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Uniform persistence and almost periodic solutions of a nonautonomous patch occupancy model

Hui Zhou^{1,5}, Jihad Alzabut², Shahram Rezapour^{3,6} and Mohammad Esmael Samei^{4*}

*Correspondence:
mesamei@gmail.com;
mesamei@basu.ac.ir

⁴Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, a nonlinear nonautonomous model in a rocky intertidal community is studied. The model is composed of two species in a rocky intertidal community and describes a patch occupancy with global dispersal of propagules and occupy each other by individual organisms. Firstly, we study the uniform persistence of the model via differential inequality techniques. Furthermore, a sharp threshold of global asymptotic stability and the existence of a unique almost periodic solution are derived. To prove the main results, we construct an appropriate Lyapunov function whose conditions are easily verified. The assumptions of the model are reasonable, and the results complement previously known ones. An example with specific values of parameters is included for demonstration of theoretical outcomes.

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1 Introduction

Nonlocal problems concerning the conditions of the behavior of different classes of solutions play an important role in the qualitative theory of ordinary differential equations [1–16]. For more precision, we refer the readers to some specific problems such as boundedness, periodicity, almost periodicity, stability in the sense of Poisson and Ulam and to the problem of the existence of limit regimes of different types, convergence, dissipativity, and so on [17–30]. On the one hand, as pointed out in [31] in the real world, the delays in differential equations of populations, ecology, and dynamic problems are usually infinite time delays; for example, Zhou *et al.* [32] studied positive almost periodic solutions for a class of Lasota–Ważewska model with infinite time delays. On the other hand, assuming that a harvesting function is a function of the delayed estimate of the true ecological and dynamic models, Zhou *et al.* [33] presented an overview of the results on the classical Nicholson's blowflies model with a linear harvesting term.

During the last few decades the question of the existence of oscillatory solutions for differential equations has been considered by many researchers. These systems arise while

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modeling several physical and natural phenomena in engineering, biology, chemistry, and ecology. Here we focus on the last application, which is ecology. The study of almost periodic models is relevant because they consider general seasonal oscillations of the weather that extend periodical variations of weather or habitat. The concept of almost periodicity was introduced by Danish mathematician Bohr around 1924–1926 [34–36] and later generalized by many other authors [24, 33, 37–40].

The almost periodic functions may capture the phenomena which may not be possible with periodic rates. For instance, the function $x \mapsto \cos x + \cos 5x$ is periodic, and this remains true when 5 is replaced with any other rational number. However, the sum of the periodic functions $x \mapsto \exp(ix)$ and $x \mapsto \exp(i\sqrt{2}x)$ is not periodic. Hence, when such functions, obtained by using a combination of periodic functions, are not periodic, they are not without properties: they are almost periodic functions. More generally, we know that when all the solutions of an autonomous linear finite dimensional system are bounded, then all these solutions are almost periodic.

The notion of almost periodicity for certain functions was introduced by Fréchet [25]. In view of the literature, one can claim that the analysis of boundedness and stability is one of the central foci in the qualitative analysis. Classical ecological research of rocky intertidal communities has already been reported by some ecologists [27, 28]. Recently the authors studied a model of species interactions by a simple patch occupancy with global dispersal of propagules [18, 20]. In [20], Benicá *et al.* showed that natural ecosystems can sustain continued fluctuations, the model exposes the rocky habitat as small patches, there are three species which habit rocky intertidal community occupy by their individual organism.

This paper is concerned with an occupancy model consisting of two species in a rocky intertidal community, which describes a patch occupancy with global dispersal of propagules, and occupy each other by individual organisms. The model can be formulated by the following system:

$$\begin{cases} \dot{x}(t) = x(t)[-m_1(t) + a(t)z(t) - b(t)y(t)], \\ \dot{y}(t) = y(t)[-m_2(t) + c(t)z(t) + b(t)x(t)], \end{cases} \tag{1}$$

and

$$z(t) = 1 - x(t) - y(t), \tag{2}$$

where x and y are the coverage (i.e., the fraction of patches occupied) by barnacles and crustose algae, respectively, and z is the coverage by bare rock. Barnacles only can cover bare rock, and crustose algae can inhabit bare rock and barnacle. The variables a, b, c, m_1 , and m_2 are all positive, where a is the colonization rate of barnacles on bare rock, b and c are colonization rates of crustose algae on barnacles and bare rock, and m_1 and m_2 are the mortality rates of barnacles and crustose algae, respectively.

Taking into account the practical significance of the species, it is always assumed that the initial conditions associated with system (1) are

$$x(0) > 0, \quad y(0) > 0. \tag{3}$$

In view of system (1) and initial conditions (3), it is easy to conclude that any solution of system (1) is positive.

Our paper is organized in four sections. In Sect. 2, we study Lyapunov function and general almost periodic functions. In Sect. 3, we provide the results on the uniform persistence and the global asymptotic stability of system (4). Furthermore, we give an example for the system. Finally, we compare main results in this work and the results in the literature in Sect. 4.

2 Essential preliminaries

For further consideration, we rewrite system (1) in the following form:

$$\begin{cases} \dot{x}(t) = x(t)[-m_1(t) + a(t) - a(t)x(t) - (a(t) + b(t))y(t)], \\ \dot{y}(t) = y(t)[-m_2(t) + c(t) - (c(t) - b(t))x(t) - c(t)y(t)]. \end{cases} \tag{4}$$

Let \mathcal{C}_{bd} be the collection of continuous and bounded functions from $\mathbb{R} \times \mathbb{R}^n$ to \mathbb{R}^n . Let $\mathbb{R}_+ = (0, +\infty)$, and define

$$f^+ = \sup_{t \in \mathbb{R}} f(t), \quad f^- = \inf_{t \in \mathbb{R}} f(t),$$

where f is a continuous and bounded function.

For the relation between colonization and mortality rates, the following assumptions are assumed to hold true throughout the remaining part of the paper:

- (A₁) $m_1^- < a^+$,
- (A₂) $m_2^- < c^+$.

Definition 1 ([24]) A Lyapunov function is a scalar function $V(x)$ defined on a region D that is continuous, positive definite, i.e., $V(x) > 0$ for all $x \neq 0$, and has continuous first-order partial derivatives at every point of D . The derivative of V with respect to the system $x' = f(x)$, written as $V^*(x)$, is defined as a dot product $V^*(x) = \nabla V(x) \cdot f(x)$.

The existence of a Lyapunov function, for which $V^*(y) \leq 0$ on some region D containing the origin, guarantees the stability of the zero solution of $y' = f(y)$, while the existence of a Lyapunov function, for which $V^*(y)$ is negative definite on some region D containing the origin, guarantees the asymptotic stability of the zero solution of $y' = f(y)$. For example, given the system

$$\begin{cases} y' = z, \\ z' = -y - 2z, \end{cases}$$

and the Lyapunov function $V(y, z) = \frac{y^2+z^2}{2}$, we obtain

$$V^*(y, z) = yz + z(-y - 2z) = -2z^2,$$

which is nonincreasing on every region containing the origin, and thus the zero solution is stable.

Definition 2 ([24]) Let $f : \mathbb{R} \rightarrow \mathbb{R}^n$ be a continuous function. Given $\varepsilon > 0$, we call $\tau > 0$ and ε -translation for f if and only if, for all $t \in \mathbb{R}$, $|f(t + \tau) - f(t)| < \varepsilon$. The set consisting of

all ε -translations for f is denoted by

$$E(f, \varepsilon) = \{ \tau \in \mathbb{R} \mid |f(t + \tau) - f(t)| < \varepsilon \}.$$

The function f is an almost periodic function if and only if, for each $\varepsilon > 0$, the set $E(f, \varepsilon)$ is relatively dense in \mathbb{R} , which means that there exists a constant $l > 0$ such that any subinterval of \mathbb{R} of length l meets $E(f, \varepsilon)$.

Definition 3 Let

$$u(t) = \begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix}$$

be one bounded positive solution of (4). Then u is said to be globally asymptotically stable if, for any positive bounded solution

$$v(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

of (4), the following is satisfied:

$$\lim_{t \rightarrow +\infty} [|x(t) - x^*(t)| + |y(t) - y^*(t)|] = 0. \tag{5}$$

We state the following essential lemma.

Lemma 4 ([23]) *The following hold:*

- (i) *If $a > 0, b > 0$, and $\dot{x}(t) \leq x(t)(b - ax(t))$, whenever $t \geq 0$ and $x(0) > 0$, then $\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b}{a}$.*
- (ii) *If $a > 0, b > 0$, and $\dot{x}(t) \geq x(t)(b - ax(t))$, whenever $t \geq 0$ and $x(0) > 0$, then $\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b}{a}$.*

Lemma 5 *For any solution $(x(t), y(t))$ of system (4) with initial conditions (3), there exists $T_1 > 0$ such that*

$$x(t) \leq M_1, \quad y(t) \leq M_2 \tag{6}$$

for all $t \geq T_1$, where $M_1 = \frac{a^+ - m_1^-}{a^-}$ and $M_2 = \frac{c^+ + b^+ M_1 - m_2^-}{c^-}$.

Proof It is clear that $x(t) > 0$ and $y(t) > 0$ for all $t \in \mathbb{R}$. Since

$$\begin{aligned} \dot{x}(t) &= x(t) [-m_1(t) + a(t) - a(t)x(t) - (a(t) + b(t))y(t)] \\ &\leq x(t) (-m_1(t) + a(t) - a(t)x(t)) \\ &\leq x(t) (-m_1^- + a^+ - a^- x(t)), \end{aligned}$$

and from (i) of Lemma 4, one has

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{a^+ - m_1^-}{a^-} = M_1. \tag{7}$$

By (7), for any $\varepsilon > 0$ small enough, there exists $T_1 > 0$ such that $x(t) < M_1 + \varepsilon$ for all $t \geq T_1$. For $t \geq T_1$, we have

$$\begin{aligned} \dot{y}(t) &= y(t)[-m_2(t) + c(t) - (c(t) - b(t))x(t) - c(t)y(t)] \\ &\leq y(t)[-m_2(t) + c(t)(1 - y(t)) + b(t)x(t)] \\ &\leq y(t)[-m_2^- + c^+ + b^+(M_1 + \varepsilon) - c^-y(t)]. \end{aligned}$$

From (i) of Lemma 4, one has

$$\limsup_{t \rightarrow +\infty} y(t) \leq \frac{c^+ + b^+(M_1 + \varepsilon) - m_2^-}{c^-}. \tag{8}$$

Setting $\varepsilon \rightarrow 0$ with the right-hand side of (8), then

$$\limsup_{t \rightarrow +\infty} y(t) \leq \frac{c^+ + b^+M_1 - m_2^-}{c^-} = M_2. \tag{9}$$

□

Remark 1 From Theorem 8.10 in [29] and Lemma 5 above, it is clear that the boundedness of solution of system (4) is independent of the initial conditions, which implies that any solution of (4) is ultimately bounded.

3 Main results

This section is devoted to the main results of this paper. We provide results on the uniform persistence and the global asymptotic stability of system (4).

In what follows, we set

$$\gamma_1^+ = a^+ + b^+, \quad \gamma_2^+ = \max_{t \in \mathbb{R}} |b(t) - c(t)|, \quad \text{and} \quad \Lambda = \frac{a^- c^-}{\gamma_1^+ \gamma_2^+}.$$

Theorem 6 *Assume that*

$$(A_3) \quad \frac{a^- - m_1^+}{\gamma_1^+} > M_2,$$

$$(A_4) \quad \frac{c^- - m_2^+}{\gamma_2^+} > M_1.$$

Then there exist $T_ > T_1$ and positive constants n_i ($i = 1, 2$) such that*

$$\liminf_{t \rightarrow +\infty} x(t) \geq n_1 > 0, \quad \liminf_{t \rightarrow +\infty} y(t) \geq n_2 > 0, \tag{10}$$

which implies that system (4) is uniformly persistent, where

$$n_1 = \frac{a^- - m_1^+ - \gamma_1^+ M_2}{a^+}$$

and

$$n_2 = \frac{c^- - m_2^+ - \gamma_2^+ M_1}{c^+}.$$

Proof From Eq. (6) and (A₃), for any $\epsilon > 0$ small enough, there exist $T_2 > T_1$ such that $y(t) < M_2 + \epsilon$ and

$$a^- > m_1^+ + \gamma_1^+(M_2 + \epsilon)$$

for all $t \geq T_2$. Also, for $t \geq T_2$, we get

$$\begin{aligned} \dot{x}(t) &= x(t)[-m_1(t) + a(t) - a(t)x(t) - (a(t) + b(t))y(t)] \\ &\geq x(t)[-m_1(t) + a(t) - a(t)x(t) - (a(t) + b(t))(M_2 + \epsilon)] \\ &\geq x(t)[a^- - m_1^+ - \gamma_1^+(M_2 + \epsilon) - a^+x(t)]. \end{aligned}$$

From (ii) of Lemma 4, it follows that

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{a^- - m_1^+ - \gamma_1^+(M_2 + \epsilon)}{a^+}. \tag{11}$$

Setting $\epsilon \rightarrow 0$ with the right-hand side of (11), then

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{a^- - m_1^+ - \gamma_1^+M_2}{a^+} = n_1. \tag{12}$$

From equations (7), (12) and (A₄), for any $\epsilon_* > 0$ small enough, there exist $T_3 > T_2$ such that $x(t) < M_1 + \epsilon_*$ and

$$m_2^+ - c^- > c^+(M_1 + \epsilon_*) + b^-(m_1 - \epsilon_*)$$

for all $t \geq T_3$. By the same arguments,

$$\begin{aligned} \dot{y}(t) &= y(t)[-m_2(t) + c(t) - (c(t) - b(t))x(t) - c(t)y(t)] \\ &\geq y(t)[-m_2(t) + c(t) - |c(t) - b(t)|(M_1 + \epsilon_*) - c(t)y(t)] \\ &\geq y(t)[-m_2^+ + c^- - \gamma_2^+(M_1 + \epsilon_*) - c^+y(t)]. \end{aligned}$$

From (ii) of Lemma 4, one has

$$\liminf_{t \rightarrow +\infty} y(t) \geq \frac{c^- - m_2^+ - \gamma_2^+(M_1 + \epsilon_*)}{c^+}. \tag{13}$$

Setting $\epsilon \rightarrow 0$ with the right-hand side of (13), then

$$\liminf_{t \rightarrow +\infty} y(t) \geq \frac{c^- - m_2^+ - \gamma_2^+M_1}{c^+} = n_2. \tag{14}$$

Let $T_* = T_3$, from equations (13) and (14), system (4) is uniformly persistent. The proof is completed. □

Lemma 7 ([19]) *Let r be a real number and f be a nonnegative function defined on $[r, +\infty)$ such that f is integrable and is uniformly continuous on $[r, +\infty)$. Then $\lim_{t \rightarrow +\infty} f(t) = 0$.*

The following theorem shows that relation (5) holds. That is, system (4) with initial condition (3) is globally asymptotically stable. First, we shall propose the following assumption:

$$\Lambda = \frac{a^- c^-}{\gamma_1^+ \gamma_2^+} > 1.$$

Theorem 8 Assume that (A_1) – (A_4) hold and $\Lambda > 1$, then system (4) with initial condition (3) is globally asymptotically stable.

Proof Suppose that

$$u(t) = \begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix}, \quad v(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

are two positive solutions of system (4). Since $\Lambda > 1$, one has $\frac{c^-}{\gamma_1^+} > \frac{\gamma_2^+}{a^-}$, then there exist positive constants α and β such that

$$\frac{c^-}{\gamma_1^+} > \frac{\alpha}{\beta} > \frac{\gamma_2^+}{a^-}. \tag{15}$$

Define a Lyapunov function by

$$V(t) = \alpha |\ln x(t) - \ln x^*(t)| + \beta |\ln y(t) - \ln y^*(t)| \tag{16}$$

for $t > T$. Calculating directly the upper right derivative of V along the solutions of model (4), we obtain that

$$\begin{aligned} D^+ V(t) &= \overline{\lim}_{h \rightarrow 0^+} \frac{V(t+h) - V(t)}{h} \\ &\leq \alpha [-a(t)|x(t) - x^*(t)| + (a(t) + b(t))|y(t) - y^*(t)|] \\ &\quad + \beta [-c(t)|y(t) - y^*(t)| + |c(t) - b(t)||x(t) - x^*(t)|] \\ &\leq \alpha [-a^-|x(t) - x^*(t)| + \gamma_1^+|y(t) - y^*(t)|] \\ &\quad + \beta [-c^-|y(t) - y^*(t)| + \gamma_2^+|x(t) - x^*(t)|] \\ &= (-\alpha a^- + \beta \gamma_2^+)|x(t) - x^*(t)| + (-\beta c^- + \alpha \gamma_1^+)|y(t) - y^*(t)|. \end{aligned}$$

From (15), it follows that there exist constant $\rho > 0$ and large enough $T > 0$ such that

$$D^+ V(t) \leq -\rho (|x(t) - x^*(t)| + |y(t) - y^*(t)|) \tag{17}$$

for $t > T$. Integrating both sides of (17) from T to t , and by the integral comparison theorem, one has

$$V(t) + \rho \int_T^t (|x(s) - x^*(s)| + |y(s) - y^*(s)|) ds \leq V(T) < +\infty$$

for each $t > T$, which yields

$$\int_T^t (|x(s) - x^*(s)| + |y(s) - y^*(s)|) ds \leq \rho^{-1} [V(T) - V(t)] < +\infty$$

for $t > T$. Then $|x(t) - x^*(t)| + |y(t) - y^*(t)|$ is integrable on $[T, +\infty)$. On the other hand, from Remark 1, $x, x^*, y,$ and y^* are ultimately bounded. It is clear that they have bounded

derivatives. It follows that $|x(t) - x^*(t)| + |y(t) - y^*(t)|$ is uniformly continuous on $[T, +\infty)$. The conditions of Lemma 7 are satisfied, thus

$$\lim_{t \rightarrow +\infty} [|x(t) - x^*| + |y(t) - y^*|] = 0.$$

The proof is complete. □

Before discussing the almost periodic system, we need to recall the following Lemma 9, which can be referred to in the monographs [24, 26]. Consider the following system:

$$\dot{x} = f(t, x), \tag{18}$$

where $f(t, x) \in \mathcal{C}_{bd}$.

Lemma 9 (Theorem 6.6 in [26]) *Assume that $f(t, x) \in \mathcal{C}_{bd}$ is almost periodic in t uniformly for $x \in \mathbb{R}^n$ and the solution of (18) is ultimately bounded. And suppose that there exists a Lyapunov function $V(t, x, y)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ which has the following conditions:*

- (1) *For continuous nondecreasing functions u and v ,*

$$u(|x - y|) \leq V(t, x, y) \leq v(|x - y|).$$

- (2) *There exists a positive constant L such that*

$$|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq L[|x_1 - x_2| + |y_1 - y_2|].$$

- (3) *If λ is a continuous and positive definite function, then $\dot{V} \leq -\lambda(|x - y|)$. Then system (4) has a unique almost periodic solution, which is globally uniformly asymptotically stable.*

Assume (A_1) – (A_4) hold and $\Lambda > 1$, from Lemma 5 and Theorem 6, for any bounded solution $(x(t), y(t))$ of system (4) with initial conditions (3). Then there exist positive constants n_1^*, n_2^*, M_1^* , and M_2^* such that $0 < n_1^* \leq x(t) \leq M_1^*$ and $0 < n_2^* \leq y(t) \leq M_2^*$. For almost periodic system (4), we consider the associated product system of (4) as follows:

$$\begin{cases} \dot{x}(t) = x(t)[-m_1(t) + a(t) - a(t)x(t) - (a(t) + b(t))y(t)], \\ \dot{y}(t) = y(t)[-m_2(t) + c(t) - (c(t) - b(t))x(t) - c(t)y(t)], \\ \dot{u}(t) = u(t)[-m_1(t) + a(t) - a(t)u(t) - (a(t) + b(t))v(t)], \\ \dot{v}(t) = v(t)[-m_2(t) + c(t) - (c(t) - b(t))u(t) - c(t)v(t)]. \end{cases} \tag{19}$$

Define a Lyapunov function

$$V(t) = \alpha |\ln x(t) - \ln u(t)| + \beta |\ln y(t) - \ln v(t)| \tag{20}$$

for $t \geq 0$, where α and β satisfy (15). We still assume that almost periodic system (4) satisfies (A_1) – (A_4) and $\Lambda > 1$. From Remark 1, almost periodic solution $(x(t), y(t))$ and

$(u(t), v(t))$ of system (19) is ultimately bounded. It is not difficult to verify that conditions (1) and (2) of Lemma 9 are satisfied. Moreover, $|y(t) - u(t)| \leq K$ and $|x(t) - v(t)| \leq K$, where

$$K = \max\{|M_2^* - n_1^*|, |M_1^* - n_2^*|\} > 0.$$

By the same arguments followed in the proof of Theorem 8, one can have that

$$\begin{aligned} D^+V(t) &\leq -\rho(|x(t) - u(t)| + |y(t) - v(t)|) \\ &= -\rho[|x(t) - y(t) + y(t) - u(t)| + |y(t) - x(t) + x(t) - v(t)|] \\ &\leq -\rho[|x(t) - y(t)| + |y(t) - u(t)| + |y(t) - x(t)| + |x(t) - v(t)|] \\ &\leq -\rho[2|x(t) - y(t)| + 2K] \\ &= -2\rho[|x(t) - y(t)| + K], \end{aligned}$$

where ρ is defined as in (17). Taking $\lambda(r) = 2\rho r + K, r \geq 0$, condition (3) of Lemma 9 is satisfied.

By virtue of Lemma 9, the following theorem is an immediate consequence.

Theorem 10 *Let (A_1) – (A_4) and $\Lambda > 1$ be satisfied and system (4) is an almost periodic. Then there exists a unique almost periodic solution of system (4). Moreover, the unique solution is globally asymptotically stable.*

Remark 2 Theorem 10 demonstrates that $\Lambda > 1$ is the sharp threshold of existence and global asymptotic stability of almost periodic solution of system (4). Based on the ultimate boundedness of almost periodic solution of system (4), the Lyapunov function in Theorem 8 can be applied to prove Theorem 10. Condition (3) of Lemma 9 is easier to be checked than the usual condition $\dot{V}(t, x, y) \leq -\rho V(t, x, y)$. If system (4) is a periodic system, then the following corollary can be concluded.

Corollary 1 *If (A_1) – (A_4) and $\Lambda > 1$ hold, then periodic system (4) has a unique positive periodic solution, which is globally asymptotically stable.*

Here, we provide an example to illustrate our main results.

Example 1 Consider system (4) with specific values for its parameters of the form

$$\begin{cases} \dot{x}(t) = \frac{|\sin t|}{|\sin t|+1} \left[-\frac{1}{\sqrt{t}} + \frac{1}{t^2} - \frac{1}{t^2} \cdot \frac{|\sin t|}{|\sin t|+1} - \left(\frac{1}{t^2} + \frac{1}{3t^2}\right) \frac{\sqrt{t}}{\sqrt{t+1}} \right], \\ \dot{y}(t) = \frac{\sqrt{t}}{\sqrt{t+1}} \left[-\frac{3}{\sqrt{t+3}} + \frac{1}{\sqrt{t}} - \left(\frac{1}{\sqrt{t}} - \frac{1}{3t^2}\right) \frac{|\sin t|}{|\sin t|+1} - \frac{1}{\sqrt{t}} \cdot \frac{\sqrt{t}}{\sqrt{t+1}} \right], \end{cases} \tag{21}$$

with initial conditions (3). Clearly,

$$\begin{aligned} x(t) &= \frac{|\sin t|}{|\sin t| + 1}, & m_1(t) &= \frac{1}{\sqrt{t}}, & a(t) &= \frac{1}{t^2}, \\ y(t) &= \frac{\sqrt{t}}{\sqrt{t} + 1}, & m_2(t) &= \frac{3}{\sqrt{t} + 3}, & c(t) &= \frac{1}{\sqrt{t}}, \end{aligned}$$

and $b(t) = \frac{1}{3t^2}$. By direct computations, one can conclude that all the assumptions of Theorem 8 hold and $\Lambda > 1$. Therefore, system (21) with initial conditions (3) is globally asymptotically stable.

4 Conclusion

Dynamic properties of nonlinear nonautonomous models are rarely considered in the literature. In this paper, we try to fill in this gap by studying a nonautonomous model in a rocky intertidal community. The uniform persistence of the model is examined prior to proceeding to the main results. We establish the threshold condition $\Lambda > 1$ to guarantee the global asymptotic stability and the existence of a unique positive almost periodic solution of the addressed system. Furthermore, an appropriate Lyapunov function is constructed to prove the main results. A particular example with specific parameters that are consistent with the theoretical assumptions is constructed for the sake of demonstrating the validity of outcomes. The authors claim that the results of this paper are new and present different approach, the main theorem of the paper improved the known results of some models [6, 13, 27]. Indeed, it is a challenging problem to search for a suitable Lyapunov function to discuss a three-dimensional model in a rocky intertidal community. Precisely saying, we described the essence of the species interactions by a simple patch occupancy model with global dispersal of propagules. The model can be used in the rocky habitat as a large set of very small patches, each of which can be empty (bare rock) or occupied by an individual organism.

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Not applicable.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Author details

¹School of Mathematics and Statistics, Hefei Normal University, Hefei, P.R. China. ²Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia. ³Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. ⁴Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran. ⁵School of Mathematics, University of Science and Technology of China, Hefei, P.R. China. ⁶Department of Medical Research, China Medical University Hospital, Taichung, Taiwan.

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