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# Existence and uniqueness for fuzzy differential equation with Hilfer–Katugampola fractional derivative

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**Abstract**

In this paper, we study a kind of fuzzy differential equation with Hilfer–Katugampola fractional derivative and nonlocal condition. By using successive approximation method, we obtain some sufficient conditions to ensure the existence and uniqueness of solution. An illustrative example is given to show the practical usefulness of the analytical results.

**Keywords:** Fuzzy differential equation; Hilfer–Katugampola fractional derivative; Existence and uniqueness

## 1 Introduction

In recent years, the definition of fractional calculus has been more suitable for describing historical dependence processes than the local limit definitions of integer ordinary differential equations or partial differential equations, and has received more and more attention in many fields. Fractional order differential equations are more accurate than integral order differential equations in describing objective laws and the nature of things. In 1695, Leibnitz discovered fractional derivatives, and after that, more and more scholars have devoted themselves to the study of fractional calculus. Riemann–Liouville calculus definition, Caputo differential definition, and Grunwald–Letnikov differential definition are the most commonly used fractional calculus definitions in basic mathematical research and engineering application research [15]. In 2011, a new fractional integration was proposed by Katugampola, which generalized the Riemann–Liouville and Hadamard integral into a single form. When a parameter was fixed at different values, it produced the above integrals as special cases [13]. In 2014, Katugampola presented the representation of the generalized derivative called Katugampola derivative [14]. In addition, Oliveira proposed a new fractional derivative, i.e., the Hilfer–Katugampola fractional derivative [18].

Recently, fuzzy analysis and fuzzy differential equations have been put forward to solve the uncertainty caused by incomplete information in some mathematical or computer models that determine real-world phenomena [2, 4–8, 10, 17, 19–21]. In [3] and [1], the concept of fuzzy type Riemann–Liouville differentiability based on Hukuhara differentiability was came up with, and by using the Hausdorff measure of noncompactness the authors studied the existence of solution for some fuzzy integral equations. In [7], based on Hukuhara differentiability or generalized Hukuhara differentiability, Bede and Stefanini introduced and studied new generalized differentiability concepts for fuzzy-valued functions.

In [11], Hoa, Lupulescu, and O'Regan considered the following fuzzy fractional differential equation with order $\alpha \in (0,1)$:

$$\begin{cases} ({}^{C}D^{\alpha}_{a+}x)(t) = f(t,x(t)), \\ x(a) = x_0 \in E, \end{cases}$$

where $f : [a,b] \times E \to E$ is a fuzzy function and $x_0 \in E$ is a nontrivial fuzzy constant. The paper presented some remarks on solutions of fractional fuzzy differential equation and proved that a fractional fuzzy differential equation and a fractional fuzzy integral equation are not equivalent generally. An appropriate condition was given so that this equivalence is valid.

In [12], Hoa, Vu, and Duc considered the Caputo–Katugampola (CK) fractional differential equations fuzzy set with the initial condition:

$$\begin{cases} ({}^{C}D^{\alpha,\rho}_{a+}x)(t) = f(t,x(t)), \\ x(a) = x_0, \end{cases}$$

where $0 < a < t \leq b$, ${}^{C}D^{\alpha,\rho}_{a+}$ is the fuzzy CK fractional generalized Hukuhara derivative, $f : [a,b] \times E \to E$ is a fuzzy function. An idea of successive approximations under generalized Lipschitz condition was used to prove the existence and uniqueness of solution.

In [9], Hoa studied the existence results for extremal solutions of interval fractional functional integro-differential equations by using the monotone iterative technique combined with the method of upper and lower solutions.

Inspired by the above discussion, in this paper, we initiate the study of the existence and uniqueness of solution for fuzzy fractional differential equation with Hilfer–Katugampola fractional derivative and nonlocal condition as follows:

$$\begin{cases} ({}^{\rho}D^{\alpha,\beta}_{a+}x)(t) = f(t,x(t)), \quad t \in [a,b], \\ ({}^{\rho}I^{1-\gamma}_{a+}x)(a) = x_0 = \sum_{i=1}^{m} C_i x(t_i), \quad \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$, $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta(1-\alpha)$ and $\rho > 0$, $f : [a,b] \times E \to E$ is a fuzzy function. Moreover, ${}^{\rho}I^{1-\gamma}_{a+}$, ${}^{\rho}D^{\alpha,\beta}_{a+}$ are the Hilfer–Katugampola fractional integral and derivative, which will be given in the next section. $t_i$ ($i = 1,\dots,m$) satisfies $a < t_1 \leq t_2 \leq \cdots \leq t_m < b$ and $c_i$ is a real number, $x_0 \in \mathbb{R}$. Here nonlocal conditions are more effective than the initial conditions $({}^{\rho}I^{1-\gamma}_{a+}x)(0) = x_0$ in terms of physical problems. $x$ is said to be a solution of (1.1).

The rest of the paper is organized as follows. In Sect. 2, we give some preliminary facts that we need in what follows. In Sect. 3, we present our main results on the existence and uniqueness of solution by using successive approximation method. An illustrative example is given to show the practical usefulness of the analytical results. Conclusion is given in Sect. 4.

## 2 Preliminaries

We denote by $E$ the space of all fuzzy numbers on $\mathbb{R}$.

For $c \in \mathbb{R}$, $1 \leq p \leq \infty$, let $X_c^p(a,b)$ denote the space of all complex-valued Lebesgue measurable functions $f$ on a finite interval [a,b] for which

$$\|f\|_{X_c^p} < \infty,$$

with the norm

$$\|f\|_{X_c^p} = \left(\int_a^b \left|t^c f(t)\right|^p \frac{dt}{t}\right)^{1/p} < \infty.$$

**Definition 2.1** (see [12]) A fuzzy number is a fuzzy set $x : \mathbb{R} \to [0,1]$ which satisfies the following conditions (i)–(iv):

(i) $x$ is normal, that is, there exists $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$;

(ii) $x$ is fuzzy convex in $\mathbb{R}$, that is, for $0 \leq \lambda \leq 1$,

$$x\big(\lambda t_1 + (1-\lambda)t_2\big) \geq \min\big\{x(t_1), x(t_2)\big\} \quad \text{for any } t_1, t_2 \in \mathbb{R};$$

(iii) $x$ is upper semicontinuous on $\mathbb{R}$;

(iv) $[x]^0 = \mathrm{cl}\{z \in \mathbb{R} \mid x(z) > 0\}$ is compact.

Denote by $C([a,b],E)$ the set of all continuous fuzzy functions and by $\mathrm{AC}([a,b],E)$ the set of all absolutely continuous fuzzy functions on the interval $[a,b]$ with values in $E$. Let $\gamma \in (0,1)$, by $C_\gamma[a,b]$ we denote the space of continuous functions defined by $C_\gamma[a,b] = \{f : (a,b] \to E : (\frac{t^\rho - a^\rho}{\rho})^{1-\gamma} f(t) \in C[a,b]\}$. Let $L([a,b],E)$ be the set of all fuzzy functions $x : [a,b] \to E$ such that the functions $t \mapsto D_0[x(t), \widehat{0}]$ belong to $L^1[a,b]$.

If $x$ is a fuzzy number on $\mathbb{R}$, we define $[x]^r = \{z \in \mathbb{R} \mid x(z) \geq r\}$ the $r$-level of $x$, with $r \in (0,1]$. From conditions (i) to (iv), it follows that the $r$-level set of $x \in E$, $[x]^r$, is a nonempty compact interval for any $r \in [0,1]$. We denote by $[\underline{x}(r), \overline{x}(r)]$ the $r$-level of a fuzzy number $x$. For $x_1, x_2 \in E$, and $\lambda \in \mathbb{R}$, the sum $x_1 + x_2$ and the product $\lambda \cdot x_1$ are defined by $[x_1 + x_2]^r = [x_1]^r + [x_2]^r$, $[\lambda \cdot x_1]^r = \lambda [x_1]^r$, $\forall r \in [0,1]$, where $[x_1]^r + [x_2]^r$ means the usual addition of two intervals of $\mathbb{R}$ and $\lambda [x_1]^r$ means the usual scalar product between $\lambda$ and an real interval. For $x \in E$, we define the diameter of the $r$-level set of $x$ as $\mathrm{diam}[u]^r = \overline{u}(r) - \underline{u}(r)$.

**Definition 2.2** (see [16]) Let $x_1, x_2 \in E$. If there exists $x_3 \in E$ such that $x_1 = x_2 + x_3$, then $x_3$ is called the Hukuhara difference of $x_1$ and $x_2$ and it is denoted by $x_1 \ominus x_2$. We note that $x_1 \ominus x_2 \neq x_1 + (-)x_2$.

**Definition 2.3** (see [16]) The distance $D_0[x_1, x_2]$ between two fuzzy numbers is defined as

$$D_0[x_1, x_2] = \sup_{r \in [0,1]} H\big([x_1]^r, [x_2]^r\big), \quad \forall x_1, x_2 \in E,$$

where $H([x_1]^r, [x_2]^r) = \max\{|\underline{u}_1(r) - \underline{u}_1(r)|, |\overline{u}_1(r) - \overline{u}_1(r)|\}$ is the Hausdorff distance between $[x_1]^r$ and $[x_2]^r$.

Triangular fuzzy numbers are defined as a fuzzy set in $E$ that is specified by an ordered triple $x = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $[x]^r = [\underline{x}(r), \overline{x}(r)]$ are the endpoints of $r$-level sets for all $r \in [0, 1]$, where $\underline{x}(r) = a + (b - a)r$ and $\overline{x}(r) = c - (c - b)r$. In general, the parametric form of a fuzzy number $x$ is a pair $[x]^r = [\underline{x}(r), \overline{x}(r)]$ of functions $\underline{x}(r)$, $\overline{x}(r)$, $r \in [0, 1]$, which satisfy the following conditions: $\underline{u}(r)$ is a monotonically increasing left-continuous function, $\overline{u}(r)$ is a monotonically decreasing left-continuous function, and $\underline{u}(r) \leq \overline{u}(r)$, $r \in [0, 1]$.

**Definition 2.4** (see [6]) The generalized Hukuhara difference of two fuzzy numbers $x, y \in E$ (gH-difference for short) is defined as follows:

$$x \ominus_{gH} y = \omega \quad \Leftrightarrow \quad x = y + \omega, \quad \text{or} \quad y = x + (-1)\omega.$$

A function $x : [a, b] \to E$ is called d-increasing (d-decreasing) on $[a, b]$ if for every $r \in [0, 1]$ the function $t \mapsto \operatorname{diam}[x(t)]^r$ is nondecreasing (nonincreasing) on $[a, b]$. If $x$ is d-increasing or d-decreasing on $[a, b]$, then we say that $x$ is d-monotone on $[a, b]$.

**Definition 2.5** (see [13]) The Katugampola left-sided fractional integral of order $\alpha > 0$, $\rho > 0$ of $x \in X_c^p(a, b)$ for $-\infty < a < t < \infty$ is defined by

$$\left({}^\rho I_{a+}^\alpha x\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} x(\tau)\, d\tau. \tag{2.1}$$

**Definition 2.6** (see [14]) The Katugampola fractional derivative associated with the generalized fractional integrals (2.1) are defined, for $0 \leq a < t < \infty$, $n = [\alpha] + 1$, by

$$\begin{aligned} \left({}^\rho D_{a+}^\alpha x\right)(t) &= \left(t^{1-\rho} \frac{d}{dt}\right)^n \left({}^\rho I_{a+}^{n-\alpha} x\right)(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{\alpha-n+1}} x(\tau)\, d\tau. \end{aligned} \tag{2.2}$$

Let $x \in L([a, b], E)$, then the Katugampola fractional integral of order $\alpha$ of the fuzzy function $x$ is defined as follows:

$$x_{\alpha,\rho}(t) = \left({}^\rho I_{a+}^\alpha x\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} x(\tau)\, d\tau, \quad t \geq a.$$

Since $[x(t)]^r = [\underline{x}(r, t), \overline{x}(r, t)]$ and $0 < \alpha < 1$, we can consider the fuzzy Katugampola fractional integral of the fuzzy function $x$ based on lower and upper functions, that is,

$$\left[\left({}^\rho I_{a+}^\alpha x\right)(t)\right]^r = \left[\left({}^\rho I_{a+}^\alpha \underline{x}\right)(r, t), \left({}^\rho I_{a+}^\alpha \overline{x}\right)(r, t)\right], \quad t \geq a,$$

where

$$\left({}^\rho I_{a+}^\alpha \underline{x}\right)(r, t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \underline{x}(r, \tau)(\tau)\, d\tau$$

and

$$\left({}^\rho I_{a+}^\alpha \overline{x}\right)(r, t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \overline{x}(r, \tau)(\tau)\, d\tau.$$

In addition, it follows that the operator $x_{\alpha,\rho}(t)$ is linear and bounded from $C([a,b],E)$ to $C([a,b],E)$. Indeed, we have

$$c \leq \|x\|_0 \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} \, d\tau = \frac{\rho^{-\alpha}\|x\|_0}{\Gamma(\alpha+1)} \left(t^\rho - a^\rho\right)^\alpha,$$

where $\|z\|_0 = \sup_{t\in[a,b]} D_0[z(t), \widehat{0}]$.

**Definition 2.7** (see [18]) Let order $\alpha$ and type $\beta$ satisfy $n-1<\alpha \leq n$ and $0 \leq \beta \leq 1$, with $n \in \mathbb{N}$. The fuzzy Hilfer–Katugampola generalized Hukuhara fractional derivative(or Hilfer–Katugampola gH-fractional derivative) (left-sided/right-sided), with respect to $t$, with $\rho > 0$ of a function $t \in C_{1-\gamma,\rho}[a,b]$, is defined by

$$\begin{aligned} \left({}^\rho D_{a+}^{\alpha,\beta} x\right)(t) &= \left({}^\rho I_{a+}^{\beta(1-\alpha)} \tau^{\rho-1} \frac{d}{d\tau} {}^\rho I_{a+}^{(1-\beta)(n-\alpha)} x\right)(t) \\ &= \left({}^\rho I_{a+}^{\beta(1-\alpha)} \delta_\rho {}^\rho I_{a+}^{(1-\beta)(1-\alpha)} x\right)(t) \end{aligned}$$

if the gH-derivative $x'_{(1-\alpha),\rho}(t)$ exists for $t \in [a,b]$, where

$$x_{(1-\alpha),\rho}(t) := \left({}^\rho I_{a+}^{(1-\alpha)} x\right)(t) = \frac{\rho^{-\alpha}}{\Gamma(1-\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{-\alpha}} x(\tau)\, d\tau, \quad t \geq a.$$

**Lemma 2.8** (see [18]) *Let* ${}^\rho I_{a+}^\alpha$ *according to Eqs.* (2.1). *Then*

$${}^\rho I_{a+}^\alpha \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta-1}, \quad \alpha \geq 0, \beta > 0.$$

**Lemma 2.9** (see [18]) *Let* $\alpha > 0$, $0 \leq \gamma < 1$. *If* $x \in C_\gamma[a,b]$ *and* ${}^\rho I_{a+}^{1-\alpha} x \in C_\gamma^1[a,b]$, *then*

$$\left({}^\rho I_{a+}^\alpha {}^\rho D_{a+}^\alpha x\right)(t) = x(t) - \frac{({}^\rho I_{a+}^{1-\alpha} x)(a)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}.$$

**Lemma 2.10** (see [18]) *Let* $x \in L^1(a,b)$. *If* ${}^\rho D_{a+}^{\beta(1-\alpha)} x$ *exists on* $L^1(a,b)$, *then*

$${}^\rho D_{a+}^{\alpha,\beta} {}^\rho I_{a+}^\alpha x = {}^\rho I_{a+}^{\beta(1-\alpha)} {}^\rho D_{a+}^{\beta(1-\alpha)} x$$

*for all* $t(a,b]$.

**Lemma 2.11** *If* $x \in \mathrm{AC}([a,b],E)$ *is a d-monotone fuzzy function, where* $[x(t)]^r = [\underline{x}(r,t), \overline{x}(r,t)]$ *for* $0 \leq r \leq 1$, $a \leq t \leq b$, *then for* $0 < \alpha < 1$ *and* $\rho > 0$ *we have that*

(i) $[({}^\rho D_{a^+}^{\alpha,\beta} x)(t)]^r = [{}^\rho D_{a^+}^{\alpha,\beta} \underline{x}(r,t), {}^\rho D_{a^+}^{\alpha,\beta} \overline{x}(r,t)]$ *for* $a \leq t \leq b$, *if* $x$ *is d-increasing;*

(ii) $[({}^\rho D_{a^+}^{\alpha,\beta} x)(t)]^r = [{}^\rho D_{a^+}^{\alpha,\beta} \overline{x}(r,t), {}^\rho D_{a^+}^{\alpha,\beta} \underline{x}(r,t)]$ *for* $a \leq t \leq b$, *if* $x$ *is d-decreasing.*

*Proof* It is well known that if $x$ is d-increasing, then $[x'(t)]^r = [\frac{d}{dt}\underline{x}(r,t), \frac{d}{dt}\overline{x}(r,t)]$. Therefore, from Definition 2.7 we have

$$\begin{aligned} \left[\left({}^\rho D_{a^+}^{\alpha,\beta} x\right)(t)\right]^r &= \left[\left({}^\rho I_{a^+}^{\beta(1-\alpha)} \delta_\rho^1 {}^\rho I_{a^+}^{(1-\beta)(1-\alpha)} \underline{x}\right)(r,t), \left({}^\rho I_{a^+}^{\beta(1-\alpha)} \delta_\rho^1 {}^\rho I_{a^+}^{(1-\beta)(1-\alpha)} \overline{x}\right)(r,t)\right] \\ &= \left[{}^\rho D_{a^+}^{\alpha,\beta} \underline{x}(r,t), {}^\rho D_{a^+}^{\alpha,\beta} \overline{x}(r,t)\right]. \end{aligned}$$

If $x$ is d-decreasing, then $[x'(t)]^r = [\frac{d}{dt}\overline{x}(r,t), \frac{d}{dt}\underline{x}(r,t)]$, and therefore we get

$$\begin{aligned}\bigl[\bigl({}^\rho D_{a^+}^{\alpha,\beta}x\bigr)(t)\bigr]^r &= \bigl[\bigl({}^\rho I_{a^+}^{\beta(1-\alpha)}\delta_\rho^1\,{}^\rho I_{a^+}^{(1-\beta)(1-\alpha)}\overline{x}\bigr)(r,t), \bigl({}^\rho I_{a^+}^{\beta(1-\alpha)}\delta_\rho^1\,{}^\rho I_{a^+}^{(1-\beta)(1-\alpha)}\underline{x}\bigr)(r,t)\bigr] \\ &= \bigl[{}^\rho D_{a^+}^{\alpha,\beta}\overline{x}(r,t), {}^\rho D_{a^+}^{\alpha,\beta}\underline{x}(r,t)\bigr].\end{aligned}$$

The proof is complete. □

**Lemma 2.12** *If $x \in \mathrm{AC}([a,b],E)$ is a d-monotone fuzzy function, $t \in (a,b]$, and $\alpha \in (0,1)$, we set $z(t) := {}^\rho I_{a+}^\alpha$ and $z_{(1-\alpha),\rho}(t)$ is d-increasing on $(a,b]$, then*

$$\bigl({}^\rho I_{a+}^\alpha\,{}^\rho D_{a+}^{\alpha,\beta}x\bigr)(t) = x(t) \ominus \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}$$

*and*

$$\bigl({}^\rho D_{a+}^{\alpha,\beta}\,{}^\rho I_{a+}^\alpha x\bigr)(t) = x(t).$$

*Proof* If $z(t)$ is d-increasing on $[a,b]$ or $z(t)$ is d-decreasing on $[a,b]$ and $z_{(1-\alpha),\rho}(t)$ is d-increasing on $(a,b]$.

By using the Definition 2.6, Definition 2.7, Lemma 2.9, and the initial condition $({}^\rho I_{a+}^{1-\gamma}x)(a) = c$, we have that

$$\begin{aligned}\bigl({}^\rho I_{a+}^\alpha\,{}^\rho D_{a+}^{\alpha,\beta}x\bigr)(t) &= \left({}^\rho I_{a+}^\alpha\,{}^\rho I_{a+}^{\beta(1-\alpha)} t^{\rho-1}\frac{d}{dt}\,{}^\rho I_{a+}^{(1-\beta)(n-\alpha)}x\right)(t) \\ &= \left({}^\rho I_{a+}^\gamma t^{\rho-1}\frac{d}{dt}\,{}^\rho I_{a+}^{1-\gamma}x\right)(t) \\ &= \bigl({}^\rho I_{a+}^\gamma\,{}^\rho D_{a+}^\gamma x\bigr)(t) \\ &= x(t) \ominus \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}.\end{aligned}$$

Now, by Lemma 2.8, Lemma 2.9, and Lemma 2.10, we get

$$\begin{aligned}\bigl({}^\rho D_{a+}^{\alpha,\beta}\,{}^\rho I_{a+}^\alpha x\bigr)(t) &= \bigl({}^\rho I_{a+}^{\beta(1-\alpha)}\,{}^\rho D_{a+}^{\beta(1-\alpha)}x\bigr)(t) \\ &= x(t) \ominus \frac{({}^\rho I_{a+}^{1-\beta(1-\alpha)}x)(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= x(t).\end{aligned}$$

On the other hand, since $x \in \mathrm{AC}([a,b],E)$, there exists a constant $K$ such that $K = \sup_{t\in[a,b]} D_0[x(t),\widehat{0}]$. Then

$$D_0\bigl[{}^\rho I_{a+}^\alpha x(t),\widehat{0}\bigr] \le K\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}}\,d\tau = \frac{\rho^{-\alpha}K}{\Gamma(\alpha+1)}\bigl(t^\rho - a^\rho\bigr)^\alpha$$

and ${}^\rho I_{a+}^\alpha x(t) = 0$ at $t = a$. The proof is complete. □

**Lemma 2.13** *Let $\psi : [a,b] \to \mathbb{R}^+$ be a continuous function on the interval $[a,b]$ and satisfy ${}^\rho D_{a+}^{\alpha,\beta}\psi(t) \le g(t,\psi(t))$, $t \ge a$, where $g \in C([a,b] \times \mathbb{R}^+, \mathbb{R}^+)$. Assume that $m(t) = m(t,a,\xi_0)$ is the maximal solution of the initial value problem*

$$
{}^\rho D_{a+}^{\alpha,\beta}\xi(t) = g(t,\xi), \qquad \left({}^\rho I_{a+}^{1-\gamma}\xi\right)(a) = \xi_0 \ge 0 \tag{2.3}
$$

*on $[a,b]$. Then, if $\psi(a) \le \xi_0$, we have $\psi(t) \le m(t)$, $t \in [a,b]$.*

*Proof* We omit the proof of this theorem as it is similar to that of Theorem 2.2 in [12]. □

**Lemma 2.14** *Consider the initial value problem as follows*:

$$
{}^\rho D_{a+}^{\alpha,\beta}\psi(t) = g\bigl(t,\psi(t)\bigr), \qquad \left({}^\rho I_{a+}^{1-\gamma}\psi\right)(a) = \psi_0 = 0, \quad t \in [a,b]. \tag{2.4}
$$

*Let $\eta > 0$ be a given constant and $B(\psi_0,\eta) = \{\psi \in \mathbb{R} : |\psi - \psi_0| \le \eta\}$. Assume that the real-valued function $g : [a,b] \times [0,\eta] \to \mathbb{R}^+$ satisfies the following conditions*:

(i) *$g \in C([a,b] \times [0,\eta], \mathbb{R}^+)$, $g(t,0) \equiv 0$, $0 \le g(t,\psi) \le M_g$ for all $(t,\psi) \in [a.b] \times [0,\eta]$;*

(ii) *$g(t,\psi)$ is nondecreasing in $\psi$ for every $t \in [a,b]$. Then problem (2.4) has at least one solution defined on $[a,b]$ and $\psi(t) \in B(\psi_0,\eta)$.*

*Proof* We omit the proof of this theorem as it is similar to that of Theorem 2.3 in [12]. □

## 3 Main results

In this section, the existence and uniqueness of solution to problem (1.1) are investigated by using successive approximations method under generalized Lipschitz condition of the right-hand side.

**Lemma 3.1** *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1$, $0 \le \beta \le 1$, and $\rho > 0$, let $f : (a,b] \times E \to E$ be a fuzzy function such that $t \longmapsto f(t,x)$ belongs to $C_{\gamma,\rho}([a,b],E)$ for any $x \in E$. Then a d-monotone fuzzy function $x \in C([a,b],E)$ is a solution of initial value problem (1.1) if and only if $x$ satisfies the integral equation*

$$
\begin{aligned}
&x(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \\
&\quad = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f\bigl(\tau, x(\tau)\bigr)\,d\tau, \quad t \in [a,b]
\end{aligned} \tag{3.1}
$$

*and the fuzzy function $t \mapsto {}^\rho I_{a+}^{1-\gamma} f(t,x)$ is d-increasing on $(a,b]$.*

*Proof* First, we prove the necessity condition. Let $x \in C([a,b],E)$ be a d-monotone solution of (1.1), and let $z(t) := x(t) \ominus_{gH} ({}^\rho I_{a+}^{1-\gamma}x)(a)$, $t \in (a,b]$. Because $x$ is d-monotone on $[a,b]$, it follows that $t \mapsto z(t)$ is d-increasing on $[a,b]$ (see [11]). From (1.1) and Lemma 2.12 we have that

$$
\left({}^\rho I_{a+}^{\alpha}\, {}^\rho D_{a+}^{\alpha,\beta} x\right)(t) = x(t) \ominus \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \tag{3.2}
$$

for $t \in [a,b]$. Since $f(t,x) \in C_\gamma([a,b],E)$ for any $x \in E$, and from (1.1), it follows that

$$\left({}^\rho I^\alpha_{a+}\, {}^\rho D^{\alpha,\beta}_{a+} x\right)(t) = {}^\rho I^\alpha_{a+} f\big(t,x(t)\big) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}} f\big(\tau,x(\tau)\big)\,d\tau \tag{3.3}$$

for $t \in [a,b]$. In addition, since $z(t)$ is d-increasing on $(a,b]$, it follows that $t \mapsto f_{\alpha,\rho}(t,x)$ is also d-increasing on $(a,b]$. Consequently, combining (3.2) and (3.3) proves the necessity condition.

Next, we prove the sufficiency. Let $x \in C([a,b],E)$ be a d-monotone fuzzy function satisfying (3.1) and such that $t \mapsto f_{\alpha,\rho}(t,x)$ is d-increasing on $(a,b]$. Because of the continuity of the fuzzy function $f$, the fuzzy function $t \mapsto f_{\alpha,\rho}(t,x)$ is continuous on $(a,b]$ and $f_{\alpha,\rho}(a,x(a)) = \lim_{t\to a^+} f_{\alpha,\rho}(t,x) = 0$. Then

$$\begin{aligned} x(t) &= \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho I^\alpha_{a+} f\big(t,x(t)\big)\right)(t), \\ {}^\rho I^{1-\gamma}_{a+} x(t) &= \sum_{i=1}^m C_i x(t_i) + \left({}^\rho I^{1-\beta(1-\alpha)}_{a+} f\big(t,x(t)\big)\right)(t), \end{aligned}$$

and

$${}^\rho I^{1-\gamma}_{a+} x(0) = \sum_{i=1}^m C_i x(t_i).$$

In addition, since $t \mapsto f_{\alpha,\rho}(t,x)$ is d-increasing on $(a,b]$, acting on the two sides of (3.1) by the operator ${}^\rho D^{\alpha,\beta}_{a+}$, we obtain

$$\begin{aligned} &{}^\rho D^{\alpha,\beta}_{a+} \left[ x(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \right] = f\big(t,x(t)\big), \\ &{}^\rho D^{\alpha,\beta}_{a+} x(t) = f\big(t,x(t)\big). \end{aligned}$$

The proof is complete. □

Let $h > 0$ be a given constant, and let $\mathbb{B}(x_0,h) = \{x \in E : D_0[x,x_0] \le h\}$.

**Theorem 3.2** *Let $f \in C([a,b] \times \mathbb{B}(x_0,h), E)$ and assume that the following conditions hold:*

(i) *There exists a positive constant $M_f$ such that $D_0[f(t,z),\widehat{0}] \le M_f$, $\forall (t,z) \in [a,b] \times \mathbb{B}(x_0,h)$;*

(ii) *For every $t \in [a,b]$ and every $z,\omega \in \mathbb{B}(x_0,h)$,*

$$D_0\big[f(t,z),f(t,\omega)\big] \le g\big(t, D_0[z,\omega]\big),$$

*where $g(t,\cdot) \in C([a,b] \times [0,\rho],\mathbb{R}^+)$ satisfies the conditions in Lemma 2.14 provided that problem (2.4) has only the solution $\psi(t) \equiv 0$ on $[a,b]$. Then, the following successive approximations given by $x^0(t) = x_0$ and for $n = 1,2,\ldots,$*

$$\begin{aligned} & x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \\ &\quad = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}} f\big(\tau, x^{n-1}(\tau)\big)\,d\tau \end{aligned} \tag{3.4}$$

*converge uniformly to a unique solution of problem* (1.1) *on some intervals* $[a,T]$ *for some* $T \in (a,b]$ *provided that the function* $t \mapsto {}^{\rho}I^{\alpha}_{a+}f(t,x^n(t))$ *is d-increasing on* $[a,T]$.

*Proof* Choose $t^* > a$ such that $t^* \leq [(\frac{h\Gamma(1+\alpha)}{M}\rho^{\alpha})^{\frac{1}{\alpha}} + a^{\rho}]^{\frac{1}{\rho}}$, where $M = \max\{M_g, M_f\}$, and put $T := \min\{t^*, b\}$. Let $\mathbb{S}$ be a set of continuous fuzzy functions $x$ such that $\omega(a) = x_0$ and $\omega(t) \in \mathbb{B}(x_0,h)$ for all $t \in [a,T]$. Next, we consider the sequence of continuous fuzzy function $\{x^n\}_{n=0}^{\infty}$ given by: $x^0(t) = x_0$, $\forall t \in [a,T]$, and for $n = 1,2,\dots,$

$$x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}} f\bigl(\tau, x^{n-1}(\tau)\bigr)\,d\tau, \quad [a,T]. \tag{3.5}$$

Step 1: First of all, we prove that $x^n(t) \in C([a,T], B(x_0,h))$. For $n \geq 1$ and for any $t_1, t_2 \in [a,T]$ with $t_1 < t_2$, we have

$$\begin{aligned} &D_0\biggl[x^n(t_1) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}, x^n(t_2) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}\biggr] \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{t_1} \tau^{\rho-1}\bigl[\bigl({t_1}^{\rho}-\tau^{\rho}\bigr)^{\alpha-1} - \bigl({t_2}^{\rho}-\tau^{\rho}\bigr)^{\alpha-1}\bigr] D_0\bigl[f\bigl(\tau, x^n(\tau)\bigr), \widehat{0}\bigr]\,d\tau \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{\tau^{\rho-1}}{({t_2}^{\rho}-\tau^{\rho})^{1-\alpha}} D_0\bigl[f\bigl(\tau, x^n(\tau)\bigr), \widehat{0}\bigr]\,d\tau. \end{aligned}$$

The second integral on the right-hand side of the last inequality has the value $\frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)}(t_2^{\rho} - t_1^{\rho})^{\alpha}$. For the first integral, it has value $\frac{\rho^{1-\alpha}}{\Gamma(\alpha+1)}[(t_1^{\rho}-a^{\rho})^{\alpha} - (t_2^{\rho}-a^{\rho})^{\alpha}]$. Hence, we get

$$\begin{aligned} D_0\bigl[x^n(t_1), x^n(t_2)\bigr] &\leq \frac{\rho^{-\alpha}M_f}{\Gamma(\alpha+1)}\bigl[\bigl(t_2^{\rho}-t_1^{\rho}\bigr)^{\alpha} + \bigl(t_2^{\rho}-t_1^{\rho}\bigr)^{\alpha} - \bigl(t_2^{\rho}-a^{\rho}\bigr)^{\alpha}\bigr] \\ &\leq \frac{2\rho^{-\alpha}M_f}{\Gamma(\alpha+1)}\bigl(t_2^{\rho}-t_1^{\rho}\bigr)^{\alpha}, \end{aligned}$$

and it follows that the last expression converges to 0 as $t_1 \to t_2$, which proves that $x^n$ is a continuous function on $[a,T]$ for all $n \geq 1$. In addition, it follows that $x^n(t) \in \mathbb{B}(x_0,h)$ for all $t \in [a,T]$ and for all $n \geq 0$ if and only if $x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}(\frac{t^{\rho}-a^{\rho}}{\rho})^{\gamma-1} \in \mathbb{B}(0,h)$ for all $t \in [a,T]$ and for all $n \geq 0$. Indeed, if we suppose that $x^{n-1}(t) \in \mathbb{S}$ for all $t \in [a,T]$ and for a given $n \geq 2$, then from

$$\begin{aligned} &D_0\biggl[x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}, \widehat{0}\biggr] \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^{\rho}-\tau^{\rho})^{1-\alpha}} D_0\bigl[f\bigl(\tau, x^{n-1}(\tau)\bigr), \widehat{0}\bigr]\,d\tau \\ &\leq \frac{M_f(t^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \leq h, \end{aligned}$$

it follows that $x^n(t) \in \mathbb{S}$ for all $t \in [a,T]$. Hence, by mathematical induction, we have that $x^n(t) \in \mathbb{S}$ for all $t \in [a,T]$ and for all $n \geq 1$. Next, we prove that the sequence $x^n(t)$ converges uniformly to a continuous function $x \in C([a,T],\mathbb{B}(x_0,h))$. By assumption (ii) and mathematical induction, we have for $t \in [a,T]$

$$D_0\left[x^{n+1}(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}, x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\right] \le \psi^n(t), \quad n = 0,1,2,\dots, \tag{3.6}$$

where $\psi^n(t)$ is defined as follows:

$$\psi^n(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} g\big(\tau, \psi^{n-1}(\tau)\big)\, d\tau$$

and $\psi^0(t) = \frac{M(t^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)}$. Thus, we have, for $t \in [a,T]$ and for $n = 0,1,2,\dots$,

$$\begin{aligned} D_0\left[{}^\rho D_{a+}^{\alpha,\beta} x^{n+1}(t), {}^\rho D_{a+}^{\alpha,\beta} x^n(t)\right] &\le D_0\left[f\big(t,x^n(t)\big), f\big(t,x^{n-1}(t)\big)\right] \\ &\le g\big(t, D_0\left[x^n(t), x^{n-1}(t)\right]\big) \\ &\le g\big(t, \psi^{n-1}(t)\big). \end{aligned}$$

Let $m \ge n$ and $t \in [a,T]$, then we can obtain

$$\begin{aligned} {}^\rho D_{a+}^{\alpha,\beta} D_0\left[x^n(t), x^m(t)\right] &\le D_0\left[{}^\rho D_{a+}^{\alpha,\beta} x^n(t), {}^\rho D_{a+}^{\alpha,\beta} x^m(t)\right] \\ &\le D_0\left[{}^\rho D_{a+}^{\alpha,\beta} x^n(t), {}^\rho D_{a+}^{\alpha,\beta} x^{n+1}(t)\right] \\ &\quad + D_0\left[{}^\rho D_{a+}^{\alpha,\beta} x^{n+1}(t), {}^\rho D_{a+}^{\alpha,\beta} x^{m+1}(t)\right] \\ &\quad + D_0\left[{}^\rho D_{a+}^{\alpha,\beta} x^{m+1}(t), {}^\rho D_{a+}^{\alpha,\beta} x^m(t)\right] \\ &\le 2g\big(t, \psi^{n-1}(t)\big) + g\big(t, D_0\left[x^n(t), x^m(t)\right]\big). \end{aligned}$$

From (ii), because we have that the solution $\psi(t) = 0$ is a unique solution of problem (2.4) and $g(\cdot, \psi^{n-1}(\cdot)) : [a,T] \to [0, M_g]$ uniformly converges to 0, for every $\varepsilon > 0$, there exists a natural number $n_0$ such that

$${}^\rho D_{a+}^{\alpha,\beta} D_0\left[x^n(t), x^m(t)\right] \le g\big(t, D_0\left[x^n(t), x^m(t)\right]\big) + \varepsilon \quad \text{for } m \ge n \ge n_0.$$

From the fact that $D_0[x^n(a), x^m(a)] = 0 < \varepsilon$ and by using Lemma 2.13, we have for $t \in [a,T]$

$$D_0\left[x^n(t), x^m(t)\right] \le \lambda_\varepsilon(t), \quad m \ge n \ge n_0, \tag{3.7}$$

where $\lambda_\varepsilon(t)$ is the maximal solution to the following IVP:

$$\big({}^\rho D_{a+}^{\alpha,\beta} \lambda_\varepsilon\big)(t) = g\big(t, \lambda_\varepsilon(t)\big) + \varepsilon, \qquad \big({}^\rho I_{a+}^{1-\gamma} \lambda_\varepsilon\big)(a) = \varepsilon.$$

Due to Lemma 2.13 one can infer that $\{\psi_\varepsilon(\cdot,\omega)\}$ converges uniformly to the maximal solution $\psi(t) \equiv 0$ of (2.4) on $[a,T]$ as $\varepsilon \to 0$. Hence, by virtue of (3.7), we can find $n_0 \in \mathbb{N}$

large enough such that, for $n, m > n_0$,

$$\sup_{t\in[a,T]} D_0\Bigg[x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}, \\ x^m(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\Bigg] \le \varepsilon. \tag{3.8}$$

Since $(E, D_0)$ is a complete metric space and (3.8) holds, it follows that $\{x^n(t)\}$ converges uniformly to $x \in C([a,b], \mathbb{B}(x_0, h))$. Hence, we obtain

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} &= \lim_{n\to\infty}\left(x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\right) \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f\big(\tau, x(\tau)\big)\, d\tau, \quad [a, T]. \end{aligned}$$

Due to Lemma 3.1 the function $x(t)$ is the solution to (1.1) on $[a, T]$.

Step 2: To show that $x$ is the unique solution, assume that $y : [a, T] \to E$ is another solution of problem (1.1) on $[a, T]$. Denote $k(t) = D_0[x(t), y(t)]$. Then $k(a) = 0$ and for every $t \in [a, T]$ we have

$${}^\rho D_{a+}^{\alpha,\beta} k(t) \le D_0\big[f\big(t, x(t)\big), f\big(t, y(t)\big)\big] \le g\big(t, k(t)\big).$$

Again applying the comparison Lemma 2.13, we obtain $k(t) \le m(t)$, where $m$ is a maximal solution of the IVP ${}^\rho D_{a+}^{\alpha,\beta} m(t) \le g(t, m(t))$, $({}^\rho I_{a+}^{1-\gamma} m)(a) = 0$. By assumption (ii), we have $m(t) = 0$ and therefore $x(t) = y(t)$, $\forall t \in [a, T]$. The proof is complete. □

**Corollary 3.3** *Let $f \in C([a,b], E)$. Assume that there exist positive constants $L$, $M_f$ such that, for every $z, \omega \in E$,*

$$D_0\big[f(t, z), f(t, \omega)\big] \le L D_0[z, \omega], \qquad D_0\big[f(t, z), \widehat{0}\big] \le M_f.$$

*Then the following successive approximations given by $u^0(t) = u_0$ and for $n = 1, 2, \ldots$*

$$x^n(t) \ominus_{gH} \frac{\sum_{i=1}^m C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f\big(\tau, x^{n-1}(\tau)\big)\, d\tau$$

*converge uniformly to a unique solution of problem (1.1) on some intervals $[a, T]$ for some $T \in (a, b]$ provided that the function $t \mapsto {}^\rho I_{a+}^\alpha f(t, x^n(t))$ is d-increasing on $[a, T]$.*

*Example* 3.4 Let $\gamma = \alpha + \beta(1 - \alpha)$, where $0 < \alpha < 1$, $0 \le \beta \le 1$, $\rho > 0$, and $\lambda \in \mathbb{R}$. We consider the linear fuzzy fractional differential equation under Hilfer–Katugampola fractional derivative and assume that the following conditions hold:

$$\begin{cases} ({}^\rho D_{a+}^{\alpha,\beta} x)(t) = \lambda x(t) + p(t), & t \in (a, b], \\ ({}^\rho I_{a+}^{1-\gamma} x)(a) = x_0 = \sum_{i=1}^m C_i x(t_i), & \gamma = \alpha + \beta(1 - \alpha). \end{cases} \tag{3.9}$$

Applying Lemma 3.1, we see that

$$\begin{aligned} &x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \\ &= \lambda \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} x(\tau)\,d\tau + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} p(\tau)\,d\tau, \quad t \in [a,b] \\ &= \lambda \left({}^\rho I_{a+}^{\alpha} x\right)(t) + \left({}^\rho I_{a+}^{\alpha} p\right)(t), \end{aligned}$$

where $p \in C((a,b],E)$, and we also assume that the diameter of the right-hand side of the above equation is increasing. We observe that $f(t,x) := \lambda x + p$ satisfies the assumptions of Corollary 3.3. To obtain an explicit solution of (3.9), we apply the method of successive approximations. Set $u^0(t) = u_0$ and

$$x^n(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} = \lambda \left({}^\rho I_{a+}^{\alpha} x^{n-1}\right)(t) + \left({}^\rho I_{a+}^{\alpha} p\right)(t), n = 1, 2, \ldots.$$

For $n = 1$ and $\lambda > 0$, if we assume that $x$ is d-increasing, then it follows that

$$x^1(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} = \sum_{i=1}^{m} C_i x(t_i) \frac{\lambda (t^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \left({}^\rho I_{a+}^{\alpha} p\right)(t).$$

On the other hand, if we assume that $\lambda < 0$ and $x$ is d-decreasing, then it follows that

$$(-1)\left(\frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \ominus_{gH} x^1(t)\right) = \sum_{i=1}^{m} C_i x(t_i) \frac{\lambda (t^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \left({}^\rho I_{a+}^{\alpha} p\right)(t).$$

For $n = 2$, we also see that

$$\begin{aligned} x^2(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} &= \sum_{i=1}^{m} C_i x(t_i) \left[\frac{\lambda (t^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \frac{\lambda^2 (t^\rho - a^\rho)^{2\alpha}}{\rho^{2\alpha} \Gamma(2\alpha+1)}\right] \\ &\quad + \left({}^\rho I_{a+}^{\alpha} p\right)(t) + \left({}^\rho I_{a+}^{2\alpha} p\right)(t), \end{aligned}$$

if $\lambda > 0$ and $x$ is d-increasing, and

$$\begin{aligned} &(-1)\left(\frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \ominus_{gH} x^2(t)\right) \\ &= \sum_{i=1}^{m} C_i x(t_i) \left[\frac{\lambda (t^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} + \frac{\lambda^2 (t^\rho - a^\rho)^{2\alpha}}{\rho^{2\alpha} \Gamma(2\alpha+1)}\right] \\ &\quad + \left({}^\rho I_{a+}^{\alpha} p\right)(t) + \left({}^\rho I_{a+}^{2\alpha} p\right)(t) \end{aligned}$$

if λ < 0 and $x$ is d-decreasing. If we proceed inductively and let $n \to \infty$, we obtain the solution

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} &= \sum_{i=1}^{m} C_i x(t_i) \sum_{j=1}^{\infty} \frac{\lambda^j (t^\rho - a^\rho)^{j\alpha}}{\rho^{j\alpha}\Gamma(j\alpha+1)} + \int_a^t \tau^{\rho-1} \sum_{j=1}^{\infty} \frac{\lambda^{j-1}(t^\rho - \tau^\rho)^{j\alpha-1}}{\rho^{j\alpha-1}\Gamma(j\alpha)} p(\tau)\,d\tau \\ &= \sum_{i=1}^{m} C_i x(t_i) \sum_{j=1}^{\infty} \frac{\lambda^j (t^\rho - a^\rho)^{j\alpha}}{\rho^{j\alpha}\Gamma(j\alpha+1)} + \int_a^t \tau^{\rho-1} \sum_{j=0}^{\infty} \frac{\lambda^{j}(t^\rho - \tau^\rho)^{j\alpha+(\alpha-1)}}{\rho^{j\alpha+(\alpha-1)}\Gamma(j\alpha+\alpha)} p(\tau)\,d\tau \\ &= \sum_{i=1}^{m} C_i x(t_i) \sum_{j=1}^{\infty} \frac{\lambda^j (t^\rho - a^\rho)^{j\alpha}}{\rho^{j\alpha}\Gamma(j\alpha+1)} + \int_a^t \tau^{\rho-1} \frac{(t^\rho - \tau^\rho)^{(\alpha-1)}}{\rho^{(\alpha-1)}} \sum_{j=0}^{\infty} \frac{\lambda^{j}(t^\rho - \tau^\rho)^{j\alpha}}{\rho^{j\alpha}\Gamma(j\alpha+\alpha)} p(\tau)\,d\tau, \end{aligned}$$

for each case of λ > 0 and $x$ is d-increasing, or λ < 0 and $x$ is d-decreasing, respectively. Then, by applying definition of Mittag-Leffler function $E_{\alpha,\beta}(x) = \sum_{j=1}^{\infty} \frac{x^k}{\Gamma(j\alpha+\beta)}$, $\alpha, \beta > 0$, the solution of problem (3.9) is expressed by

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} &= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\left(\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha}\right) \\ &\quad + \frac{1}{\rho^{\alpha-1}} \int_a^t \tau^{\rho-1}\left(t^\rho - \tau^\rho\right)^{(\alpha-1)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha}\right) p(\tau)\,d\tau \end{aligned}$$

for the case of λ > 0 and $x$ is d-increasing. On the other hand, if λ < 0 and $x$ is d-decreasing, then we obtain the solution of problem (3.9)

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} &= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\left(\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha}\right) \ominus (-1) \frac{1}{\rho^{\alpha-1}} \int_a^t \tau^{\rho-1}\left(t^\rho - \tau^\rho\right)^{(\alpha-1)} \\ &\quad \times E_{\alpha,\alpha}\left(\lambda\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha}\right) p(\tau)\,d\tau. \end{aligned}$$

*Remark* 3.5 In problem (3.9), suppose that λ > 0 and the solution of (3.9) is d-increasing. We observe that the solutions of problem (3.9) admit particular cases as follows: if $\beta = 0$, then we obtain the solution of problem (3.9) with the Caputo–Katugampola fractional derivative as follows:

$$\begin{aligned} x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} &= \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\left(\lambda\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha}\right) \\ &\quad + \frac{1}{\rho^{\alpha-1}} \int_a^t \tau^{\rho-1}\left(t^\rho - \tau^\rho\right)^{(\alpha-1)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha}\right) p(\tau)\,d\tau. \end{aligned}$$

If the value of $\rho$ tends to 1 and $\beta = 0$, then we obtain the solution of problem (3.9) with the Caputo fractional derivative as follows:

$$\begin{aligned} &x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)}(t-a)^{\alpha-1} \\ &\quad = \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\big(\lambda(t-a)^{\alpha}\big) + \int_a^t (t-\tau)^{(\alpha-1)} E_{\alpha,\alpha}\big(\lambda(t-\tau)^{\alpha}\big) p(\tau)\,d\tau. \end{aligned}$$

In addition, if the value of $\rho$ tends to $0^+$ and $\beta = 0$, then we obtain the following solution of problem (3.9) with the Caputo–Hadamard fractional derivative:

$$\begin{aligned} &x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} \\ &\quad = \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\left(\lambda\left(\log\frac{t}{a}\right)^{\alpha}\right) + \int_a^t \left(\log\frac{t}{\tau}\right)^{(\alpha-1)} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{\tau}\right)^{\alpha}\right)\frac{p(\tau)}{\tau}\,d\tau. \end{aligned}$$

*Remark* 3.6 In problem (3.9), suppose that $\lambda < 0$ and the solution of (3.9) is d-decreasing. We observe that the solutions of problem (3.9) admit particular cases as follows: if $\beta = 0$, then we obtain the solution of problem (3.9) with the Caputo–Katugampola fractional derivative as follows:

$$\begin{aligned} &x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1} \\ &\quad = \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\left(\lambda\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right) \ominus (-1)\frac{1}{\rho^{\alpha-1}}\int_a^t \tau^{\rho-1}\big(t^{\rho}-\tau^{\rho}\big)^{(\alpha-1)} \\ &\qquad \times E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha}\right) p(\tau)\,d\tau. \end{aligned}$$

If the value of $\rho$ tends to 1 and $\beta = 0$, then we obtain the solution of problem (3.9) with the Caputo fractional derivative as follows:

$$\begin{aligned} &x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)}(t-a)^{\alpha-1} \\ &\quad = \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\big(\lambda(t-a)^{\alpha}\big) \ominus (-1)\int_a^t (t-\tau)^{(\alpha-1)} E_{\alpha,\alpha}\big(\lambda(t-\tau)^{\alpha}\big) p(\tau)\,d\tau. \end{aligned}$$

In addition, if the value of $\rho$ tends to $0^+$ and $\beta = 0$, then we obtain the following solution of problem (3.9) with the Caputo–Hadamard fractional derivative:

$$\begin{aligned} &x(t) \ominus_{gH} \frac{\sum_{i=1}^{m} C_i x(t_i)}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1} \\ &\quad = \sum_{i=1}^{m} C_i x(t_i) E_{\alpha,1}\left(\lambda\left(\log\frac{t}{a}\right)^{\alpha}\right) \ominus (-1)\int_a^t \left(\log\frac{t}{\tau}\right)^{(\alpha-1)} E_{\alpha,\alpha}\left(\lambda\left(\log\frac{t}{\tau}\right)^{\alpha}\right)\frac{p(\tau)}{\tau}\,d\tau. \end{aligned}$$

## 4 Conclusion

In this paper, the existence and uniqueness results of solution of a kind of fuzzy differential equation with Hilfer–Katugampola fractional derivative and nonlocal condition were investigated. To obtain our main result, we established some necessary comparison theorems. By using the method of successive approximations under generalized Lipschitz condition, we obtained some sufficient conditions to ensure the existence and uniqueness of solution. An illustrative example is given to show the practical usefulness of the analytical results.

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The authors declare that they have no competing interest.

**Authors' contributions**
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