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Existence of traveling wave solutions with critical speed in a delayed diffusive epidemic model

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Abstract

In this paper, we prove the existence of a critical traveling wave solution for a delayed diffusive SIR epidemic model with saturated incidence. Moreover, we establish the nonexistence of traveling wave solutions with nonpositive wave speed for this model. Our results solve some open problems left in the recent paper (Z. Xu in Nonlinear Anal. 111:66–81, 2014).

Keywords: SIR epidemic model; Upper and lower solutions; Critical traveling wave

1 Introduction

In the past few decades, more research has focused on spatial propagation of communicable diseases in mathematical epidemiology and more reaction-diffusion SIR models have been proposed to describe the transmission of communicable diseases [1, 4, 6–9, 13, 19–26, 29–31]. For most epidemic diseases models, the traveling wave solutions can describe the phase transmission from a disease-free state to an infective state. The existence and non-existence of the traveling wave solutions can predict whether or not the epidemic disease transmits in the population and how fast it spreads geographically [2, 3, 10–14, 16–18, 25]. Recently, Xu [26] considered the following delayed diffusive SIR model:

$$\begin{cases} \frac{\partial}{\partial t}S(t,x) = d_1\frac{\partial^2 S(t,x)}{\partial x^2} - \frac{\beta S(t,x)I(t-\tau,x)}{1+\alpha I(t-\tau,x)}, \\ \frac{\partial}{\partial t}I(t,x) = d_2\frac{\partial^2 I(t,x)}{\partial x^2} + \frac{\beta S(t,x)I(t-\tau,x)}{1+\alpha I(t-\tau,x)} - \gamma I(t,x), \\ \frac{\partial}{\partial t}R(t,x) = d_3\frac{\partial^2 R(t,x)}{\partial x^2} + \gamma I(t,x), \end{cases} \tag{1.1}$$

where $S(t,x)$, $I(t,x)$, and $R(t,x)$ denote the densities of the susceptible, infected, and recovered individuals at time t and location x, respectively. The constants $d_i > 0$ $(i = 1,2,3)$ are the diffusion rates, $\tau > 0$ is the incubation period, $\beta > 0$ is the transmission coefficient, and $\gamma > 0$ represents the recovery rate. The nonlinear incidence $\frac{\beta SI}{1+\alpha I}$ $(\alpha > 0)$ is called a saturated incidence [5, 15, 30]. Since the third equation in (1.1) is decoupled with the first two, the author studied the subsystem of (1.1) for S and I. In [26] he proved that if $\mathcal{R}_0 = \beta S_0/\gamma > 1$ and $c > c^*$ (c^* is the critical wave speed), then the subsystem of (1.1) admits a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying the wave system

$$\begin{cases} d_1 S''(\xi) - cS'(\xi) - \frac{\beta S(\xi) I(\xi - c\tau)}{1+\alpha I(\xi - c\tau)} = 0, \\ d_2 I''(\xi) - cI'(\xi) + \frac{\beta S(\xi) I(\xi - c\tau)}{1+\alpha I(\xi - c\tau)} - \gamma I(\xi) = 0, \\ S(-\infty) = S_0, \qquad S(\infty) \in [0, S_0), \qquad I(\pm\infty) = 0, \end{cases} \tag{1.2}$$

where $\xi = x + ct$ is the moving coordinate, $c \in \mathbb{R}$ is the wave speed, and $S_0 > 0$ is a given constant. On the other hand, he showed that if $\mathcal{R}_0 < 1$ and $c \geq 0$ or $\mathcal{R}_0 > 1$ and $c \in (0, c^*)$, then the subsystem of (1.1) has no nontrivial and nonnegative traveling wave solutions.

Observing his results in [26], one can find that there exist some open problems listed as follows:

(P1) Does the traveling wave solution of (1.1) exist if (i) $\mathcal{R}_0 > 1$ and $c = c^*$; (ii) $\mathcal{R}_0 = 1$ and $c \in \mathbb{R}$; (iii) $\mathcal{R}_0 > 1$ and $c \leq 0$?

(P2) How does the R-component in (1.1) change?

As was discussed in [19], traveling wave solutions with the critical speed play an important role in the research of epidemic spread. However, it is challenging to investigate the existence of critical traveling wave solutions. There has been some work on the existence of critical traveling wave solutions for diffusive epidemic systems [2, 7, 14, 19, 23, 25, 27, 28, 30]. In this paper, we solve problems (P1) and (P2). For our purpose, we need the following lemma which is established in Lemma 3.1 of [26].

Lemma 1.1 *Assume that* $\mathcal{R}_0 = \beta S_0/\gamma > 1$, *and let*

$$\Delta(\lambda, c) := d_2 \lambda^2 - c\lambda - \gamma + \beta S_0 e^{-\lambda c \tau}. \tag{1.3}$$

Then there exist $c^* > 0$ *and* $\lambda^* > 0$ *such that*

$$\Delta(\lambda^*, c^*) = d_2 (\lambda^*)^2 - c^* \lambda^* - \gamma + \beta S_0 e^{-\lambda^* c^* \tau} = 0 \tag{1.4}$$

and

$$\left.\frac{\partial \Delta(\lambda, c)}{\partial \lambda}\right|_{(\lambda^*, c^*)} = 2 d_2 \lambda^* - c^* - \beta S_0 c^* \tau e^{-\lambda^* c^* \tau} = 0. \tag{1.5}$$

Proof Due to $\mathcal{R}_0 > 1$, for every $c > 0$, at $\lambda = 0$, it is obvious to get that

$$\begin{aligned} &\Delta(0, c) = \beta S_0 - \gamma > 0 \\ &\left.\frac{\partial \Delta(\lambda, c)}{\partial \lambda}\right|_{\lambda=0} = 2 d_2 \lambda - c - \beta S_0 c \tau e^{-\lambda c \tau}|_{\lambda=0} < 0, \\ &\left.\frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2}\right|_{\lambda=0} = 2 d_2 + \beta S_0 (c\tau)^2 e^{-\lambda c \tau}|_{\lambda=0} > 0, \\ &\Delta(+\infty, c) = +\infty. \end{aligned}$$

For every $\lambda > 0$, we can get that

$$\Delta(\lambda, 0) = d_2\lambda^2 - \gamma + \beta S_0 > 0,$$
$$\frac{\partial \Delta(\lambda, c)}{\partial c} = -\lambda - \lambda\beta S_0 \lambda\tau e^{-\lambda c\tau} < 0, \quad c > 0,$$
$$\Delta(\lambda, +\infty) = -\infty.$$

Making full use of the above computations and the rough graphs of $\Delta(\lambda, c)$, we obtain the desired results of this lemma. □

Now, we state our strategies and organization as follows. In Sect. 2, we state our results and some remarks. In Sect. 3, we construct a pair of upper and lower solutions of the wave system and apply Schauder's fixed point theorem to derive the existence of a critical traveling wave solution for (1.1). In addition, employing the subtle analysis and a limiting approach, we obtain the asymptotic boundary conditions of the traveling wave solution and its other properties. In Sect. 4, by contradictory arguments, we establish the non-existence of the traveling wave solutions for the cases $\mathcal{R}_0 = 1$ and $c \in \mathbb{R}$ or $\mathcal{R}_0 > 1$ and $c \leq 0$. In Sect. 5, we make a brief conclusion.

2 Main results

Now we introduce the definition concerning critical traveling wave solutions of (1.1).

Definition 1 A critical traveling solution of (1.1) is a special solution in the form of $(S(\xi), I(\xi), R(\xi)) = (S(x + c^*t), I(x + c^*t), R(x + c^*t))$, where $\xi := x + c^*t$ and c^* is the critical wave speed (see Lemma 1.1). Meanwhile, $(S(\xi), I(\xi), R(\xi)) \in C^2(\mathbb{R}, \mathbb{R}^3)$ is the wave profile that propagates in the one-dimension spatial domain at the constant critical wave speed and connects the initial disease-free equilibrium $(S(-\infty), I(-\infty), R(-\infty))$ to the final disease-free equilibrium $(S(\infty), I(\infty), R(\infty))$.

Next, we mainly consider the following critical wave system:

$$\begin{cases} d_1 S''(\xi) - c^* S'(\xi) - \frac{\beta S(\xi) I(\xi - c^*\tau)}{1 + \alpha I(\xi - c^*\tau)} = 0, \\ d_2 I''(\xi) - c^* I'(\xi) + \frac{\beta S(\xi) I(\xi - c^*\tau)}{1 + \alpha I(\xi - c^*\tau)} - \gamma I(\xi) = 0, \\ d_3 R''(\xi) - c^* R'(\xi) + \gamma I(\xi) = 0. \end{cases} \tag{2.1}$$

For all $\xi \in \mathbb{R}$, we will show the existence and non-existence of the critical traveling wave solution $(S(\xi), I(\xi), R(\xi))$ of (1.1) satisfying the asymptotic boundary conditions

$$(S, I, R)(-\infty) = (S_0, 0, 0), \qquad (S, I, R)(\infty) = (\varepsilon_0, 0, S_0 - \varepsilon_0), \tag{2.2}$$

where ε_0 is some constant and $0 \leq \varepsilon_0 < S_0$. Now we state our results.

Theorem 2.1 *If $\mathcal{R}_0 > 1$ and $c = c^*$, then system (1.1) admits a critical traveling wave solution $(S(\xi), I(\xi), R(\xi))$ satisfying (2.2). Moreover,*

(1) *$S(\xi) > 0$, $I(\xi) > 0$, and $R(\xi) > 0$ on $\mathbb{R}$;*
(2) *$S(-\infty) = S_0$, $I(-\infty) = 0$, $R(-\infty) = 0$, and $I(\xi) = O(-\xi e^{\lambda^*\xi})$ as $\xi \to -\infty$;*

(3) *$S(\xi)$ is strictly decreasing and $R(\xi)$ is strictly increasing on $\mathbb{R}$; $S(\infty) = \varepsilon_0 < S_0$, $I(\infty) = 0$ and $R(\infty) = S_0 - \varepsilon_0$; $S'(\xi)$, $I'(\xi)$, $R'(\xi)$, $S''(\xi)$, $I''(\xi)$, $R''(\xi) \to 0$ as $\xi \to \pm\infty$; $\gamma \int_{\mathbb{R}} I(\xi)\,d\xi = \beta \int_{\mathbb{R}} \frac{S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}\,d\xi = c^*(S_0 - \varepsilon_0)$;*
(4) *$S(\xi) < S_0$, $I(\xi) < \frac{1}{\alpha}(\frac{\beta S_0}{\gamma} - 1)$, and $R(\xi) < (S_0 - \varepsilon_0)$ on $\mathbb{R}$.*

Theorem 2.2 *Assume that $\mathcal{R}_0 = 1$ and $c \in \mathbb{R}$ or $\mathcal{R}_0 > 1$ and $c \le 0$, then system (1.1) admits no positive traveling wave solutions $(S(\xi), I(\xi), R(\xi))$ satisfying (2.2).*

Remark 1 It is necessary to point out that in Theorems 2.1 and 2.2 we have solved open problems (P1) and (P2). Our method adopted here can be used to improve the corresponding results for super-critical traveling wave solutions in [26].

Remark 2 In order to address the change of the number for R-component in (1.1), we study the three equations in (1) together, please refer to our construction of upper-lower solutions. In Theorem 2.1, we proved the existence of the traveling wave solutions, meanwhile we obtained a lot of nice properties of the traveling wave solutions for (1.1).

3 Proof of Theorem 2.1

3.1 Upper and lower solutions

First, we give the definition of upper and lower solutions of (2.1).

Definition 2 The pair of continuous functions $(\bar{S}(\xi), \bar{I}(\xi), \bar{R}(\xi))$ and $(\underline{S}(\xi), \underline{I}(\xi), \underline{R}(\xi))$ is called a pair of upper and lower solutions for (2.1) if they satisfy

$$d_1\bar{S}''(\xi) - c^*\bar{S}'(\xi) - \frac{\beta\bar{S}(\xi)\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} \le 0, \tag{3.1}$$

$$d_2\bar{I}''(\xi) - c^*\bar{I}'(\xi) + \frac{\beta\bar{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} - \gamma\bar{I}(\xi) \le 0, \tag{3.2}$$

$$d_3\bar{R}''(\xi) - c^*\bar{R}'(\xi) + \gamma\bar{I}(\xi) \le 0, \tag{3.3}$$

$$d_1\underline{S}''(\xi) - c^*\underline{S}'(\xi) - \frac{\beta\underline{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} \ge 0, \tag{3.4}$$

$$d_2\underline{I}''(\xi) - c^*\underline{I}'(\xi) + \frac{\beta\underline{S}(\xi)\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} - \gamma\underline{I}(\xi) \ge 0, \tag{3.5}$$

$$d_3\underline{R}''(\xi) - c^*\underline{R}'(\xi) + \gamma\underline{I}(\xi) \ge 0 \tag{3.6}$$

except for finitely many points of ξ on $\mathbb{R}$.

For $\xi \in \mathbb{R}$, define the following nonnegative continuous functions:

$$\bar{S}(\xi) := S_0,$$

$$\underline{S}(\xi) := \begin{cases} S_0(1 - \sigma_2^{-1}e^{\sigma_2\xi}), & \xi < \xi_2, \\ 0, & \xi \ge \xi_2, \end{cases}$$

$$\bar{I}(\xi) := \begin{cases} -L_1\xi e^{\lambda^*\xi}, & \xi < \xi_1, \\ \frac{1}{\alpha}(\frac{\beta S_0}{\gamma} - 1), & \xi \ge \xi_1, \end{cases}$$

$$\underline{I}(\xi) := \begin{cases} -L_1\xi e^{\lambda^*\xi} - L_2(-\xi)^{\frac{1}{2}} e^{\lambda^*\xi}, & \xi < \xi_3, \\ 0, & \xi \geq \xi_3, \end{cases}$$

$$\bar{R}(\xi) := L_3 e^{\sigma_1 \xi}, \qquad \underline{R}(\xi) := 0,$$

where λ^* is defined in Lemma 1.1,

$$\xi_1 = -\frac{1}{\lambda^*}, \qquad \xi_2 = \frac{\ln\sigma_2}{\sigma_2}, \qquad \xi_3 = -\frac{L_2^2}{L_1^2}, \qquad L_1 = \frac{e\lambda^*}{\alpha}\left(\frac{\beta S_0}{\gamma} - 1\right),$$

and the constants σ_1, σ_2, L_2, and $L_3 \in \mathbb{R}^+$ are to be determined later. In the next lemma, we will prove that $(\bar{S}(\xi), \bar{I}(\xi), \bar{R}(\xi))$ and $(\underline{S}(\xi), \underline{I}(\xi), \underline{R}(\xi))$ are a pair of upper and lower solutions of (2.1).

Lemma 3.1 *The function* $\bar{S}(\xi)$ *satisfies the inequality*

$$d_1\bar{S}''(\xi) - c^*\bar{S}'(\xi) - \frac{\beta\bar{S}(\xi)\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} \leq 0$$

and the function $\underline{R}(\xi)$ *satisfies the inequality*

$$d_3\underline{R}''(\xi) - c^*\underline{R}'(\xi) + \gamma\underline{I}(\xi) \geq 0$$

for all $\xi \in \mathbb{R}$.

Proof By the definitions of $\bar{S}(\xi)$, $\underline{R}(\xi)$, and $\underline{I}(\xi)$ on $\mathbb{R}$, one can get

$$d_1\bar{S}''(\xi) - c^*\bar{S}'(\xi) - \frac{\beta\bar{S}(\xi)\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} = -\frac{\beta S_0\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} \leq 0 \tag{3.7}$$

and

$$d_3\underline{R}''(\xi) - c^*\underline{R}'(\xi) + \gamma\underline{I}(\xi) = \gamma\underline{I}(\xi) \geq 0, \tag{3.8}$$

which completes the proof. □

Lemma 3.2 *The function* $\bar{I}(\xi)$ *satisfies the inequality*

$$d_2\bar{I}''(\xi) - c^*\bar{I}'(\xi) + \frac{\beta\bar{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} - \gamma\bar{I}(\xi) \leq 0$$

for all $\xi \neq \xi_1$.

Proof When $\xi < \xi_1$, $\bar{I}(\xi) = -L_1\xi e^{\lambda^*\xi}$, then it follows that

$$\begin{aligned} & d_2\bar{I}''(\xi) - c^*\bar{I}'(\xi) + \frac{\beta\bar{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} - \gamma\bar{I}(\xi) \\ & \quad \leq d_2\bar{I}''(\xi) - c^*\bar{I}'(\xi) + \beta S_0\bar{I}\left(\xi - c^*\tau\right) - \gamma\bar{I}(\xi) \\ & \quad = -L_1 e^{\lambda^*\xi}\left[\Delta\left(\lambda^*, c^*\right) + \frac{\partial\Delta}{\partial\lambda}\left(\lambda^*, c^*\right)\right] = 0, \quad \xi < \xi_1. \end{aligned}$$

When $\xi > \xi_1$, $\bar{I}(\xi - c^*\tau) < \bar{I}(\xi) = \frac{1}{\alpha}(\frac{\beta S_0}{\gamma} - 1)$, we have that

$$\begin{aligned} & d_2\bar{I}''(\xi) - c^*\bar{I}'(\xi) + \frac{\beta\bar{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} - \gamma\bar{I}(\xi) \\ &\quad = \frac{\beta\bar{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} - \gamma\bar{I}(\xi) \\ &\quad \le \frac{\beta S_0\bar{I}(\xi)}{1 + \alpha\bar{I}(\xi)} - \gamma\bar{I}(\xi) = 0. \end{aligned}$$

This is the end of the proof. □

Lemma 3.3 *The function $\bar{R}(\xi)$ satisfies the inequality*

$$d_3\bar{R}''(\xi) - c^*\bar{R}'(\xi) + \gamma\bar{I}(\xi) \le 0$$

for all $\xi \in \mathbb{R}$.

Proof Choose a sufficiently large $L_3 > 0$ and a sufficiently small $\sigma_1 \in (0, \min\{\lambda^*, c^*/d_3\})$ such that

$$d_3\sigma_1^2 - c^*\sigma_1 - \gamma L_1 {L_3}^{-1}\xi e^{(\lambda^* - \sigma_1)\xi} < 0, \quad \xi < \xi_1 \tag{3.9}$$

and

$$d_3\sigma_1^2 - c^*\sigma_1 + \frac{\gamma}{\alpha L_3}\left(\frac{\beta S_0}{\gamma} - 1\right)e^{-\sigma_1\xi} < 0, \quad \xi \ge \xi_1. \tag{3.10}$$

When $\xi < \xi_1$, $\bar{I}(\xi) = -L_1\xi e^{\lambda^*\xi}$. Then, by using (3.9), we obtain that

$$\begin{aligned} & d_3\bar{R}''(\xi) - c^*\bar{R}'(\xi) + \gamma\bar{I}(\xi) \\ &\quad = d_3{\sigma_1}^2 L_3 e^{\sigma_1\xi} - c^* L_3\sigma_1 e^{\sigma_1\xi} + \gamma(-L_1)\xi e^{\lambda^*\xi} \\ &\quad = L_3 e^{\sigma_1\xi}\left[d_3{\sigma_1}^2 - c^*\sigma_1 - \gamma L_1 {L_3}^{-1}\xi e^{(\lambda^* - \sigma_1)\xi}\right] \\ &\quad \le 0, \quad \xi < \xi_1. \end{aligned}$$

When $\xi \ge \xi_1$, $\bar{I}(\xi) = \frac{1}{\alpha}(\frac{\beta S_0}{\gamma} - 1)$. Using (3.10), we get that

$$\begin{aligned} & d_3\bar{R}''(\xi) - c^*\bar{R}'(\xi) + \gamma\bar{I}(\xi) \\ &\quad = d_3{\sigma_1}^2 L_3 e^{\sigma_1\xi} - c^* L_3\sigma_1 e^{\sigma_1\xi} + \frac{\gamma}{\alpha}\left(\frac{\beta S_0}{\gamma} - 1\right) \\ &\quad = L_3 e^{\sigma_1\xi}\left[d_3{\sigma_1}^2 - c^*\sigma_1 + \frac{\gamma}{\alpha L_3}\left(\frac{\beta S_0}{\gamma} - 1\right)e^{-\sigma_1\xi}\right] \le 0, \quad \xi \ge \xi_1. \end{aligned}$$

The proof of this lemma is finished. □

Lemma 3.4 *The function $\underline{S}(\xi)$ satisfies the inequality*

$$d_1\underline{S}''(\xi) - c^*\underline{S}'(\xi) - \frac{\beta\underline{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} \geq 0$$

for all $\xi \neq \xi_2$.

Proof Select σ_2 to be small enough such that $\sigma_2 \in (0, \min\{\lambda^*, c^*/d_1\})$, $\xi_2 < \xi_1$, and

$$-d_1\sigma_2 + c^* + \beta L_1(\xi - c^*\tau)(1 - \sigma_2{}^{-1}e^{\sigma_2\xi})e^{(\lambda^*-\sigma_2)\xi - \lambda^*c^*\tau} \geq 0, \quad \xi < \xi_2. \tag{3.11}$$

If $\xi < \xi_2$, then $\underline{S}(\xi) = S_0(1 - \sigma_2^{-1}e^{\sigma_2\xi})$ and $\bar{I}(\xi) = -L_1\xi e^{\lambda^*\xi}$. Utilizing (3.11), we deduce that

$$\begin{aligned}
&d_1\underline{S}''(\xi) - c^*\underline{S}'(\xi) - \frac{\beta\underline{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} \\
&\quad \geq d_1\underline{S}''(\xi) - c^*\underline{S}'(\xi) - \beta\underline{S}(\xi)\bar{I}(\xi - c^*\tau) \\
&\quad = -d_1S_0\sigma_2e^{\sigma_2\xi} + c^*S_0e^{\sigma_2\xi} + L_1\beta S_0(1 - \sigma_2^{-1}e^{\sigma_2\xi})(\xi - c^*\tau)e^{\lambda^*(\xi - c^*\tau)} \\
&\quad = S_0e^{\sigma_2\xi}\bigl[-d_1\sigma_2 + c^* + \beta L_1(\xi - c^*\tau)(1 - \sigma_2{}^{-1}e^{\sigma_2\xi})e^{(\lambda^*-\sigma_2)\xi - \lambda^*c^*\tau}\bigr] \\
&\quad \geq 0, \quad \xi < \xi_2.
\end{aligned}$$

If $\xi > \xi_2$, then $\underline{S}(\xi) = 0$ and

$$d_1\underline{S}''(\xi) - c^*\underline{S}'(\xi) - \frac{\beta\underline{S}(\xi)\bar{I}(\xi - c^*\tau)}{1 + \alpha\bar{I}(\xi - c^*\tau)} = 0$$

holds naturally. □

Lemma 3.5 *The function $\underline{I}(\xi)$ satisfies the inequality*

$$d_2\underline{I}''(\xi) - c^*\underline{I}'(\xi) + \frac{\beta\underline{S}(\xi)\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} - \gamma\underline{I}(\xi) \geq 0$$

for all $\xi \neq \xi_3$.

Proof Choosing large enough $L_2 > 0$ such that $\xi_3 < \min\{\xi_2, -c^*\tau\}$ and recalling $\xi_2 < \xi_1$, then for $\xi < \xi_3$ we have that

$$S_0(1 - \sigma_2{}^{-1}e^{\sigma_2\xi}) < S_0, \tag{3.12}$$

$$\underline{I}(\xi) = \bar{I}(\xi) - L_2(-\xi)^{\frac{1}{2}}e^{\lambda^*\xi}, \tag{3.13}$$

$$1 + c^*\tau\xi^{-1} > 0, \tag{3.14}$$

and

$$\frac{1}{16}L_2(c^*\tau)^2 - \sigma_2^{-1}L_1(-\xi)^{\frac{5}{2}}e^{\sigma_2\xi} - \alpha L_1^2(-\xi)^{\frac{7}{2}}e^{\lambda^*(\xi - c^*\tau)} \geq 0. \tag{3.15}$$

From (3.13), for $\xi < \xi_3$, we can get that

$$\begin{aligned} d_2\underline{I}''(\xi) &= d_2\bar{I}''(\xi) + d_2L_2e^{\lambda^*\xi}\left[\lambda^*(-\xi)^{-\frac{1}{2}} - (\lambda^*)^2(-\xi)^{\frac{1}{2}} + \frac{1}{4}(-\xi)^{-\frac{3}{2}}\right] \\ &\geq d_2\bar{I}''(\xi) + d_2L_2e^{\lambda^*\xi}\left[\lambda^*(-\xi)^{-\frac{1}{2}} - (\lambda^*)^2(-\xi)^{\frac{1}{2}}\right] \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} -c^*\underline{I}'(\xi) - \gamma\underline{I}(\xi) &= -c^*\bar{I}'(\xi) - c^*L_2\left[\frac{1}{2}(-\xi)^{-\frac{1}{2}} - \lambda^*(-\xi)^{\frac{1}{2}}\right]e^{\lambda^*\xi} \\ &\quad - \gamma\bar{I}(\xi) + \gamma L_2(-\xi)^{\frac{1}{2}}e^{\lambda^*\xi}. \end{aligned} \tag{3.17}$$

Using the inequality $\frac{x}{1+\alpha x} \geq x(1-\alpha x)$ for $x \geq 0$ and $\alpha > 0$, we obtain from (3.12) that

$$\begin{aligned} &\frac{\beta\underline{S}(\xi)\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} \\ &\quad \geq \beta\underline{S}(\xi)\underline{I}(\xi - c^*\tau)\left[1 - \alpha\underline{I}(\xi - c^*\tau)\right] \\ &\quad \geq \beta\underline{S}(\xi)\underline{I}(\xi - c^*\tau) - \alpha\beta\underline{S}(\xi)\left(\bar{I}(\xi - c^*\tau)\right)^2 \\ &\quad \geq \beta S_0\left(1 - \sigma_2^{-1}e^{\sigma_2\xi}\right)\left[\bar{I}(\xi - c^*\tau) - L_2(-\xi + c^*\tau)^{\frac{1}{2}}e^{\lambda^*(\xi - c^*\tau)}\right] \\ &\qquad - \alpha\beta S_0L_1^2\xi^2e^{2\lambda^*(\xi - c^*\tau)} \\ &\quad \geq \beta S_0\left[\bar{I}(\xi - c^*\tau) - L_2(-\xi + c^*\tau)^{\frac{1}{2}}e^{\lambda^*(\xi - c^*\tau)}\right. \\ &\qquad \left. + \sigma_2^{-1}L_1(\xi - c^*\tau)e^{\sigma_2\xi + \lambda^*(\xi - c^*\tau)} - \alpha L_1^2\xi^2e^{2\lambda^*(\xi - c^*\tau)}\right] \end{aligned} \tag{3.18}$$

for $\xi < \xi_3$. By Taylor's formula, for $\xi < \xi_3$, we have that

$$(-\xi + c^*\tau)^{\frac{1}{2}} \leq (-\xi)^{\frac{1}{2}} + \frac{1}{2}(-\xi)^{-\frac{1}{2}}c^*\tau - \frac{1}{8}(-\xi)^{-\frac{3}{2}}(c^*\tau)^2 + \frac{1}{16}(-\xi)^{-\frac{5}{2}}(c^*\tau)^3. \tag{3.19}$$

From (3.12)–(3.19), (1.4), and (1.5), we deduce that

$$\begin{aligned} &d_2\underline{I}''(\xi) - c^*\underline{I}'(\xi) + \frac{\beta\underline{S}(\xi)\underline{I}(\xi - c^*\tau)}{1 + \alpha\underline{I}(\xi - c^*\tau)} - \gamma\underline{I}(\xi) \\ &\quad \geq d_2\bar{I}''(\xi) + d_2L_2e^{\lambda^*\xi}\left[\lambda^*(-\xi)^{-\frac{1}{2}} - (\lambda^*)^2(-\xi)^{\frac{1}{2}}\right] \\ &\qquad - c^*\bar{I}'(\xi) - c^*L_2\left[\frac{1}{2}(-\xi)^{-\frac{1}{2}} - \lambda^*(-\xi)^{\frac{1}{2}}\right]e^{\lambda^*\xi} \\ &\qquad - \gamma\bar{I}(\xi) + \gamma L_2(-\xi)^{\frac{1}{2}}e^{\lambda^*\xi} + \beta S_0\left[\bar{I}(\xi - c^*\tau) - L_2(-\xi + c^*\tau)^{\frac{1}{2}}e^{\lambda^*(\xi - c^*\tau)}\right. \\ &\qquad \left. + \sigma_2^{-1}L_1(\xi - c^*\tau)e^{\sigma_2\xi + \lambda^*(\xi - c^*\tau)} - \alpha L_1^2\xi^2e^{2\lambda^*(\xi - c^*\tau)}\right] \\ &\quad \geq -L_1e^{\lambda^*\xi}\left[\xi\left(d_2(\lambda^*)^2 - c^*\lambda^* - \gamma + \beta S_0e^{-\lambda^*c^*\tau}\right) + \left(2d_2\lambda^* - c^* - \beta S_0c^*\tau e^{-\lambda^*c^*\tau}\right)\right] \\ &\qquad + L_1\beta S_0(\xi - c^*\tau)e^{\lambda^*(\xi - c^*\tau)} - L_2e^{\lambda^*\xi}(-\xi)^{\frac{1}{2}}\left[d_2(\lambda^*)^2 - c^*\lambda^* - \gamma + \beta S_0e^{-\lambda^*c^*\tau}\right] \\ &\qquad + L_2\beta S_0(-\xi)^{\frac{1}{2}}e^{\lambda^*(\xi - c^*\tau)} + \frac{1}{2}L_2e^{\lambda^*\xi}(-\xi)^{-\frac{1}{2}}\left[2d_2\lambda^* - c^* - \beta S_0c^*\tau e^{-\lambda^*c^*\tau}\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}L_2\beta S_0 c^*\tau(-\xi)^{-\frac{1}{2}}e^{\lambda^*(\xi-c^*\tau)}+\beta S_0\bigl[\bar{I}(\xi-c^*\tau)-L_2(-\xi+c^*\tau)^{\frac{1}{2}}e^{\lambda^*(\xi-c^*\tau)}\\
&+\sigma_2^{-1}L_1(\xi-c^*\tau)e^{\sigma_2\xi+\lambda^*(\xi-c^*\tau)}-\alpha L_1^2\xi^2e^{2\lambda^*(\xi-c^*\tau)}\bigr]\\
\geq{}& -L_1e^{\lambda^*\xi}\left[\Delta(\lambda^*,c^*)+\frac{\partial\Delta}{\partial\lambda}(\lambda^*,c^*)\right]-L_2e^{\lambda^*\xi}(-\xi)^{\frac{1}{2}}\bigl[\Delta(\lambda^*,c^*)\bigr]\\
&+\frac{1}{2}L_2e^{\lambda^*\xi}(-\xi)^{\frac{1}{2}}\left[\frac{\partial\Delta}{\partial\lambda}(\lambda^*,c^*)\right]\\
&+\beta S_0\left\{L_2e^{\lambda^*(\xi-c^*\tau)}\left[\frac{1}{8}(-\xi)^{-\frac{3}{2}}(c^*\tau)^2-\frac{1}{16}(-\xi)^{-\frac{5}{2}}(c^*\tau)^3\right]\right.\\
&\left.+\sigma_2^{-1}L_1\xi e^{\sigma_2\xi+\lambda^*(\xi-c^*\tau)}-\alpha L_1^2\xi^2e^{2\lambda^*(\xi-c^*\tau)}\right\}\\
={}&\beta S_0(-\xi)^{-\frac{3}{2}}e^{\lambda^*(\xi-c^*\tau)}\left[\frac{1}{16}L_2(c^*\tau)^2-\sigma_2^{-1}L_1(-\xi)^{\frac{5}{2}}e^{\sigma_2\xi}-\alpha L_1^2(-\xi)^{\frac{7}{2}}e^{\lambda^*(\xi-c^*\tau)}\right]\\
&+\frac{1}{16}\beta S_0L_2(c^*\tau)^2(-\xi)^{-\frac{3}{2}}e^{\lambda^*(\xi-c^*\tau)}\bigl(1+c^*\tau\xi^{-1}\bigr)\\
\geq{}&0.
\end{aligned}
$$

If $\xi > \xi_3$, then $\underline{I}(\xi) = 0$, and the inequality

$$
d_2\underline{I}''(\xi)-c^*\underline{I}'(\xi)+\frac{\beta\underline{S}(\xi)\underline{I}(\xi-c^*\tau)}{1+\alpha\underline{I}(\xi-c^*\tau)}-\gamma\underline{I}(\xi)=\frac{\beta\underline{S}(\xi)\underline{I}(\xi-c^*\tau)}{1+\alpha\underline{I}(\xi-c^*\tau)}\geq 0
$$

holds trivially. The proof of this lemma is finished. □

3.2 Application of Schauder's fixed point theorem

Introduce a functional space

$$
B_\mu(\mathbb{R},\mathbb{R}^3):=\left\{\varphi(\xi)=\bigl(\varphi_1(\xi),\varphi_2(\xi),\varphi_3(\xi)\bigr)\in C(\mathbb{R},\mathbb{R}^3):\sup_{\xi\in\mathbb{R}}\bigl|\varphi_i(\xi)\bigr|e^{-\mu|\xi|}<\infty, i=1,2,3\right\}
$$

equipped with the norm $|\varphi|_\mu := \max\{\sup_{\xi\in\mathbb{R}}|\varphi_i(\xi)|e^{-\mu|\xi|}, i=1,2,3\}$, where $\mu\in(\sigma_1,\mu_0)$ is a constant and μ_0 is also a constant that will be specified later. Define a cone by

$$
\mathcal{S}:=\left\{\bigl(S(\xi),I(\xi),R(\xi)\bigr)\in B_\mu(\mathbb{R},\mathbb{R}^3)\;\middle|\;\begin{array}{l}\underline{S}(\xi)\leq S(\xi)\leq\bar{S}(\xi),\\ \underline{I}(\xi)\leq I(\xi)\leq\bar{I}(\xi),\\ \underline{R}(\xi)\leq R(\xi)\leq\bar{R}(\xi)\end{array}\right\}.
$$

It is easy to see that $\mathcal{S}$ is nonempty, bounded, closed, and convex in $B_\mu(\mathbb{R},\mathbb{R}^3)$. Choosing a constant m to be satisfied $m>\max\{\alpha^{-1}\beta,\beta\}$ and noting that $\beta>\gamma$, one can get that:

$$
H_1[S,I,R](\xi):=mS(\xi)-\frac{\beta S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}
$$

is increasing with respect to S and decreasing with respect to I;

$$
H_2[S,I,R](\xi):=\frac{\beta S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}+(m-\gamma)I(z)
$$

is increasing in both S and I;

$$H_3[S,I,R](\xi) := mR(\xi) + \gamma I(\xi)$$

is increasing in both I and R. For any $(S,I,R) \in \mathcal{S}$, define a nonlinear operator $\mathcal{M} := (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ on the space $B_\mu(\mathbb{R}, \mathbb{R}^3)$ by

$$\mathcal{M}_i[S,I,R](\xi) := \frac{1}{\Lambda_i}\left\{\int_{-\infty}^{\xi} e^{r_{i1}(\xi-\eta)} H_i[S,I,R](\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_{i2}(\xi-\eta)} H_i[S,I,R](\eta)\,d\eta\right\},$$

where

$$r_{i1} = \frac{c^* - \sqrt{(c^*)^2 + 4md_i}}{2d_i}, \qquad r_{i2} = \frac{c^* + \sqrt{(c^*)^2 + 4md_i}}{2d_i}, \qquad \Lambda_i = d_i(r_{i2} - r_{i1})$$

for $i = 1,2,3$. Note that any fixed point of $\mathcal{M}$ is a solution of (2.1).

Lemma 3.6 $\mathcal{M}(\mathcal{S}) \subset \mathcal{S}$.

Proof Clearly, $(\mathcal{M}_1[S,I,R](\xi), \mathcal{M}_2[S,I,R](\xi), \mathcal{M}_3[S,I,R](\xi)) \in B_\mu(\mathbb{R}, \mathbb{R}^3)$ for any $(S,I,R) \in \mathcal{S}$. Then, by the monotonicity of H_i $(i = 1,2,3)$, we need to prove that

$$\underline{S}(\xi) \le \mathcal{M}_1[\underline{S},\bar{I},R](\xi) \le \mathcal{M}_1[S,I,R](\xi) \le \mathcal{M}_1[\bar{S},\underline{I},R](\xi) \le \bar{S}(\xi), \tag{3.20}$$

$$\underline{I}(\xi) \le \mathcal{M}_2[\underline{S},\underline{I},R](\xi) \le \mathcal{M}_2[S,I,R](\xi) \le \mathcal{M}_2[\bar{S},\bar{I},R](\xi) \le \bar{I}(\xi), \tag{3.21}$$

$$\underline{R}(\xi) \le \mathcal{M}_3[S,\underline{I},\underline{R}](\xi) \le \mathcal{M}_3[S,I,R](\xi) \le \mathcal{M}_3[S,\bar{I},\bar{R}](\xi) \le \bar{R}(\xi) \tag{3.22}$$

for any $(S,I,R) \in \mathcal{S}$.

Proof of (3.20). Using (3.1) and $\bar{S}(\xi) = S_0$, we derive that

$$\begin{aligned}
&\mathcal{M}_1[\bar{S},\underline{I},R](\xi)\\
&\quad = \frac{1}{\Lambda_1}\left\{\int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)} H_1[\bar{S},\underline{I},R](\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)} H_1[\bar{S},\underline{I},R](\eta)\,d\eta\right\}\\
&\quad \le \frac{1}{\Lambda_1}\left\{\int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)}\left[m\bar{S}(\eta) + c^*\bar{S}'(\eta) - d_1\bar{S}''(\eta)\right]d\eta\right.\\
&\qquad \left. + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)}\left[m\bar{S}(\eta) + c^*\bar{S}'(\eta) - d_1\bar{S}''(\eta)\right]d\eta\right\}\\
&\quad = \frac{mS_0}{\Lambda_1}\left[\int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)}\,d\eta + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)}\,d\eta\right]\\
&\quad = S_0, \quad \xi \in \mathbb{R}.
\end{aligned}$$

It follows from (3.4) that

$$\begin{aligned}
&\mathcal{M}_1[\underline{S},\bar{I},R](\xi)\\
&\quad = \frac{1}{\Lambda_1}\left\{\int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)} H_1[\underline{S},\bar{I},R](\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)} H_1[\underline{S},\bar{I},R](\eta)\,d\eta\right\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\Lambda_1}\left\{\int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)}\big[m\underline{S}(\eta)+c^*\underline{S}'(\eta)-d_1\underline{S}''(\eta)\big]\,d\eta\right.\\
&\quad\left.+\int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)}\big[m\underline{S}(\eta)+c^*\underline{S}'(\eta)-d_1\underline{S}-''(\eta)\big]\,d\eta\right\}\\
&=\underline{S}(\xi)+\frac{e^{r_{11}(\xi-\xi_2)}[\underline{S}'(\xi_2+0)-\underline{S}'(\xi_2-0)]}{r_{12}-r_{11}}\\
&\geq \underline{S}(\xi),\quad \xi\neq\xi_2.
\end{aligned}
$$

By the continuity of both $\mathcal{M}_1[\underline{S},\bar{I},R](\xi)$ and $\underline{S}(\xi)$ on $\mathbb{R}$, we get that

$$
\mathcal{M}_1[\underline{S},\bar{I},R](\xi)\geq\underline{S}(\xi),\quad \xi\in\mathbb{R}.
$$

Proof of (3.21). From (3.2) and (3.5), we get that

$$
\begin{aligned}
&\mathcal{M}_2[\bar{S},\bar{I},R](\xi)\\
&=\frac{1}{\Lambda_2}\left\{\int_{-\infty}^{\xi} e^{r_{21}(\xi-\eta)}H_2[\bar{S},\bar{I},R](\eta)\,d\eta+\int_{\xi}^{\infty} e^{r_{22}(\xi-\eta)}H_2[\bar{S},\bar{I},R](\eta)\,d\eta\right\}\\
&\leq\frac{1}{\Lambda_2}\left\{\int_{-\infty}^{\xi} e^{r_{21}(\xi-\eta)}\big[m\bar{I}(\eta)+c^*\bar{I}'(\eta)-d_2\bar{I}''(\eta)\big]\,d\eta\right.\\
&\quad\left.+\int_{\xi}^{\infty} e^{r_{22}(\xi-\eta)}\big[m\bar{I}(\eta)+c^*\bar{I}'(\eta)-d_2\bar{I}''(\eta)\big]\,d\eta\right\}\\
&=\bar{I}(\xi)+\frac{e^{r_{21}(\xi-\xi_1)}[\bar{I}'(\xi_1+0)-\bar{I}'(\xi_1-0)]}{r_{22}-r_{21}}\\
&\leq\bar{I}(\xi),\quad \xi\neq\xi_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{M}_2[\underline{S},\underline{I},R](\xi)\\
&=\frac{1}{\Lambda_2}\left\{\int_{-\infty}^{\xi} e^{r_{21}(\xi-\eta)}H_2[\underline{S},\underline{I},R](\eta)\,d\eta+\int_{\xi}^{\infty} e^{r_{22}(\xi-\eta)}H_2[\underline{S},\underline{I},R](\eta)\,d\eta\right\}\\
&\geq\frac{1}{\Lambda_2}\left\{\int_{-\infty}^{\xi} e^{r_{21}(\xi-\eta)}\big[m\underline{I}(\eta)+c^*\underline{I}'(\eta)-d_2\underline{I}''(\eta)\big]\,d\eta\right.\\
&\quad\left.+\int_{\xi}^{\xi_3} e^{r_{22}(\xi-\eta)}\big[m\underline{I}(\eta)+c^*\underline{I}'(\eta)-d_2\underline{I}''(\eta)\big]\,d\eta\right\}\\
&=\bar{I}(\xi)+\frac{e^{r_{21}(\xi-\xi_3)}[\bar{I}'(\xi_3+0)-\bar{I}'(\xi_3-0)]}{r_{22}-r_{21}}\\
&\geq\underline{I}(\xi),\quad \xi\neq\xi_3.
\end{aligned}
$$

Using the continuity of both $\mathcal{M}_2[\bar{S},\bar{I},R](\xi)$, $\mathcal{M}_2[\underline{S},\underline{I},R](\xi)$, $\bar{I}(\xi)$ and $\underline{I}(\xi)$ on $\mathbb{R}$, we obtain

$$
\mathcal{M}_2[\bar{S},\bar{I},R](\xi)\leq\bar{I}(\xi),\qquad \mathcal{M}_2[\underline{S},\underline{I},R](\xi)\geq\underline{I}(\xi),\quad \xi\in\mathbb{R}.
$$

Proof of (3.22). From (3.3), (3.6), and the expressions of $\bar{R}(\xi)$ and $\underline{R}(\xi)$, we deduce that

$$\begin{aligned}
&\mathcal{M}_3[S,\bar{I},\bar{R}](\xi)\\
&\quad= \frac{1}{\Lambda_3}\left\{\int_{-\infty}^{\xi} e^{r_{31}(\xi-\eta)}H_3[S,\bar{I},\bar{R}](\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_{32}(\xi-\eta)}H_3[S,\bar{I},\bar{R}](\eta)\,d\eta\right\}\\
&\quad\le \frac{1}{\Lambda_3}\left\{\int_{-\infty}^{\xi} e^{r_{31}(\xi-\eta)}\bigl[m\bar{R}(\eta)+c^*\bar{R}'(\eta)-d_3\bar{R}''(\eta)\bigr]\,d\eta\right.\\
&\qquad\left.+\int_{\xi}^{\infty} e^{r_{32}(\xi-\eta)}\bigl[m\bar{R}(\eta)+c^*\bar{R}'(\eta)-d_3\bar{R}''(\eta)\bigr]\,d\eta\right\}\\
&\quad= \bar{R}(\xi),\quad \xi\in\mathbb{R},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{M}_3[S,\underline{I},\underline{R}](\xi)\\
&\quad= \frac{1}{\Lambda_3}\left\{\int_{-\infty}^{\xi} e^{r_{31}(\xi-\eta)}H_3[S,\underline{I},\underline{R}](\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_{32}(\xi-\eta)}H_3[S,\underline{I},\underline{R}](\eta)\,d\eta\right\}\\
&\quad\ge \frac{1}{\Lambda_3}\left\{\int_{-\infty}^{\xi} e^{r_{31}(\xi-\eta)}\bigl[m\underline{R}(\eta)+c^*\underline{R}'(\eta)-d_3\underline{R}''(\eta)\bigr]\,d\eta\right.\\
&\qquad\left.+\int_{\xi}^{\infty} e^{r_{32}(\xi-\eta)}\bigl[m\underline{R}(\eta)+c^*\underline{R}'(\eta)-d_3\underline{R}''(\eta)\bigr]\,d\eta\right\}\\
&\quad= \underline{R}(\xi),\quad \xi\in\mathbb{R}.
\end{aligned}$$

The proof of this lemma is finished. □

Lemma 3.7 *The operator $\mathcal{M} := (\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)$ is completely continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$.*

Proof First, we show that $\mathcal{M}$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$. For any $\Psi_1=(S_1,I_1,R_1)\in\mathcal{S}$ and $\Psi_2=(S_2,I_2,R_2)\in\mathcal{S}$, we derive that

$$\begin{aligned}
&\bigl|H_1(S_1,I_1,R_1)(\xi)-H_1(S_2,I_2,R_2)(\xi)\bigr|e^{-\mu|\xi|}\\
&\quad\le\left(m+\frac{\beta}{\alpha}\right)\bigl|S_1(\xi)-S_2(\xi)\bigr|e^{-\mu|\xi|}+\beta S_0\bigl|I_1\bigl(\xi-c^*\tau\bigr)-I_2\bigl(\xi-c^*\tau\bigr)\bigr|e^{-\mu|\xi|}\\
&\quad\le\left(m+\frac{\beta}{\alpha}\right)|S_1-S_2|_\mu+\beta S_0e^{\mu c^*\tau}|I_1-I_2|_\mu\\
&\quad\le l|\Psi_1-\Psi_2|_\mu,\\
&\bigl|H_2(S_1,I_1,R_1)(\xi)-H_2(S_2,I_2,R_2)(\xi)\bigr|e^{-\mu|\xi|}\\
&\quad\le\frac{\beta}{\alpha}|S_1-S_2|_\mu+\bigl(m+\beta S_0e^{\mu c^*\tau}-\gamma\bigr)|I_1-I_2|_\mu\\
&\quad\le l|\Psi_1-\Psi_2|_\mu,
\end{aligned}$$

and

$$
\begin{aligned}
&\bigl|H_3(S_1,I_1,R_1)(\xi)-H_3(S_2,I_2,R_2)(\xi)\bigr|e^{-\mu|\xi|}\\
&\quad\le m|R_1-R_2|_\mu+\gamma|I_1-I_2|_\mu\\
&\quad\le l|\Psi_1-\Psi_2|_\mu,
\end{aligned}
$$

where $l=m+\frac{\beta}{\alpha}+\gamma+\beta S_0e^{\mu c^*\tau}$. Then, choosing $\mu\in(\sigma_2,-r_{i1})$, we have that

$$
\begin{aligned}
&\bigl|\mathcal{M}_i[S_1,I_1,R_1](\xi)-\mathcal{M}_i[S_2,I_2,R_2](\xi)\bigr|e^{-\mu|\xi|}\\
&\quad\le\frac{1}{\Lambda_i}\bigl|H_i(S_1,I_1,R_1)-H_i(S_2,I_2,R_2)\bigr|_\mu\biggl[\int_{-\infty}^{\xi}e^{r_{i1}(\xi-\eta)}e^{\mu|\eta|-\mu|\xi|}\,d\eta\\
&\qquad+\int_{\xi}^{\infty}e^{r_{i2}(\xi-\eta)}e^{\mu|\eta|-\mu|\xi|}\,d\eta\biggr]\\
&\quad\le\frac{l}{\Lambda_i}|\Psi_1-\Psi_2|_\mu\biggl[\int_{-\infty}^{\xi}e^{r_{i1}(\xi-\eta)}e^{\mu|\eta-\xi|}\,d\eta+\int_{\xi}^{\infty}e^{r_{i2}(\xi-\eta)}e^{\mu|\eta-\xi|}\,d\eta\biggr]\\
&\quad=\frac{l(2\mu+r_{i1}-r_{i2})}{d_i(r_{i2}-r_{i1})(r_{i2}-\mu)(r_{i1}+\mu)}|\Psi_1-\Psi_2|_\mu,\quad i=1,2,3,
\end{aligned}
$$

which implies that $\mathcal{M}$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$.

Now we turn to proving that $\mathcal{M}$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$. For any $(S,I,R)\in\mathcal{S}$, we deduce for $\xi\in\mathbb{R}$ that

$$
\begin{aligned}
&\biggl|\frac{d\mathcal{M}_1[S,I,R](\xi)}{d\xi}\biggr|\\
&\quad=\biggl|\frac{r_{11}}{\Lambda_1}\int_{-\infty}^{\xi}e^{r_{11}(\xi-\eta)}H_1[S,I,R](\eta)\,d\eta+\frac{r_{12}}{\Lambda_1}\int_{\xi}^{\infty}e^{r_{12}(\xi-\eta)}H_1[S,I,R](\eta)\,d\eta\biggr|\\
&\quad\le-\frac{r_{11}}{\Lambda_1}\int_{-\infty}^{\xi}e^{r_{11}(\xi-\eta)}H_1[S,I,R](\eta)\,d\eta+\frac{r_{12}}{\Lambda_1}\int_{\xi}^{\infty}e^{r_{12}(\xi-\eta)}H_1[S,I,R](\eta)\,d\eta\\
&\quad\le-\frac{r_{11}mS_0}{\Lambda_1}\int_{-\infty}^{\xi}e^{r_{11}(\xi-\eta)}\,d\eta+\frac{r_{12}mS_0}{\Lambda_1}\int_{\xi}^{\infty}e^{r_{12}(\xi-\eta)}\,d\eta\\
&\quad=\frac{2mS_0}{\Lambda_1},
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\biggl|\frac{d\mathcal{M}_2[S,I,R](\xi)}{d\xi}\biggr|&=\biggl|\frac{r_{21}}{\Lambda_2}\int_{-\infty}^{\xi}e^{r_{21}(\xi-\eta)}H_2[S,I,R](\eta)\,d\eta\\
&\quad+\frac{r_{22}}{\Lambda_2}\int_{\xi}^{\infty}e^{r_{22}(\xi-\eta)}H_2[S,I,R](\eta)\,d\eta\biggr|\\
&\le-\frac{r_{21}}{\Lambda_2}\int_{-\infty}^{\xi}e^{r_{21}(\xi-\eta)}H_2[S,I,R](\eta)\,d\eta\\
&\quad+\frac{r_{22}}{\Lambda_2}\int_{\xi}^{\infty}e^{r_{22}(\xi-\eta)}H_2[S,I,R](\eta)\,d\eta\\
&\le-\frac{r_{21}(m+\beta S_0-\gamma)\bar{I}}{\Lambda_2}\int_{-\infty}^{\xi}e^{r_{21}(\xi-\eta)}\,d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{r_{22}(m+\beta S_0-\gamma)\bar{I}}{\Lambda_2}\int_{\xi}^{\infty} e^{r_{22}(\xi-\eta)}\,d\eta \\
&=\frac{2(m+\beta S_0-\gamma)\bar{I}}{\Lambda_2},
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
\left|\frac{d\mathcal{M}_3[S,I,R](\xi)}{d\xi}\right| &= \left|\frac{r_{31}}{\Lambda_3}\int_{-\infty}^{\xi} e^{r_{31}(\xi-\eta)}H_3[S,I,R](\eta)\,d\eta \right.\\
&\quad \left. +\frac{r_{32}}{\Lambda_3}\int_{\xi}^{\infty} e^{r_{32}(\xi-\eta)}H_3[S,I,R](\eta)\,d\eta\right| \\
&\le -\frac{r_{31}}{\Lambda_3}\int_{-\infty}^{\xi} e^{r_{31}(\xi-\eta)}\bigl(mL_3e^{\sigma_1\eta}+\gamma\bar{I}\bigr)\,d\eta \\
&\quad +\frac{r_{32}}{\Lambda_3}\int_{\xi}^{\infty} e^{r_{32}(\xi-\eta)}\bigl(mL_3e^{\sigma_1\eta}+\gamma\bar{I}\bigr)\,d\eta \\
&\le \frac{mL_3|2r_{31}r_{32}-\sigma_1(r_{31}+r_{32})|}{\Lambda_3|(\sigma_1-r_{31})(\sigma_1-r_{32})|}e^{\sigma_1\xi}+\frac{2\gamma\bar{I}}{\Lambda_3}.
\end{aligned}
\tag{3.25}
$$

It follows from Lemma 3.6 that $|\mathcal{M}_1[S,I,R](\xi)|+|\mathcal{M}_2[S,I,R](\xi)|+|\mathcal{M}_3[S,I,R](\xi)|\le S_0+\bar{I}+L_3e^{\sigma_1\xi}$ on $\mathbb{R}$. Recall that $\mu>\sigma_1$. Then, for any $\varepsilon>0$, there is a sufficiently large number $N>0$ such that

$$
\begin{aligned}
&\bigl\{\bigl|\mathcal{M}_1[S,I,R](\xi)\bigr|+\bigl|\mathcal{M}_2[S,I,R](\xi)\bigr|+\bigl|\mathcal{M}_3[S,I,R](\xi)\bigr|\bigr\}e^{-\mu|\xi|} \\
&\quad\le \bigl(S_0+\bar{I}+L_3e^{\sigma_1\xi}\bigr)e^{-\mu|\xi|} \\
&\quad< (S_0+\bar{I})e^{-\mu N}+L_3e^{(\sigma_1-\mu)N} \\
&\quad< \varepsilon,\quad |\xi|>N.
\end{aligned}
\tag{3.26}
$$

Utilizing (3.23)–(3.25) and Arzerà–Ascoli theorem, we can select finite elements in $\mathcal{M}(\mathcal{S})$ such that they are a finite ε-net of $\mathcal{M}(\mathcal{S})(\xi)$ on $[-N,N]$ with the supremum norm, a finite ε-net of $\mathcal{M}(\mathcal{S})(\xi)$ on $\mathbb{R}$ with the norm $|\cdot|_\mu$ (see (3.26)). Thus $\mathcal{M}$ is compact with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$. The proof of this lemma is completed. □

By Lemma 3.6, Lemma 3.7, and Schauder's fixed point theorem, we deduce that the operator $\mathcal{M}$ has a fixed point $(S(\xi),I(\xi),R(\xi))\in\mathcal{S}$, which is a solution of the system

$$
\begin{cases}
d_1S''(\xi)-c^*S'(\xi)-\frac{\beta S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}=0,\\
d_2I''(\xi)-c^*I'(\xi)+\frac{\beta S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}-\gamma I(\xi)=0,\\
d_3R''(\xi)-c^*R'(\xi)+\gamma I(\xi)=0.
\end{cases}
\tag{3.27}
$$

Based on the above analysis, we have the following results.

Proposition 3.1 *If $\mathcal{R}_0>1$ and $c=c^*$, then system (1.1) admits a traveling wave solution $(S(\xi),I(\xi),R(\xi))$ such that*

$$
\underline{S}(\xi)\le S(\xi)\le\bar{S}(\xi),\qquad \underline{I}(\xi)\le I(\xi)\le\bar{I}(\xi),\qquad \underline{R}(\xi)\le R(\xi)\le\bar{R}(\xi),\quad \xi\in\mathbb{R}.
\tag{3.28}
$$

3.3 Properties of the critical traveling wave solutions

In this section, we focus on some properties of the critical traveling wave solution of (2.1), that is, the proof of the four properties in Theorem 2.1.

Proof (1) By contradiction, suppose that $I(\hat{\xi}) = 0$ for some $\hat{\xi} \in \mathbb{R}$. Then there are two constants $a, b \in \mathbb{R}$ such that $a < \xi_3 \leq b$ and $a < \hat{\xi} < b$, which implies that $I(\xi)$ attains its minimum in (a, b). It follows from the second equation in (3.27) that $-d_2 I''(\xi) + c^* I'(\xi) + \gamma I(\xi) \geq 0$ for $\xi \in [a, b]$. By the strong maximum principle, we deduce that $I(\xi) \equiv 0$ for $\xi \in [a, b]$, which contradicts the fact that $I(\xi) \geq I_-(\xi) > 0$ for $\xi \in [a, \xi_3)$. Thus, $I(\xi) > 0$ on $\mathbb{R}$. Similarly, one can obtain $S(\xi) > 0$ on $\mathbb{R}$. Assume that $R(\tilde{\xi}) = 0$ for some $\tilde{\xi} \in \mathbb{R}$, then $R'(\tilde{\xi}) = 0$ and $R''(\tilde{\xi}) \geq 0$. We infer from the third equation in (3.28) that $I(\tilde{\xi}) \leq 0$, which contradicts the positiveness of $I(\xi)$ on $\mathbb{R}$. This implies that $R(\xi) > 0$ on $\mathbb{R}$.

(2) From (3.28), we get

$$
\begin{aligned}
&S_0\left(1 - \sigma_2^{-1} e^{\sigma_2 \xi}\right) \leq S(\xi) \leq S_0, \\
&\left[-L_1 \xi - L_2 (-\xi)^{\frac{1}{2}}\right] e^{\lambda^* \xi} \leq I(\xi) \leq -L_1 \xi e^{\lambda^* \xi}, \\
&0 \leq R(\xi) \leq L_3 e^{\sigma_1 \xi}, \quad \xi \in \mathbb{R}.
\end{aligned}
$$

Then using the squeeze rule yields that

$$
S(-\infty) = S_0, \qquad I(-\infty) = 0, \qquad R(-\infty) = 0 \quad \text{and} \quad I(\xi) = O\left(-\xi e^{\lambda^* \xi}\right) \tag{3.29}
$$

as $\xi \to -\infty$.

(3) Since $S(\xi)$ and $I(\xi)$ are uniformly bounded on $\mathbb{R}$, we have from the first two equations in (3.27) that

$$
\begin{cases}
S(\xi) = \frac{1}{\Lambda_1}\{\int_{-\infty}^{\xi} e^{r_{11}(\xi-\eta)} H_1[S,I,R](\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_{12}(\xi-\eta)} H_1[S,I,R](\eta)\,d\eta\}, \\
I(\xi) = \frac{1}{\Lambda_2}\{\int_{-\infty}^{\xi} e^{r_{21}(\xi-\eta)} H_2[S,I,R](\eta)\,d\eta + \int_{\xi}^{\infty} e^{r_{22}(\xi-\eta)} H_2[S,I,R](\eta)\,d\eta\},
\end{cases} \tag{3.30}
$$

where

$$
H_1[S,I,R](\eta) = mS(\eta) - \frac{\beta S(\eta) I(\eta - c^*\tau)}{1 + \alpha I(\eta - c^*\tau)}
$$

and

$$
H_2[S,I,R](\eta) = \frac{\beta S(\eta) I(\eta - c^*\tau)}{1 + \alpha I(\eta - c^*\tau)} + (m - \gamma) I(\eta).
$$

Using L'Hôpital rule in (3.30) gives

$$
S'(\pm\infty) = 0 \quad \text{and} \quad I'(\pm\infty) = 0. \tag{3.31}
$$

Integrating the first equation in (3.27) from $-\infty$ to ξ and using (3.29) and (3.31), we have that

$$
\beta \int_{-\infty}^{\xi} \frac{S(\eta) I(\eta - c^*\tau)}{1 + \alpha I(\eta - c^*\tau)}\,d\eta = c^*\left[S(\xi) - S_0\right] - d_1 S'(\xi) \leq c^* S_0 - d_1 S'(\xi), \quad \xi \in \mathbb{R}. \tag{3.32}
$$

Again, integrating the second equation in (3.27) from $-\infty$ to ξ and utilizing (3.29), (3.31), and (3.32), we get that

$$\begin{aligned}\gamma\int_{-\infty}^{\xi} I(\eta)\,d\eta &= d_2I'(\xi)-c^*I(\xi)+\beta\int_{-\infty}^{\xi}\frac{S(\eta)I(\eta-c^*\tau)}{1+\alpha I(\eta-c^*\tau)}\,d\eta\\ &\le d_2I'(\xi)+\beta\int_{-\infty}^{\xi}\frac{S(\eta)I(\eta-c^*\tau)}{1+\alpha I(\eta-c^*\tau)}\,d\eta\\ &\le d_2I'(\xi)+c^*S_0-d_1S'(\xi),\quad \xi\in\mathbb{R}.\end{aligned}$$

Then, by the virtue of (3.31), we further obtain $\int_{\mathbb{R}} I(\xi)\,d\xi<\infty$, which together with the boundedness of $I'(\xi)$ on $\mathbb{R}$ (see (3.31)) implies that

$$I(\infty)=0. \tag{3.33}$$

It follows from the first equation in (3.27) that

$$\big[e^{-\frac{c^*}{d_1}\xi}S'(\xi)\big]'=\frac{\beta}{d_1}e^{-\frac{c^*}{d_1}\xi}\frac{S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}. \tag{3.34}$$

Integrating (3.34) from ξ to ∞, utilizing $S'(\infty)=0$ and $S(\xi),I(\xi)>0$ on $\mathbb{R}$, we deduce

$$S'(\xi)=-\frac{\beta}{d_1}\int_{\xi}^{\infty}e^{\frac{c^*}{d_1}(\xi-\eta)}\frac{S(\eta)I(\eta-c^*\tau)}{1+\alpha I(\eta-c^*\tau)}\,d\eta<0, \tag{3.35}$$

which means that $S(\xi)$ is strictly decreasing on $\mathbb{R}$. This together with $S(\xi)>0$ on $\mathbb{R}$ gives that the limit $S(\infty)$ exists and $S(\infty):=\varepsilon_0<S_0$. Moreover, an integration of the first equation in (3.27) over $\mathbb{R}$ gives

$$\beta\int_{-\infty}^{\infty}\frac{S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}\,d\xi=c^*(S_0-\varepsilon_0), \tag{3.36}$$

where we have used (3.29) and (3.31). Another integration of the second equation in (3.27) over $\mathbb{R}$ yields

$$\gamma\int_{-\infty}^{\infty}I(\xi)\,d\xi=\beta\int_{-\infty}^{\infty}\frac{S(\xi)I(\xi-c^*\tau)}{1+\alpha I(\xi-c^*\tau)}\,d\xi, \tag{3.37}$$

since $I(\pm\infty)=I'(\pm\infty)=0$. Solving the third equation in (3.27) and using $R(-\infty)=0$ lead to

$$R(\xi)=Ce^{\frac{c^*}{d_3}\xi}+\frac{\gamma}{c^*}\int_{-\infty}^{\xi}I(\eta)\,d\eta+\frac{\gamma}{c^*}\int_{\xi}^{+\infty}e^{\frac{c^*}{d_3}(\xi-\eta)}I(\eta)\,d\eta,$$

where C is a constant of integration. Since $R(\xi)\le L_3e^{\sigma_1\xi}$ and $\sigma_1<c^*/d_3$ (see the proof of Lemma 3.4), we obtain

$$R(\xi)=\frac{\gamma}{c^*}\int_{-\infty}^{\xi}I(\eta)\,d\eta+\frac{\gamma}{c^*}\int_{\xi}^{\infty}e^{\frac{c^*}{d_3}(\xi-\eta)}I(\eta)\,d\eta. \tag{3.38}$$

We infer from (3.36)–(3.38) and L'Hôpital's rule that

$$R(\infty) = \frac{\gamma}{c^*}\int_{-\infty}^{\infty} I(\xi)\,d\xi = S_0 - \varepsilon_0. \tag{3.39}$$

Differentiating (3.38) with respect to ξ and using $I(\xi) > 0$ on $\mathbb{R}$, we have

$$R'(\xi) = \frac{\gamma}{d_3}\int_{\xi}^{\infty} e^{\frac{c^*}{d_3}(\xi-\eta)} I(\eta)\,d\eta > 0, \tag{3.40}$$

which means that $R(\xi)$ is strictly increasing on $\mathbb{R}$. Combining (3.40), $I(\pm\infty) = 0$, and L'Hôpital's rule yields

$$R'(\pm\infty) = 0. \tag{3.41}$$

Note from (3.29), (3.31), (3.32), (3.39), and (3.41) that

$$S''(\pm\infty) = 0, \qquad I''(\pm\infty) = 0, \quad \text{and} \quad R''(\pm\infty) = 0. \tag{3.42}$$

(4) Since $S(\xi)$ is strictly decreasing and $R(\xi)$ is strictly increasing on $\mathbb{R}$, we obtain $S(\xi) < S_0$ and $R(\xi) < S_0$ for $\xi \in \mathbb{R}$. Now we claim that $I(\xi) < \frac{1}{\alpha}(\frac{\beta S_0}{\gamma} - 1)$ on $\mathbb{R}$. For contradiction, we assume that $I(\acute{\xi}) = \frac{1}{\alpha}(\frac{\beta S_0}{\gamma} - 1)$ for some $\acute{\xi} \in \mathbb{R}$, which results in $I'(\acute{\xi}) = 0$ and $I''(\acute{\xi}) \le 0$. By the second equation in (3.27) and $S(\acute{\xi}) < S_0$, we deduce that

$$\begin{aligned}
0 &= d_2 I''(\acute{\xi}) - c^* I'(\acute{\xi}) + \frac{\beta S(\acute{\xi}) I(\acute{\xi} - c^*\tau)}{1 + \alpha I(\acute{\xi} - c^*\tau)} - \gamma I(\acute{\xi}) \\
&\le \frac{\beta S(\acute{\xi}) I(\acute{\xi} - c^*\tau)}{1 + \alpha I(\acute{\xi} - c^*\tau)} - \gamma I(\acute{\xi}) \\
&< \frac{\frac{\beta S_0}{\alpha}(\frac{\beta S_0}{\gamma} - 1)}{1 + (\frac{\beta S_0}{\gamma} - 1)} - \frac{\gamma}{\alpha}\left(\frac{\beta S_0}{\gamma} - 1\right) \\
&= 0,
\end{aligned}$$

leading to a contradiction. Thus $I(\xi) < \frac{1}{\alpha}(\frac{\beta S_0}{\gamma} - 1)$ on $\mathbb{R}$. The proof is completed. □

4 Proof of Theorem 2.2

This proof is based on the contradictory argument. Suppose that the pair of continuous positive functions $(S(\xi), I(\xi), R(\xi))$ $(\xi \in \mathbb{R})$ is a solution of the wave system of (1.1)

$$\begin{cases} d_1 S''(\xi) - cS'(\xi) - \frac{\beta S(\xi) I(\xi - c\tau)}{1 + \alpha I(\xi - c\tau)} = 0, \\ d_2 I''(\xi) - cI'(\xi) + \frac{\beta S(\xi) I(\xi - c\tau)}{1 + \alpha I(\xi - c\tau)} - \gamma I(\xi) = 0, \\ d_3 R''(\xi) - cR'(\xi) + \gamma I(\xi) = 0, \end{cases} \tag{4.1}$$

satisfying the asymptotic boundary conditions

$$(S, I, R)(-\infty) = (S_0, 0, 0), \qquad (S, I, R)(\infty) = (\varepsilon, 0, S_0 - \varepsilon), \tag{4.2}$$

where $c \in \mathbb{R}$ is the wave speed. The proof of Theorem 2.2 is divided into two cases: the one is $\mathcal{R}_0 = 1$ and $c \in \mathbb{R}$; the other one is $\mathcal{R}_0 > 1$ and $c \le 0$.

4.1 Case 1: $\mathcal{R}_0 = 1$ and $c \in \mathbb{R}$

From the second equation of (4.1), we get

$$I(\xi) = \frac{\beta}{d_2(\lambda^+ - \lambda^-)}\left[\int_{-\infty}^{\xi} e^{\lambda^-(\xi-\eta)}\frac{S(\eta)I(\eta-c\tau)}{1+\alpha I(\eta-c\tau)}\,d\eta + \int_{\xi}^{\infty} e^{\lambda^+(\xi-\eta)}\frac{S(\eta)I(\eta-c\tau)}{1+\alpha I(\eta-c\tau)}\,d\eta\right], \tag{4.3}$$

where

$$\lambda^- = \frac{c-\sqrt{c^2+4d_2\gamma}}{2d_2} \quad \text{and} \quad \lambda^+ = \frac{c+\sqrt{c^2+4d_2\gamma}}{2d_2}.$$

Applying L'Hôpital rule in (4.3) yields $I'(\pm\infty) = 0$. Then, integrating the second equation in (4.1) over $\mathbb{R}$ and using $\mathcal{R}_0 = 1$, that is, $\beta S_0 = \gamma$, we obtain

$$\gamma\int_{-\infty}^{\infty} I(\xi)\,d\xi = \beta\int_{-\infty}^{\infty}\frac{S(\xi)I(\xi-c\tau)}{1+\alpha I(\xi-c\tau)}\,d\xi. \tag{4.4}$$

By $\int_{\mathbb{R}} I(\xi)\,d\xi = \int_{\mathbb{R}} I(\xi-c\tau)\,d\xi$ and $\sup_{\xi\in\mathbb{R}} S(\xi) \le S_0$ and (4.4), we have

$$\begin{aligned}
0 &= \gamma\int_{-\infty}^{\infty} I(\xi)\,d\xi - \beta\int_{-\infty}^{\infty}\frac{S(\xi)I(\xi-c\tau)}{1+\alpha I(\xi-c\tau)}\,d\xi \\
&= \gamma\int_{-\infty}^{\infty} I(\xi)\,d\xi - \beta\int_{-\infty}^{\infty}\frac{S(\xi+c\tau)I(\xi)}{1+\alpha I(\xi)}\,d\xi \\
&\ge \beta\int_{-\infty}^{\infty}\bigl(S_0 - S(\xi+c\tau)\bigr)I(\xi)\,d\xi \\
&\ge 0,
\end{aligned}$$

which leads to

$$\beta\int_{-\infty}^{\infty}\bigl(S_0 - S(\xi+c\tau)\bigr)I(\xi)\,d\xi = 0. \tag{4.5}$$

By a similar argument as that in (3.35) and the fact $\sup_{\xi\in\mathbb{R}} S(\xi) \le S_0$, we get from (4.5) that

$$\bigl(S_0 - S(\xi+c\tau)\bigr)I(\xi) = 0, \quad \xi \in \mathbb{R}, \tag{4.6}$$

which together with $I(\xi) > 0$ implies that

$$S(\xi) = S_0, \quad \xi \in \mathbb{R}. \tag{4.7}$$

A contradiction appears. The proof is completed.

4.2 Case 2: $\mathcal{R}_0 > 1$ and $c \le 0$

Due to $S(-\infty) = S_0$ and $I(-\infty) = 0$, we have

$$\lim_{\xi\to-\infty}\frac{\beta S(\xi)}{1+\alpha I(\xi-c\tau)} = \beta S_0.$$

Then there exists a sufficiently small constant $\xi^* < 0$ such that

$$\frac{\beta S(\xi)}{1+\alpha I(\xi - c\tau)} > \frac{\beta S_0 + \gamma}{2}, \quad \xi < \xi^*.$$

Thus, from the second equation in (3.27), we obtain

$$\begin{aligned} cI'(\xi) &= d_2 I''(\xi) + \frac{\beta S(\xi) I(\xi - c\tau)}{1+\alpha I(\xi - c\tau)} - \gamma I(\xi) \\ &\geq d_2 I''(\xi) + \frac{\beta S_0 + \gamma}{2}\big(I(\xi - c\tau) - I(\xi)\big) + \frac{\beta S_0 - \gamma}{2} I(\xi), \quad \xi < \xi^*. \end{aligned} \tag{4.8}$$

By the integrability of $I(\xi)$ on $\mathbb{R}$, we can define

$$Q(\xi) := \int_{-\infty}^{\xi} I(x)\,dx, \quad \xi \in \mathbb{R}. \tag{4.9}$$

Since $I(\xi) > 0$ in $\mathbb{R}$, one can see that $Q(\xi)$ is strictly increasing on $\mathbb{R}$. Integrating (4.8) from $-\infty$ to ξ ($\xi < \xi^*$) and using $I(-\infty) = 0$ and $I'(-\infty) = 0$ yield that

$$cI(\xi) \geq d_2 I'(\xi) + \frac{\beta S_0 + \gamma}{2}\big(Q(\xi - c\tau) - Q(\xi)\big) + \frac{\beta S_0 - \gamma}{2} Q(\xi), \quad \xi < \xi^*. \tag{4.10}$$

Integrating (4.10) from $-\infty$ to ξ, for $\xi < \xi^*$, we get that

$$cQ(\xi) \geq d_2 I(\xi) + \frac{\beta S_0 + \gamma}{2}\int_{-\infty}^{\xi}\big(Q(x - c\tau) - Q(x)\big)\,dx + \frac{\beta S_0 - \gamma}{2}\int_{-\infty}^{\xi} Q(x)\,dx. \tag{4.11}$$

Noting that $c \leq 0$, $\tau > 0$, and $Q(\xi)$ is strictly increasing in $\mathbb{R}$, we obtain from (4.11) that

$$\begin{aligned} 0 &\geq cQ(\xi) \\ &\geq d_2 I(\xi) + \frac{\beta S_0 + \gamma}{2}\int_{-\infty}^{\xi}\big(Q(x - c\tau) - Q(x)\big)\,dx + \frac{\beta S_0 - \gamma}{2}\int_{-\infty}^{\xi} Q(x)\,dx \\ &> 0, \xi < \xi^*. \end{aligned} \tag{4.12}$$

A contradiction occurs. The proof is finished.

5 Conclusion

In this paper we have solved the open problems raised in the introduction, which are different from those in [26]. In the proof of the existence of critical traveling waves, we constructed a new pair of upper and lower solutions, which was an innovation of the paper. Then we mainly used the contradictory arguments and subtle analysis to establish the non-existence of traveling wave solutions for the cases: (i) $\mathcal{R}_0 = 1$ and $c \in \mathbb{R}$; (ii) $\mathcal{R}_0 > 1$ and $c \leq 0$. In order to address the change of the number for R-component in (1.1), we study the three equations together, which is helpful to describe the whole transmission behavior of the epidemic model. In Theorem 2.1, we obtained a lot of nice properties of the traveling wave solutions for (1.1). Our method adopted here can be used to improve the corresponding results for super-critical traveling wave solutions in [26] and also be helpful to the study of critical traveling wave solutions.

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