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# An enhanced stability criterion for linear time-delayed systems via new Lyapunov–Krasovskii functionals

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**Abstract**

The stability problem of linear systems with time-varying delays is studied by improving a Lyapunov–Krasovskii functional (LKF). Based on the newly developed LKF, a less conservative stability criterion than some previous ones is derived. Firstly, to avoid introducing the terms with $h^2(t)$, which are not convenient to directly use the convexity of linear matrix inequality (LMI), the type of integral terms $\{\int_s^t \dot{x}(u)\,du, \int_{t-h}^s \dot{x}(u)\,du\}$ is used in the LKF instead of $\{\int_s^t x(u)\,du, \int_{t-h}^s x(u)\,du\}$. Secondly, two couples of integral terms $\{\int_s^t \dot{x}(u)\,du, \int_{t-h(t)}^s \dot{x}(u)\,du\}$, and $\{\int_s^{t-h(t)} \dot{x}(u)\,du, \int_{t-h}^s \dot{x}(u)\,du\}$ are supplemented in the integral functionals $\int_{t-h(t)}^t \dot{x}(u)\,du$ and $\int_{t-h}^{t-h(t)} \dot{x}(u)\,du$, respectively, so that the time delay, its derivative, and information between them can be fully utilized. Thirdly, the LKF is further augmented by two delay-dependent non-integral items. Finally, three numerical examples are presented under two different delay sets, to show the effectiveness of the proposed approach.

**Keywords:** Delay-dependent stability; Lyapunov–Krasovskii functional; Linear matrix inequalities; Time-delayed system; Time-varying delay

## 1 Introduction

Time delays are of frequent occurrence in many practical systems, which often results in the major source of poor performance and instability. The stability problems of time-delayed systems have been a hot research topic. In this paper, the stability problems of time-delayed linear systems will be further analyzed via the LKF method application. The time-delayed linear system is described as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)), \\ x(s) = \psi(s), \quad s \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system. $A$ and $A_d$ are real constant matrices with appropriate dimensions. $h(t)$ is the time-varying delay, which is a continuous and differentiable functional satisfying the following constraint condition:

$$0 \le h(t) \le h, \qquad \left|\dot{h}(t)\right| \le \mu < 1, \quad \forall t \ge 0, \tag{2}$$

where $h$ and $\mu$ are positive constants.

To analyze the stability problem of system (1) based on the Lyapunov theorem [1–5], the main efforts are concentrated on the following several directions: one is finding an appropriate LKF, for example, LKF with delay partitioning approach [6–9], LKF with augmented terms [10–12], LKF with triple-integral and quadruple-integral terms [13, 14], and so on. The other is reducing the upper bounds of the time derivative of LKF as much as possible by developing various inequality techniques, such as Jensen inequality [15], Wirtinger-based inequality [16], auxiliary function based inequality [17], Bessel–Legendre inequality [18], and so on. Besides, further increasing the freedom of solving LMI, additional free-weighting-matrix technique is frequently introduced into the derivatives of LKF, for instance, the generalized zero equality [19], the one or second-order reciprocally convex combination [20–22], the free-weighting-matrix approach [23], and so on. Inspired by the research of [16–18, 24, 25], the tighter inequality technique seems to lead to less conservative stability criteria. Recently, Zhang [26] considered the effect of the LKFs while discussing the relationship between the inequality technique and the conservatism of results. The results illustrate that the integral inequality that makes the upper bound closer to the true value does not always reduce the conservatism of the corresponding stability results unless an appropriate LKF is constructed. Thus, it is very important to construct a proper LKF. Recently, a novel LKF with the single integral items deliberately augmented by adding state derivative-related integral terms was proposed by Lee et al. [27] as follows:

$$\begin{aligned} V(t) &= \eta_1(t)^T P \eta_1(t) + \int_{t-h(t)}^{t} \eta_2(t,s)^T Q \eta_2(t,s)\,ds \\ &\quad + \int_{t-h}^{t-h(t)} \eta_2(t,s)^T S \eta_2(t,s)\,ds + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s)\,ds\,d\theta, \end{aligned} \tag{3}$$

where $P > 0$, $Q > 0$, $S > 0$, and $R > 0$ are the corresponding Lyapunov matrices which need to be determined and

$$\begin{aligned} \eta_1^T(t) = \Big[ & x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h) \quad \textstyle\int_{t-h(t)}^{t} x(s)\,ds \quad \int_{t-h}^{t-h(t)} x(s)\,ds \\ & \tfrac{1}{h(t)} \textstyle\int_{-h(t)}^{0}\int_{t+\theta}^{t} x(s)\,ds\,d\theta \quad \tfrac{1}{h-h(t)} \int_{-h}^{-h(t)}\int_{t+\theta}^{t-h(t)} x(s)\,ds\,d\theta \Big], \\ \eta_2^T(t,s) = \Big[ & x^T(s) \quad \dot{x}^T(s) \quad \textstyle\int_{s}^{t} \dot{x}^T(u)\,du \quad \int_{t-h}^{s} \dot{x}^T(u)\,du \Big]. \end{aligned}$$

Compared with the LKFs proposed in [28], $\eta_1(t)$ included the state-related vectors, that is, single- and double-integral terms, to coordinate with the application of the second-order affine Bessel–Legendre inequality; and to avoid introducing the terms with $h^2(t)$, the integral items $\int_s^t \dot{x}(u)\,du$ and $\int_{t-h}^s \dot{x}(u)\,du$ were added into the integrand vector $\eta_2(t,s)$ instead of $\int_s^t x(u)\,du$ and $\int_{t-h}^s x(u)\,du$, which can be solved conveniently by Matlab LMI-Tool box. A less conservative stability condition than some previous ones was obtained in [27] via LKF (3). However, if the integral terms $\int_s^t \dot{x}(u)\,du$ and $\int_{t-h}^s \dot{x}(u)\,du$ are divided into

$\{\int_s^t \dot{x}(u)\,du, \int_{t-h(t)}^s \dot{x}(u)\,du\}$, and $\{\int_s^{t-h(t)} \dot{x}(u)\,du, \int_{t-h}^s \dot{x}(u)\,du\}$, respectively, rather than considering about them directly, the relationships of part $\eta_2(t,s)$-related states can be tightly characterized by correlative Lyapunov matrices. So, there is still room for improvement in the conservatism of stability conditions by constructing a new LKF based on the above analysis.

This paper contributes to the stability problem of time-delayed linear systems via a new LKF application. Firstly, the type of integral terms $\{\int_s^t \dot{x}(u)\,du, \int_{t-h}^s \dot{x}(u)\,du\}$ is chosen in the LKF instead of $\{\int_s^t x(u)\,du, \int_{t-h}^s x(u)\,du\}$. Secondly, two couples of integral terms $\{\int_s^t \dot{x}(u)\,du, \int_{t-h(t)}^s \dot{x}(u)\,du\}$, and $\{\int_s^{t-h(t)} \dot{x}(u)\,du, \int_{t-h}^s \dot{x}(u)\,du\}$ are involved in $\eta_2(t,s)$, respectively, such that the time delay, its derivative and some single integral-related states information between them can be tightly characterized by correlative Lyapunov matrices. Finally, a less conservative stability criterion than some existing ones is given based on the new LKF.

For intuitive and simple understanding, the main contributions are summed up as follows:

- Two integral terms $\{\int_s^t \dot{x}(u)\,du, \int_{t-h(t)}^s \dot{x}(u)\,du\}$, and $\{\int_s^{t-h(t)} \dot{x}(u)\,du, \int_{t-h}^s \dot{x}(u)\,du\}$ are supplemented in the integrand vectors $\eta_2(t,s)$, respectively, which can be seen as a complement to the terms $\{\int_{t-h(t)}^t \dot{x}(u)\,du$ and $\int_{t-h}^{t-h(t)} \dot{x}(u)\,du\}$. The time delay, its derivative, and some single integral-related states information between them can be fully utilized through the Lyapunov matrices $Q_1$ and $Q_2$.
- The type of integral terms $\{\int_s^t \dot{x}(u)\,du, \int_{t-h}^s \dot{x}(u)\,du\}$ is chosen in the integrand vectors $\eta_2(t,s)$ of the LKF instead of $\{\int_s^t x(u)\,du, \int_{t-h}^s x(u)\,du\}$, which can be solved conveniently by using the convexity of LMI without introducing any additional inequalities.
- The affine Bessel–Legendre inequality proposed in [27] is used to bound the derivative of the LKF instead of Bessel–Legendre inequality proposed in [18], because the former is the affine version of the length of the integral interval not the reciprocal of the integral interval, that is, $b-a$ is linear in affine Bessel–Legendre inequality, which can be easily solved by the convex property.

**Notation** $P>0$ $(<0)$ means that matrix $P$ is a positive (negative) definite matrix. $I_n$ and $0_n$ represent an $n$-dimensional unit matrix and an $n$-dimensional zero matrix. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix, and $e_i$ $(i=1,\ldots,m)$ are block entry matrices with $e_3^T = [0\,0\,I\,\underbrace{0\cdots0}_{m-3}]$, where $m$ is the length of the vector $\xi(t)$ in theorems and corollaries. $*$ denotes the symmetric terms in a block matrix. $F[h(t),d(t)]$, $G[x(t)]$ denote $F$, $G$ are the functions of $h(t)$, $d(t)$, and $x(t)$, respectively. $\mathrm{Sym}\{B\} = B + B^T$.

## 2 Problem formulation

This paper mainly derives a new stability criterion for the time-delayed linear system (1) satisfying condition (2) via a modified LKF application. To achieve this purpose, the following lemma is very important.

**Lemma 1** (Affine Bessel–Legendre inequality [27]) *For given matrices $R>0$, $X$ and a continuous and differentiable function $\{x(s) \mid s\in[a,b]\}$, the following integral inequality holds* *for all integer* $N \in \mathbb{N}$:

$$\int_a^b \dot{x}^T(s)R\dot{x}(s)\,ds \geq \xi_N^T\bigl[XH_N + H_N^TX^T - (b-a)X\tilde{R}^{-1}X^T\bigr]\xi_N, \tag{4}$$

*where*

$$\xi_N = \begin{cases} [x^T(b)\ x^T(a)]^T, & \text{if } N = 0,\\ [x^T(b)\ x^T(a)\ \frac{1}{b-a}\Omega_0^T\ \cdots\ \frac{1}{b-a}\Omega_{N-1}^T]^T, & \text{if } N > 0,\end{cases}$$

$$\Gamma_N(k) = \begin{cases} [I_n\ -I_n], & \text{if } N = 0,\\ [I_n\ (-1)^{k+1}I_n\ \gamma_{Nk}^0 I_n\ \cdots\ \gamma_{Nk}^{N-1}I_n], & \text{if } N > 0,\end{cases}$$

$$\Omega_k = \int_a^b L_k(s)x(s)\,ds, \qquad \gamma_{Nk}^i = \begin{cases} -(2i+1)(1-(-1)^{k+i}), & \text{if } i \leq k,\\ 0, & \text{if } i \geq k+1,\end{cases}$$

$$L_k(s) = (-1)^k\sum_{l=0}^{k}\left[(-1)^l\binom{k}{l}\binom{k+l}{l}\right]\left(\frac{s-a}{b-a}\right)^l,$$

$$H_N = \bigl[\Gamma_N^T(0)\quad \Gamma_N^T(1)\quad \cdots\quad \Gamma_N^T(N)\bigr]^T, \qquad \tilde{R} = \operatorname{diag}\bigl\{R, 3R, \ldots, (2N+1)R\bigr\}.$$

## 3 Main results

### 3.1 A modified LKF

For simplicity of presentation, we define the following notations:

$$h_d = 1 - \dot{h}(t), \qquad h_{12} = h - h(t), \qquad v_1(t) = \int_{t-h(t)}^{t}\frac{x(s)}{h(t)}\,ds, \qquad v_2(t) = \int_{t-h}^{t-h(t)}\frac{x(s)}{h_{12}}\,ds,$$

$$u_1(t) = \int_{-h(t)}^{0}\int_{t+\theta}^{t}\frac{x(s)}{h^2(t)}\,ds\,d\theta, \qquad u_2(t) = \int_{-h}^{-h(t)}\int_{t+\theta}^{t-h(t)}\frac{x(s)}{h_{12}^2}\,ds\,d\theta,$$

$$\zeta^T(t) = \bigl[x^T(t)\quad x^T(t-h(t))\quad x^T(t-h)\quad h(t)v_1^T(t)\quad h_{12}v_2^T(t)\quad h(t)u_1^T(t)\quad h_{12}u_2^T(t)\bigr],$$

$$\zeta_a^T(t) = \bigl[x^T(t)\quad x^T(t-h(t))\quad x^T(t-h)\quad v_1^T(t)\quad u_1^T(t)\bigr],$$

$$\zeta_b^T(t) = \bigl[x^T(t)\quad x^T(t-h(t))\quad x^T(t-h)\quad v_2^T(t)\quad u_2^T(t)\bigr],$$

$$\eta_1^T(t,s) = \bigl[x^T(s)\quad \dot{x}^T(s)\quad \textstyle\int_s^t \dot{x}^T(u)\,du\quad \int_{t-h(t)}^s \dot{x}^T(u)\,du\bigr],$$

$$\eta_2^T(t,s) = \bigl[x^T(s)\quad \dot{x}^T(s)\quad \textstyle\int_s^{t-h(t)} \dot{x}^T(u)\,du\quad \int_{t-h}^s \dot{x}^T(u)\,du\bigr],$$

$$\xi^T(t) = \bigl[x^T(t)\quad x^T(t-h(t))\quad x^T(t-h)\quad \dot{x}^T(t)\quad \dot{x}^T(t-h(t))\quad \dot{x}^T(t-h)\quad v_1^T(t)\quad v_2^T(t)\quad u_1^T(t)\quad u_2^T(t)\bigr].$$

Now, we construct the following new LKF:

$$\begin{aligned} V(t) &= \zeta(t)^TP\zeta(t) + h(t)\zeta(t)_a^TP_a\zeta_a(t) + h_{12}\zeta_b(t)^TP_b\zeta_b(t) + \int_{t-h(t)}^{t}\eta_1(t,s)^TQ_1\eta_1(t,s)\,ds\\ &\quad + \int_{t-h}^{t-h(t)}\eta_2(t,s)^TQ_2\eta_2(t,s)\,ds + \int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds\,d\theta, \end{aligned} \tag{5}$$

where $P > 0$, $P_a > 0$, $P_b > 0$, $Q_1 > 0$, $Q_2 > 0$, and $R > 0$ are the corresponding Lyapunov matrices which need to be determined.

*Remark* 1 Compared with LKFs (3) in this paper and (7) in [29], two delay-dependent terms $h(t)\zeta(t)_a^T P_a \zeta_a(t)$, $h_{12}\zeta_b(t)^T P_b \zeta_b(t)$ are added into the non-integral items and two different vectors $\eta_1(t,s)$ and $\eta_2(t,s)$ in the new LKF (5) are further augmented by supplementing $\int_{t-h(t)}^{s} \dot{x}(u)\,du$ and $\int_{s}^{t-h(t)} \dot{x}(u)\,du$ corresponding to different integral intervals, respectively, so that two different integral intervals $[t-h(t),t]$ and $[t-h,t-h(t)]$ correspond to two different integral vectors $\eta_1(t,s)$ and $\eta_2(t,s)$, not just the same integral vector $[x^T(s)\ \dot{x}^T(s)\ \int_s^t \dot{x}^T(u)\,du\ \int_{t-h}^{s} \dot{x}^T(u)\,du]$. Moreover, for the domain of integration, the whole integral domain of $\int_{t-h(t)}^{t} \dot{x}(s)\,ds$ and $\int_{t-h}^{t-h(t)} \dot{x}(s)\,ds$ is just the sum of the two integral domains of the integral items $\int_{t-h(t)}^{s} \dot{x}(u)\,du$ and $\int_s^t \dot{x}(u)\,du$ and the sum of the two integral domains of the integral items $\int_{t-h}^{s} \dot{x}(u)\,du$ and $\int_{s}^{t-h(t)} \dot{x}(u)\,du$, respectively. Thus, the two supplementary integral items can be seen as the complements to the terms $\int_s^t \dot{x}(u)\,du$ and $\int_{t-h}^{s} \dot{x}(u)\,du$, where the time delay, its derivative, and some single integral-related states coupling information between them can be tightly characterized through the positive-definite matrices $Q_1$ and $Q_2$. Three numerical examples in Sect. 4 will show that it is very helpful for the two complementary terms to reduce the conservatism of the stability criterion.

*Remark* 2 Recently, LKFs with $\int_a^b x(u)\,du$ in $\eta_1(t,s)$ and $\eta_2(t,s)$ instead of $\int_a^b \dot{x}(u)\,du$ were constructed in [11, 28], which was aimed at coordinating with the second-order B-L integral inequality with the integral items $u_1(t)$ and $u_2(t)$. However, they had to introduce the terms with $h^2(t)$ when bounding the derivative of the LKFs, which led to adding some additional inequality constraints into the main results. Indeed, the coupling relationship between the two integral items $u_1(t)$ and $u_2(t)$ already included in the derivative of $\zeta(t)^T P \zeta(t)$. Thus, to avoid introducing the terms with $h^2(t)$, the type of $\int_a^b \dot{x}(u)\,du$ integral items is chosen in $\eta_1(t,s)$ and $\eta_2(t,s)$, respectively, which can be solved conveniently by using the convexity of LMI without introducing any additional inequality constraints.

*Remark* 3 The time delay concerned in this paper is time varying, differentiable and its change rate should be smaller than 1. Indeed, the delay may be undifferentiable for some practical systems [30, 31]. Thus, our methods can be extended to the case of undifferentiable delay by reconstructing some augmented terms in our LKF. For example, $\zeta^T(t) = [x^T(t)\ x^T(t-h)\ \int_{t-h}^{t} x^T(s)\,ds\ \int_{-h}^{0}\int_{t+\theta}^{t} x^T(s)\,ds\,d\theta]$, $\int_{t-h}^{t} \eta_1^T(t,s) Q \eta_1(t,s)$, and so on.

### 3.2 Stability conditions

**Theorem 1** *For given values of* $h \geq 0$, $\mu < 1$, *if there exist real positive definite matrices* $P \in \mathbb{R}^{7n\times 7n}$, $(P_a, P_b \in \mathbb{R}^{5n\times 5n})$, $(Q_i \in \mathbb{R}^{4n\times 4n})$, $(R \in \mathbb{R}^{n\times n})$ *and any matrices* $(\bar{U} \in \mathbb{R}^{3n\times n})$, $X_i \in \mathbb{R}^{4n\times 3n}$ $(i = 1,2)$ *such that the following LMIs hold for* $d \in \{-\mu, \mu\}$, *then system* (1) *is stable under constraint conditions* (2):

$$\begin{bmatrix} \Pi[0,d] & hE_2X_2 \\ * & -h\tilde{R} \end{bmatrix} < 0, \tag{6}$$

$$\begin{bmatrix} \Pi[h,d] & hE_1X_1 \\ * & -h\tilde{R} \end{bmatrix} < 0, \tag{7}$$

*where*

$$\begin{aligned}
\Pi\big[h(t),\dot{h}(t)\big] &= \mathrm{Sym}\big\{\Pi_1\big[h(t),\dot{h}(t)\big]\big\} + \Pi_2\big[\dot{h}(t)\big] + \Pi_3,\\
\Pi_1\big[h(t),\dot{h}(t)\big] &= \Delta_1 P_a \Omega_1^T + \Delta_2 P_b \Omega_2^T + G_0\big[h(t)\big] P G_1^T\big[\dot{h}(t)\big]\\
&\quad + \Pi_1 \bar{U} \Pi_2^T + G_2\big[h(t)\big] Q_1 G_3^T\big[\dot{h}(t)\big] + G_4\big[h(t)\big] Q_2 G_5^T\big[\dot{h}(t)\big],\\
\Delta_1 &= \begin{bmatrix} e_1 & e_2 & e_3 & e_7 & e_9 \end{bmatrix}, \qquad \Delta_2 = \begin{bmatrix} e_1 & e_2 & e_3 & e_8 & e_{10} \end{bmatrix},\\
\Omega_1 &= \begin{bmatrix} h(t)e_4 & h(t)h_d e_5 & h(t)e_6 & e_1 - h_d e_2 - \dot{h}(t)e_7 & e_1 - h_d e_7 - 2\dot{h}(t)e_9 \end{bmatrix},\\
\Omega_2 &= \begin{bmatrix} h_{12}e_4 & h_{12}h_d e_5 & h_{12}e_6 & h_d e_2 - e_3 + \dot{h}(t)e_8 & h_d e_2 - e_8 + 2\dot{h}(t)e_{10} \end{bmatrix},\\
G_0\big[h(t)\big] &= \begin{bmatrix} e_1 & e_2 & e_3 & h(t)e_7 & h_{12}e_8 & h(t)e_9 & h_{12}e_{10} \end{bmatrix},\\
G_1\big[\dot{h}(t)\big] &= \Big[ e_4 \quad h_d e_5 \quad e_6 \quad e_1 - h_d e_2 \quad h_d e_2 - e_3 \quad e_1 - h_d e_7 - \dot{h}(t)e_9\\
&\qquad h_d e_2 - e_8 + \dot{h}(t)e_{10} \Big],\\
G_2\big[h(t)\big] &= \begin{bmatrix} h(t)e_7 & e_1 - e_2 & h(t)(e_1 - e_7) & h(t)(e_7 - e_2) \end{bmatrix},\\
G_3\big[\dot{h}(t)\big] &= \begin{bmatrix} 0 & 0 & e_4 & -h_d e_5 \end{bmatrix},\\
G_4\big[h(t)\big] &= \begin{bmatrix} h_{12}e_8 & e_2 - e_3 & h_{12}(e_2 - e_8) & h_{12}(e_8 - e_3) \end{bmatrix},\\
G_5\big[\dot{h}(t)\big] &= \begin{bmatrix} 0 & 0 & h_d e_5 & -e_6 \end{bmatrix},\\
\Pi_2\big[\dot{h}(t)\big] &= \dot{h}(t)\Delta_1 P_a \Delta_1^T - \dot{h}(t)\Delta_2 P_b \Delta_2^T\\
&\quad + \begin{bmatrix} e_1 & e_4 & 0 & e_1 - e_2 \end{bmatrix} Q_1 \begin{bmatrix} e_1 & e_4 & 0 & e_1 - e_2 \end{bmatrix}^T\\
&\quad - h_d \begin{bmatrix} e_2 & e_5 & e_1 - e_2 & 0 \end{bmatrix} Q_1 \begin{bmatrix} e_2 & e_5 & e_1 - e_2 & 0 \end{bmatrix}^T\\
&\quad + h_d \begin{bmatrix} e_2 & e_5 & 0 & e_2 - e_3 \end{bmatrix} Q_2 \begin{bmatrix} e_2 & e_5 & 0 & e_2 - e_3 \end{bmatrix}^T\\
&\quad - \begin{bmatrix} e_3 & e_6 & e_2 - e_3 & 0 \end{bmatrix} Q_2 \begin{bmatrix} e_3 & e_6 & e_2 - e_3 & 0 \end{bmatrix}^T,\\
\Pi_3 &= h e_4 R e_4^T - E_1\big[X_1 H + H^T X_1^T\big]E_1^T - E_2\big[X_2 H + H^T X_2^T\big]E_2^T,\\
E_1 &= \begin{bmatrix} e_1 & e_2 & e_7 & e_9 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} e_2 & e_3 & e_8 & e_{10} \end{bmatrix},\\
\tilde{R} &= \mathrm{diag}\{R, 3R, 5R\}, \qquad \Pi_1 = \begin{bmatrix} e_1 & e_2 & e_4 \end{bmatrix}, \qquad \Pi_2 = e_1 A^T + e_2 A_d^T - e_4,\\
H &= \begin{bmatrix} I_n & -I_n & 0_n & 0_n \\ I_n & I_n & -2I_n & 0_n \\ I_n & -I_n & 6I_n & -12I_n \end{bmatrix}.
\end{aligned}$$

*Proof* Construct *LKF* (5). The time derivative of $V(t)$ with respect to time is as follows:

$$\begin{aligned}
\dot{V}(t) &= 2\zeta^T(t)P\dot{\zeta}(t) + 2h(t)\zeta_a^T(t)P_a\dot{\zeta}_a(t) + 2h_{12}\zeta_b^T(t)P_b\dot{\zeta}_b(t)\\
&\quad + \dot{h}(t)\zeta_a^T(t)P_a\zeta_a(t) - \dot{h}(t)\zeta_b^T(t)P_b\zeta_b(t)
\end{aligned}$$

$$\begin{aligned}&+\eta_1^T(t,t)Q_1\eta_1(t,t)-h_d\eta_1^T\big(t,t-h(t)\big)Q_1\eta_1\big(t,t-h(t)\big)\\&+h_d\eta_2^T\big(t,t-h(t)\big)Q_2\eta_1\big(t,t-h(t)\big)-\eta_2^T(t,t-h)Q_2\eta_2(t,t-h)\\&+2\int_{t-h(t)}^{t}\eta_1^T(t,s)Q_1\frac{\partial(\eta_1(t,s))}{\partial t}\,ds+2\int_{t-h}^{t-h(t)}\eta_2^T(t,s)Q_2\frac{\partial(\eta_2(t,s))}{\partial t}\,ds\\&+h\dot{x}^T(t)R\dot{x}(t)-\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds.\end{aligned}$$

We can get the following facts:

$$\zeta^T(t)=\xi^T(t)\Big[e_1\quad e_2\quad e_3\quad h(t)e_7\quad h_{12}e_8\quad h(t)e_9\quad h_{12}e_{10}\Big], \tag{8}$$

$$\begin{aligned}\dot{\zeta}(t)=\Big[&e_4\quad h_de_5\quad e_6\quad e_1-h_de_2\quad h_de_2-e_3\quad e_1-h_de_7-\dot{h}(t)e_9\\&h_de_2-e_8+\dot{h}(t)e_{10}\Big]^T\xi(t),\end{aligned} \tag{9}$$

$$\begin{aligned}&\zeta_a^T(t)=\xi^T(t)\Big[e_1\quad e_2\quad e_3\quad e_7\quad e_9\Big],\\&\zeta_b^T(t)=\xi^T(t)\Big[e_1\quad e_2\quad e_3\quad e_8\quad e_{10}\Big],\end{aligned} \tag{10}$$

$$h(t)\dot{\zeta}_a(t)=\Big[h(t)e_4\quad h(t)h_de_5\quad h(t)e_6\quad e_1-h_de_2-\dot{h}(t)e_7\quad e_1-h_de_7-2\dot{h}(t)e_9\Big]^T\xi(t), \tag{11}$$

$$h_{12}\dot{\zeta}_a(t)=\Big[h_{12}e_4\quad h_{12}h_de_5\quad h_{12}e_6\quad h_de_2-e_3+\dot{h}(t)e_8\quad h_de_2-e_8+2\dot{h}(t)e_{10}\Big]^T\xi(t), \tag{12}$$

$$\begin{aligned}&\eta_1^T(t,t)=\xi^T(t)\Big[e_1\quad e_4\quad 0\quad e_1-e_2\Big],\\&\eta_1^T\big(t,t-h(t)\big)=\xi^T(t)\Big[e_2\quad e_5\quad e_1-e_2\quad 0\Big],\end{aligned} \tag{13}$$

$$\begin{aligned}&\eta_2^T\big(t,t-h(t)\big)=\xi^T(t)\Big[e_2\quad e_5\quad 0\quad e_2-e_3\Big],\\&\eta_2^T(t,t-h)=\xi^T(t)\Big[e_3\quad e_6\quad e_2-e_3\quad 0\Big],\end{aligned} \tag{14}$$

$$\int_{t-h(t)}^{t}\eta_1^T(t,s)\,ds=\xi^T(t)\Big[h(t)e_7\quad e_1-e_2\quad h(t)(e_1-e_7)\quad h(t)(e_7-e_2)\Big], \tag{15}$$

$$\int_{t-h}^{t-h(t)}\eta_2^T(t,s)\,ds=\xi^T(t)\Big[h_{12}e_8\quad e_2-e_3\quad h_{12}(e_2-e_8)\quad h_{12}(e_8-e_3)\Big], \tag{16}$$

$$\frac{\partial(\eta_1(t,s))}{\partial t}=\Big[0\quad 0\quad e_4\quad -h_de_5\Big]^T\xi(t), \tag{17}$$

$$\frac{\partial(\eta_2(t,s))}{\partial t}=\Big[0\quad 0\quad h_de_5\quad -e_6\Big]^T\xi(t). \tag{18}$$

For any appropriately dimensioned matrices $\bar{U}^T=[U_1^T\ U_2^T\ U_3^T]$, it is true that

$$0=2\xi^T(t)\Big[e_1\quad e_2\quad e_4\Big]\bar{U}\big[\bar{A}x(t)+\bar{A}_dx\big(t-h(t)\big)-\dot{x}(t)\big]. \tag{19}$$

It follows from Lemma 1 with $N = 2$ that

$$-\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds = -\int_{t-h(t)}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s)R\dot{x}(s)\,ds$$
$$\leq -\xi^T(t)E_1\big[X_1H + H^TX_1^T - h(t)X_1\tilde{R}^{-1}X_1^T\big]E_1^T\xi(t)$$
$$- \xi^T(t)E_2\big[X_2H + H^TX_2^T - h_{12}X_2\tilde{R}^{-1}X_2^T\big]E_2^T\xi(t).$$

Finally, from the above derivation, we have

$$\dot{V}(t) \leq \xi^T(t)\big\{\Pi\big[h(t),\dot{h}(t)\big] + h(t)E_1X_1\tilde{R}^{-1}X_1^TE_1^T + h_{12}E_2X_2^T\tilde{R}^{-1}X_2E_2^T\big\}\xi(t). \tag{20}$$

Then one can see that $\Pi[h(t),\dot{h}(t)] + h(t)E_1X_1\tilde{R}^{-1}X_1^TE_1^T + h_{12}E_2X_2^T\tilde{R}^{-1}X_2E_2^T$ is linear on the two time delay variables $h(t)$ and $\dot{h}(t)$. Thus, inequalities (6)–(7) hold for $h(t) \in [0,h]$, $\dot{h}(t) \in [-\mu,\mu]$, which implies that $\dot{V}(t) < 0$ by the transformation of *Schur complement equivalence*. This shows that system (1) is stable from Lyapunov stability theory, which completes the proof. □

*Remark* 4 Indeed, Theorem 1 can be generalized to an $N$-dependent stability criterion based the $N$-dependent affine Bessel–Legendre inequality. For the sake of simplicity, $N = 2$ is chosen in this paper. Therefore, in the case of $N > 2$, an appropriate LKF can be obtained by adding the following form of state vectors in $\zeta^T(t)$, $\zeta_a^T(t)$, and $\zeta_b^T(t)$: $\nu_N(t) = \int_{-h(t)}^{0}\int_{t+\theta_1}^{t}\int_{t+\theta_2}^{t}\cdots\int_{t+\theta_{N-1}}^{t}\frac{x(s)}{h^{N-1}(t)}\,ds\,d\theta_{N-1}\,d\theta_{N-2}\cdots d\theta_1$ and $\bar{\nu}_N(t) = \int_{-h}^{-h(t)}\int_{t+\theta_1}^{t-h(t)}\int_{t+\theta_2}^{t-h(t)}\cdots\int_{t+\theta_{N-1}}^{t-h(t)}\frac{x(s)}{h_{12}^{N-1}(t)}\,ds\,d\theta_{N-1}\,d\theta_{N-2}\cdots d\theta_1$. So the stability criterion derived via the $N$-dependent LKF is also hierarchy of LMI conditions, that is, the conservatism of the stability criterion decreases as $N$ increases.

*Remark* 5 It is worth noting that the author of [32] pointed out that the delay set is a polyhedral set, and two main characterizations of the allowable delay set were given, that is, the usual assumptive delay set $\mathcal{H}_1$ satisfying $[h(t),\dot{h}(t)] \in \mathcal{H}_1 = [0,h] \times [-\mu,\mu]$ and another new allowable delay set $\mathcal{H}_2$ satisfying $[h(t),\dot{h}(t)] \in \mathcal{H}_2 = \{(0,0),(0,\mu),(h,0),(h,-\mu)\}$. Figure 1 depicts the graphical interpretation of the above two delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$, where

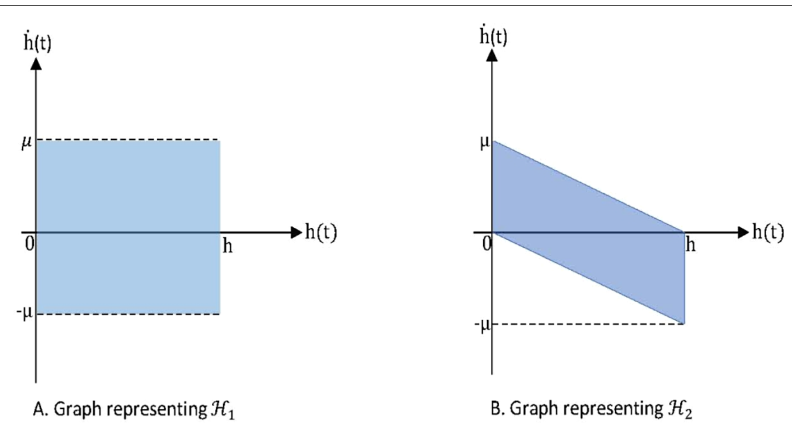


**Figure 1** Graphical interpretation of $\mathcal{H}_1$ and $\mathcal{H}_2$

we can find that once the values of $h$, $\mu$ are given, $\mathcal{H}_2$ is included in $\mathcal{H}_1$. In the next section of this paper, the two allowable delay sets, that is, the usual delay set $\mathcal{H}_1$ and the refined allowable delay set $\mathcal{H}_2$, will be used to show the effectiveness of Theorem 1.

In addition, the original forms of inequalities (6) and (7) are not LMIs because of their dependence on the two time-varying delay variables $h(t)$ and $\dot{h}(t)$. Indeed, the conditions can be rearranged as the following form:

$$\Xi_1 + \dot{h}(t)\big[\Xi_2 + h(t)\Xi_3\big] < 0, \tag{21}$$

where $\Xi_i$, $i = 1, 2, 3$, are time-independent matrix-combinations. According to the convex combination technique [33], the original forms of inequalities (6) and (7) hold if the following LMIs hold for the above two allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively:

$$\mathcal{H}_1: \quad \Xi_1 + \dot{h}(t)\big[\Xi_2 + h(t)\Xi_3\big]_{\{[h(t),\dot{h}(t)]=[0,h]\times[-\mu,\mu]\}} < 0, \tag{22}$$

$$\mathcal{H}_2: \quad \Xi_1 + \dot{h}(t)\big[\Xi_2 + h(t)\Xi_3\big]_{\{(h(t),\dot{h}(t))=\{(0,0),(0,\mu),(h,0),(h,-\mu)\}\}} < 0, \tag{23}$$

which implies that the solution of inequalities (6)–(7) becomes the feasibility-checking of the LMIs.

## 4 Numerical example

In the following content, the maximum allowable upper bounds (MAUBs) in three numerical examples will be calculated by Theorem 1 under the two delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$. The main program tools for obtaining the MAUBs is Matlab LMI-based toolbox. And the corresponding values of MAUB and the numbers of decision variables (NoVs) will be carefully compared with some recent methods, which is provided to illustrate the effectiveness of Theorem 1.

*Example* 1 Consider the following systems:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x\big(t - h(t)\big). \tag{24}$$

The corresponding values of MAUB and NoV compared among some recent previous results and Theorem 1 are listed in Table 1. To confirm the obtained result ($h = 3.118$),

**Table 1** *MAUBs h* under different $\mu$ for Example 1

| Delay sets | Methods\\$\mu$ | 0.05 | 0.1 | 0.3 | 0.5 | NoVs |
|---|---|---|---|---|---|---|
| $\mathcal{H}_1$ | [34] (Th. 1) | 2.613 | 2.424 | 2.131 | 1.793 | $27n^2 + 4n$ |
| | [24] (Th. 2 C2.) | 2.598 | 2.397 | 2.128 | 1.787 | $23n^2 + 4n$ |
| | [35] (Th. 1) | 2.573 | 2.420 | 2.133 | 2.005 | $142n^2 + 18n$ |
| | [36] (Th. 1) | 2.575 | 2.425 | 2.230 | 2.019 | $114n^2 + 18n$ |
| | [37] (Th. 3) | 2.590 | 2.438 | 2.240 | 2.026 | $70n^2 + 12n$ |
| | [38] (Th. 3) | 2.602 | 2.475 | 2.230 | 2.102 | $91.5n^2 + 4.5n$ |
| | Theorem 1 | 2.821 | 2.746 | 2.549 | 2.437 | $90n^2 + 13n$ |
| | Percentage over [38] | 8.42% | 10.95% | 14.30% | 15.94% | – |
| $\mathcal{H}_2$ | Theorem 1 | 3.118 | 3.109 | 3.084 | 3.058 | $90n^2 + 13n$ |
| | Percentage over [38] | 19.83% | 25.62% | 38.30% | 45.48% | – |
| | Percentage over $\mathcal{H}_1$ | 10.53% | 13.22% | 20.99% | 25.48% | – |

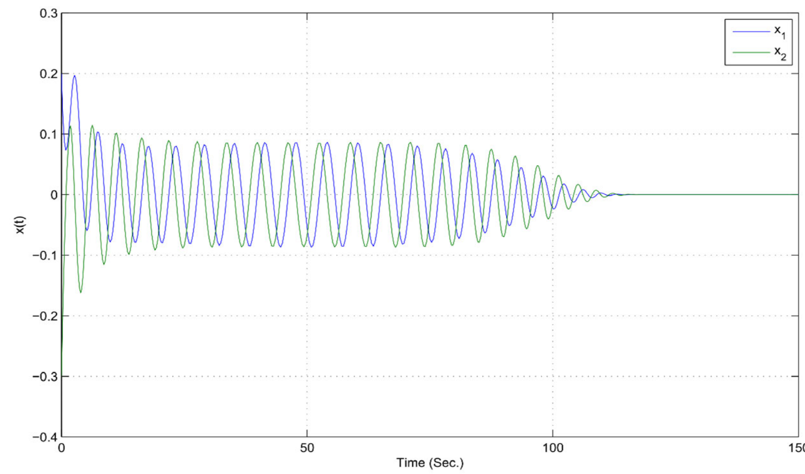


**Figure 2** The state responses for Example 1

**Table 2** *MAUBs* under different $\mu$ for Example 2

| Delay sets | Methods\\$\mu$ | 0.1 | 0.2 | 0.5 | 0.8 | NoVs |
|---|---|---|---|---|---|---|
| $\mathcal{H}_1$ | [24] (Th. 2 C2.) | 6.6103 | 4.0034 | 1.6875 | 1.0287 | $23n^2 + 4n$ |
| | [35] (Th. 1) | 7.1672 | 4.5179 | 2.4158 | 1.8384 | $142n^2 + 18n$ |
| | [36] (Th. 1) | 7.1765 | 4.5438 | 2.4963 | 1.9225 | $114n^2 + 18n$ |
| | [39] (Th. 3) | 7.2030 | 4.5126 | 2.3860 | 1.8476 | $203n^2 + 9n$ |
| | [37] (Th. 3) | 7.1905 | 4.5275 | 2.4473 | 1.8562 | $70n^2 + 12n$ |
| | [10] (Pro. 1) | 7.2734 | 4.6213 | 2.6505 | 2.0612 | $78.5n^2 + 2.5n$ |
| | [11] (Th. 1) | 7.4001 | 4.7954 | 2.7175 | 2.0894 | $108n^2 + 12n$ |
| | [38] (Th. 3) | 8.6565 | 5.8907 | 3.1754 | 2.3953 | $91.5n^2 + 4.5n$ |
| | Theorem 1 | 9.1713 | 7.0501 | 3.7790 | 2.6594 | $90n^2 + 13n$ |
| | Percentage over [38] | 5.95% | 19.68% | 19.01% | 10.84% | – |
| $\mathcal{H}_2$ | Theorem 1 | 20.8822 | 14.3184 | 9.1853 | 7.5645 | $90n^2 + 13n$ |
| | Percentage over [38] | 141.23% | 143.07% | 189.26% | 215.78% | – |
| | Percentage over $\mathcal{H}_1$ | 127.69% | 103.09% | 143.06% | 184.93% | – |

the simulation result is shown in Fig. 2. As you can see from Fig. 2, the state responses of system (24) with $h(t) = \frac{3.118}{2} + \frac{3.118}{2}\sin(\frac{0.1t}{3.118})$ and the initial vector $[0.2 - 0.3]^T$ converge to zero.

*Example* 2 Consider another system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x\bigl(t - h(t)\bigr). \tag{25}$$

Table 2 shows the MAUBs calculated by using Theorem 1 and other methods in the references under different bounds of delay derivative. To confirm the obtained result ($h$ = 20.8822), the simulation result is shown in Fig. 3. We can find from the figure that the state responses of system (25) with $h(t) = \frac{20.8822}{2} + \frac{20.8822}{2}\sin(\frac{0.2t}{20.8822})$ and the initial vector $[0.50.3]^T$ converge to zero.

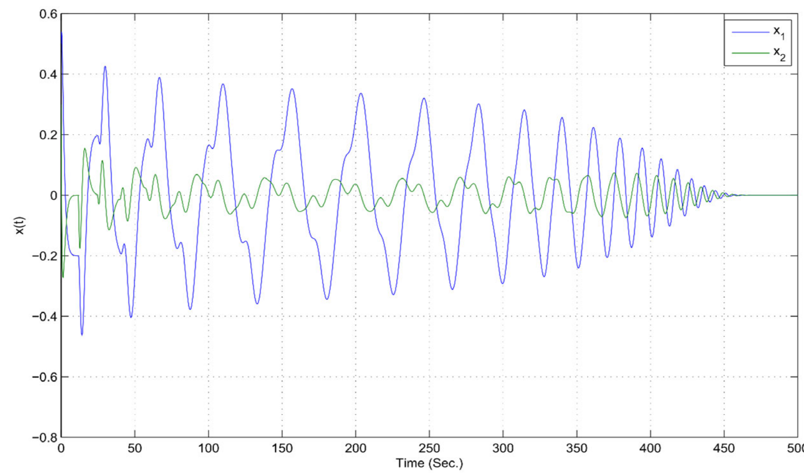


**Figure 3** The state responses for Example 2

**Table 3** *MAUBs h* for different $\mu$ (Example 3)

| Delay sets | Methods\ $\mu$ | 0.1 | 0.2 | 0.5 | 0.8 | NoVs |
|---|---|---|---|---|---|---|
| $\mathcal{H}_1$ | [34] (Th. 1) | 4.753 | 3.873 | 2.429 | 2.183 | $27n^2 + 4n$ |
| | [24] (Th. 2 C2.) | 4.714 | 3.855 | 2.608 | 2.375 | $23n^2 + 4n$ |
| | [23] (Th. 1) | 4.788 | 4.065 | 3.055 | 2.615 | $65n^2 + 11n$ |
| | [26] (Th. 2 C2.) | 4.809 | 4.091 | 3.109 | 2.710 | $25n^2 + 7n$ |
| | [35] (Th. 1) | 4.829 | 4.139 | 3.155 | 2.730 | $142n^2 + 18n$ |
| | [36] (Th. 1) | 4.831 | 4.142 | 3.148 | 2.713 | $114n^2 + 18n$ |
| | [37] (Th. 3) | 4.844 | 4.142 | 3.117 | 2.698 | $70n^2 + 12n$ |
| | [38] (Th. 1) | 4.883 | 4.167 | 3.163 | 2.730 | $91.5n^2 + 4.5n$ |
| | [28] (Pro. 1) | 4.910 | 4.233 | 3.309 | 2.882 | $54.5n^2 + 6.5n$ |
| | [32] (Th. 8 $N = 2$) | 4.930 | 4.220 | 3.090 | 2.660 | $62.5n^2 + 6.5n$ |
| | [27] (Th. 2 $N = 2$) | 4.900 | 4.190 | 3.160 | 2.730 | $65n^2 + 8n$ |
| | [11] (Th. 1) | 4.942 | 4.234 | 3.309 | 2.882 | $108n^2 + 12n$ |
| | [39] (Th. 3) | 4.944 | 4.274 | 3.305 | 2.850 | $203n^2 + 9n$ |
| | Theorem 1 | 5.102 | 4.402 | 3.411 | 2.981 | $90n^2 + 13n$ |
| | Percentage over [39] | 3.20% | 3.00% | 3.21% | 4.60% | – |
| $\mathcal{H}_2$ | [32] (Th. 8 $N = 2$) | 6.172 | 6.164 | 5.07 | 3.94 | $62.5n^2 + 6.5n$ |
| | Theorem 1 | 6.168 | 6.168 | 5.97 | 5.43 | $90n^2 + 13n$ |
| | Percentage over [39] | 24.76% | 44.31% | 80.64% | 90.53% | – |
| | Percentage over [32] | −0.07% | 0.06% | 17.75% | 37.82% | – |
| | Percentage over $\mathcal{H}_1$ | 13.36% | 24.75% | 30.64% | 19.00% | – |

*Example* 3 Consider the following system:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x\bigl(t - h(t)\bigr). \tag{26}$$

[40] gave the MAUB $h_{\max} = 6.1725$ by delay sweeping techniques (see, for instance, [40]). The comparative results among some other methods in the references and Theorem 1 are listed in Table 3.

From Tables 1–3, we can find the following five observations:

- For the delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$, it is obvious that the MAUBs obtained by Theorem 1 are all larger than those obtained by other methods proposed in [11, 23, 24, 26–28, 34–39], especially for the delay set $\mathcal{H}_2$.

- It is seen that the NoVs in our criterion is larger than those in [24, 26, 34], smaller than those in [10, 11, 23, 28, 32, 35–39], and the same as that in [27].
- From Table 3, for the case of slow-varying delays and the delay set $\mathcal{H}_2$, only Theorem 8 in [32] is less conservative than Theorem 1. However, Theorem 1 becomes less conservative than the condition in [32] with the smaller NoVs for the case of fast-varying delays.
- The effectiveness of Theorem 1 is more obvious under the fast-varying delays than the slow-varying delays by comparing the corresponding increase percentages.
- The choice of the delay set, such as $\mathcal{H}_2$, has a great effect on increasing the MAUBs, which matches the description in [32].

## 5 Conclusions

This paper considers the stability problem of time-delayed linear systems. A modified LKF is proposed with two couples of integral terms supplemented in the integral functionals. A less conservative stability criterion is derived based on the new LKF and Lemma 1. Three numerical examples illustrate the effectiveness of the proposed result.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
WD conceived the research idea and co-wrote the paper. YL conducted the numerical simulations. JC conducted the English expression and format, and he recalculated part of the data in Sect. 4. All authors read and approved the final manuscript.

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