\right|ds \\ &\geq d_1 - \frac{1}{2}\int_0^{\omega}\left|u'(s)\right|ds \\ &\geq d_1 - \frac{\omega^{\frac{m-1}{m}}}{2}\left(\int_0^{\omega}\left|u'(s)\right|^m ds\right)^{\frac{1}{m}} \\ &\geq d_1 - \frac{\omega^{\frac{m-1}{m}}}{2}\left(\frac{\|e\|_{\frac{m-1}{m}}}{\alpha}\right)^{\frac{1}{m-1}} := M_3 > 0, \end{aligned}$$

since $\|e\|_{\frac{m-1}{m}} < 2\alpha d_1^{m-1}\omega^m$, we obtain that $\frac{\omega^{\frac{m-1}{m}}}{2}(\frac{\|e\|_{\frac{m-1}{m}}}{\alpha})^{\frac{1}{m-1}} < d_1$. The remaining part of the proof is the same as that of Theorem 2.1. □

By Theorem 2.2, we obtain the following corollary.

Corollary 2.2 *Assume that conditions* (H_1') *and* (H_2) *hold. If* $\|e\|_{\frac{m-1}{m}} < 2\alpha d_1^{m-1}\omega^m$, *then equation* (1.1) *has at least one periodic solution.*

Comparing Theorem 2.1 to 2.2, Theorem 2.2 is applicable to weak as well as strong singularities, whereas Theorem 2.1 is only applicable to a strong singularity. Besides, equation (1.2) is relatively weaker than condition $\|e\|_{\frac{m-1}{m}} < 2^{m-1}\alpha d_1^{m-1}\omega^m$. On the other hand, Theorems 2.1 and 2.2 require that g satisfies a singularity of repulsive type (i.e., $\lim_{u\to 0^+} g(u) = -\infty$). In the following, we consider that g satisfies a singularity of attractive type (i.e., $\lim_{u\to 0^+} g(u) = +\infty$). It is obvious that attractive condition and equation (1.2), (H_2) contradict each other. Therefore, we have to find another conditions to consider equation (1.1) with singularity of attractive type.

Theorem 2.3 *Assume that* (H_1) *holds. Furthermore, suppose that the following conditions hold*:

- (H_3) *There exist two positive constants* d_3, d_4 *with* $d_3 < d_4$ *such that* $g(u) - e(t) > 0$ *for* $(t,u) \in [0,\omega] \times (0,d_3)$ *and* $g(u) - e(t) < 0$ *for* $(t,u) \in [0,\omega] \times (d_4,+\infty)$.
- (H_4) (*Strong singularity of attractive type*)

$$\lim_{u\to 0^+} g(u) = +\infty \quad \text{and} \quad \lim_{u\to 0^+}\int_u^1 g(v)\,dv = -\infty.$$

Then equation (1.1) *has at least one periodic solution.*

Proof We follow the same strategy and notation as in the proof of Theorem 2.1. Next, we consider $\int_0^T |g(u(t))|\,dt$. From equations (2.12) and (2.13), we see that

$$\begin{aligned}\int_0^\omega \bigl|g\bigl(u(t)\bigr)\bigr|\,dt &= \int_{g(u(t))\geq 0} g\bigl(u(t)\bigr)\,dt - \int_{g(u(t))\leq 0} g\bigl(u(t)\bigr)\,dt \\ &= -2\int_{g(u(t))\leq 0} g^-\bigl(u(t)\bigr)\,dt - \int_0^\omega f\bigl(t,u'(t)\bigr)\,dt + \int_0^\omega e(t)\,dt \\ &\leq 2\int_0^\omega \bigl|g^-\bigl(u(t)\bigr)\bigr|\,dt + \int_0^\omega \bigl|f\bigl(t,u'(t)\bigr)\bigr|\,dt + \int_0^\omega \bigl|e(t)\bigr|\,dt, \end{aligned} \tag{2.16}$$

where $g^-(u) := \min\{g(u), 0\}$. Since $g^-(u(t)) \leq 0$, from conditions (H_3) and (H_4), we know that there exists a positive constant d_4^* with $d_4^* > d_3$ such that $u(t) \geq d_4^*$. Therefore, from equations (2.9) and (2.10), equation (2.16) implies

$$\int_0^\omega \bigl|g\bigl(u(t)\bigr)\bigr|\,dt \leq 2\omega \bigl\|g_{M_1}^-\bigr\| + \omega\|f_{M_2}\| + \omega^{\frac{1}{m}}\|e\|_{\frac{m-1}{m}},$$

where $\|g_{M_1}^-\| := \max_{d_4^*\leq u\leq M_1} |g^-(x)|$. The remaining part of the proof is the same as that of Theorem 2.1. □

By Theorem 2.3, we obtain the following corollary.

Corollary 2.3 *Assume that conditions (H_1'), (H_3), and (H_4) hold. Then equation (1.1) has at least one periodic solution.*

By Theorems 2.2 and 2.3, we obtain the following conclusion.

Theorem 2.4 *Assume that conditions (H_1) and (H_3) hold. If $\|e\|_{\frac{m-1}{m}} < 2^{m-1}\alpha d_3^{m-1}\omega^m$, then equation (1.1) has at least one periodic solution.*

By Theorem 2.4, we get the following corollary.

Corollary 2.4 *Assume that conditions (H_1') and (H_3) hold. If $\|e\|_{\frac{m-1}{m}} < 2^{m-1}\alpha d_3^{m-1}\omega^m$, then equation (1.1) has at least one periodic solution.*

Finally, we illustrate our results with two numerical examples.

Example 2.1 Consider the following prescribed mean curvature Rayleigh equation with strong singularity of attractive type:

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + (\cos t + 4)\bigl(u'(t)\bigr)^3 - \sum_{i=1}^n u^i(t) + \frac{5}{u^\rho(t)} = e^{\sin t}, \tag{2.17}$$

where ρ is a positive constant and $\rho \geq 1$, n is a positive integer.

It is clear that $\omega = 2\pi$, $f(t,v) = (\cos t + 4)v^3$, $g(u) = -\sum_{i=1}^n u^i + \frac{5}{u^\rho}$, $e(t) = e^{\sin t}$. We know that $f(t,v)v = (\cos t + 4)v^4 \geq 3v^4$. Take $\alpha = 3$, $m = 4$, $d_3 = 0.01$, $d_4 = 3$. Then conditions (H_1) and (H_3) hold. Since $\lim_{u\to 0^+}\int_u^1 g(v)\,dv = \lim_{u\to 0^+}\int_u^1 (-\sum_{i=1}^n v^i + \frac{6}{v^\rho})\,dv = -\infty$, condition (H_4) is satisfied. Therefore, by Theorem 2.3, equation (2.17) has at least one 2π-periodic solution.

Example 2.2 Consider the following prescribed mean curvature Rayleigh equation with a weak singularity of repulsive type:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + (\sin 2t+5)\bigl(u'(t)\bigr) + u^5(t) = \frac{4}{u^{\frac{1}{2}}(t)} + \cos 2t, \tag{2.18}$$

where $p > 1$.

It is obvious that $T = \pi$, $f(t,v) = (\sin 2t+5)v$, $g(u) = u^5 - \frac{4}{u^{\frac{1}{2}}}$, $e(t) = \cos 2t$. Take $\alpha = 4$, $m = 2$, $d_1 = 0.09$, $d_2 = 4$, conditions (H_1) and (H_2) are satisfied. Furthermore, we consider

$$2^{m-1}\alpha d_1^{m-1}\omega^m = 2\times 4\times 0.09\times\pi^2 = 0.72\pi^2 > \pi^{\frac{1}{2}}.$$

Hence, applying Theorem 2.2, equation (2.18) has at least one π-periodic solution.

3 Periodic solution for equation (1.1) in the case that $p \neq 2$

In the following, by Lemma 2.1 and Theorem 2.1, we prove the existence of a periodic solution for equation (1.1) with singularity of repulsive type.

Theorem 3.1 *Assume that conditions* (H_1), (H_2), *and* $p \neq 2$ *hold. Then equation* (1.1) *has at least periodic solution.*

Proof Let $\underline{t}, \bar{t} \in (0,\omega)$ be minimum and maximum points of $u(t)$, and $u'(\underline{t}) = u'(\bar{t}) = 0$. Besides, we claim that there exists a positive constant ε such that

$$u'(t) \leq 0 \quad \text{for } t \in (\underline{t}-\varepsilon, \underline{t}+\varepsilon). \tag{3.1}$$

Assume, by way of contradiction, that equation (3.1) does not hold. Then $u'(t) > 0$ for $t \in (\underline{t}-\varepsilon, \underline{t}+\varepsilon)$. Therefore, $u(t)$ is strictly increasing for $t \in (\underline{t}-\varepsilon, \underline{t}+\varepsilon)$, this contradicts the definition of $\underline{t}$. Hence, equation (3.1) is true. Since

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' = \left(\left|\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right|^{p-2}\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)'. \tag{3.2}$$

Applying equation (3.1) into (3.2), we get

$$\begin{aligned}\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' &= \left(\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)^{p-1}\right)' \\ &= (p-1)\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)^{p-2}\left(\frac{2u''(t)+u''(t)(u'(t))^2}{\sqrt{1+(u'(t))^2}}\right)\end{aligned} \tag{3.3}$$

for $t \in (\underline{t}-\varepsilon, \underline{t}+\varepsilon)$. From equation (3.3) and $p \neq 2$, we obtain

$$\left(\phi_p\left(\frac{u'(\underline{t})}{\sqrt{1+(u'(\underline{t}))^2}}\right)\right)' = 0. \tag{3.4}$$

From equations (2.2) and (3.4), we have

$$g\bigl(\underline{t}, u(\underline{t})\bigr) - e(\underline{t}) = 0,$$

since $f(t,0) \equiv 0$. By condition (H_2), we get

$$d_1 \le u(\underline{t}) \le d_2. \tag{3.5}$$

Similarly, by condition (H_2), we obtain

$$d_1 \le u(\bar{t}) \le d_2. \tag{3.6}$$

Therefore, from equations (3.5) and (3.6), we see that

$$d_1 \le u(t) \le d_2 \quad \text{for } t \in \mathbb{R}. \tag{3.7}$$

The remaining part of the proof is the same as that of Theorem 2.1. □

By Theorem 3.1, we get the following corollary.

Corollary 3.1 *Assume that conditions (H_1'), (H_2), and $p \neq 2$ hold. Then equation (1.1) has at least one periodic solution.*

Comparing Theorems 2.1 and 3.1, Theorem 3.1 is applicable to weak and strong singularities. Theorem 2.1 is only applicable to a strong singularity. However, Theorem 3.1 does not cover the case of $p = 2$, Theorem 2.1 covers the case of $p = 2$. Therefore, Theorem 2.1 can be more general. Besides, Theorem 3.1 requires that g satisfies a singularity of repulsive type. In the following, we consider that g satisfies a singularity of attractive type. It is obvious that the attractive condition and (H_2) contradict each other. By Theorems 2.3 and 3.1, we obtain the following conclusion.

Theorem 3.2 *Assume that conditions (H_1), (H_3), and $p \neq 2$ hold. Then equation (1.1) has at least one periodic solution.*

By Theorem 3.2, we get the following corollary.

Corollary 3.2 *Assume that conditions (H_1'), (H_3), and $p \neq 2$ hold. Then equation (1.1) has at least one periodic solution.*

It is worth mentioning that the method of Theorem 3.1 is also applicable to the case where g satisfies nonautonomous, i.e., $g(u(t)) = g(t,u(t))$. Then equation (1.1) is rewritten as the following form:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + f(t,u'(t) + g\bigl(t,u(t)\bigr) = e(t). \tag{3.8}$$

Applying Lemma 2.1 and Theorem 3.1, we obtain the following conclusion.

Theorem 3.3 *Assume that conditions (H_1) and $p \neq 2$ hold. Furthermore, suppose that the following conditions hold:*

(H_5) *There exist two positive constants d_5, d_6 with $d_5 < d_6$ such that $g(t,u) - e(t) < 0$ for $(t,u) \in [0,\omega] \times (0,d_5)$ and $g(t,u) - e(t) > 0$ for $(t,u) \in [0,\omega] \times (d_6,+\infty)$.*

Then equation (3.8) has at least one periodic solution.

Proof Consider the following equation:

$$\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + \lambda f\big(t,u'(t)\big) + \lambda g\big(t,u(t)\big) = \lambda e(t), \tag{3.9}$$

where $\lambda \in (0,1)$. From equation (3.7) and (H_5), we get

$$d_5 \le u(t) \le d_6 \quad \text{for } t \in \mathbb{R}. \tag{3.10}$$

Multiplying both sides of equation (3.9) by $u'(t)$ and integrating from 0 to T, we have

$$\begin{aligned}
&\int_0^\omega \left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' u'(t)\,dt + \lambda \int_0^\omega f\big(t,u'(t)\big)u'(t)\,dt + \lambda \int_0^\omega g\big(t,u(t)\big)u'(t)\,dt \\
&\quad = \lambda \int_0^\omega e(t)u'(t)\,dt.
\end{aligned} \tag{3.11}$$

Substituting $\int_0^\omega (\phi_p(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}))' u'(t)\,dt = 0$ into equation (3.11), it is clear that

$$\left|\int_0^\omega f\big(t,u'(t)\big)u'(t)\,dt\right| = \left|-\int_0^\omega g\big(t,u(t)\big)u'(t)\,dt + \int_0^\omega e(t)u'(t)\,dt\right|.$$

By condition (H_1) and equation (3.10), the above equation implies

$$\begin{aligned}
\alpha \int_0^T \left|u'(t)\right|^m dt &\le \left|\int_0^\omega f\big(t,u'(t)\big)u'(t)\,dt\right| \\
&\le \int_0^\omega \left|g\big(t,u(t)\big)\right|\left|u'(t)\right| dt + \int_0^\omega \left|e(t)\right|\left|u'(t)\right| dt \\
&\le \|g_1\|\omega^{\frac{m-1}{m}}\left(\int_0^\omega \left|u'(t)\right|^m dt\right)^{\frac{1}{m}} \\
&\quad + \left(\int_0^T \left|e(t)\right|^{\frac{m}{m-1}} dt\right)^{\frac{m-1}{m}} \left(\int_0^\omega \left|u'(t)\right|^m dt\right)^{\frac{1}{m}} \\
&\le \left(\|g_1\|\omega^{\frac{m-1}{m}} + \|e\|_{\frac{m-1}{m}}\right)\left(\int_0^\omega \left|u'(t)\right|^m dt\right)^{\frac{1}{m}},
\end{aligned}$$

where $\|g_1\| := \max_{d_5 \le u(t) \le d_6} |g(t,u)|$. Since $\int_0^\omega |u'(t)|^m\,dt \ne 0$ and $\gamma > 0$, we arrive at

$$\left(\int_0^T \left|u'(t)\right|^m dt\right)^{\frac{m-1}{m}} \le \frac{\|g_1\|\omega^{\frac{m-1}{m}} + \|e\|_{\frac{m-1}{m}}}{\alpha} := M_2''. \tag{3.12}$$

From equation (3.12), using the Hölder inequality, we get

$$\|u'\| = \frac{1}{\omega}\int_0^\omega \|u'\|\,dt \le \omega^{-\frac{1}{m}}\left(\int_0^\omega \|u'\|^m\,dt\right)^{\frac{1}{m}} \le T^{-\frac{1}{m}}\left(M_2''\right)^{\frac{1}{m-1}} := M_2^{**}. \tag{3.13}$$

The remaining part of the proof is the same as that of Theorem 2.1. □

By Theorem 3.3, we get the following corollary.

Corollary 3.3 *Assume that conditions* (H_1'), (H_5), *and* $p \neq 2$ *hold. Then equation* (3.8) *has at least one periodic solution.*

Theorem 3.3 requires that g of equation (3.8) satisfies a singularity of repulsive type. In the following, by Theorems 2.3 and 3.3, we discuss equation (3.8) with singularity of attractive type.

Theorem 3.4 *Assume that conditions* (H_1) *and* $p \neq 2$ *hold. Furthermore, suppose that the following conditions hold*:

(H_6) *There exist two positive constants* d_7, d_8 *with* $d_7 < d_8$ *such that* $g(t,u) - e(t) > 0$ *for* $(t,u) \in [0,\omega] \times (0,d_7)$ *and* $g(t,u) - e(t) < 0$ *for* $(t,u) \in [0,\omega] \times (d_8,+\infty)$.

Then equation (3.8) *has at least one periodic solution.*

By Theorem 3.4, we get the following corollary.

Corollary 3.4 *Assume that conditions* (H_1'), (H_6), *and* $p \neq 2$ *hold. Then equation* (3.8) *has at least one periodic solution.*

Finally, we illustrate our results with one numerical example.

Example 3.1 Consider the following prescribed mean curvature Rayleigh equation with a weak singularity of attractive type:

$$\begin{aligned}&\left(\phi_p\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)\right)' + \left(\cos^2 t + 3\right)\left(u'(t)\right)^7 + \frac{\cos^2 t + 1}{u^{\frac{1}{5}}(t)}\\&\quad = \left(\sin^2 t + 2\right)u^3(t) + e^{\sin 2t},\end{aligned} \tag{3.14}$$

where $p = 5$.

It is clear that $T = \pi$, $f(t,v) = (\cos^2 t + 3)v^7$, $g(t,u) = -(\sin^2 t + 2)u^3(t) + \frac{\cos^2 t+1}{u^{\frac{1}{5}}(t)}$, $e(t) = e^{\sin 2t}$. Take $\alpha = 1$, $m = 8$, $d_7 = 0.01$, $d_8 = 3$. Then conditions (H_1) and (H_6) hold. Therefore, by Theorem 3.4, equation (3.14) has at least one π-periodic solution.

4 Conclusions

In this paper, applying an extension of Mawhin's continuation theorem, we first investigate the existence of a periodic solution for equation (1.1) in the case that $p > 1$, where g satisfies weak and strong singularities of attractive type or weak and strong singularities of repulsive type, and g may satisfy sub-linearity, semi-linearity, and super-linearity conditions at infinity. After that, we consider the existence of a periodic solution for equation (1.1) in the case that $p > 1$ and $p \neq 2$. Our results are more general than those in [1, 9, 18].

Acknowledgements
YX and GXH are grateful to anonymous referees for their constructive comments and suggestions which have greatly improved this paper.

Funding
This work was supported by the Education Department of Henan Province project (No. 16B110006), Fundamental Research Funds for the Universities of Henan Province (NSFRF170302).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
YX and GXH contributed to each part of this study equally and declare that they have no competing interests.

Competing interests
YX and GXH declare that they have no competing interests.

Consent for publication
YX and GXH read and approved the final version of the manuscript.

Authors' contributions
YX and GXH contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 8 February 2020 Accepted: 21 May 2020 Published online: 05 June 2020

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