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Existence and uniqueness of periodic solutions for a system of differential equations via operator methods

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Abstract

This article investigates the existence and uniqueness of periodic solutions for a new system of differential equations. By employing fixed point theorems for increasing φ-(h,τ)-concave operators, we establish the existence of unique periodic solution for our differential system and then give a monotone iterative scheme to approximate the unique periodic solution. Some examples are presented in the end to illustrate the validity of our main results.

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Keywords: Differential system; Periodic solution; Existence and uniqueness; φ-(h,τ)-concave operator

1 Introduction

In this article, we investigate the existence and uniqueness of periodic solutions or positive periodic solutions for the following system of differential equations:

$$\begin{cases} x'(t) = a(t)x(t) - f_1(t,x(t),y(t)) + g_1(t), \\ y'(t) = -b(t)y(t) + f_2(t,x(t),y(t)) - g_2(t), \end{cases} \tag{1.1}$$

where $a,b \in C(\mathbf{R},\mathbf{R}_+)$ are ω-periodic for some $\omega > 0$, $f_1(t,x,y), f_2(t,x,y) \in C(\mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+,\mathbf{R}_+)$ and $g_1(t),g_2(t) \in C(\mathbf{R},\mathbf{R}_+)$ are ω-periodic functions in t with $g_i(t) \leq 1$, $i = 1,2$. Here we remark (1.1) is a new system in the context of one-order differential equations. By using recent fixed point theorems for increasing φ-(h,τ)-concave operators, we not only get the existence and uniqueness of periodic solutions for (1.1), but also we can give convergent sequences which can approximate the unique solution. This is a significant improvement compared with some results in the literature. Different from other articles, we discuss a differential equation system by new operator methods.

During the past decades, many people have studied the theories of differential equations in economical, population dynamics, control, ecology and epidemiology; see the monographs [1–5] for example. Owing to its theoretical and practical significance, the study of periodic solutions for differential equations has been paid much attention to and it has a fast development in ordinary and partial differential equations; see the papers [6–22] and the references therein. In these papers, some good results have been established on the existence of periodic solutions. Very recently, there were some articles reported on the existence of periodic solutions for several systems of differential equations; see [23–26] for example. In [23], Radu Precup discussed the existence of multiple positive periodic solutions for the following differential system:

$$\begin{cases} u_1'(t) = -a_1(t)u_1(t) + \epsilon_1 f_1(t, u_1(t), u_2(t)), \\ u_2'(t) = -a_2(t)u_2(t) + \epsilon_2 f_2(t, u_1(t), u_2(t)), \end{cases} \tag{1.2}$$

where for $i \in \{1,2\} : a_i \in C(\mathbf{R}, \mathbf{R})$, $\int_0^\omega a_i\,dt \neq 0$, $\epsilon_i = \operatorname{sign} \int_0^\omega a_i(t)\,dt$, $f_i \in C(\mathbf{R} \times \mathbf{R}_+^2, \mathbf{R}_+)$, and $a_i, f_i(\cdot, u_1, u_2)$ are ω-periodic functions for some $\omega > 0$. The method used to resolve (1.2) is a different version of Krasnosel'skii's fixed point theorem in cones.

In [25], the authors studied the following system of differential equations:

$$\begin{cases} u_i'(t) = u_i(t)[a_i(t) - f_i(t, u(t), v(t))], \quad i = 1, 2, \dots, n, \\ v_j'(t) = v_j(t)[b_j(t) + g_j(t, u(t), v(t))], \quad j = 1, 2, \dots, m, \end{cases} \tag{1.3}$$

where $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$, $v(t) = (v_1(t), v_2(t), \dots, v_m(t))^T$, and f_i, g_j for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, are ω-periodic functions in t. By applying a fixed point theorem, they gave the existence of positive periodic solutions for system (1.3).

However, there are still few papers that studied periodic solutions for systems of differential equations and the uniqueness of solutions is seldom obtained in literature. Motivated by some recently published articles [27–34], we will study the uniqueness of periodic solutions for system (1.1). We will give the existence and uniqueness of periodic solutions or positive periodic solutions for system (1.1) and construct an iterative to approximate the unique solution. Also, we can get the existence and uniqueness of periodic solutions or positive periodic solutions for the following system:

$$\begin{cases} x'(t) = -a(t)x(t) + f_1(t, x(t), y(t)) - g_1(t), \\ y'(t) = b(t)y(t) - f_2(t, x(t), y(t)) + g_2(t), \end{cases} \tag{1.4}$$

where a, b, $f_1(t,x,y)$, $f_2(t,x,y)$ and $g_1(t)$, $g_2(t)$ are the same as in (1.1).

From the articles mentioned above, we know that $f_i - g_i$ $(i = 1, 2)$ in (1.1) and (1.4) may be nonnegative or negative, while the f_i $(i = 1, 2, \dots, n)$, g_j $(j = 1, 2, \dots, m)$ in (1.2) and (1.3) are always nonnegative. So systems (1.1), (1.4) are different from (1.2), (1.3) and other ones in the literature. Moreover, our results indicate that the unique periodic solution exists in a product set, and can be approximated by making an iterative sequence for any initial point in the product set.

2 Preliminaries

We shall find a unique periodic solution for system (1.1) and for this purpose we use operator methods as in [35]. For ω-periodic functions $a, b \in C(\mathbf{R}, \mathbf{R}_+)$ and $f_1, f_2 \in C(\mathbf{R}, \mathbf{R})$, from

[23], the unique ω-periodic solution (x,y) of the system

$$\begin{cases} x'(t) = a(t)x(t) - f_1(t), \\ y'(t) = -b(t)y(t) + f_2(t), \end{cases} \tag{2.1}$$

can be written as

$$\begin{cases} x(t) = \int_t^{t+\omega} H_1(t,s)f_1(s)\,ds, \\ y(t) = \int_t^{t+\omega} H_2(t,s)f_2(s)\,ds, \end{cases} \tag{2.2}$$

where

$$H_1(t,s) = \frac{e^{-\int_t^s a(\xi)\,d\xi}}{1 - e^{-\int_0^\omega a(\xi)\,d\xi}}, \qquad H_2(t,s) = \frac{e^{\int_t^s b(\xi)\,d\xi}}{e^{\int_0^\omega b(\xi)\,d\xi} - 1}, \quad (t,s) \in (\mathbf{R},\mathbf{R}). \tag{2.3}$$

Set

$$\begin{aligned} m_1 &= \min_{t\in[0,\omega]} \int_t^{t+\omega} H_1(t,s)\,ds, \qquad m_2 = \min_{t\in[0,\omega]} \int_t^{t+\omega} H_2(t,s)\,ds, \\ M_1 &= \max_{t\in[0,\omega]} \int_t^{t+\omega} H_1(t,s)\,ds, \qquad M_2 = \max_{t\in[0,\omega]} \int_t^{t+\omega} H_2(t,s)\,ds. \end{aligned}$$

Clearly, (x,y) is a periodic solution of system (1.1) if and only if (x,y) is a solution of the following integral equation system:

$$\begin{cases} x(t) = \int_t^{t+\omega} H_1(t,s)f_1(s,x(s),y(s))\,ds - \int_t^{t+\omega} H_1(t,s)g_1(s)\,ds, \\ y(t) = \int_t^{t+\omega} H_2(t,s)f_2(s,x(s),y(s))\,ds - \int_t^{t+\omega} H_2(t,s)g_2(s)\,ds, \end{cases} \tag{2.4}$$

which can be regarded as an operator equation.

Now we present some notations, concepts and lemmas which have already become known in previous work; see [35–37] and the references therein. Let $(E,\|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$. For any $x,y \in E$, $x \sim y$ means that there are $\alpha > 0$ and $\beta > 0$ such that $\alpha x \le y \le \beta x$. Take $h > \theta$ (i.e., $h \ge \theta$ and $h \ne \theta$), we consider a set $P_h = \{x \in E \mid x \sim h\}$. Clearly, $P_h \subset P$. Take another element $\tau \in P$ with $\theta \le \tau \le h$, we define $P_{h,\tau} = \{x \in E \mid x + \tau \in P_h\}$.

Next we list the definition of φ-(h,τ)-concave operators and fixed point theorems for such operators, which are fundamental to our proofs of our results.

Definition 2.1 (See [35]) Suppose that $N\colon P_{h,\tau} \to E$ is an operator which satisfies: for any $x \in P_{h,\tau}$ and $\lambda \in (0,1)$, there exists $\varphi(\lambda) > \lambda$ such that $N(\lambda x + (\lambda - 1)\tau) \ge \varphi(\lambda)Nx + (\varphi(\lambda) - 1)\tau$. Then we call N a φ-(h,τ)-concave operator.

Lemma 2.1 (See [35]) *Assume that P is a normal cone and N is an increasing φ-(h,τ)-concave operator satisfying $Nh \in P_{h,\tau}$. Then N has a unique fixed point x^* in $P_{h,\tau}$. In addition, for any $w_0 \in P_{h,\tau}$, constructing the sequence $w_n = Nw_{n-1}$, $n = 1,2,\dots$, then $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

Lemma 2.2 (See [36]) *Assume that P is normal and N is an increasing φ-(h,θ)-concave operator satisfying $Nh \in P_h$. Then N has a unique fixed point x^* in P_h. In addition, for any $w_0 \in P_h$, constructing the sequence $w_n = Nw_{n-1}$, $n = 1,2,\ldots$, then $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

For $h_1, h_2 \in P$ with $h_1, h_2 \neq \theta$. Let $h = (h_1, h_2)$, then $h \in \overline{P} := P \times P$. Take $\theta \leq \tau_1 \leq h_1$, $\theta \leq \tau_2 \leq h_2$, and denote $\overline{\theta} = (\theta,\theta)$, $\tau = (\tau_1, \tau_2)$. Then $\overline{\theta} = (\theta,\theta) \leq (\tau_1,\tau_2) \leq (h_1,h_2) = h$. That is, $\overline{\theta} \leq \tau \leq h$. If P is normal, then $\overline{P} = P \times P$ is normal (see [37]).

Lemma 2.3 ([27]) $\overline{P}_h = P_{h_1} \times P_{h_2}$.

Lemma 2.4 ([28]) $\overline{P}_{h,\tau} = P_{h_1,\tau_1} \times P_{h_2,\tau_2}$.

3 Existence and uniqueness of periodic solutions

In this section, we will prove the existence and uniqueness of periodic solutions for system (1.1). Let $E = \{x \in C(\mathbf{R},\mathbf{R}) : x(t) = (t+\omega)$ for every $t \in \mathbf{R}\}$, then E is a Banach space under the norm

$$\|x\|_\infty = \max_{t\in[0,\omega]} \bigl|x(t)\bigr|.$$

We will discuss (1.1) in $E \times E$. For $(x,y) \in E \times E$, let $\|(x,y)\| = \|x\|_\infty + \|y\|_\infty$. Then $(E \times E, \|(x,y)\|)$ is also a Banach space. Moreover, let

$$\overline{P} = \bigl\{(x,y) \in E \times E : x(t) \geq 0, y(t) \geq 0, t \in \mathbf{R}\bigr\}, \qquad P = \bigl\{x \in E : x(t) \geq 0, t \in \mathbf{R}\bigr\},$$

then $\overline{P} \subset E \times E$ and $\overline{P} = P \times P$ is normal and $E \times E$ has a partial order: $(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow x_1(t) \leq x_2(t)$, $y_1(t) \leq y_2(t)$, $t \in \mathbf{R}$.

For $(x,y) \in E \times E$, we define an operator $N = (N_1, N_2)$ with

$$N_1(x,y)(t) = \int_t^{t+\omega} H_1(t,s) f_1\bigl(s,x(s),y(s)\bigr)\,ds - \int_t^{t+\omega} H_1(t,s) g_1(s)\,ds,$$
$$N_2(x,y)(t) = \int_t^{t+\omega} H_2(t,s) f_2\bigl(s,x(s),y(s)\bigr)\,ds - \int_t^{t+\omega} H_2(t,s) g_2(s)\,ds.$$

Then $N_1, N_2 : E \times E \to E$ and $N : E \times E \to E \times E$. From (2.4), (x,y) is an ω-periodic solution of system (1.1) if and only if (x,y) is a fixed point of operator N.

To obtain our results, we first define several functions:

$$\tau_1(t) = \int_t^{t+\omega} H_1(t,s) g_1(s)\,ds, \qquad \tau_2(t) = \int_t^{t+\omega} H_2(t,s) g_2(s)\,ds, \tag{3.1}$$
$$h_1(t) = \int_t^{t+\omega} H_1(t,s)\,ds, \qquad h_2(t) = \int_t^{t+\omega} H_2(t,s)\,ds. \tag{3.2}$$

Remark 3.1 From (2.3), we can prove that $\tau_1(t)$, $\tau_2(t)$, $h_1(t)$ and $h_2(t)$ are ω-periodic functions. Moreover, it is easy to show $\tau_1, \tau_2, h_1, h_2 \in P$.

Theorem 3.1 *Let τ_1, τ_2, h_1, h_2 be given as in* (3.1) *and* (3.2). *Moreover, for $i = 1,2$,*

(H_1) *$f_i(t,x,y)$: $\mathbf{R}\times[-\tau_1^*,+\infty)\times[-\tau_2^*,+\infty)\to\mathbf{R}$ is ω-periodic with respect to first variable, and increasing with respect to the second, third variables, where $\tau_i^*=\max\{\tau_i(t):t\in[0,\omega]\}$;*

(H_2) *for $\lambda\in(0,1)$, there exists $\varphi(\lambda)>\lambda$ such that*

$$f_i\big(t,\lambda x_1+(\lambda-1)x_2,\lambda y_1+(\lambda-1)y_2\big)\ge\varphi(\lambda)f_i(t,x_1,y_1),$$
$$t,x_1,y_1\in\mathbf{R},x_2\in\big[0,\tau_1^*\big],y_2\in\big[0,\tau_2^*\big];$$

(H_3) *$f_i(t,0,0)\ge 0$ with $f_i(t,0,0)\not\equiv 0$ for $t\in[0,\omega]$.*

Then:

(1) *system (1.1) has a unique periodic solution (x^*,y^*) in $\overline{P}_{h,\tau}$, where*

$$\tau(t)=\big(\tau_1(t),\tau_2(t)\big),\qquad h(t)=\big(h_1(t),h_2(t)\big),\quad t\in[0,\omega];$$

(2) *for any point $(x_0,y_0)\in\overline{P}_{h,\tau}$, we construct the following sequences*:

$$x_{n+1}(t)=\int_t^{t+\omega}H_1(t,s)f_1\big(s,x_n(s),y_n(s)\big)\,ds-\int_t^{t+\omega}H_1(t,s)g_1(s)\,ds,$$
$$y_{n+1}(t)=\int_t^{t+\omega}H_2(t,s)f_2\big(s,x_n(s),y_n(s)\big)\,ds-\int_t^{t+\omega}H_2(t,s)g_2(s)\,ds,$$

$n=0,1,2,\dots$, and then we obtain $x_{n+1}(t)\to x^*(t)$, $y_{n+1}(t)\to y^*(t)$ as $n\to\infty$.

Proof From Remark 3.1, $\tau=(\tau_1,\tau_2)\in\overline{P}$, $h=(h_1,h_2)\in\overline{P}$. Due to $g_i(t)\le 1$, $i=1,2$. For $t\in\mathbf{R}$,

$$\tau_1(t)=\int_t^{t+\omega}g_1(s)H_1(t,s)\,ds\le\int_t^{t+\omega}H_1(t,s)\,ds=h_1(t),$$
$$\tau_2(t)=\int_t^{t+\omega}g_2(s)H_2(t,s)\,ds\le\int_t^{t+\omega}H_2(t,s)\,ds=h_2(t).$$

So we get $\tau_1\le h_1$, $\tau_2\le h_2$ and thus $\tau=(\tau_1,\tau_2)\le(h_1,h_2)=h$.

Now we show that operator $N:\overline{P}_{h,\tau}\to E\times E$ is a φ-(h,τ)-concave operator. For $(x,y)\in\overline{P}_{h,\tau}$ and $\lambda\in(0,1)$, we get

$$\begin{aligned}N\big(\lambda(x,y)+(\lambda-1)\tau\big)(t)&=N\big(\lambda(x,y)+(\lambda-1)\tau\big)(t)\\&=\big(N_1\big(\lambda(x,y)+(\lambda-1)\tau\big),N_2\big(\lambda(x,y)+(\lambda-1)\tau\big)\big)(t).\end{aligned}$$

Hence we need to discuss $N_1(\lambda(x,y)+(\lambda-1)\tau)(t)$ and $N_2(\lambda(x,y)+(\lambda-1)\tau)(t)$, respectively. By considering (H_2),

$$\begin{aligned}&N_1\big(\lambda(x,y)+(\lambda-1)\tau\big)(t)\\&\quad=N_1\big(\lambda x+(\lambda-1)\tau_1,\lambda y+(\lambda-1)\tau_2\big)(t)\\&\quad=\int_t^{t+\omega}H_1(t,s)f_1\big(s,\lambda x(s)+(\lambda-1)\tau_1(s),\lambda y(s)+(\lambda-1)\tau_2(s)\big)\,ds-\tau_1(t)\\&\quad\ge\varphi(\lambda)\int_t^{t+\omega}H_1(t,s)f_1\big(s,x(s),y(s)\big)\,ds-\tau_1(t)\end{aligned}$$

$$
\begin{aligned}
&= \varphi(\lambda)\left[\int_t^{t+\omega} H_1(t,s)f_1\bigl(s,x(s),y(s)\bigr)\,ds - \tau_1(t)\right] + \varphi(\lambda)\tau_1(t) - \tau_1(t) \\
&= \varphi(\lambda)N_1(x,y)(t) + \bigl[\varphi(\lambda)-1\bigr]\tau_1(t).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&N_2\bigl(\lambda(x,y)+(\lambda-1)\tau\bigr)(t) \\
&\quad= N_2\bigl(\lambda x+(\lambda-1)\tau_1, \lambda y+(\lambda-1)\tau_2\bigr)(t) \\
&\quad= \int_t^{t+\omega} H_2(t,s)f_2\bigl(s,\lambda x(s)+(\lambda-1)\tau_1(s), \lambda y(s)+(\lambda-1)\tau_2(s)\bigr)\,ds - \tau_2(t) \\
&\quad\geq \varphi(\lambda)\int_t^{t+\omega} H_2(t,s)f_2\bigl(s,x(s),y(s)\bigr)\,ds - \tau_2(t) \\
&\quad= \varphi(\lambda)\left[\int_t^{t+\omega} H_2(t,s)f_2\bigl(s,x(s),y(s)\bigr)\,ds - \tau_2(t)\right] + \varphi(\lambda)\tau_2(t) - \tau_2(t) \\
&\quad= \varphi(\lambda)N_2(x,y)(t) + \bigl[\varphi(\lambda)-1\bigr]\tau_2(t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&N\bigl(\lambda(x,y)+(\lambda-1)\tau\bigr)(t) \\
&\quad\geq \bigl(\varphi(\lambda)N_1(x,y)(t)+\bigl[\varphi(\lambda)-1\bigr]\tau_1(t), \varphi(\lambda)N_2(x,y)(t)+\bigl[\varphi(\lambda)-1\bigr]\tau_2(t)\bigr) \\
&\quad= \bigl(\varphi(\lambda)N_1(x,y)(t), \varphi(\lambda)N_2(x,y)(t)\bigr) + \bigl(\bigl(\varphi(\lambda)-1\bigr)\tau_1(t), \bigl(\varphi(\lambda)-1\bigr)\tau_2(t)\bigr) \\
&\quad= \varphi(\lambda)\bigl(N_1(x,y)(t), N_2(x,y)(t)\bigr) + \bigl(\varphi(\lambda)-1\bigr)\bigl(\tau_1(t),\tau_2(t)\bigr) \\
&\quad= \varphi(\lambda)N(x,y)(t) + \bigl(\varphi(\lambda)-1\bigr)\tau(t).
\end{aligned}
$$

That is,

$$
N\bigl(\lambda(x,y)+(\lambda-1)\tau\bigr) \geq \varphi(\lambda)N(x,y) + \bigl[\varphi(\lambda)-1\bigr]\tau, \quad (x,y)\in \overline{P}_{h,\tau}, \lambda\in(0,1).
$$

Therefore, we find that N is a φ-(h,τ)-concave operator.

In the following, we prove that $N : \overline{P}_{h,\tau} \to E\times E$ is increasing. For $(x,y)\in\overline{P}_{h,\tau}$, one has $(x,y)+\tau\in\overline{P}_h$. From Lemma 2.3, $(x+\tau_1, y+\tau_2)\in P_{h_1}\times P_{h_2}$, which means that there exist $\lambda_1,\lambda_2>0$ such that

$$
x(t)+\tau_1(t) \geq \lambda_1 h_1(t), \qquad y(t)+\tau_2(t)\geq \lambda_2 h_2(t), \quad t\in\mathbf{R}.
$$

Consequently, $x(t)\geq \lambda_1 h_1(t)-\tau_1(t)\geq -\tau_1(t)\geq -\tau_1^*$, $y(t)\geq\lambda_2 h_2(t)-\tau_2(t)\geq -\tau_2(t)\geq -\tau_2^*$. By considering (H_1) and the definitions of N_1, N_2, we know that $N : \overline{P}_{h,\tau}\to E\times E$ is increasing.

Now we show that the important condition $Nh\in\overline{P}_{h,\tau}$ is also satisfied. That is, we need to prove $Nh+\tau\in\overline{P}_h$. For any $t\in\mathbf{R}$,

$$
Nh(t)+\tau(t) = N(h_1,h_2)(t)+\tau(t)
$$

$$\begin{aligned}
&= \big(N_1(h_1,h_2)(t), N_2(h_1,h_2)(t)\big) + \big(\tau_1(t), \tau_2(t)\big) \\
&= \big(N_1(h_1,h_2)(t) + \tau_1(t), N_2(h_1,h_2)(t) + \tau_2(t)\big).
\end{aligned}$$

Clearly, we need to discuss $N_1(h_1,h_2)(t) + \tau_1(t)$, $N_2(h_1,h_2)(t) + \tau_2(t)$, respectively. For convenience, we set

$$r_i = \min_{t\in[0,\omega]}\big\{f_i(t,m_1,m_2)\big\}, \qquad R_i = \min_{t\in[0,\omega]}\big\{f_i(t,M_1,M_2)\big\}, \quad i = 1,2.$$

By (H_1) and (H_3), $R_1 \geq r_1 > 0$, $R_2 \geq r_2 > 0$. Note that $m_i \leq h_i(t) \leq M_i$, $i = 1,2$, and from (H_1),

$$\begin{aligned}
N_1(h_1,h_2)(t) + \tau_1(t) &= \int_t^{t+\omega} H_1(t,s) f_1\big(s,h_1(s),h_2(s)\big)\,ds \\
&\geq \int_t^{t+\omega} H_1(t,s) f_1(s,m_1,m_2)\,ds \\
&= r_1 \int_t^{t+\omega} H_1(t,s)\,ds = r_1 h_1(t)
\end{aligned}$$

and

$$\begin{aligned}
N_1(h_1,h_2)(t) + \tau_1(t) &= \int_t^{t+\omega} H_1(t,s) f_1\big(s,h_1(s),h_2(s)\big)\,ds \\
&\leq \int_t^{t+\omega} H_1(t,s) f_1(s,M_1,M_2)\,ds \\
&= R_1 \int_t^{t+\omega} H_1(t,s)\,ds = R_1 h_1(t).
\end{aligned}$$

So we get $r_1 h_1 \leq N_1(h_1,h_2) + \tau_1 \leq R_1 h_1$, *i.e.*, $N_1(h_1,h_2) + \tau_1 \in P_{h_1}$. Similarly, we can obtain $N_2(h_1,h_2) + \tau_2 \in P_{h_2}$. Hence, by Lemma 2.4,

$$N(h_1,h_2) + \tau = \big(N_1(h_1,h_2) + \tau_1, N_2(h_1,h_2) + \tau_2\big) \in P_{h_1} \times P_{h_2} = \overline{P_h}.$$

Finally, an application of Lemma 2.1 implies that N has a unique fixed point $(x^*,y^*) \in \overline{P}_{h,\tau}$. And, for any given $(x_0,y_0) \in \overline{P}_{h,\tau}$, the sequence

$$(x_n,y_n) = N(x_{n-1},y_{n-1}) = \big(N_1(x_{n-1},y_{n-1}), N_2(x_{n-1},y_{n-1})\big), \quad n = 1,2,\ldots,$$

converges to (x^*,y^*) as $n \to \infty$. Therefore, system (1.1) has a unique periodic solution (x^*,y^*) in $\overline{P}_{h,\tau}$; and choosing any initial point $(x_0,y_0) \in \overline{P}_{h,\tau}$, we have the following sequences:

$$\begin{aligned}
x_n(t) &= \int_t^{t+\omega} H_1(t,s) f_1\big(s,x_{n-1}(s),y_{n-1}(s)\big)\,ds - \int_t^{t+\omega} H_1(t,s) g_1(s)\,ds, \\
y_n(t) &= \int_t^{t+\omega} H_2(t,s) f_2\big(s,x_{n-1}(s),y_{n-1}(s)\big)\,ds - \int_t^{t+\omega} H_2(t,s) g_2(s)\,ds,
\end{aligned}$$

$n = 1,2,\ldots$, satisfying $x_{n+1} \to x^*$, $y_{n+1} \to y^*$ as $n \to \infty$. □

Let

$$H_1(t,s) = \frac{e^{\int_t^s a(\xi)\,d\xi}}{e^{\int_0^\omega a(\xi)\,d\xi} - 1}, \qquad H_2(t,s) = \frac{e^{-\int_t^s b(\xi)\,d\xi}}{1 - e^{-\int_0^\omega b(\xi)\,d\xi}}, \quad (t,s) \in (\mathbf{R},\mathbf{R}). \tag{3.3}$$

Similar to the proof of Theorem 3.1, we can obtain the following conclusion.

Theorem 3.2 *Let τ_1, τ_2, h_1, h_2 be given as in (3.1) and (3.2) with H_1, H_2 are replaced by (3.3). Assume that the conditions (H_1)–(H_3) hold. Then:*

(1) *system (1.4) has a unique periodic solution (x^*,y^*) in $\overline{P}_{h,\tau}$, where*

$$\tau(t) = \bigl(\tau_1(t),\tau_2(t)\bigr), \qquad h(t) = \bigl(h_1(t),h_2(t)\bigr), \quad t \in [0,\omega];$$

(2) *for any point $(x_0,y_0) \in \overline{P}_{h,\tau}$, construct the following sequences:*

$$x_{n+1}(t) = \int_t^{t+\omega} H_1(t,s) f_1\bigl(s,x_n(s),y_n(s)\bigr)\,ds - \int_t^{t+\omega} H_1(t,s) g_1(s)\,ds,$$
$$y_{n+1}(t) = \int_t^{t+\omega} H_2(t,s) f_2\bigl(s,x_n(s),y_n(s)\bigr)\,ds - \int_t^{t+\omega} H_2(t,s) g_2(s)\,ds,$$

$n = 0,1,2,\ldots$, and then one has $x_{n+1}(t) \to x^*(t)$, $y_{n+1}(t) \to y^*(t)$ as $n \to \infty$.

When $g_1(t) = g_2(t) \equiv 0$, we can get the uniqueness of positive periodic solutions for systems (1.1) and (1.4) by using Lemma 2.2. The proofs are similar to Theorem 3.1.

Corollary 3.1 *Let h_1, h_2 be given as in (3.2). Moreover, for $i = 1,2$,*

(H_4) *$f_i(t,x,y)$: $\mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is ω-periodic with respect to first variable, and increasing with respect to the second, third variables;*

(H_5) *for $\lambda \in (0,1)$, there exists $\varphi(\lambda) > \lambda$ such that*

$$f_i(t,\lambda x,\lambda y) \geq \varphi(\lambda) f_i(t,x,y), \quad t,x,y \in \mathbf{R};$$

(H_6) *$f_i(t,0,0) \not\equiv 0$ for $t \in [0,\omega]$.*

Then system (1.1) has a unique positive periodic solution (x^*,y^*) in $\overline{P}_h$, where $h(t) = (h_1(t),h_2(t))$, $t \in \mathbf{R}$. Further, for any point $(x_0,y_0) \in \overline{P}_h$, make the following sequences:

$$x_{n+1}(t) = \int_t^{t+\omega} H_1(t,s) f_1\bigl(s,x_n(s),y_n(s)\bigr)\,ds,$$
$$y_{n+1}(t) = \int_t^{t+\omega} H_2(t,s) f_2\bigl(s,x_n(s),y_n(s)\bigr)\,ds,$$

$n = 0,1,2,\ldots$, and then we get $x_{n+1}(t) \to x^*(t)$, $y_{n+1}(t) \to y^*(t)$ as $n \to \infty$.

Corollary 3.2 *Let h_1, h_2 be given as in (3.2) with H_1, H_2 are replaced by (3.3). Assume that the conditions (H_4)–(H_6) hold. Then system (1.4) has a unique positive periodic solution (x^*,y^*) in $\overline{P}_h$, where $h(t) = (h_1(t),h_2(t))$, $t \in \mathbf{R}$. Further, for any point $(x_0,y_0) \in \overline{P}_h$, put the following sequences:*

$$x_{n+1}(t) = \int_t^{t+\omega} H_1(t,s) f_1\bigl(s,x_n(s),y_n(s)\bigr)\,ds,$$

$$y_{n+1}(t) = \int_t^{t+\omega} H_2(t,s) f_2\bigl(s, x_n(s), y_n(s)\bigr)\, ds,$$

$n = 0, 1, 2, \ldots$, *and then we obtain* $x_{n+1}(t) \to x^*(t)$, $y_{n+1}(t) \to y^*(t)$ *as* $n \to \infty$.

Remark 3.2 The form of differential system (1.1) is more general. Our method is new to the study of nonlinear systems of differential equations, which gives the existence and uniqueness of periodic solutions. Moreover, the unique periodic solution can be approximated by an iteration.

Remark 3.3 By using the same discussion as with Theorem 3.1 and Corollary 3.1, we can consider the following differential equation:

$$x'(t) = a(t)x(t) - f\bigl(t, x(t)\bigr) + g(t),$$

where $a \in C(\mathbf{R}, \mathbf{R}_+)$ is ω-periodic for some $\omega > 0$, $f(t,x) \in C(\mathbf{R} \times \mathbf{R}_+, \mathbf{R}_+)$ and $g(t) \in C(\mathbf{R}, \mathbf{R}_+)$ are ω-periodic functions in t with $g(t) \le 1$. We can also give the existence and uniqueness of periodic solutions or positive periodic solutions. Moreover, the unique periodic solution can be also approximated by making an iterative sequence.

4 Examples

In this section, we present two simple examples to illustrate the main results.

Example 4.1 Consider the simple system of differential equations:

$$\begin{cases} x'(t) = ax(t) - [y(t) + \frac{1}{4b}]^{\frac{1}{3}} \sin^2 t + \frac{1}{2}, \\ y'(t) = -by(t) + [x(t) + \frac{1}{2a}]^{\frac{1}{3}} \cos^2 t - \frac{1}{4}, \end{cases} \tag{4.1}$$

where $a, b > 0$. In this example, we let

$$f_1(t,y) = \left(y + \frac{1}{4b}\right)^{\frac{1}{3}} \sin^2 t, \qquad f_2(t,x) = \left(x + \frac{1}{2a}\right)^{\frac{1}{3}} \cos^2 t, \qquad g_1(t) = \frac{1}{2}, \qquad g_2(t) = \frac{1}{4}$$

and they are π-periodic functions in t. By direct calculation,

$$H_1(t,s) = \frac{e^{-a(s-t)}}{1 - e^{-a\omega}}, \qquad H_2(t,s) = \frac{e^{b(s-t)}}{e^{b\omega} - 1},$$
$$\tau_1(t) = \int_t^{t+\pi} H_1(t,s) g_1(s)\, ds = \frac{1}{2a}, \qquad \tau_2(t) = \int_t^{t+\pi} H_2(t,s) g_2(s)\, ds = \frac{1}{4b},$$
$$h_1(t) = \int_t^{t+\pi} H_1(t,s)\, ds = \frac{1}{a}, \qquad h_2(t) = \int_t^{t+\pi} H_2(t,s)\, ds = \frac{1}{b},$$

and thus $\tau_1 \le h_1$, $\tau_2 \le h_2$, $\tau_1^* = \frac{1}{2a}$, $\tau_2^* = \frac{1}{4b}$. It is easy to see that $f_1(t,y) : \mathbf{R} \times [-\frac{1}{4b}, +\infty) \to \mathbf{R}$ and $f_2(t,x) : \mathbf{R} \times [-\frac{1}{2a}, +\infty) \to \mathbf{R}$ are π-periodic with respect to first variable and increasing with respect to the second variable. In addition,

$$f_1(t,0) = \left(\frac{1}{4b}\right)^{\frac{1}{3}} \sin^2 t \not\equiv 0, \qquad f_2(t,0) = \left(\frac{1}{2a}\right)^{\frac{1}{3}} \cos^2 t \not\equiv 0, \quad t \in \mathbf{R}.$$

Hence, the conditions (H_1), (H_3) are satisfied.

In the following, we show that (H_2) holds. Let $\varphi(\lambda) = \lambda^{\frac{1}{3}}$, then $\varphi(\lambda) > \lambda$ for $\lambda \in (0,1)$, and for $x_1, y_1 \in \mathbf{R}$, $x_2 \in [0, \frac{1}{2a}]$, $y_2 \in [0, \frac{1}{4b}]$,

$$\begin{aligned}
f_1\bigl(t, \lambda y_1 + (\lambda-1)y_2\bigr) &= \left[\lambda y_1 + (\lambda-1)y_2 + \frac{1}{4b}\right]^{\frac{1}{3}} \sin^2 t \\
&= \lambda^{\frac{1}{3}}\left[y_1 + \left(1 - \frac{1}{\lambda}\right)y_2 + \frac{1}{\lambda}\frac{1}{4b}\right]^{\frac{1}{3}} \sin^2 t \\
&\geq \lambda^{\frac{1}{3}}\left[y_1 + \left(1 - \frac{1}{\lambda}\right)\frac{1}{4b} + \frac{1}{\lambda}\frac{1}{4b}\right]^{\frac{1}{3}} \sin^2 t \\
&= \lambda^{\frac{1}{3}}\left[y_1 + \frac{1}{4b}\right]^{\frac{1}{3}} \sin^2 t \\
&= \varphi(\lambda) f_1(t, y_1), \\
f_2\bigl(t, \lambda x_1 + (\lambda-1)x_2\bigr) &= \left[\lambda x_1 + (\lambda-1)x_2 + \frac{1}{2a}\right]^{\frac{1}{3}} \cos^2 t \\
&= \lambda^{\frac{1}{3}}\left[x_1 + \left(1 - \frac{1}{\lambda}\right)x_2 + \frac{1}{\lambda}\frac{1}{2a}\right]^{\frac{1}{3}} \cos^2 t \\
&\geq \lambda^{\frac{1}{3}}\left[x_1 + \left(1 - \frac{1}{\lambda}\right)\frac{1}{2a} + \frac{1}{\lambda}\frac{1}{2a}\right]^{\frac{1}{3}} \cos^2 t \\
&= \lambda^{\frac{1}{3}}\left[x_1 + \frac{1}{2a}\right]^{\frac{1}{3}} \cos^2 t \\
&= \varphi(\lambda) f_2(t, x_1).
\end{aligned}$$

So, the condition (H_2) is satisfied. By Theorem 3.1, system (4.1) has a unique periodic solution (x^*, y^*) in $\overline{P}_{h,\tau}$, where

$$\tau(t) = \bigl(\tau_1(t), \tau_2(t)\bigr) = \left(\frac{1}{2a}, \frac{1}{4b}\right), \qquad h(t) = \bigl(h_1(t), h_2(t)\bigr) = \left(\frac{1}{a}, \frac{1}{b}\right).$$

Take any initial point $(x_0, y_0) \in \overline{P}_{h,\tau}$, making the sequences:

$$\begin{aligned}
x_{n+1}(t) &= \int_t^{t+\pi} \frac{e^{-a(s-t)}}{1 - e^{-a\omega}} \left\{\left[y_n(s) + \frac{1}{4b}\right]^{\frac{1}{3}} \sin^2 t - \frac{1}{2}\right\} ds, \\
y_{n+1}(t) &= \int_t^{t+\pi} \frac{e^{b(s-t)}}{e^{b\omega} - 1} \left\{\left[x_n(s) + \frac{1}{2a}\right]^{\frac{1}{3}} \cos^2 t - \frac{1}{4}\right\} ds,
\end{aligned}$$

$n = 0, 1, 2, \ldots$, one has $x_{n+1} \to x^*(t)$, $y_{n+1} \to y^*(t)$ as $n \to \infty$.

Example 4.2 Consider the following system of differential equations:

$$\begin{cases} x'(t) = a(t)x(t) - [x(t) + y^2(t) + 1]^{\frac{1}{4}} \sin^2 t, \\ y'(t) = -b(t)y(t) - [x^2(t) + y^3(t) + 2]^{\frac{1}{6}} \cos^2 t, \end{cases} \tag{4.2}$$

where $a(t), b(t) \geq 0$ and $a(t)$, $b(t)$ are π-periodic in t. In this example, we let

$$f_1(t,x,y) = \left[x(t) + y^2(t) + 1\right]^{\frac{1}{4}} \sin^2 t, \qquad f_2(t,x,y) = \left[x^2(t) + y^3(t) + 2\right]^{\frac{1}{6}} \cos^2 t,$$

and they are π-periodic functions in t. Moreover,

$$f_1(t,0,0) = \sin^2 t \not\equiv 0, \qquad f_2(t,0,0) = 2\cos^2 t \not\equiv 0, \quad t \in \mathbf{R}.$$

So the $f_i(t,x,y)$: $\mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ satisfy (H_4) and (H_6) in Corollary 3.1. Next, we prove that the condition (H_5) also holds. Let $\varphi(\lambda) = \lambda^{\frac{1}{2}}$, then $\varphi(\lambda) > \lambda$, $\lambda \in (0,1)$. And for $\lambda \in (0,1)$ and $x, y \geq 0$,

$$\begin{aligned} f_1(t,\lambda x,\lambda y) &= \left(\lambda x + \lambda^2 y^2 + 1\right)^{\frac{1}{4}} \sin^2 t \geq \lambda^{\frac{1}{2}} \left(x + y^2 + 1\right) \sin^2 t = \varphi(\lambda) f_1(t,x,y), \\ f_2(t,\lambda x,\lambda y) &= \left(\lambda^2 x^2 + \lambda^3 y^3 + 2\right)^{\frac{1}{6}} \cos^2 t \geq \lambda^{\frac{1}{2}} \left(x^2 + y^3 + 2\right) \cos^2 t = \varphi(\lambda) f_2(t,x,y). \end{aligned}$$

Hence the condition (H_5) is satisfied. By Corollary 3.1, system (4.2) has a unique positive periodic solution (x^*, y^*) in $\overline{P}_h$, where $h(t) = (h_1(t), h_2(t))$, $h_1(t) = \int_t^{t+\pi} H_1(t,s)\,ds$, $h_2(t) = \int_t^{t+\pi} H_2(t,s)\,ds$ with

$$H_1(t,s) = \frac{e^{-\int_s^t a(s)\,ds}}{1 - e^{-\int_0^\pi a(s)\,ds}}, \qquad H_2(t,s) = \frac{e^{\int_s^t a(s)\,ds}}{e^{-\int_0^\pi b(s)\,ds} - 1}.$$

Further, for any point $(x_0, y_0) \in \overline{P}_h$, put the following sequences:

$$\begin{aligned} x_{n+1}(t) &= \int_t^{t+\pi} H_1(t,s)\left[x_n(s) + y_n^2(s) + 1\right]^{\frac{1}{4}} \sin^2 s\,ds, \\ y_{n+1}(t) &= \int_t^{t+\pi} H_2(t,s)\left[x_n^2(s) + y_n^3(s) + 2\right]^{\frac{1}{6}} \cos^2 s\,ds, \end{aligned}$$

$n = 0, 1, 2, \ldots$, and then we get $x_{n+1}(t) \to x^*(t)$, $y_{n+1}(t) \to y^*(t)$ as $n \to \infty$.

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