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A fourth order non-polynomial quintic spline collocation technique for solving time fractional superdiffusion equations

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Abstract

The purpose of this article is to present a technique for the numerical solution of Caputo time fractional superdiffusion equation. The central difference approximation is used to discretize the time derivative, while non-polynomial quintic spline is employed as an interpolating function in the spatial direction. The proposed method is shown to be unconditionally stable and $O(h^4 + \Delta t^2)$ accurate. In order to check the feasibility of the proposed technique, some test examples have been considered and the simulation results are compared with those available in the existing literature.

Keywords: Non-polynomial quintic spline; Finite central difference approach; Superdiffusion equation; Caputo time fractional derivative

1 Introduction

In this article, we consider the following time fractional fourth order superdiffusion equation [1]:

$$\frac{\partial^\alpha y}{\partial t^\alpha} + \gamma \frac{\partial^4 y}{\partial z^4} = f(z,t), \quad 0 \le z \le L, 0 \le t \le T, \tag{1}$$

with the initial/end conditions

$$\begin{aligned}
&y(z,0) = \phi(z), \qquad y_t(z,0) = \psi(z),\\
&y(0,t) = y(L,t) = 0,\\
&y_{zz}(0,t) = y_{zz}(L,t) = 0,
\end{aligned}$$

where $\alpha \in (1,2]$ denotes the order of time fractional derivative, γ is a constant, and $\phi(z)$ is continuous on $[0,L]$.

Fourth order time fractional partial differential equations (PDEs) arise in mathematical modeling of several plate-like objects [2]. Most of the analytical techniques for solving fractional order PDEs are based on Laplace and Fourier transforms, while others involve the separation of variables technique [3, 4]. Some semi-analytic methods have also been employed to explore series solution for fractional order PDEs. These included homotopy analysis method [5], Adomian decomposition method [6, 7], homotopy perturbation method [8], variational iteration method [9], and fractional differential transformation method [10].

In recent years, spline functions have also been frequently employed for the numerical solution of fractional order PDEs. These functions have advantages over the usual finite difference methods as they can provide a continuous differentiable approximation to the unknown function in the spatial domain with good accuracy. The simple and straightforward application of spline functions provides enough motivation to employ them for the numerical study of fractional PDEs. Zahra and Elkholy [11] employed cubic spline functions combined with shooting method for solving fractional Bagley–Torvik equation. Talaat and Danaf [12] applied the quadratic spline method for numerical investigation of fractional diffusion equation. Siddiqi and Arshed [13] used the quintic B-spline collocation method for numerical solution of time fractional fourth order PDEs. In [14], the authors introduced new fractional order spline functions to study the numerical solution of fractional Bagely–Torvik equation. Mohyu-Din *et al.* [15] investigated the extended B-spline solution of time fractional advection diffusion equation by means of a fully implicit finite difference scheme. Li *et al.* [16] developed a non-polynomial spline scheme to solve time fractional nonlinear Schrodinger equation. In [17], Pezza and Pitolli used a fractional spline collocation Galerkin scheme to develop series solution for time fractional diffusion equation. Khalid *et al.* [18] utilized the non-polynomial quintic spline collocation method to explore the numerical solution of fourth order fractional boundary value problem, involving product terms. In [19], Amin *et al.* employed the quintic non-polynomial spline collocation scheme for solving time fractional fourth order PDEs.

There are several techniques to deal with the fractional differentiation but Riemann–Liouville's and Caputo's approaches are the most common. Here, we utilize Caputo's definition as it allows us to use the ordinary initial/boundary constraints. The Caputo time fractional derivative of order α is expressed as

$$\frac{\partial^\alpha y(z,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 y(z,s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha}-1}, & 1 < \alpha < 2, \\ \frac{\partial^2 y(z,t)}{\partial t^2}, & \alpha = 2. \end{cases}$$

This paper aims to develop a spline collocation approach for numerical solution of fourth order time fractional superdiffusion problem. For spatial discretization, a non-polynomial quintic spline interpolant, composed of trigonometric and polynomial components, is used. For temporal discretization, a central difference approximation is used.

The outline of this paper is as follows: A short description of the non-polynomial quintic spline technique is given in Sect. 2. The consistency relations between the spline approximation and its spatial derivatives at the grid points are derived in this section. In Sect. 3, the application of Caputo's definition and finite central difference formulation for temporal discretization is shown. The stability and convergence of the proposed problem is discussed in Sect. 4. The discretization in space direction is given in Sect. 5. The approximate results are discussed in Sect. 6, and the conclusion of the proposed study is given in Sect. 7.

2 Non-polynomial quintic spline functions

In this section, we construct the formulation and derive the truncation error of non-polynomial quintic spline functions.

2.1 Formulation

Let $z_i = a + ih$ be the grid points of a uniform partition of $[0, L]$ dividing it into the subintervals $[z_i, z_{i+1}]$, where $h = \frac{L}{N}$ and $0 \le i \le N$. We consider that $y(z)$ is sufficiently differentiable in $[0, L]$ and $Y(z)$ is its quintic non-polynomial spline approximation. Let each non-polynomial spline segment $S_i(z)$ have the following form [18, 19]:

$$S_i(z) = a_i \cos\eta(z - z_i) + b_i \sin\eta(z - z_i) + c_i(z - z_i)^3 + d_i(z - z_i)^2 + e_i(z - z_i) + f_i, \quad 0 \le i \le N, \tag{2}$$

where a_i, b_i, c_i, d_i, e_i, and f_i are constants to be determined and η denotes the frequency of the trigonometric functions. Moreover,

$$S_i(z) \in C^\infty[0, L]$$

and

$$Y(z) = S_i(z), \quad \forall z \in [z_i, z_{i+1}], i = 0, 1, 2, \dots, N. \tag{3}$$

Let

$$Y_i = S_i(z_i), \qquad M_i = Y''(z_i), \qquad F_i = Y^{(4)}(z_i).$$

The constants involved in $S_i(z)$ can be expressed as follows:

$$\begin{aligned}
a_i &= \frac{h^4}{\theta^4} F_i, \\
b_i &= \frac{h^4}{\theta^4 \sin(\theta)} \big(F_{i+1} - F_i \cos(\theta)\big), \\
c_i &= \frac{1}{6h}(M_{i+1} - M_i) + \frac{h}{6\theta^2}(F_{i+1} - F_i), \\
d_i &= \frac{1}{2} M_i + \frac{h^2}{2\theta^2} F_i, \\
e_i &= \frac{1}{h}(Y_{i+1} - Y_i) + \left(\frac{h^3}{\theta^4} - \frac{h^3}{3\theta^2}\right) F_i - \left(\frac{h^3}{\theta^4}, + \frac{h^3}{6\theta^2}\right) F_{i+1} - \frac{h}{6}(M_{i+1} + 2M_i), \\
f_i &= Y_i - \frac{h^4}{\theta^4} F_i,
\end{aligned}$$

where $\theta = \eta h$ and $i = 0 \le i \le N$. Using 1st and 3rd derivative continuities at the knots, $S_{i-1}^{(\tau)}(z_i) = S_i^{(\tau)}(z_i)$ for $\tau = 1, 3$, the following important relations can be derived:

$$\begin{aligned}
M_{i+1} + 4M_i + M_{i-1} = {} & \frac{6}{h^2}(Y_{i+1} - 2Y_i + Y_{i-1}) + \frac{6h^2}{\theta^2}\left(\frac{1}{\theta \sin(\theta)} - \frac{1}{\theta^2} - \frac{1}{6}\right)(F_{i+1} + F_{i-1}) \\
& + \frac{6h^2}{\theta^2}\left(\frac{2}{\theta^2} - \frac{2\cos(\theta)}{\theta \sin(\theta)} - \frac{4}{6}\right) F_i
\end{aligned} \tag{4}$$

and

$$M_{i+1} - 2M_i + M_{i-1} = h^2\left(\frac{1}{\theta \sin(\theta)} - \frac{1}{\theta^2}\right)(F_{i+1} + F_{i-1}) + 2h^2\left(\frac{1}{\theta^2} - \frac{\cos(\theta)}{\theta \sin(\theta)}\right) F_i. \tag{5}$$

Solving (4) and (5) yields

$$M_i = \frac{1}{h^2}(Y_{i+1} - 2Y_i + Y_{i-1}) + h^2\left(\frac{1}{\theta^3 \sin(\theta)} - \frac{1}{6\theta \sin(\theta)} - \frac{1}{\theta^4}\right)(F_{i+1} + F_{i-1}).$$
$$+ h^2\left(\frac{2}{\theta^4} - \frac{2\cos(\theta)}{\theta^3 \sin(\theta)} + \frac{2\cos(\theta)}{6\theta \sin(\theta)} - \frac{1}{\theta^2}\right)(F_i). \tag{6}$$

Using (5) and (6), we obtain the following consistency relation involving F_i and Y_i for $i = 2, 3, \dots, N-2$:

$$Y_{i+2} - 4Y_{i+1} + 6Y_i - 4Y_{i-1} + Y_{i-2} = h^4(\mu_1 F_{i-2} + \nu_1 F_{i-1} + \kappa_1 F_i + \nu_1 F_{i+1} + \mu_1 F_{i+2}), \tag{7}$$

where

$$\mu_1 = \frac{1}{\theta^4} + \frac{1}{6\theta \sin(\theta)} - \frac{1}{\theta^3 \sin(\theta)}, \qquad \nu_1 = \frac{2 + 2\cos(\theta)}{\theta^3 \sin(\theta)} + \frac{2 - \cos(\theta)}{3\theta \sin(\theta)} - \frac{4}{\theta^4},$$
$$\kappa_1 = \frac{1 - 4\cos(\theta)}{3\theta \sin(\theta)} - \frac{2 + 4\cos(\theta)}{\theta^3 \sin(\theta)} + \frac{6}{\theta^4}.$$

Relation (7) produces $(N-3)$ algebraic equations involving $(N-1)$ unknowns, Y_i, $i = 1, 2, \dots, N-1$. In order to solve this system, we obtain two more conditions as follows:

Setting $i = 1, 2$ in (4), we have

$$M_0 + 4M_1 + M_2 = \frac{6}{h^2}(Y_0 - 2Y_1 + Y_2) + \tilde{\tau}(F_0 + F_2) + \tilde{\kappa} F_1 \tag{8}$$

and

$$M_1 + 4M_2 + M_3 = \frac{6}{h^2}(Y_1 - 2Y_2 + Y_3) + \tilde{\tau}(F_1 + F_3) + \tilde{\kappa} F_2. \tag{9}$$

Similarly, using (5), the following two expressions can be derived with $i = 1, 2$:

$$M_0 - 2M_1 + M_2 = \approx{\tau}(F_0 + F_2) + \approx{\kappa} F_1 \tag{10}$$

and

$$M_1 - 2M_2 + M_3 = \approx{\tau}(F_1 + F_3) + \approx{\kappa} F_2, \tag{11}$$

where

$$\tilde{\tau} = \frac{6h^2}{\theta^2}\left(\frac{1}{\theta \sin\theta} - \frac{1}{\theta^2} - \frac{1}{6}\right), \qquad \tilde{\kappa} = \frac{6h^2}{\theta^2}\left(\frac{2}{\theta^2} - \frac{2\cos(\theta)}{\theta \sin(\theta)} - \frac{4}{6}\right),$$
$$\approx{\tau} = h^2\left(\frac{1}{\theta \sin(\theta)} - \frac{1}{\theta^2}\right) \quad \text{and} \quad \approx{\kappa} = 2h^2\left(\frac{1}{\theta^2} - \frac{\cos(\theta)}{\theta \sin(\theta)}\right).$$

From (8) and (10), we have

$$M_1 = \frac{1}{h^2}(Y_0 - 2Y_1 + Y_2) + \frac{\tilde{\tau} - \approx{\tau}}{6}(F_0 + F_2) + \frac{\tilde{\kappa} - \approx{\kappa}}{6} F_1. \tag{12}$$

Similarly, using (9) and (11), we obtain

$$M_2 = \frac{1}{h^2}(Y_1 - 2Y_2 + Y_3) + \frac{\widetilde{\tau} - \approx{\tau}}{6}(F_1 + F_3) + \frac{\widetilde{\kappa} - \approx{\kappa}}{6}F_2. \tag{13}$$

Substituting (12) and (13) into (8) for $i = 1$ yields

$$2Y_0 - 5Y_1 + 4Y_2 - Y_3 = h^2 M_0 - h^4(\varpi_0 F_0 + \varpi_1 F_1 + \varpi_2 F_2 + \varpi_3 F_3). \tag{14}$$

Also, for $i = n$, the following relation can be established:

$$\begin{aligned} &Y_{N-3} - 4Y_{N-2} + 5Y_{N-1} - 2Y_N \\ &\quad = -h^2 M_N + h^4(\varpi_3 F_{N-3} + \varpi_2 F_{N-2} + \varpi_1 F_{N-1} + \varpi_0 F_N), \end{aligned} \tag{15}$$

where

$$\varpi_0 = \frac{2}{\theta^3\sin(\theta)} - \frac{2}{\theta^4} + \frac{4}{6\theta\sin(\theta)} - \frac{1}{\theta^2}, \qquad \varpi_1 = \frac{1 - 8\cos(\theta)}{6\theta\sin(\theta)} - \frac{1 + 4\cos(\theta)}{\theta^3\sin(\theta)} + \frac{5}{\theta^4},$$
$$\varpi_2 = \frac{2 + 2\cos(\theta)}{\theta^3\sin(\theta)} + \frac{2 - \cos(\theta)}{3\theta\sin(\theta)} - \frac{4}{\theta^4}, \qquad \varpi_3 = \frac{1}{6\theta\sin(\theta)} - \frac{1}{\theta^3\sin(\theta)} + \frac{1}{\theta^4}.$$

2.2 Truncation error

To calculate $\widetilde{t}_i$, $1 \le i \le N-1$, for the current scheme, we rewrite (7), (14), and (15) as follows:

$$\begin{aligned} \widetilde{t}_1 &= -2y_0 + 5y_1 - 4y_2 + y_3 + h^2 M_0 - h^4\bigl(\varpi_0 y_0^{(4)} + \varpi_1 y_1^{(4)} + \varpi_2 y_2^{(4)} + \varpi_3 y_3^{(4)}\bigr), \\ \widetilde{t}_i &= y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} - h^4\bigl(\alpha_1 y_{i-2}^{(4)} + \beta_1 y_{i-1}^{(4)} + \gamma_1 y_i^{(4)} + \beta_1 y_{i+1}^{(4)} + \alpha_1 y_{i+2}^{(4)}\bigr), \\ \widetilde{t}_{N-1} &= y_{N-3} - 4y_{N-2} + 5y_{N-1} - 2y_n + h^2 M_N \\ &\quad + h^4\bigl(\varpi_3 y_{N-3}^{(4)} + \varpi_2 y_{N-2}^{(4)} + \varpi_1 y_{N-1}^{(4)} + \varpi_0 y_N^{(4)}\bigr). \end{aligned}$$

The following relations for $\widetilde{t}_i$, $1 \le i \le N-1$, can be established by expanding y_0, y_1, $y_1^{(4)}$, y_2, $y_2^{(4)}$, y_3, $y_3^{(4)}$, etc. about the points z_i, $1 \le i \le N-1$, by means of Taylor's series:

$$\widetilde{t}_i = \begin{cases} (\frac{11}{12} - \varpi_0 - \varpi_1 - \varpi_2 - \varpi_3)h^4 y_i^{(4)} + (\frac{1}{12} + \varpi_0 - \varpi_2 - 2\varpi_3)h^5 y_i^{(5)} \\ \quad + (\frac{11}{90} - \frac{1}{2}\varpi_0 - \frac{1}{2}\varpi_2 - 2\varpi_3)h^6 y_i^{(6)} + (\frac{1}{60} + \frac{1}{6}\varpi_0 - \frac{1}{6}\varpi_2 - \frac{4}{3}\varpi_3)h^7 y_i^{(7)} \\ \quad + (\frac{17}{2240} - \frac{1}{24}\varpi_0 - \frac{1}{24}\varpi_2 - \frac{2}{3}\varpi_3)h^8 y_i^{(8)} + O(h^9), \\ \quad i = 1 \\ (1 - 2\alpha_1 - 2\beta_1 - \gamma_1)h^4 y_i^{(4)} + (\frac{1}{6} - 4\alpha_1 - \beta_1)h^6 y_i^{(6)} \\ \quad + (\frac{1}{180} - \frac{4}{3}\alpha_1 - \frac{1}{12}\beta_1)h^8 y_i^{(8)} + (\frac{17}{30240} - \frac{8}{45}\alpha_1 - \frac{1}{360}\beta_1)h^{10} y_i^{(10)} + O(h^{11}), \\ \quad i = 2(1)N-2 \\ (\frac{11}{12} - \varpi_0 - \varpi_1 - \varpi_2 - \varpi_3)h^4 y_i^{(4)} + (\frac{1}{12} + \varpi_0 - \varpi_2 - 2\varpi_3)h^5 y_i^{(5)} \\ \quad + (\frac{11}{90} - \frac{1}{2}\varpi_0 - \frac{1}{2}\varpi_2 - 2\varpi_3)h^6 y_i^{(6)} + (\frac{1}{60} + \frac{1}{6}\varpi_0 - \frac{1}{6}\varpi_2 - \frac{4}{3}\varpi_3)h^7 y_i^{(7)} \\ \quad + (\frac{17}{2240} - \frac{1}{24}\varpi_0 - \frac{1}{24}\varpi_2 - \frac{2}{3}\varpi_3)h^8 y_i^{(8)} + O(h^9), \\ \quad i = N-1. \end{cases} \tag{16}$$

Comparing the coefficients of $y_i^{(\tau)}$ for $\tau = 4,5,6,7$, we obtain

$$\mu_1 = -\frac{1}{720}, \qquad \nu_1 = \frac{31}{180}, \qquad \kappa_1 = \frac{79}{120}, \qquad \varpi_0 = \frac{7}{90},$$
$$\varpi_1 = \frac{49}{72}, \qquad \varpi_2 = \frac{7}{45} \quad \text{and} \quad \varpi_3 = \frac{1}{360}.$$

Finally we get

$$\tilde{t}_i = \begin{cases} -\frac{241}{60480} h^8 y_i^{(8)} + O(h^9), & i = 1, \\ \frac{1}{3024} h^{10} y_i^{(10)} + O(h^{11}), & i = 2(1)N-2, \\ -\frac{241}{60480} h^8 y_i^{(8)} + O(h^9), & i = N-1. \end{cases} \tag{17}$$

Following [19], the formulation and truncation error given in Sects. 2.1 and 2.2 respectively are reproduced here for the sake of completeness.

3 Time discretization

Let $t_m = m\Delta t$, where $\Delta t = \frac{T}{K}$ is the step size in the time direction for $m = 0,1,2,\ldots,K$. To discretize the Caputo fractional time derivative at $t = t_{m+1}$, the usual central difference approach is used as follows [20]:

$$\frac{\partial^\alpha y(z,t_{m+1})}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_{k+1}} \frac{\partial^2 y(z,w)}{\partial w^2} (t_{m+1}-w)^{-\alpha+1}\,dw,$$

$$\begin{aligned} &\frac{\partial^\alpha y(z,t_{m+1})}{\partial t^\alpha} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} \frac{\partial^2 y(z,w)}{\partial w^2} (t_{m+1}-w)^{-\alpha+1}\,dw \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{m} \frac{y(z,t_{k+1}) - 2y(z,t_k) + y(z,t_{k-1})}{\Delta t^2} \int_{t_k}^{t_{k+1}} (t_{m+1}-w)^{-\alpha+1}\,dw + l_{\Delta t}^{m+1} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{m} \frac{y(z,t_{k+1}) - 2y(z,t_k) + y(z,t_{k-1})}{\Delta t^2} \int_{t_{m-k}}^{t_{m-k+1}} (\upsilon)^{-\alpha+1}\,d\upsilon + l_{\Delta t}^{m+1} \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{m} \frac{y(z,t_{m-k+1}) - 2y(z,t_{m-k}) + y(z,t_{m-k-1})}{\Delta t^2} \int_{t_k}^{t_{k+1}} (\upsilon)^{-\alpha}\,d\upsilon + l_{\Delta t}^{m+1} \\ &= \frac{1}{\Gamma(3-\alpha)} \sum_{k=0}^{m} \frac{y(z,t_{m-k+1}) - 2y(z,t_{m-k}) + y(z,t_{m-k-1})}{\Delta t^\alpha} \bigl((k+1)^{2-\alpha} - k^{2-\alpha}\bigr) + l_{\Delta t}^{m+1} \\ &= \frac{1}{\Gamma(3-\alpha)} \sum_{k=0}^{m} d_k \frac{y(z,t_{m-k+1}) - 2y(z,t_{m-k}) + y(z,t_{m-k-1})}{\Delta t^\alpha} + l_{\Delta t}^{m+1}, \end{aligned} \tag{18}$$

where $d_k = (k+1)^{2-\alpha} - k^{2-\alpha}$ and $\upsilon = (t_{m+1} - w)$.

Now, introduce a fractional differential operator Ω_t^α:

$$\Omega_t^\alpha y(z,t_{k+1}) = \frac{1}{\Gamma(3-\alpha)} \sum_{k=0}^{m} d_k \frac{y(z,t_{m-k+1}) - 2y(z,t_{m-k}) + y(z,t_{m-k-1})}{\Delta t^\alpha}.$$

Equation (18) can be rewritten as follows:

$$\frac{\partial^\alpha y(z,t_{m+1})}{\partial t^\alpha} = \Omega_t^\alpha y(z,t_{k+1}) + l_{\Delta t}^{m+1}. \tag{19}$$

Here, $l_{\Delta t}^{m+1}$ denotes the truncation error between $\frac{\partial^\alpha}{\partial t^\alpha} y(z,t_{m+1})$ and $\Omega_t^\alpha y(z,t_{m+1})$. Equation (1) can be written as

$$\Omega_t^\alpha y(z,t_{m+1}) + \gamma \frac{\partial^4}{\partial z^4} y(z,t_{m+1}) = f(z,t_{m+1}), \tag{20}$$

where $\Omega_t^\alpha y(z,t_{m+1})$ denotes the Caputo fractional time derivative approximation at $t = t_{m+1}$. Using (18), expression (20) takes the following form:

$$\begin{aligned} y^{m+1}(z) + \beta\gamma y_{xxxx}^{m+1} &= -d_m y^{-1}(z) + (2d_m - d_{m-1})y^0(z) + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})y^{m-k}(z) \\ &\quad + (2d_0 - d_1)y^m(z) + \beta f^{m+1}(z), \quad m = 1,2,3,\dots,K-1, \end{aligned} \tag{21}$$

where $\beta = \Gamma(3-\alpha)\Delta t^\alpha$ and $y^{m+1}(z) = y(z,t^{m+1})$ and the initial conditions are imposed as follows:

$$y(z,t_0) = \phi(z), \quad 0 \le z \le L, \tag{22}$$

$$\frac{\partial y(z,t_0)}{\partial t} = \psi(z), \quad 0 \le z \le L. \tag{23}$$

Moreover, the constants $d_k s$ appearing in (18) possess the following properties:

- $d_0 = 1$ and $\forall k$, $d_k s > 0$,
- $(2-d_1) - \sum_{k=1}^{m-1}(d_{k+1} - 2d_{k+1} + d_{k-1}) + (2d_m - d_{m-1}) - d_m = 1$.

The truncation error in (19) is bounded, i.e.,

$$\left| l_{\Delta t}^{m+1} \right| \le \zeta \Delta t^2. \tag{24}$$

Here, ζ is a constant depending on y.

To implement this scheme, first we calculate y^{-1} as follows:

$$y^{-1}(z) = y(z,t_0) - \Delta t y_t(z,t_0),$$

$$y^{-1}(z) = \phi(z) - \Delta t \psi(z).$$

For $m = 0$, (21) takes the following form:

$$y^1(z) + \beta\gamma y_{zzzz}^1 = -d_0 y^{-1}(z) + 2d_0 y_0(z) + \beta f^1(z). \tag{25}$$

Now, Eqs. (21) and (25) together with initial/boundary conditions become a complete set of semi-discrete problem for (1).

Also, l^{m+1}, the error at $t = t_{m+1}$, is given by [21]

$$l^{m+1} = \beta\left(\frac{\partial^\alpha}{\partial t^\alpha} y(z,t_{m+1}) - G_t^\alpha y(z,t_{m+1})\right). \tag{26}$$

From Eqs. (19) and (24), the above expression can be written as

$$\left|l^{m+1}\right| = \left|l_{\Delta t}^{m+1}\right| \leq \zeta \Delta t^2. \tag{27}$$

Some relevant functional spaces, their inner product and standard norms are defined as follows:

$$\Upsilon^2(\eta) = \left\{u \in L^2(\eta), u_z, u_{zz} \in L^2(\eta)\right\},$$
$$\Upsilon_0^2(\eta) = \left\{u \in \Upsilon^2(\eta), u|_{\partial\eta} = 0, u_z|_{\partial\eta} = 0\right\},$$
$$\Upsilon^n(\eta) = \left\{u \in L^2(\eta), u_z^{(r)}, \forall r \leq N\right\},$$

where $L^2(\eta)$ denotes the space of those measurable functions whose squares are Lebesgue integrable in η. The norm and inner product of $L^2(\eta)$ are given by

$$\langle v, u\rangle = \int_\eta vu\,dz, \quad \|u\|_0 = \langle u, u\rangle^{\frac{1}{2}}.$$

Also, for $\Upsilon^2(\eta)$, we take

$$\langle v, u\rangle_2 = \langle v, u\rangle + \langle v_z, u_z\rangle + \langle v_{zz}, u_{zz}\rangle, \quad \|u\|_2 = \sqrt{\langle u, u\rangle_2}.$$

The norm $\|\cdot\|$ of the space $\Upsilon^N(\eta)$ is defined as

$$\|u\|_N = \sqrt{\sum_{r=0}^{N} \left\|u_x^{(r)}\right\|_0^2}. \tag{28}$$

$\|\cdot\|_2$ is defined as

$$\|u\|_2 = \sqrt{\|u\|_0^2 + \beta\gamma \left\|u_x^{(2)}\right\|_0^2}. \tag{29}$$

Now, to carry out the stability and convergence analysis, we need to find $y^{m+1} \in \Upsilon_0^2(\eta)$ such that $\forall u \in \Upsilon_0^2(\eta)$. From (21) and (25), we have

$$\begin{aligned}
&\left\langle y^{m+1}, u\right\rangle + \beta\gamma\left\langle y_{zzzz}^{m+1}, u\right\rangle \\
&\quad = -d_m\left\langle y^{-1}, u\right\rangle + (2d_m - d_{m-1})\left\langle y^0, u\right\rangle \\
&\qquad + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\left\langle y^{m-k}, u\right\rangle + (2d_0 - d_1)\left\langle y^m, u\right\rangle + \beta\left\langle f^{m+1}, u\right\rangle
\end{aligned} \tag{30}$$

and

$$\left\langle y^1, u\right\rangle + \beta\gamma\left\langle y_{zzzz}^1, u\right\rangle = -d_0\left\langle y^{-1}, u\right\rangle + 2d_0\left\langle y^0, u\right\rangle + \beta\left\langle f^1, u\right\rangle. \tag{31}$$

Definition 1 Let g_m and h_m, $m = 1, 2, \ldots, N$, be the sequences which satisfy the inequality

$$g_m \leq \left(\sum_{i=1}^{m-1} g_i h_i + \kappa\right), \quad m = 1, 2, \ldots, N,$$

where $g_m \geq 0$, $\kappa \geq 0$, then the following discrete Gronwall inequality holds:

$$g_m \leq \kappa . \exp\left(\sum_{i=1}^{m-1} g_i\right), \quad m = 1, 2, \ldots, N. \tag{32}$$

4 Stability and convergence

The approach used in this section follows the general approach used in [19]. The stability and convergence analysis for semi-discrete problem are described in the following Theorems 1 and 2, respectively.

Theorem 1 $\forall \Delta t > 0$, *the semi-discrete problem is unconditionally stable if*

$$\|y^{m+1}\|_2 \leq c\big(\|\phi\|_0 + \Delta t \|\psi\|_0 + \beta \|f^{k+1}\|_0\big), \quad 0 \leq m \leq K - 1. \tag{33}$$

Proof We apply mathematical induction to prove this theorem.

For $m = 0$ and $u = y^1$, Eq. (30) can be expressed as

$$\langle y^1, y^1 \rangle + \beta\gamma \langle y^1_{xxxx}, y^1 \rangle = -d_0 \langle y^{-1}, y^1 \rangle + 2\langle y^0, y^1 \rangle + \beta \langle f^1, y^1 \rangle$$

or

$$\langle y^1, y^1 \rangle + \beta\gamma \langle y^1_{zz}, y^1_{zz} \rangle = -d_0 \langle y^{-1}, y^1 \rangle + 2d_0 \langle y^0, y^1 \rangle + \beta \langle f^1, y^1 \rangle. \tag{34}$$

Here, all the boundary related contributions vanish because of the boundary constraints on u. Using $\|u\|_0 \leq \|u\|_2$ and Schwarz's inequality, Eq. (34) becomes

$$\begin{aligned}
\|y^1\|_2^2 &\leq \|y^{-1}\|_0 \|y^1\|_0 + 2\|y^0\|_0 \|y^1\|_0 + \beta \|f^1\|_0 \|y^1\|_0 \\
&\leq \|y^{-1}\|_0 \|y^1\|_2 + 2\|y^0\|_0 \|y^1\|_2 + \beta \|f^1\|_0 \|y^1\|_2, \\
\|y^1\|_2 &\leq \|y^{-1}\|_0 + 2\|y^0\|_0 + \beta \|f^1\|_0, \\
\|y^1\|_2 &\leq \big(\|\phi\|_0 - \Delta t \|\psi\|_0\big) + 2\|\phi_0\|_0 + \beta \|f^1\|_0, \\
\|y^1\|_2 &\leq 3\big(\|\phi\|_0 - \Delta t \|\psi\|_0 + \beta \|f^1\|_0\big).
\end{aligned}$$

Hence

$$\|y^1\|_2 \leq c\big(\|\phi\|_0 - \Delta t \|\psi\|_0 + \beta \|f^1\|_0\big).$$

We assume that the result is true for $u = y^k$, i.e.,

$$\|y^k\|_2 \leq c\big(\|\phi_0\|_0 + \Delta t \|\psi_0\|_0 + \beta \|f^k\|_0\big), \quad k = 2, 3, \ldots, m. \tag{35}$$

Let $u = y^{m+1}$ in Eq. (30)

$$\begin{aligned}
&\langle y^{m+1}, y^{m+1} \rangle + \beta\gamma \langle y^{m+1}_{zzzz}, y^{m+1} \rangle \\
&\quad = -d_m \langle y^{-1}, y^{m+1} \rangle + (2d_m - d_{m-1}) \langle y^0, y^{m+1} \rangle \\
&\qquad + \sum_{k=1}^{m-1} (-d_{k-1} + 2d_k - d_{k+1}) \langle y^{m-k}, y^{m+1} \rangle + (2d_0 - d_1) \langle y^m, y^{m+1} \rangle + \beta \langle f^{m+1}, y^{m+1} \rangle.
\end{aligned} \tag{36}$$

Integrating by parts gives

$$\begin{aligned}&\langle y^{m+1}, y^{m+1}\rangle + \beta\gamma\langle y_{zz}^{m+1}, y_{zz}^{m+1}\rangle \\ &\quad = -d_m\langle y^{-1}, y^{m+1}\rangle + (2d_m - d_{m-1})\langle y^0, y^{m+1}\rangle \\ &\qquad + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\langle y^{m-k}, y^{m+1}\rangle + (2d_0 - d_1)\langle y^m, y^{m+1}\rangle + \beta\langle f^{m+1}, y^{m+1}\rangle. \qquad (37)\end{aligned}$$

Using $\|u\|_0 \le \|u\|_2$ and Schwarz's inequality, the above expression then takes the following form:

$$\begin{aligned}\|y^{m+1}\|_2^2 &\le d_m\|y^{-1}\|_0\|y^{m+1}\|_0 + (2d_m - d_{m-1})\|y^0\|_0\|y^{m+1}\|_0 + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k \\ &\quad - d_{k+1})\|y^{m-k}\|_0\|y^{m+1}\|_0 + (2d_0 - d_1)\|y^m\|_0\|y^{m+1}\|_0 + \beta\|f^{k+1}\|_0\|y^{m+1}\|_0,\end{aligned}$$

or

$$\begin{aligned}\|y^{m+1}\|_2^2 &\le d_m\|y^{-1}\|_0\|y^{m+1}\|_2 + (2d_m - d_{m-1})\|y^0\|_0\|y^{m+1}\|_2 + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k \\ &\quad - d_{k+1})\|y^{m-k}\|_0\|y^{m+1}\|_2 + (2d_0 - d_1)\|y^m\|_0\|y^{m+1}\|_2 + \beta\|f^{k+1}\|_0\|y^{m+1}\|_2, \\ \|y^{m+1}\|^2 &\le d_m\|y^{-1}\|_0 + (2d_m - d_{m-1})\|y^0\|_0 + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\|y^{m-k}\|_0 \\ &\quad + (2d_0 - d_1)\|y^m\|_0 + \beta\|f^{k+1}\|_0.\end{aligned}$$

Using (32), the above relation can be written as follows:

$$\begin{aligned}\|y^{m+1}\|_2 &\le \big((2d_m - d_{m-1})\|y^0\|_0 + d_m\|y^{-1}\|_0 + \beta\|f^{m+1}\|_0\big)\exp\Bigg[(2d_0 - d_1) \\ &\quad + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\Bigg], \\ \|y^{m+1}\|_2 &\le \big(\|y^0\|_0 + \|y^{-1}\|_0 + \beta\|f^{k+1}\|_0\big)\exp(1 + d_{m-1} - d_m), \\ \|y^{m+1}\|_2 &\le \big(\|\phi\|_0 + \|\phi\|_0 - \Delta t\|\psi\|_0 + \beta\|f^{k+1}\|_0\big)\exp(1 + d_{m-1} - d_m), \\ \|y^{m+1}\|_2 &\le c\big(\|\phi\|_0 - \Delta t\|\psi\|_0 + \beta\|f^{k+1}\|_0\big).\end{aligned}$$

□

Theorem 2 *The numerical solution obtained by the proposed method converges to the exact solution if the following relation holds*:

$$\|y(t_m) - y^m\|_2 \le \zeta\Delta t^2, \quad m = 1, 2, \dots, K, \qquad (38)$$

where ζ is constant and $\zeta = (1 + d_{m-1})\exp(1 + d_{m-1} - d_m)$.

Proof Consider $e^m = y(z, t_m) - y^m(z)$ for $m = 1$, using Eqs. (1) and (30), we have

$$\langle e^1, u\rangle + \beta\gamma\langle e_{zz}^1, u_{zz}\rangle = \langle e^{-1}, u\rangle + 2d_0\langle e^0, u\rangle + \langle l^1, u\rangle, \quad \forall u \in H_0^2(\eta).$$

Again using $\|u\|_0 \le \|u\|_2$, Schwarz's inequality, $u = e^1$, and $e^0 = 0$, we get

$$\begin{aligned}
&\|e^1\|_2^2 \le \|e^{-1}\|_0 + \|e^1\|_0 + \|l^1\|_0\|e^1\|_0,\\
&\|e^1\|_2^2 \le \|e^{-1}\|_0 + \|e^1\|_2 + \|l^1\|_0\|e^1\|_2,\\
&\|e^1\|_2 \le \|e^{-1}\|_0 + \|l^1\|_0.
\end{aligned}$$

Since $\|e^{-1}\| \le \Delta t^2$, using Eq. (27) leads to

$$\begin{aligned}
&\|y(t_1) - y^1\|_2 \le 2\Delta t^2,\\
&\|y(t_1) - y^1\|_2 \le \zeta\Delta t^2.
\end{aligned}$$

For $m = 1$, Eq. (38) is true.

Now, consider (38) is satisfied for $m = (1)r$, i.e.,

$$\|y(t_m) - y^m\|_2 \le \zeta\Delta t^2. \tag{39}$$

Using (1), (29), (30) and for $m = r + 1$, the error equation is derived as follows:

$$\begin{aligned}
&\langle e^{m+1}, u\rangle + \beta\gamma\langle e_{zz}^{m+1}, u_{zz}\rangle\\
&\quad= -d_m\langle e^{-1}, u\rangle + (2d_m - d_{m-1})\langle e^0, u\rangle\\
&\qquad+ \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\langle e^{m-k}, u\rangle + (2d_0 - d_1)\langle e^m, u\rangle + \langle l^{m+1}, u\rangle.
\end{aligned} \tag{40}$$

Now, using the induction assumption and taking $u = e^{m+1}$, Eq. (40) can be written as follows:

$$\begin{aligned}
\|e^{m+1}\|_2^2 \le{}& -d_m\|e^{-1}\|_0\|e^{m+1}\|_0 + (2d_m - d_{m-1}) + \|e^0\|_0\|e^{m+1}\|_0\\
&+ \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\|e^{m-k}\|_0\|e^{m+1}\|_0 + (2d_0 - d_1)\|e^m\|_0\|e^{m+1}\|_0\\
&+ \|l^{m+1}\|_0\|e^{m+1}\|_0,\\
\|e^{m+1}\|_2 \le{}& -d_m\|e^{-1}\|_0 + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\|e^{m-k}\|_2 + (2d_0 - d_1)\|e^m\|_2 + \|l^{m+1}\|_0.
\end{aligned}$$

Using (34), we have

$$\|e^{m+1}\|_2 \le \left(d_m\|e^{-1}\|_0 + \|l^{m+1}\|_0\right)\exp\left[\sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1}) + 2d_0 - d_1\right],$$

or

$$\begin{aligned}
&\|e^{m+1}\|_2 \le \left(d_m\|e^{-1}\|_0 + \|l^{m+1}\|_0\right)\exp(1 + d_{k-1} - d_k),\\
&\|e^{m+1}\|_2 \le \left(d_m\Delta t^2 + \Delta t^2\right) + \exp(1 + d_{k-1} - d_k),\\
&\|e^{m+1}\|_2 \le \zeta\Delta t^2.
\end{aligned}$$

Hence, proved. □

5 Spatial discretization

The approach used in this section follows the general approach used in [19].

Let (z_i, t_m) be the grid points of a uniform mesh to discretize the region $[0,L] \times [0,T]$, where $z_i = ih$, $i = 0,1,2,\dots,N$, and $h = \frac{L}{N}$. The spatial discretization of Eq. (21) using quintic non-polynomial spline is formulated as follows:

$$\begin{aligned} &Y_i^{m+1}(z) + \beta\gamma F^{m+1} \\ &\quad = -d_m Y^{-1}(z) + (2d_m - d_{m-1})Y^0(z) + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})Y^{m-k}(z) \\ &\qquad + (2d_0 - d_1)Y^m(z) + \beta f^{m+1}(z), \quad m = 1,2,3,\dots,K-1. \end{aligned} \tag{41}$$

The operator φ is defined as

$$\varphi Y_k = \mu_1 Y_{k-2} + \nu_1 Y_{k-1} + \kappa_1 Y_k + \nu_1 Y_{k+1} + \mu_1 Y_{k+2}. \tag{42}$$

Now, Eq. (7) takes the following form:

$$\varphi F_i = \frac{1}{h^4}(Y_{i-2} - 4Y_{i-1} + 6Y_i - 4Y_{i+1} + Y_{i+2}). \tag{43}$$

Applying φ on Eq. (41), we obtain

$$\begin{aligned} &\bigl(\mu_1 Y_{i-2}^{m+1} + \nu_1 Y_{i-1}^{m+1} + \kappa_1 Y_i^{m+1} + \nu_1 Y_{i+1}^{m+1} + \mu_1 Y_{i+2}^{m+1}\bigr) + \frac{\beta\gamma}{h^4}\bigl(Y_{i-2}^{m+1} - 4Y_{i-1}^{m+1} + 6Y_i^{m+1} \\ &\quad - 4Y_{i+1}^{m+1} + Y_{i+2}^{m+1}\bigr) \\ &\quad = -d_m\bigl(\mu_1 Y_{i-2}^{-1} + \nu_1 Y_{i-1}^{-1} + \kappa_1 Y_i^{-1} + \nu_1 Y_{i+1}^{-1} + \mu_1 Y_{i+2}^{-1}\bigr) + (2d_m \\ &\qquad - d_{m-1})\bigl(\mu_1 Y_{i-2}^0 + \nu_1 Y_{i-1}^0 + \kappa_1 Y_i^0 + \nu_1 Y_{i+1}^0 + \mu_1 Y_{i+2}^0\bigr) \\ &\qquad + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\bigl(\mu_1 Y_{i-2}^{m-k} \\ &\qquad + \nu_1 Y_{i-1}^{m-k} + \kappa_1 Y_i^{m-k} + \nu_1 Y_{i+1}^{m-k} + \mu_1 Y_{i+2}^{m-k}\bigr) \\ &\qquad + (2d_0 - d_1)\bigl(\mu_1 Y_{i-2}^m + \nu_1 Y_{i-1}^m + \kappa_1 Y_i^m + \nu_1 Y_{i+1}^m \\ &\qquad + \mu_1 Y_{i+2}^m\bigr) + \beta\bigl(\mu_1 f_{i-2}^{m+1} + \nu_1 f_{i-1}^{m+1} + \kappa_1 f_i^{m+1} + \nu_1 f_{i+1}^{m+1} + \mu_1 f_{i+2}^{m+1}\bigr), \\ &\quad 1 \le m \le K-1. \end{aligned} \tag{44}$$

After simplification, (44) takes the following form:

$$\begin{aligned} &\left(\mu_1 + \frac{\beta\gamma}{h^4}\right)Y_{i-2}^{m+1} + \left(\nu_1 - 4\frac{\beta\gamma}{h^4}\right)Y_{i-1}^{m+1} + \left(\kappa_1 + 6\frac{\beta\gamma}{h^4}\right)Y_i^{m+1} \\ &\qquad + \left(\nu_1 - 4\frac{\beta\gamma}{h^4}\right)Y_{i+1}^{m+1} + \left(\mu_1 + \frac{\beta\gamma}{h^4}\right)Y_{i+2}^{m+1} \\ &\quad = Q_i, \quad 2 \le i \le N-2, 1 \le m \le K-1, \end{aligned} \tag{45}$$

where

$$Q_i = -d_m\big(\mu_1 Y_{i-2}^{-1} + \nu_1 Y_{i-1}^{-1} + \kappa_1 Y_i^{-1} + \nu_1 Y_{i+1}^{-1} + \mu_1 Y_{i+2}^{-1}\big) + (2d_m - d_{m-1})\big(\mu_1 Y_{i-2}^{0} + \nu_1 Y_{i-1}^{0} + \kappa_1 Y_i^{0} + \nu_1 Y_{i+1}^{0} + \mu_1 Y_{i+2}^{0}\big) + \sum_{k=1}^{m-1}(-d_{k-1} + 2d_k - d_{k+1})\big(\mu_1 Y_{i-2}^{m-k} + \nu_1 Y_{i-1}^{m-k} + \kappa_1 Y_i^{m-k} + \nu_1 Y_{i+1}^{m-k} + \mu_1 Y_{i+2}^{m-k}\big) + (2d_0 - d_1)\big(\mu_1 Y_{i-2}^{m} + \nu_1 Y_{i-1}^{m} + \kappa_1 Y_i^{m} + \nu_1 Y_{i+1}^{m} + \mu_1 Y_{i+2}^{m}\big) + \beta\big(\mu_1 f_{i-2}^{m+1} + \nu_1 f_{i-1}^{m+1} + \kappa_1 f_i^{m+1} + \nu_1 f_{i+1}^{m+1} + \mu_1 f_{i+2}^{m+1}\big), \quad 1 \le m \le K-1.$$

The above system yields $(N-3)$ equations involving $(N-1)$ unknowns Y_i^{m+1}, $1 \le i \le N-1$. We extract two more equations from boundary conditions as follows:

$$\left(\varpi_0 - 2\frac{\beta\gamma}{h^4}\right)Y_0^{m+1} + \left(\varpi_1 + 5\frac{\beta\gamma}{h^4}\right)Y_1^{m+1} + \left(\varpi_2 - 4\frac{\beta\gamma}{h^4}\right)Y_2^{m+1} + \left(\varpi_3 + \frac{\beta\gamma}{h^4}\right)Y_3^{m+1} = (2d_0 - d_1)\big(\varpi_0 Y_0^m + \varpi_1 Y_1^m + \varpi_2 Y_2^m + \varpi_3 Y_3^m\big) + \sum_{k=1}^{m-1}(d_k - d_{k+1})\big(\varpi_0 Y_0^{m-k} + \varpi_1 Y_1^{m-k} + \varpi_2 Y_2^{m-k} + \varpi_3 Y_3^{m-k}\big) + (d_m - d_{m-1})(\varpi_0\phi_0 + \varpi_1\phi_1 + \varpi_2\phi_2 + \varpi_3\phi_3) + \Delta d_m(\varpi_0\psi_0 + \varpi_1\psi_1 + \varpi_2\psi_2 + \varpi_3\psi_3) + \beta\big(\varpi_0 f_0^{m+1} + \varpi_1 f_1^{m+1} + \varpi_2 f_2^{m+1} + \varpi_3 f_3^{m+1}\big). \tag{46}$$

Similarly,

$$\left(\varpi_3 + \frac{\beta\gamma}{h^4}\right)Y_{N-3}^{m+1} + \left(\varpi_2 - 4\frac{\beta\gamma}{h^4}\right)Y_{N-2}^{m+1} + \left(\varpi_1 5\frac{\beta\gamma}{h^4}\right)Y_{N-1}^{m+1} + \left(\varpi_0 - 2\frac{\beta\gamma}{h^4}\right)Y_N^{m+1} = (2d_0 - d_1)\big(\varpi_3 Y_{N-3}^m + \varpi_2 Y_{N-2}^m + \varpi_1 Y_{N-1}^m + \varpi_0 Y_N^m\big) + \sum_{k=1}^{m-1}(d_k - d_{k+1})\big(\varpi_3 Y_{N-3}^{m-k} + \varpi_2 Y_{N-2}^{m-k} + \varpi_1 Y_{N-1}^{m-k} + \varpi_0 Y_N^{m-k}\big) + (d_m - d_{m-1})(\varpi_3\phi_{N-3} + \varpi_2\phi_{N-2} + \varpi_1\phi_{N-1} + \varpi_0\phi_N) + \Delta d_m(\varpi_3\psi_{N-3} + \varpi_2\psi_{N-2} + \varpi_1\psi_{N-1} + \varpi_0\psi_N) + \beta\big(\varpi_3 f_{N-3}^{m+1} + \varpi_2 f_{N-2}^{m+1} + \varpi_1 f_{N-1}^{m+1} + \varpi_0 f_N^{m+1}\big). \tag{47}$$

In order to apply this scheme, $Y^2 = [Y_1^2, Y_2^2, Y_3^2, \ldots, Y_{N-1}^2]^T$ and $Y^1 = [Y_1^1, Y_2^1, Y_3^1, \ldots, Y_{N-1}^1]^T$ are required. Solving (25) and utilizing the quintic non-polynomial spline, Y^1 is calculated as follows:

$$\left(\mu_1 + \frac{\beta\gamma}{h^4}\right)Y_{i-2}^1 + \left(\nu_1 - 4\frac{\beta\gamma}{h^4}\right)Y_{i-1}^1 + \left(\kappa_1 + 6\frac{\beta\gamma}{h^4}\right)Y_i^1 + \left(\nu_1 - 4\frac{\beta\gamma}{h^4}\right)Y_{i+1}^1 + \left(\mu_1 + \frac{\beta\gamma}{h^4}\right)Y_{i+2}^1 = J_i, \quad 2 \le i \le N-2, \tag{48}$$

where

$$
\begin{aligned}
J_i &= (\mu_1\phi_{i-2} + \nu_1\phi_{i-1} + \kappa_1\phi_i + \nu_1\phi_{i+1} + \mu_1\phi_{i+2}) + \Delta t\big(\mu_1\psi^1_{i-2} + \nu_1\psi^1_{i-1} + \kappa_1\psi^1_i \\
&\quad + \nu_1\psi^1_{i+1} + \mu_1\psi^1_{i+2}\big) + \beta\big(\mu_1 f^1_{i-2} + \nu_1 f^1_{i-1} + \kappa_1 f^1_i + \nu_1 f^1_{i+1} + \mu_1 f^1_{i+2}\big).
\end{aligned}
$$

System (48) contains $(N-1)$ unknowns Y^1_i, $1 \leq i \leq N-1$, involved in $(N-3)$ equations. We extract two more equations from the end conditions as follows:

$$
\begin{aligned}
&\left(\varpi_0 - 2\frac{\beta\gamma}{h^4}\right)Y^1_0 + \left(\varpi_1 + 5\frac{\beta\gamma}{h^4}\right)Y^1_1 + \left(\varpi_2 - 4\frac{\beta\gamma}{h^4}\right)Y^1_2 + \left(\varpi_3 + \frac{\beta\gamma}{h^4}\right)Y^1_3 \\
&\quad = (\varpi_0\phi_0 + \varpi_1\phi_1 + \varpi_2\phi_2 + \varpi_3\phi_3) + \Delta t(\varpi_0\psi_0 + \varpi_1\psi_1 + \varpi_2\psi_2 + \varpi_3\psi_3) \\
&\qquad + \beta\big(\varpi_0 f^1_0 + \varpi_1 f^1_1 + \varpi_2 f^1_2 + \varpi_3 f^1_3\big)
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
&\left(\varpi_3 + \frac{\beta\gamma}{h^4}\right)Y^1_{N-3} + \left(\varpi_2 - 4\frac{\beta\gamma}{h^4}\right)Y^1_{N-2} + \left(\varpi_1 + 5\frac{\beta\gamma}{h^4}\right)Y^1_{N-1} + \left(\varpi_0 - 2\frac{\beta\gamma}{h^4}\right)Y^1_N \\
&\quad = (\varpi_3\phi_{N-3} + \varpi_2\phi_{N-2} + \varpi_1\phi_{N-1} + \varpi_0\phi_N) + \Delta t(\varpi_3\psi_{N-3} + \varpi_2\psi_{N-2} \\
&\qquad + \varpi_1\psi_{N-1} + \varpi_0\psi_N) + \beta\big(\varpi_3 f^1_{N-3} + \varpi_2 f^1_{N-2} + \varpi_1 f^1_{N-1} + \varpi_0 f^1_N\big).
\end{aligned}
\tag{50}
$$

Suppose $\phi = [\phi_1, \phi_2, \ldots, \phi_{N-1}]^T$, $f = [f_1, f_2, \ldots, f_{N-1}]^T$, $\widetilde{\phi} = [\phi_0, 0, \ldots, 0, \phi_N]^T$, and $\widetilde{f} = [f_0, 0, \ldots, 0, f_N]^T$ are column matrices with $(N-1)$ entries. The system in (48)–(50) can be expressed as

$$
AY^1 = B\big(\phi + \Delta t\psi + \beta f^1\big) + C(\widetilde{\phi} + \Delta t\widetilde{\psi} + \beta\widetilde{f}) - D.
\tag{51}
$$

Here, A,B, and C are square matrices of order $n-1$.

$$
A = \begin{pmatrix}
\varpi_1 + 5\frac{\beta\gamma}{h^4} & \varpi_2 - 4\frac{\beta\gamma}{h^4} & \varpi_3 + \frac{\beta\gamma}{h^4} & 0 & 0 & 0 & \cdots & 0 \\
\nu_1 - 4\frac{\beta\gamma}{h^4} & \kappa_1 + 6\frac{\beta\gamma}{h^4} & \nu_1 - 4\frac{\beta\gamma}{h^4} & \mu_1 + \frac{\beta\gamma}{h^4} & 0 & 0 & \cdots & 0 \\
\mu_1 + \frac{\beta\gamma}{h^4} & \nu_1 - 4\frac{\beta\gamma}{h^4} & \kappa_1 + 6\frac{\beta\gamma}{h^4} & \nu_1 - 4\frac{\beta\gamma}{h^4} & \mu_1 + \frac{\beta\gamma}{h^4} & 0 & \cdots & 0 \\
 & \ddots & & & \ddots & & \ddots & \\
0 & \cdots & 0 & \mu_1 + \frac{\beta\gamma}{h^4} & \nu_1 - 4\frac{\beta\gamma}{h^4} & \kappa_1 + 6\frac{\beta\gamma}{h^4} & 0\nu_1 - 4\frac{\beta\gamma}{h^4} & \mu_1 + \frac{\beta\gamma}{h^4} \\
0 & \cdots & 0 & & \mu_1 + \frac{\beta\gamma}{h^4} & \nu_1 - 4\frac{\beta\gamma}{h^4} & \kappa_1 + 6\frac{\beta\gamma}{h^4} & \nu_1 - 4\frac{\beta\gamma}{h^4} \\
0 & \cdots & 0 & 0 & 0 & \varpi_3 + \frac{\beta\gamma}{h^4} & \varpi_2 - 4\frac{\beta\gamma}{h^4} & \varpi_1 + 5\frac{\beta\gamma}{h^4}
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
\varpi_1 & \varpi_2 & \varpi_3 & 0 & 0 & 0 & \cdots & 0 \\
\nu_1 & \kappa_1 & \nu_1 & \mu_1 & 0 & 0 & \cdots & 0 \\
\mu_1 & \nu_1 & \kappa_1 & \nu_1 & \mu_1 & 0 & \cdots & 0 \\
 & \ddots & & & \ddots & & \ddots & \\
0 & \cdots & 0 & \mu_1 & \nu_1 & \kappa_1 & \nu_1 & \mu_1 \\
0 & \cdots & 0 & 0 & \mu_1 & \nu_1 & \kappa_1 & \nu_1 \\
0 & \cdots & 0 & 0 & 0 & \varpi_3 & \varpi_2 & \varpi_1
\end{pmatrix} \quad \text{and}
$$

$$C=\begin{pmatrix}\varpi_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\ & \ddots & & & \ddots & & \ddots & & \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \varpi_0\end{pmatrix}.$$

$$D=\left[-M_0h^2,0,0,0,\dots,-M_Nh^2\right]^T.$$

6 Numerical results and discussion

In this section, we present the simulation results for two test examples in order to validate the proposed numerical algorithm. All calculations are performed in Mathematica 10.0. The accuracy of the current scheme is verified by the error norms L_∞, L_2 and experimental order of convergence (EOC) [22, 23]:

$$L_\infty=\max|y_j-Y_j|,\qquad L_2=\sqrt{\frac{\sum_{j=0}^N|y_j-Y_j|^2}{\sum_{j=0}^N|y_j|^2}},\qquad EOC=\frac{1}{\log(2)}\log\left[\frac{L_\infty(2N)}{L_\infty(N)}\right],$$

where y_j, Y_j are the exact analytical and quintic non-polynomial spline solutions at *jth* nodal point respectively.

Problem 1 Consider problem (1) with $\gamma=0.05$ [1].

The exact analytical solution of this problem is

$$y(z,t)=2(t+1)\sin^2(z).$$

The maximum and Euclidean error norms corresponding to four different values of Δt are reported in Table 1 using $N=100$ and $\alpha=1.75$. It is clear that the presented scheme approximates the exact analytical solution more precisely as compared to the method used

Table 1 Error norms for Problem 1 at $t=1$ with $N=100$, $\alpha=1.75$

Δt	Method in [1]		Proposed method	
	L_∞	L_2	L_∞	L_2
0.001	1.8221×10^{-3}	1.1556×10^{-4}	1.5535×10^{-6}	2.3182×10^{-7}
0.0005	5.6177×10^{-4}	2.9965×10^{-5}	5.3380×10^{-7}	6.2981×10^{-8}
0.00025	1.5380×10^{-4}	9.9200×10^{-6}	1.9124×10^{-7}	2.1050×10^{-8}
0.000125	3.9312×10^{-5}	1.9910×10^{-6}	5.6215×10^{-8}	6.4940×10^{-9}

Table 2 Error norms for Problem 1, with $N=80$ and $\Delta t=0.001$

α		$t=0.25$	$t=0.5$	$t=1$
1.25	L_2	2.5869×10^{-7}	2.0680×10^{-7}	1.5465×10^{-7}
	L_∞	2.0113×10^{-6}	1.4491×10^{-6}	9.9176×10^{-7}
1.50	L_2	2.4923×10^{-7}	2.2555×10^{-7}	1.9223×10^{-7}
	L_∞	2.0700×10^{-6}	1.6562×10^{-6}	1.2714×10^{-6}
1.75	L_2	2.3392×10^{-7}	2.3887×10^{-7}	2.3182×10^{-7}
	L_∞	2.0669×10^{-6}	1.8185×10^{-6}	1.5535×10^{-6}

Table 3 Experimental order of convergence for Problem 1, when $\alpha = 1.50$, $N = 80$, $\Delta t = 0.001$

N	L_∞	EOC	L_2	EOC
10	9.8950×10^{-2}	–	2.5950×10^{-2}	–
20	6.0451×10^{-3}	4.0329	1.4778×10^{-3}	4.1342
40	3.7842×10^{-4}	3.9977	8.5142×10^{-5}	4.1174
80	2.1927×10^{-5}	4.1092	6.0168×10^{-6}	3.8228

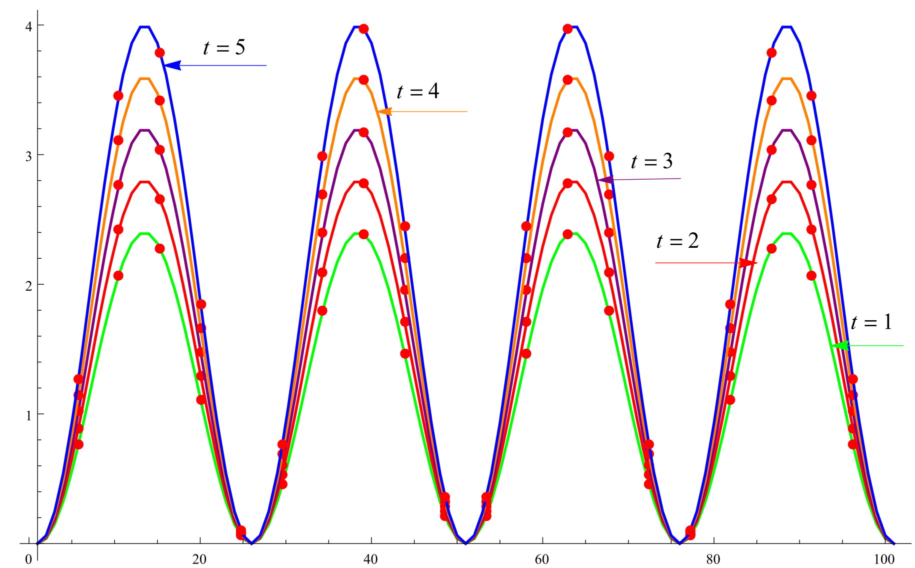


Figure 1 Approximate solution for Problem 1 at different time levels when $\Delta t = 0.001$, $N = 80$, $\alpha = 1.75$

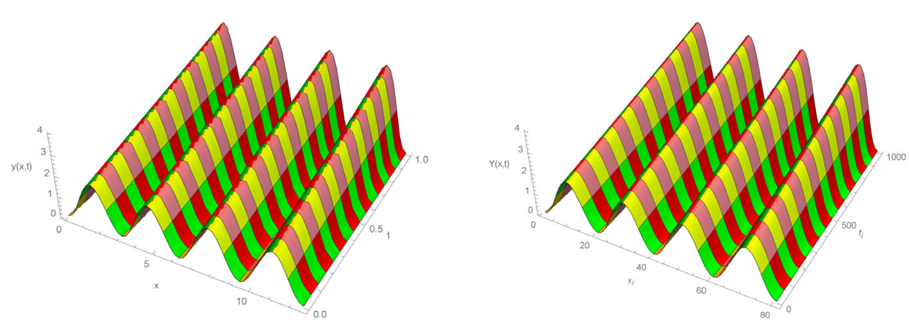


Figure 2 Exact and numerical solution for Problem 1 with $N = 80$, $\Delta t = 0.001$, and $\alpha = 1.50$

in [1]. In Table 2, the error norms L_∞ and L_2 are listed at $t = 0.25, 0.5, 1$ using different values of α. The calculations of slope rate of convergence in spatial direction are presented in Table 3 when error norms are calculated for $\Delta t = 0.001$, $\alpha = 1.50$. The computational rate of convergence of the proposed method in spatial direction is in line with theoretical results even with a larger time step. Figure 1 displays the physical behaviour of numerically approximated solution at various time stages. The three-dimensional visuals of exact analytical and non-polynomial quintic spline solutions are shown in Fig. 2 using $\alpha = 1.50$, $N = 100$, and $\Delta t = 0.001$. From Fig. 2, it is clear that the numerical solution is consistent

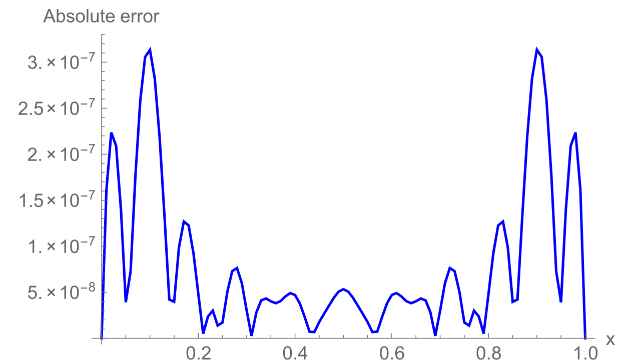


Figure 3 Absolute error for Problem 1 when $N = 100$, $\alpha = 1.75$, and $\Delta t = 0.0005$

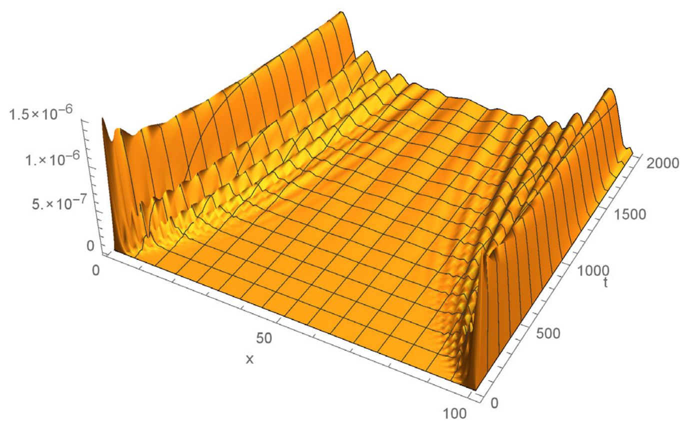


Figure 4 Absolute error for Problem 1 when $N = 100$, $\alpha = 1.5$ and $t = 1$

with the exact solution, which indicates that the proposed method is effective. Figure 3 displays the absolute error at $t = 1$ with $\alpha = 1.75$ and $N = 100$, whereas Fig. 4 represents 3D error plot using $N = 100$, $\alpha = 1.5$, and $\Delta t = 0.001$.

Problem 2 As a second experiment, consider the following fourth order superdiffusion equation [1]:

$$\frac{\partial^\alpha y}{\partial t^\alpha} + \gamma \frac{\partial^4 y}{\partial z^4} = f(z,t), \quad z \in [0,1], t \in [0,T],$$

the initial/end conditions are

$$y(z,0) = \frac{1}{\pi^5}\bigl(\pi^{10}\sin(\pi z) + \cos(\pi z) - \cos(3\pi z)\bigr),$$

$$y(0,t) = y(1,t) = 0,$$

$$y_{zz}(0,t) = \frac{1}{\pi^3}8(1+t), \qquad y_{zz}(1,t) = -\frac{1}{\pi^3}8(1+t).$$

Table 4 Error norms for Problem 2 at $t = 1$ with $N = 80$, $\alpha = 1.50$

Δt	Method in [1]		Proposed method	
	L_∞	L_2	L_∞	L_2
0.001	1.8221×10^{-3}	1.1556×10^{-4}	7.4454×10^{-8}	5.2661×10^{-8}
0.0005	5.6177×10^{-4}	2.9965×10^{-5}	2.1479×10^{-8}	1.6911×10^{-8}
0.00025	1.5380×10^{-4}	9.9200×10^{-6}	5.4476×10^{-9}	5.1349×10^{-9}
0.000125	3.9312×10^{-5}	1.9910×10^{-6}	1.9831×10^{-9}	1.4734×10^{-9}

Table 5 Error norms for Problem 2, when $N = 80$ and $\Delta t = 0.001$

α		$t = 0.25$	$t = 0.50$	$t = 1$
1.25	L_∞	2.8042×10^{-9}	5.0116×10^{-9}	7.8473×10^{-9}
	L_2	1.8445×10^{-9}	3.4181×10^{-9}	5.3792×10^{-9}
1.50	L_∞	2.4671×10^{-8}	3.8931×10^{-8}	7.4454×10^{-8}
	L_2	1.4233×10^{-8}	2.7534×10^{-8}	5.2661×10^{-8}
1.75	L_∞	2.1815×10^{-8}	6.7973×10^{-8}	1.7391×10^{-8}
	L_2	1.4870×10^{-8}	4.7793×10^{-8}	1.2290×10^{-7}

Table 6 Experimental order of convergence for Problem 2, when $N = 80$, $\alpha = 1.50$, $\Delta t = 0.001$

N	L_∞	EOC	L_2	EOC
10	4.2119×10^{-4}	–	1.6769×10^{-4}	–
20	2.2034×10^{-5}	4.2566	9.9991×10^{-6}	4.0678
40	1.8694×10^{-6}	4.1846	6.9693×10^{-7}	3.8452
80	7.4454×10^{-8}	4.0245	5.2661×10^{-8}	3.8605

The exact analytical solution to this problem is

$$y(z,t) = \frac{1}{\pi^5}\bigl(\pi^{10}\sin(\pi z) + \cos(\pi z) - \cos(3\pi z)\bigr)(t+1).$$

The computational error norms corresponding to four different selections of Δt are given in Table 4 with $\gamma = 0.05$ and $N = 80$. It can be seen that our presented computational strategy yields more accurate solutions as compared to the technique used in [1]. Table 5 shows the maximum and Euclidian error norms at different time levels. The experimental rate of convergence is tabulated in Table 6 when error norms are calculated for $N = 80$, $\alpha = 1.50$, and $\Delta t = 0.001$. It is clear that the experimental results support the theoretical estimation. Figure 5 displays the approximate solution at $t = 1, 2, 3, 4, 5$. The three-dimensional plots of analytical exact and numerical solutions are portrayed in Fig. 6 in order to showcase their physical behaviour. In Fig. 7, the absolute computational error is presented at $t = 1$ using $\alpha = 1.50$ and $N = 80$.

7 Conclusion

1. An algorithm utilizing quintic non-polynomial spline functions has been developed for the numerical treatment of time fractional fourth order superdiffusion equation.
2. The discretization in space and time directions has been achieved by using quintic non-polynomial spline functions and finite central difference formulation respectively.
3. The unconditional stability of the proposed scheme in temporal direction has been proved.
4. Theoretically, the presented technique is proved to be $O(h^4 + \Delta t^2)$ accurate.

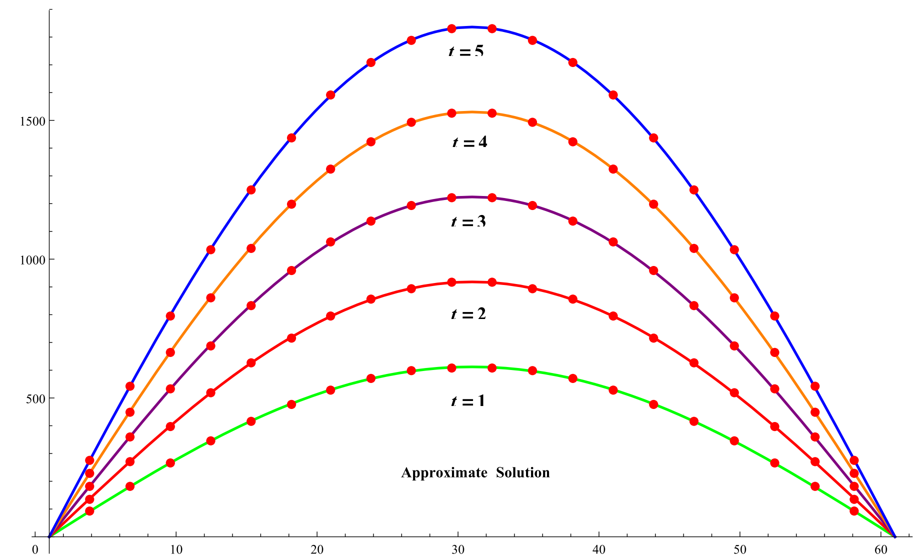
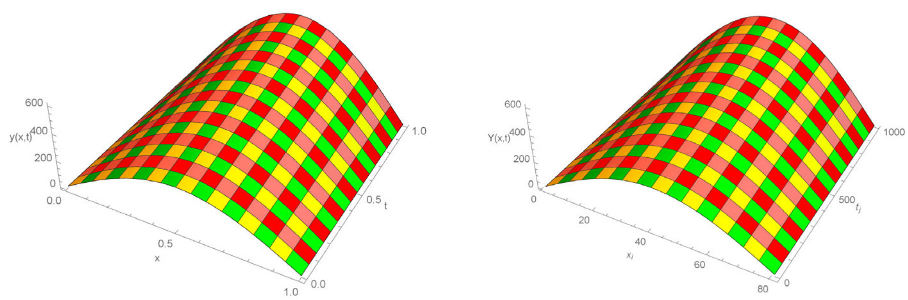


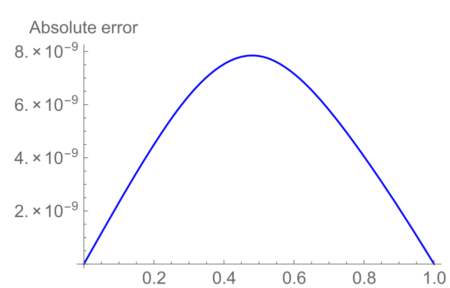
Figure 5 Approximate solution for Problem 2 at different time levels when $\Delta t = 0.001$, $N = 60$, $\alpha = 1.25$



(a) Exact (b) Approximate

Figure 6 Exact and numerical solution for Problem 2 when $N = 80$, $\alpha = 1.25$, $\Delta t = 0.001$

Figure 7 Absolute error for Problem 2 when $N = 80$, $\Delta t = 0.001$ and $\alpha = 1.25$



5. The experimental order of convergence is found to conform with the theoretical expectations.
6. The comparison of maximum and Euclidian error norms indicates the superiority of the present scheme over the method used in [1] even with larger grid spacing in time direction.

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