Ntouyas and Samei *Advances in Difference Equations* (2019) 2019:475
https://doi.org/10.1186/s13662-019-2414-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus

Sotiris K. Ntouyas[1,2] and Mohammad Esmael Samei[3*]

*Correspondence:
mesamei@gmail.com;
mesamei@basu.ac.ir
[3]Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan, Iran
Full list of author information is available at the end of the article

Abstract

In this investigation, by applying the definition of the fractional q-derivative of the Caputo type and the fractional q-integral of the Riemann–Liouville type, we study the existence and uniqueness of solutions for a multi-term nonlinear fractional q-integro-differential equations under some boundary conditions ${}^cD_q^\alpha x(t) = w(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), {}^cD_q^{\beta_1} x(t), {}^cD_q^{\beta_2} x(t), \ldots, {}^cD_q^{\beta_n} x(t))$. Our results are based on some classical fixed point techniques, as Schauder's fixed point theorem and Banach contraction mapping principle. Besides, some instances are exhibited to illustrate our results and we report all algorithms required along with the numerical result obtained.

MSC: Primary 34A08; 39A13; secondary 34K37

Keywords: Multi-term fractional q-integro-differential equation; Caputo q-derivative; Quantum calculus; Fixed point

1 Introduction

The subjects of fractional calculus and q-calculus are one of the significant branches in mathematical analysis. In 1910, the subject of q-difference equations was introduced by Jackson [1]. After that, at the beginning of the last century, studies on the q-difference equation appeared in much work, especially in [2–6]. For some earlier work on the topic, we refer to [7, 8], whereas the preliminary concepts on q-fractional calculus can be found in [9], as indicated: *Perhaps Leibniz did not expect this number of applications when he sent a letter in 1695 to L'Hopital asking about the meaning of the derivative of order half.* For countless applications on the q-fractional calculus, see for example [10–17].

In the recent years, there have appeared many papers about differential and integro-differential equations and inclusions which are valuable tools in the modeling of many phenomena in various fields of science [18–25]. In 2012, Ahmad *et al.* [26] discussed the existence and uniqueness of solutions for the fractional q-difference equations ${}^cD_q^\alpha u(t) = T(t, u(t))$, $\alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1)$ and $\alpha_2 u(1) - \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, for $t \in I$, where $\alpha \in (1, 2]$, α_i, β_i, γ_i, η_i are real numbers, for $i = 1, 2$ and $T \in C(J \times \mathbb{R}, \mathbb{R})$. In 2013, Zhao *el al.* [27] reviewed the q-integral problem $(D_q^\alpha u)(t) + f(t, u(t)) = 0$, with the conditions that $u(1)$, $u(0)$ are equal to $\mu I_q^\beta u(\eta)$, 0, respectively, for almost all $t \in (0, 1)$, where $q \in (0, 1)$ and α, β, η belong to $(1, 2]$, $(0, 2]$, $(0, 1)$, respectively, μ is positive real number, D_q^α is the

q-derivative of Riemann–Liouville and we have a real-valued continuous map u defined on $I \times [0,\infty)$. In 2014, Ahmad *et al.* [28] considered the problem

$$\begin{cases} {}^cD_q^{\beta}({}^cD_q^{\gamma} + \lambda)u(t) = pf(t,u(t)) + kI_q^{\xi}g(t,u(t)), \\ \alpha_1 u(0) - \beta_1(t^{(1-\gamma)}D_q u(0))|_{t=0} = \sigma_1 u(\eta_1), \qquad \alpha_2 u(1) + \beta_2 D_q u(1) = \sigma_2 u(\eta_2), \end{cases}$$

for $t, q \in [0,1]$, where ${}^cD_q^{\beta}$ and ${}^cD_q^{\gamma}$ denote the fractional q-derivative of the Caputo type, $0 < \beta$, $\gamma \leq 1$, $I_q^{\xi}(.)$ denotes the Riemann–Liouville integral with $\xi \in (0,1)$, f, g are given continuous functions, λ and p, k are real constants and $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}$, $\eta_i \in (0,1)$, $i = 1,2$. Also, one may refer to some research of Ahmad *et al.*, in the recent years in [12, 14, 29–31]. In 2016, Abdeljawad *et al.* [32] stated and proved a new discrete q-fractional version of Gronwall inequality, ${}_qC_a^{\alpha}u(t) = T(t,u(t))$, where $u(a) = \gamma$, such that $\alpha \in (0,1]$, $a \in \mathbb{T}_q = \{q^n : n \in \mathbb{Z}\}$, t belongs to $\mathbb{T}_a = [0,\infty)_q = \{q^{-i}a : i = 0,1,2,\ldots\}$, ${}_qC_a^{\alpha}$ means the Caputo fractional difference of order α and $T(t,x)$ fulfills a Lipschitz condition for all t and x. In 2019, Samei *et al.* [25] investigated the existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions

In this article, motivated by these achievements and the following results, we are working to stretch out solutions for the multi-term nonlinear fractional q-integro-differential equation with boundary conditions,

$$ {}^cD_q^{\alpha}x(t) = w\big(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), {}^cD_q^{\beta_1}x(t), {}^cD_q^{\beta_2}x(t), \ldots, {}^cD_q^{\beta_n}x(t)\big), \tag{1}$$

under conditions $x(0) + ax(1) = 0$ and $x'(0) + bx'(1) = 0$, for $t \in J := [0,1]$ and all $q \in (0,1)$, where $1 < \alpha < 2$, $\beta_i \in (0,1)$ with $i = 1,2,\ldots,n$, $a, b \neq -1$, $w : J \times \mathbb{R}^{n+3} \to \mathbb{R}$ is continuous for all variables and the mappings γ_j map $J \times J$ to $\mathbb{R}^+$ such that $\sup_{t\in J} |\int_0^t \gamma_j(t,s)\, d_qs|$, where $j = 1,2$, are finite, the maps φ_j, where $j = 1,2$, are defined by $(\varphi_j u)(t) = \int_0^t \gamma_j(t,s)u(s)\, d_qs$.

The rest of the paper is arranged as follows: in Sect. 2, we recall some preliminary concepts and fundamental results of q-calculus. Section 3 is devoted to the main results, while examples illustrating the obtained results and algorithm for the problems are presented in Sect. 4.

2 Preliminaries

First of all, we point out some of the materials on the fractional q-calculus and fundamental results of it which needed in the next sections (for more information, consider [1, 9–11, 33]). Then, some well-known theorems of fixed point theorems are presented.

Assume that $q \in (0,1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [1]. The power function $(x-y)_q^n$ with $n \in \mathbb{N}_0$ is $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x - yq^k)$ and $(x-y)_q^{(0)} = 1$ where $x, y \in \mathbb{R}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [10]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have $(x-y)_q^{(\alpha)} = x^{\alpha}\prod_{k=0}^{\infty}(x - yq^k)/(x - yq^{\alpha+k})$. If $y = 0$, then it is clear that $x^{(\alpha)} = x^{\alpha}$ (Algorithm 1). The q-Gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}$ [1]. Note that $\Gamma_q(z+1) = [z]_q\Gamma_q(z)$. The value of the q-Gamma function, $\Gamma_q(z)$, for input values q and z with counting the number of sentences n in summation is addressed by a simplifying analysis. For this design, we present a pseudo-code description of the technique for estimating q-Gamma function of order n which show in Algorithm 2. The q-derivative of the function f is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$, which is shown in Algorithm 3 [4]. Also,

Algorithm 1 The proposed method for calculation of $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q

$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^\alpha * s$
end if

Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculation of $\Gamma_q(x)$

Input: $n, q \in (0,1), x \in \mathbb{R}\backslash\{0,-1,2,\ldots\}$

$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1-q^{k+1})(1-q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$

Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculation of $(D_q f)(x)$

Input: $q \in (0,1), f(x), x$

syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q * z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q * x))/((1-q)x)$
end if

Output: $(D_q f)(x)$

the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [4]. The q-integral of a function f defined on $[0,b]$ is defined by

$$I_q f(x) = \int_0^x f(s)\, d_q s = x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right),$$

for $0 \leq x \leq b$, provided that the series absolutely converges [4]. For any positive number α and β, the q-Beta function is defined by [33]

$$B_q(\alpha, \beta) = \int_0^1 (1-qs)_q^{(\alpha-1)} s^{\beta-1}\, d_q s. \tag{2}$$

Algorithm 4 The proposed method for calculation of $\int_a^b f(r)\,d_q r$

Input: $q \in (0,1)$, α, n, $f(x)$, a, b

$s \leftarrow 0$

for $i = 0 : n$ **do**

  $s \leftarrow s + q^i * (b * f(b * q^i) - a * f(a * q^i))$

end for

$g \leftarrow (1-q) * s$

Output: $\int_a^b f(r)\,d_q r$

The q-derivative of the function f is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$, which is shown in Algorithm 3 [4, 11, 33]. If a is in $[0,b]$, then

$$\int_a^b f(u)\,d_q u = I_q f(b) - I_q f(a) = (1-q)\sum_{k=0}^{\infty} q^k \bigl[bf\bigl(bq^k\bigr) - af\bigl(aq^k\bigr)\bigr],$$

whenever the series exists, which is shown in Algorithm 4. The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and

$$\bigl(I_q^n h\bigr)(x) = \bigl(I_q\bigl(I_q^{n-1}h\bigr)\bigr)(x),$$

for $n \geq 1$ and $g \in C([0,b])$ [4]. It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [4]. The fractional Riemann–Liouville type q-integral of the function f on J, of $\alpha \geq 0$ is given by $(I_q^0 f)(t) = f(t)$ and

$$\bigl(I_q^\alpha f\bigr)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)\,d_q s,$$

for $t \in J$ and $\alpha > 0$ [31, 34]. Also, the fractional Caputo type q-derivative of the function f is given by

$$\begin{aligned}\bigl({}^c D_q^\alpha f\bigr)(t) &= \bigl(I_q^{[\alpha]-\alpha}\bigl(D_q^{[\alpha]} f\bigr)\bigr)(t)\\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)}\int_0^t (t-qs)^{([\alpha]-\alpha-1)}\bigl(D_q^{[\alpha]} f\bigr)(s)\,d_q s,\end{aligned} \tag{3}$$

for $t \in J$ and $\alpha > 0$ [31, 34]. It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x)$ and $(D_q^\alpha(I_q^\alpha f))(x) = f(x)$, where $\alpha, \beta \geq 0$ [34]. By using Algorithm 2, we can calculate $(I_q^\alpha f)(x)$ which is shown in Algorithm 5.

Theorem 1 (Schauder's fixed point theorem [35]) *Let E be a closed, convex and bounded subset of a Banach space X and self-map T defined on E be continuous. Then T has a fixed point in E whenever $T(E)$ is a relatively compact subset of X.*

3 Main results

Here, we investigate the inclusion of fractional q-derivative (1). First, we recall the following key result.

Algorithm 5 The proposed method for calculation of $(I_q^{\alpha} f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x

1: $s \leftarrow 0$
2: **for** $i = 0$ to n **do**
3: $\quad pf \leftarrow (1 - q^{i+1})^{\alpha-1}$
4: $\quad s \leftarrow s + pf * q^i * f(x * q^i)$
5: **end for**
6: $g \leftarrow (x^{\alpha} * (1-q) * s)/(\Gamma_q(x))$

Output: $(I_q^{\alpha} f)(x)$

Lemma 2 ([17]) *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then $I_q^{\alpha}\,{}^cD_q^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t + \cdots + c_{n-1} t^{n-1}$, where $c_0, c_1, \ldots, c_{n-1}$ belong to $\mathbb{R}$.*

Let us define the set X of all $f \in C(I)$, such that ${}^cD_q^{\beta_i} x$ belongs to $C(I)$ $(i = 1, 2, \ldots, n)$ and $q \in (0,1)$, where $0 < \beta_i < 1$. It is known that $(X, \|\cdot\|)$ with the norm $\|x\| = \max_{t\in J} |x(t)| + \sum_{i=1}^{n} \max_{t\in J} |{}^cD_q^{\beta_i} x(t)|$, is a Banach space.

Lemma 3 *Suppose that f in $C(J)$ and $\alpha \in (1,2)$. Then the boundary value problem*

$$\begin{cases} {}^cD_q^{\alpha} x(t) = f(t), & t \in J, \\ x(0) + ax(1) = 0, & x'(0) + bx'(1) = 0, \end{cases}$$

is equivalent to the following q-integral equation:

$$x(t) = I_q^{\alpha} f(t) - I_q^{\alpha} f(1) + \frac{ab - b(1+a)t}{(1+a)(1+b)} I_q^{\alpha-1} f(1). \tag{4}$$

Proof First of all, we see that Lemma 2 implies that

$$x(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\,d_q s + c_1 t + c_2, \tag{5}$$

where c_1, c_2 are arbitrary constants. By applying the boundary conditions we find

$$\begin{aligned} c_1 &= -\frac{b}{1+b} I_q^{\alpha-1} f(1), \\ c_2 &= -\frac{a}{1+a} I_q^{\alpha} f(1) + \frac{ab}{(1+a)(1+b)} I_q^{\alpha-1} f(1). \end{aligned}$$

Substituting c_1 and c_2 in (5) we get (4). The converse follows by direct computation. The proof is completed. □

Theorem 4 *Let $\ell \in L^{\frac{1}{\kappa}}(J, \mathbb{R}^+)$, $0 < \kappa < \alpha - 1$ such that*

$$|F_{t,x_i,u_i} - F_{t,x_i',v_i}| \le \ell(t) \left(\sum_{i=1}^{3} |x_i - x_i'| + \sum_{i=1}^{n} |u_i - v_i| \right),$$

for each $t \in J$, x_i, x_i', with $i = 1, 2, 3$ and $u_1, u_2, \ldots, u_n$, $v_1, v_2, \ldots, v_n \in \mathbb{R}$, where $F_{t,x_i,u_i} = w(t, x_1, x_2, x_3, u_1, u_2, \ldots, u_n)$ and $F_{t,x_i',v_i} = w(t, x_1', x_2', x_3', v_1, v_2, \ldots, v_n)$. Then the problem (1) *has a unique solution provided*

$$\begin{aligned}\Delta &= (1 + {}_0\lambda_1 + {}_0\lambda_2)\Bigg[\frac{(1+2a)\ell^* k_1}{(1+a)\Gamma_q(\alpha)} + \frac{b\ell^* k_2}{(1+a)(1+b)\Gamma_q(\alpha-1)} \\ &\quad + \sum_{i=1}^{n}\bigg(\frac{\Gamma_q(\alpha-\kappa)\ell^* k_2}{\Gamma_q(\alpha-1)\Gamma_q(\alpha-\beta_i-\kappa+1)} + \frac{b\ell^* k_2}{(1+b)\Gamma_q(2-\beta_i)\Gamma_q(\alpha-1)}\bigg)\Bigg] \\ &< 1, \end{aligned} \tag{6}$$

where

$${}_0\lambda_i = \sup_{t\in J}\left|\int_0^t \gamma_i(t,s)\,d_qs\right|, \quad i = 1,2, \qquad \ell^* = \left(\int_0^1 \big(\ell(s)\big)^{\frac{1}{\kappa}}\,d_qs\right)^{\kappa},$$

$k_1 = (\frac{1-\kappa}{\alpha-\kappa})^{1-\kappa}$ *and* $k_2 = (\frac{1-\kappa}{\alpha-\kappa-1})^{1-\kappa}$.

Proof Briefly, we put

$$F_{u(s)} = w\big(s, u(s), (\varphi_1 u)(s), (\varphi_2 u)(s), {}^cD_q^{\beta_1}u(s), {}^cD_q^{\beta_2}u(s), \dots, {}^cD_q^{\beta_n}u(s)\big),$$

and using Lemma 3, we define a self-map T on X by

$$(Tu)(t) = I_q\alpha F_{u(t)} - I_q^{\alpha}F_{u(1)} + g(t)I_q^{\alpha-1}F_{u(1)},$$

where $a_0 = \frac{a}{1+a}$ and $g(t) = \frac{ab-b(1+a)t}{(1+a)(1+b)}$ is a real-valued function on J. At present, by using the Hölder inequality, for each $u, v \in X$ and $t \in J$, we get

$$\begin{aligned}\big|(Tu)(t) - (Tv)(t)\big| &= \big|I_q^{\alpha}(F_{u(t)} - F_{v(t)}) - a_0 I_q^{\alpha}(F_{u(1)} - F_{v(1)}) \\ &\quad + g(t)I_q^{\alpha-1}(F_{u(1)} - F_{v(1)})\big| \\ &\le I_q^{\alpha}|F_{u(t)} - F_{v(t)}| + a_1 I_q^{\alpha}|F_{u(1)} - F_{v(1)}| \\ &\quad + \big|g(t)\big|I_q^{\alpha-1}|F_{u(1)} - F_{v(1)}| \\ &\le I_q^{\alpha}\Bigg(\ell(t)\Bigg(\big|u(t) - v(t)\big| + \sum_{i=1}^{2}\big|(\varphi_i u)(t) - (\varphi_i v)(t)\big| \\ &\quad + \sum_{i=1}^{n}\big|{}^cD_q^{\beta_i}u(t) - {}^cD_q^{\beta_i}v(t)\big|\Bigg)\Bigg) \\ &\quad + a_1 I_q^{\alpha}\Bigg(\ell(1)\Bigg(\big|u(1) - v(1)\big| \\ &\quad + \sum_{i=1}^{2}\big|(\varphi_i u)(1) - (\varphi_i v)(1)\big| \\ &\quad + \sum_{i=1}^{n}\big|{}^cD_q^{\beta_i}u(1) - {}^cD_q^{\beta_i}v(1)\big|\Bigg)\Bigg) \\ &\quad + a_2\big(1 + 2|a|\big)I_q^{\alpha-1}\Bigg(\ell(1)\Bigg(\big|u(1) - v(1)\big| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{2}\bigl|(\varphi_i u)(1)-(\varphi_i v)(1)\bigr| \\
&+\sum_{i=1}^{n}\bigl|{}^{c}D_q^{\beta_i}u(1)-{}^{c}D_q^{\beta_i}v(1)\bigr|\biggr)\biggr) \\
\leq{}& b_1 d\int_0^t (t-qs)^{(\alpha-1)}\ell(s)\,d_qs \\
&+a_1b_1d\int_0^1(1-qs)^{(\alpha-1)}\ell(s)\,d_qs \\
&+a_2b_2\bigl(1+2|a|\bigr)d\int_0^1(1-qs)^{(\alpha-2)}\ell(s)\,d_qs \\
\leq{}& b_1d\biggl(\int_0^t\bigl((t-qs)^{(\alpha-1)}\bigr)^{\frac{1}{1-\kappa}}\,d_qs\biggr)^{1-\kappa}\biggl(\int_0^t\bigl(\ell(s)\bigr)^{\frac{1}{\kappa}}\,d_qs\biggr)^{\kappa} \\
&+a_1b_1d\biggl(\int_0^1\bigl((1-qs)^{(\alpha-1)}\bigr)^{\frac{1}{1-\kappa}}\,d_qs\biggr)^{1-\kappa} \\
&\times\biggl(\int_0^1\bigl(\ell(s)\bigr)^{\frac{1}{\kappa}}\,d_qs\biggr)^{\kappa} \\
&+a_2b_2\bigl(1+2|a|\bigr)d\biggl(\int_0^1\bigl((1-qs)^{(\alpha-2)}\bigr)^{\frac{1}{1-\kappa}}\,d_qs\biggr)^{1-\kappa} \\
&\times\biggl(\int_0^1\bigl(\ell(s)\bigr)^{\frac{1}{\kappa}}\,d_qs\biggr)^{\kappa} \\
\leq{}& b_1\ell^*dk_1+a_1b_1\ell^*dk_1+a_2b_2\bigl(1+2|a|\bigr)\ell^*dk_2 \\
\leq{}& \biggl[\frac{(1+2|a|)b_1\ell^*}{|1+a|}k_1+a_2\bigl(1+2|a|\bigr)b_2\ell^*k_2\biggr]d,
\end{aligned}
$$

where $d=\|u-v\|$, $a_1=\frac{|a|}{|1+a|}$, $a_2=\frac{|b|}{|1+a||1+b|}$, $b_1=\frac{1+{}_0\lambda_1+{}_0\lambda_2}{\Gamma_q(\alpha)}$ and $b_2=\frac{1+{}_0\lambda_1+{}_0\lambda_2}{\Gamma_q(\alpha-1)}$. Also, we have

$$
\begin{aligned}
\bigl|{}^{c}D_q^{\beta_i}(Tu)(t)-{}^{c}D_q^{\beta_i}(Tv)(t)\bigr| &= \bigl|I_q^{1-\beta_i}(Tu)'(t)-I_q^{1-\beta_i}(Tv)'(t)\bigr| \\
&= \biggl|I_q^{1-\beta_i}\biggl(I_q^{\alpha-1}F_{u(s)}-\frac{b}{1+b}I_q^{\alpha-1}F_{u(1)}\biggr) \\
&\quad -I_q^{1-\beta_i}\biggl(I_q^{\alpha-1}F_{v(s)}-\frac{b}{1+b}I_q^{\alpha-1}F_{v(1)}\biggr)\biggr| \\
&\leq I_q^{1-\beta_i}\bigl(I_q^{\alpha-1}|F_{u(s)}-F_{v(s)}|\bigr) \\
&\quad +a_3I_q^{1-\beta_i}\bigl(I_q^{\alpha-1}|F_{u(s)}-F_{v(s)}|\bigr) \\
&\leq b_1dI_q^{1-\beta_i}\biggl(\int_0^s(s-q\tau)^{(\alpha-2)}\ell(\tau)\,d_q\tau\biggr)\,d_qs \\
&\quad +a_3b_2dI_q^{1-\beta_i}\biggl(\int_0^1(1-q\tau)^{(\alpha-2)}\ell(\tau)\,d_q\tau\biggr)\,d_qs \\
&\leq \frac{b_2\ell^*d}{\Gamma_q(1-\beta_i)}k_2\int_0^t(t-qs)^{(-\beta_i)}s^{\alpha-\kappa-1}\,d_qs \\
&\quad +\frac{a_3b_2\ell^*d}{\Gamma_q(1-\beta_i)}k_2\int_0^t(t-qs)^{(-\beta_i)}\,d_qs
\end{aligned}
$$

$$\leq \frac{b_2\ell^* d}{\Gamma_q(1-\beta_i)} k_2 \int_0^1 (1-qs)^{(-\beta_i)} s^{\alpha-\kappa-1}\, d_q s + \frac{a_3 b_2 \ell^* d}{|1+b|)\Gamma_q(2-\beta_i)} k_2,$$

where $a_3 = \frac{|b|}{|b+1|}$. Since

$$B_q(\alpha-\kappa, 1-\beta_i) = \int_0^1 (1-qs)^{(-\beta_i)} s^{\alpha-\kappa-1}\, d_q s = \frac{\Gamma_q(\alpha-\kappa)\Gamma_q(1-\beta_i)}{\Gamma_q(\alpha-\beta_i-\kappa+1)},$$

we obtain

$$\left|{}^c D_q^{\beta_i}(Tu)(t) - {}^c D_q^{\beta_i}(Tv)(t)\right| \leq \left[\frac{b_2 \Gamma(\alpha-\kappa)\ell^*}{\Gamma_q(\alpha-\beta_i-\kappa+1)} k_2 + \frac{a_3 b_2 \ell^*}{\Gamma_q(2-\beta_i)} k_2\right] d,$$

for all $i = 1, 2, \dots, n$. Hence, we get

$$\begin{aligned}\|Tu - Tv\| &\leq \Bigg[\frac{b_1(1+2|a|)\ell^*}{|1+a|} k_1 + a_2 b_2 \ell^* k_2 \\ &\quad + \sum_{i=1}^n \left(\frac{b_2 \ell^* \Gamma_q(\alpha-\kappa)}{\Gamma_q(\alpha-\beta_i-\kappa+1)} k_2 + \frac{a_3 b_2 \ell^*}{\Gamma_q(2-\beta_i)} k_2\right)\Bigg] d \\ &= \Delta d.\end{aligned}$$

By assumption, $\Delta < 1$, thus the mapping F is a contraction and so by using the Banach contraction mapping principle, F has a unique fixed point which is the unique solution of the problem (1). This completes the proof. □

Corollary 1 *Assume that there exists $M > 0$ such that*

$$|F_{t,x_i,u_i} - F_{t,x'_i,v_i}| \leq M\left[\sum_{i=1}^3 |x_i - x'_i| + \sum_{i=1}^n |u_i - v_i|\right],$$

for each $t \in J$ and real numbers x_i, x'_i for $i = 1, 2, 3$, u_i, v_i for $i = 1, 2, \dots, n$, where $F_{t,x_i,u_i} = f(t, x_1, x_2, x_3, u_1, u_2, \dots, u_n)$, and $F_{t,x'_i,v_i} = f(t, x'_1, x'_2, x'_3, v_1, v_2, \dots, v_n)$. Then the problem (1) has a unique solution whenever

$$\begin{aligned}&(1 + {}_0\lambda_1 + {}_0\lambda_2)\Bigg[\frac{[(1+2a)(1+b) + b\alpha]M}{(1+a)(1+b)\Gamma_q(\alpha+1)} \\ &\quad + \sum_{i=1}^n \left(\frac{M}{\Gamma_q(\alpha-\beta_i+1)} + \frac{bM}{(1+b)\Gamma_q(2-\beta_i)\Gamma_q(\alpha)}\right)\Bigg] < 1,\end{aligned}$$

where ${}_0\lambda_i = \sup_{t\in J} |\int_0^t \gamma_i(t,s)\, d_q s|$, $i = 1, 2$.

Theorem 5 *Let $f : J \times \mathbb{R}^{n+3} \to \mathbb{R}$ be a continuous function. In addition, we assume that there exist a positive constant $\kappa < \alpha - 1$ and a function $\ell \in L^{\frac{1}{\kappa}}(J, \mathbb{R}^+)$. Then problem (1) has* *a solution whenever*

$$|F_{t,x_j,u_i}| \le \ell(t) + \sum_{j=1}^{3} c_j |x_j|^{\nu_j} + \sum_{i=1}^{n} d_i |u_i|^{\eta_i}, \tag{7}$$

where c_j, ν_j *belong to* $[0,\infty)$, $(0,1)$, *respectively, for* $j = 1,2,3$ *and* d_i, η_i *belong to* $[0,\infty)$, $(0,1)$, *respectively, for* $i = 1,2,\dots,n$, *or whenever*

$$|F_{t,x_j,u_i}| \le \sum_{i=1}^{3} c_j |x_j|^{\nu_j} + \sum_{i=1}^{n} d_i |u_i|^{\eta_i}, \tag{8}$$

where c_j, ν_j *belong to* $(0,\infty)$, $(1,\infty)$, *respectively, for* $j = 1,2,3$ *and* d_i, η_i *belong to* $(0,\infty)$, $(1,\infty)$, *respectively, for* $i = 1,2,\dots,n$.

Proof First, we assume that the condition (7) is satisfied. Recall that $k_1 = (\frac{1-\kappa}{\alpha-\kappa})^{1-\kappa}$ and $k_2 = (\frac{1-\kappa}{\alpha-\kappa-1})^{1-\kappa}$. Let B_r is the set of all $u \in X$ such that $\|u\|$ less than or equal to r; here

$$\begin{aligned} r \ge \max\Big\{ &\big((n+4)A_0 c_1\big)^{\frac{1}{1-\nu_1}}, \big((n+4)A_0 c_{20} \lambda_1^{\nu_2}\big)^{\frac{1}{1-\nu_2}}, \\ &\big((n+4)A_0 c_{30} \lambda_2^{\nu_3}\big)^{\frac{1}{1-\nu_3}}, \max_i \big((n+4)A_0 d_i\big)^{\frac{1}{1-\eta_i}}, (n+4)K_0 \Big\}, \\ A_0 = {} & \frac{(1+2|a|)[1+(1+a)|b|]}{(1+a)(1+b)\Gamma_q(\alpha+1)} \\ &+ \sum_{i=1}^{n} \left(\frac{1}{\Gamma_q(\alpha-\beta_i+1)} + \frac{|b|}{|1+b|\Gamma_q(\alpha)\Gamma_q(2-\beta_i)} \right), \\ K_0 = {} & \frac{(1+2|a|)\ell^*}{|1+a|\Gamma_q(\alpha)} k_1 + \frac{|b|(1+2|a|)\ell^*}{|1+a||1+b|\Gamma_q(\alpha-1)} k_2 \\ &+ \frac{1}{\Gamma_q(\alpha-1)} \sum_{i=1}^{n} \left(\frac{\Gamma_q(\alpha-l)\ell^*}{\Gamma_q(\alpha-\beta_i-\kappa+1)} k_2 + \frac{|b|\ell^*}{|1+b|\Gamma_q(2-\beta_i)} k_2 \right), \end{aligned}$$

and $\ell^* = (\int_0^1 (\ell(t))^{\frac{1}{\kappa}}\, d_q s)^{\kappa}$. Note that B_r is a closed, bounded and convex subset of the Banach space X. For each $u \in B_r$, we obtain

$$\begin{aligned} \big|(Tu)(t)\big| &= \big| I_q^{\alpha} F_{u(t)} - a_0 I_q^{\alpha} F_{u(1)} + g(t) I_q^{\alpha-1} F_{u(1)} \big| \\ &\le I_q^{\alpha} |F_{u(t)}| + \frac{|a|}{|1+a|} I_q^{\alpha} |F_{u(1)}| + \frac{|b|(1+2|a|)}{|1+a||1+b|} I_q^{\alpha-1} |F_{u(1)}| \\ &\le I_q^{\alpha} \ell(t) + A_r I_q^{\alpha}(1) + \frac{|a|}{|1+a|} I_q^{\alpha} \ell(1) + \frac{|a|}{|1+a|} A_r I_q^{\alpha}(1) \\ &\quad + \frac{|b|(1+2|a|)}{|1+a||1+b|} I_q^{\alpha-1} \ell(1) + \frac{|b|(1+2|a|)}{|1+a||1+b|} A_r I_q^{\alpha-1}(1) \\ &\le \frac{1}{\Gamma_q(\alpha)} \left(\int_0^t \big((t-qs)^{(\alpha-1)}\big)^{\frac{1}{1-l}}\, d_q s \right)^{1-l} \\ &\quad \times \left(\int_0^t \big(\ell(s)\big)^{\frac{1}{l}}\, d_q s \right)^{l} \end{aligned}$$

$$
\begin{aligned}
&+\frac{|a|}{|1+a|\Gamma_q(\alpha)}\left(\int_0^1\left((1-qs)^{(\alpha-1)}\right)^{\frac{1}{1-l}}\,d_qs\right)^{1-l}\\
&\times\left(\int_0^1\left(m(s)\right)^{\frac{1}{l}}\,d_qs\right)^{l}\\
&+\frac{|b|(1+2|a|)}{|1+a||1+b|\Gamma_q(\alpha-1)}\left(\int_0^1\left((1-qs)^{(\alpha-2)}\right)^{\frac{1}{1-l}}\,d_qs\right)^{1-l}\\
&\times\left(\int_0^1\left(m(s)\right)^{\frac{1}{l}}\,d_qs\right)^{l}\\
&+\frac{(1+2|a|)(1+(1+\alpha)|b|)}{|1+a||1+b|\Gamma_q(\alpha+1)}A_r\\
\leq{}&\frac{(1+2|a|)\ell^*}{|1+a|\Gamma_q(\alpha)}k_1+\frac{|b|(1+2|a|)\ell^*}{|1+a||1+b|\Gamma_q(\alpha-1)}k_2\\
&+\frac{(1+2|a|)(1+(1+\alpha)|b|)}{|1+a||1+b|\Gamma_q(\alpha+1)}A_r,
\end{aligned}
$$

where a_0 and $g(t)$ as defined in Theorem 4 (i.e. $a_0=\frac{a}{1+a}$ and $g(t)=\frac{ab-b(1+a)t}{(1+a)(1+b)}$, $t\in J$),

$$F_{u(s)}=f\left(s,u(s),(\varphi_1u)(s),(\varphi_2u)(s),{}^cD^{\beta_1}u(s),{}^cD^{\beta_2}u(s),\dots,{}^cD^{\beta_n}u(s)\right)$$

and $A_r=c_1r_1^{\nu}+c_{20}\lambda_1^{\nu_2}r^{\nu_2}+c_{30}\lambda_2^{\nu_3}r^{\nu_3}+\sum_{i=1}^n d_ir^{\eta_i}$. Also, we have

$$
\begin{aligned}
\left|{}^cD_q^{\beta_i}(Tu)(t)\right|&=\left|I_q^{1-\beta_i}(Tu)'(t)\right|\\
&=\left|I_q^{1-\beta_i}\left(I_q^{\alpha-1}F_{u(s)}-\frac{b}{1+b}I_q^{\alpha-1}F_{u(1)}\right)\right|\\
&\leq\int_0^t\frac{(t-qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\left(\int_0^s\frac{(s-q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|F_{u(\tau)}|\,d_q\tau\right)d_qs\\
&\quad+\frac{|b|}{|1+b|}\int_0^t\frac{(t-qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\left(\int_0^1\frac{(1-q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|F_{u(\tau)}|\,d_q\tau\right)d_qs\\
&\leq\int_0^t\frac{(t-qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\left(\int_0^s\frac{(s-q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\ell(\tau)\,d_q\tau\right)d_qs\\
&\quad+A_r\int_0^t\frac{(t-qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\left(\int_0^s\frac{(s-q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\,d_q\tau\right)d_qs\\
&\quad+\frac{|b|}{|1+b|}\int_0^t\frac{(t-qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\left(\int_0^1\frac{(1-q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\ell(\tau)\,d_q\tau\right)d_qs\\
&\quad+\frac{|b|}{|1+b|}A_r\int_0^t\frac{(t-qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\left(\int_0^1\frac{(1-q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\,d_q\tau\right)d_qs\\
&\leq\frac{1}{\Gamma_q(\alpha-1)\Gamma_q(1-\beta_i)}\int_0^t(t-qs)^{(-\beta_i)}\\
&\quad\times\left[\left(\int_0^s\left((s-q\tau)^{(\alpha-2)}\right)^{\frac{1}{1-l}}\,d_q\tau\right)^{1-l}\left(\int_0^s\left(\ell(\tau)\right)^{\frac{1}{l}}\,d_q\tau\right)^{l}\right]d_qs\\
&\quad+\frac{A_r}{\Gamma_q(\alpha)\Gamma_q(1-\beta_i)}\int_0^t(t-qs)^{(-\beta_i)}s^{\alpha-1}\,d_qs
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|b|}{|1+b|\Gamma_q(\alpha-1)\Gamma_q(1-\beta_i)} \int_0^t (t-qs)^{(-\beta_i)} \\
&\times \left[\left(\int_0^1 \left((1-q\tau)^{(\alpha-2)}\right)^{\frac{1}{1-l}} d_q\tau \right)^{1-l} \left(\int_0^1 \left(\ell(\tau)\right)^{\frac{1}{l}} d_q\tau \right)^{l} \right] d_qs \\
&+ \frac{|b|}{|1+b|\Gamma_q(\alpha)\Gamma_q(2-\beta_i)} A_r \\
\le{}& \frac{\ell^*}{\Gamma_q(\alpha-1)\Gamma_q(1-\beta_i)} k_2 \int_0^t (t-qs)^{(-\beta_i)} s^{\alpha-l-1} \, d_qs \\
&+ \frac{1}{\Gamma_q(\alpha)\Gamma_q(1-\beta_i)} A_r \int_0^t (t-qs)^{(-\beta_i)} s^{\alpha-1} \, d_qs \\
&+ \frac{|b|\ell^*}{|1+b|\Gamma_q(\alpha-1)\Gamma_q(1-\beta_i)} k_2 \int_0^t (t-qs)^{(-\beta_i)} \, d_qs \\
&+ \frac{|b|}{|1+b|\Gamma_q(\alpha)\Gamma_q(2-\beta_i)} A_r \\
\le{}& \frac{\ell^*}{\Gamma_q(\alpha-1)\Gamma_q(1-\beta_i)} k_2 \int_0^1 (1-qs)^{(-\beta_i)} s^{\alpha-l-1} \, d_qs \\
&+ \frac{1}{\Gamma_q(\alpha)\Gamma_q(1-\beta_i)} A_r \int_0^1 (1-qs)^{(-\beta_i)} s^{\alpha-1} \, d_qs \\
&+ \frac{|b|\ell^*}{|1+b|\Gamma_q(\alpha-1)\Gamma_q(2-\beta_i)} k_2 + \frac{|b|}{|1+b|\Gamma_q(\alpha)\Gamma_q(2-\beta_i)} A_r.
\end{aligned}
$$

Since, by considering Eq. (2),

$$
B_q(\alpha-l, 1-\beta_i) = \int_0^1 (1-qs)^{(-\beta_i)} s^{\alpha-\kappa-1} \, d_qs = \frac{\Gamma_q(\alpha-l)\Gamma_q(1-\beta_i)}{\Gamma_q(\alpha-\beta_i-\kappa+1)}
$$

and on the other hand

$$
B_q(\alpha, 1-\beta_i) = \int_0^1 (1-q\xi)^{(-\beta_i)} \xi^{\alpha-1} \, d_q\xi = \frac{\Gamma_q(\alpha)\Gamma_q(1-\beta_i)}{\Gamma_q(\alpha-\beta_i+1)},
$$

we conclude that

$$
\begin{aligned}
\left| {}^cD_q^{\beta_i}(Tu)(t) \right| \le{}& \frac{\Gamma_q(\alpha-l)\ell^*}{\Gamma_q(\alpha-1)\Gamma_q(\alpha-\beta_i-\kappa+1)} k_2 \\
&+ \frac{|b|\ell^*}{|1+b|\Gamma_q(\alpha-1)\Gamma_q(2-\beta_i)} k_2 + \frac{1}{\Gamma_q(\alpha-\beta_i+1)} A_r \\
&+ \frac{|b|}{|1+b|A_r\Gamma_q(\alpha)\Gamma_q(2-\beta_i)} A_r
\end{aligned}
$$

for each $i = 1, 2, \dots, n$. Hence,

$$
\begin{aligned}
\|Tu\| \le{}& \frac{(1+2|a|)\ell^*}{|1+a|)\Gamma_q(\alpha)} k_1 + \frac{|b|(1+2|a|)\ell^*}{|1+a||1+b|\Gamma_q(\alpha-1)} k_2 \\
&+ \frac{1}{\Gamma_q(\alpha-1)} \sum_{i=1}^{n} \left[\frac{\Gamma_q(\alpha-l)\ell^*}{\Gamma_q(\alpha-\beta_i-l+1)} k_2 \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|b|\ell^*}{|1+b|\Gamma_q(\alpha-1)\Gamma_q(2-\beta_i)}k_2\Bigg] \\
&+ A_r\Bigg(\frac{(1+2|a|)(1+(1+\alpha)|b|)}{|1+a||1+b|\Gamma_q(\alpha+1)} \\
&+ \sum_{i=1}^{n}\bigg[\frac{1}{\Gamma_q(\alpha-\beta_i+1)} + \frac{|b|}{|1+b|\Gamma_q(\alpha)\Gamma_q(2-\beta_i)}\bigg]\Bigg) \\
&= K_0 + A_r A_0 \le \frac{r}{n+4}(n+4) = r.
\end{aligned}
$$

Hence, T maps B_r into B_r. Now, suppose that T satisfy the condition (8). In this case, choose

$$
\begin{aligned}
0 &< r \\
&\le \min\Bigg\{\bigg(\frac{1}{(n+3)A_0c_1}\bigg)^{\frac{1}{\nu_1-1}}, \bigg(\frac{1}{(n+3)A_0c_{20}\lambda_1^{\nu_2}}\bigg)^{\frac{1}{\nu_2-1}}, \\
&\quad \bigg(\frac{1}{(n+3)A_0c_{30}\lambda_2^{\nu_2}}\bigg)^{\frac{1}{\nu_2-1}}, \max_i\bigg(\frac{1}{(n+3)A_0d_i}\bigg)^{\frac{1}{\eta_i-1}}\Bigg\}.
\end{aligned}
$$

By applying a similar argument, one can prove that $\|Tu\| \le r$ and so T is a self-map on B_r. Also, one can easy to check that T is continuous, because w is continuous. For each $u \in B_r$, take

$$
N = \max_{t\in J}\big|w\big(t, u(t), (\varphi_1 u)(t), (\varphi_2 u)(t), {}^cD_q^{\beta_1}u(t), {}^cD_q^{\beta_2}u(t), \ldots, {}^cD_q^{\beta_n}u(t)\big)\big| + 1.
$$

Thus, for each $0 < t_1 < t_2 < 1$, we have

$$
\begin{aligned}
\big|(Tu)(t_2) - (Tu)(t_1)\big| &= \bigg|I_q^{\alpha}F_{u(t_2)} - I_q^{\alpha}F_{u(t_1)} + \frac{b(t_1-t_2)}{1+b}I_q^{\alpha-1}F_{u(1)}\bigg| \\
&\le \int_0^{t_1}\frac{(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|F_{u(s)}|\,d_qs \\
&\quad + \int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|F_{u(s)}|\,d_qs \\
&\quad + \frac{|b|(t_2-t_1)}{|1+b|}\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|F_{u(s)}|\,d_qs \\
&\le N\int_0^{t_1}\frac{(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\,d_qs \\
&\quad + N\int_{t_1}^{t_2}\frac{(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\,d_qs \\
&\quad + \frac{N|b|(t_2-t_1)}{|1+b|}\int_0^1\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\,d_qs \\
&= \frac{N}{\Gamma_q(\alpha+1)}\big(t_2^{\alpha} - t_1^{\alpha}\big) + \frac{N|b|}{|1+b|\Gamma_q(\alpha)}(t_2-t_1).
\end{aligned}
$$

Furthermore, for all $i = 1, 2, \dots, n$, we obtain

$$\begin{aligned}
&\bigl|{}^cD_q^{\beta_i}(Tu)(t_2) - {}^cD_q^{\beta_i}(Tu)(t_1)\bigr| \\
&= \biggl| \int_0^{t_2} \frac{(t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}(Tu)'(s)\,d_qs - \int_0^{t_1} \frac{(t_1 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}(Tu)'(s)\,d_qs \biggr| \\
&\leq \int_0^{t_1} \frac{(t_1 - qs)^{(-\beta_i)} - (t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)} \biggl(\int_0^s \frac{(s - q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |F_{u(\tau)}|\,d_q\tau \biggr) d_qs \\
&\quad + \frac{|b|}{|1+b|} \int_0^{t_1} \frac{(t_1 - qs)^{(-\beta_i)} - (t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)} \\
&\quad \times \biggl(\int_0^1 \frac{(1 - q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |F_{u(\tau)}|\,d_q\tau \biggr) d_qs \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)} \biggl(\int_0^s \frac{(s - q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |F_{u(\tau)}|\,d_q\tau \biggr) d_qs \\
&\quad + \frac{|b|}{|1+b|} \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)} \biggl(\int_0^1 \frac{(1 - q\tau)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} |F_{u(\tau)}|\,d_q\tau \biggr) d_qs \\
&\leq \frac{N}{\Gamma_q(\alpha)} \int_0^{t_1} \frac{(t_1 - qs)^{(-\beta_i)} - (t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)} s^{\alpha-1}\,d_qs \\
&\quad + \frac{N|b|}{|1+b|\Gamma_q(\alpha)} \int_0^{t_1} \frac{(t_1 - qs)^{(-\beta_i)} - (t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\,d_qs \\
&\quad + \frac{N}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)} s^{\alpha-1}\,d_qs \\
&\quad + \frac{N|b|}{|1+b|\Gamma_q(\alpha)} \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\,d_qs \\
&\leq \frac{(1+2|b|)N}{|1+b|\Gamma_q(\alpha)} \int_0^{t_1} \frac{(t_1 - qs)^{(-\beta_i)} - (t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\,d_qs \\
&\quad + \frac{(1+2|b|)N}{|1+b|\Gamma_q(\alpha)} \int_{t_1}^{t_2} \frac{(t_2 - qs)^{(-\beta_i)}}{\Gamma_q(1-\beta_i)}\,d_qs \\
&\leq \frac{(1+2|b|)N}{|1+b|\Gamma_q(\alpha)\Gamma_q(2-\beta_i)} \bigl[\bigl(t_2^{1-\beta_i} - t_1^{1-\beta_i}\bigr) + 2(t_2 - t_1)^{1-\beta_i} \bigr].
\end{aligned}$$

Hence,

$$\begin{aligned}
\bigl\| Tu(t_2) - Tu(t_1) \bigr\| &\leq \frac{N}{\Gamma_q(\alpha+1)} \bigl(t_2^\alpha - t_1^\alpha\bigr) + \frac{N|b|}{|1+b|\Gamma_q(\alpha)}(t_2 - t_1) \\
&\quad + \sum_{i=1}^n \frac{(1+2|b|)N}{|1+b|\Gamma_q(\alpha)\Gamma_q(2-\beta_i)} \bigl[\bigl(t_2^{1-\beta_i} - t_1^{1-\beta_i}\bigr) \\
&\quad + 2(t_2 - t_1)^{1-\beta_i} \bigr],
\end{aligned}$$

which implies that $\|Tu(t_2) - Tu(t_1)\| \to 0$ as $t_1 \to t_2$. Thus, T is uniformly bounded and equicontinuous and so the theorem of Arzelá–Ascoli implies that T is completely continuous. At present, from Theorem 1, T has a fixed point in B_r. Finally, the problem (1) has a solution. □

Corollary 2 *Assume that a real-valued function f defined on $J\times\mathbb{R}^{n+3}$ is continuous. Then the problem* (1) *has at least one solution whenever there exist a positive constant $l<\alpha-1$ and a real-valued function $\ell\in L^{\frac{1}{l}}(J,\mathbb{R}^+)$ such that $|w(t,x_1,x_2,x_3,u_1,u_2,\dots,u_n)|\leq\ell(t)$, for each t in J, and x_j, with $j=1,2,3$, u_i, with $1\leq i\leq n$, in $\mathbb{R}$.*

4 Examples illustrative for the problems with algorithms

In this part, we give complete computational techniques for checking working to illustrate of the problem (1), in our theorems, such that it covers all the problems and we present numerical examples which entail perfect solutions. Foremost, we present a simplified analysis that can be executed to calculate the value of the q-Gamma function, $\Gamma_q(x)$, for input q, x and different values of n. To this aim, we consider a pseudo-code description of the method for calculating the q-Gamma function of order n in Algorithm 2 (for more details, see the link https://en.wikipedia.org/wiki/Q-gamma_function). Now we give the following examples to illustrate our results.

Example 1 Consider the multi-term nonlinear fractional q-integro-differential equation

$$\begin{cases} {}^cD_q^{\frac{8}{5}}u(t)=\frac{e^{-\pi t}}{30\sqrt{\pi}+e^{-\pi t}}[\frac{\cos t+e^t}{1+t^4}+\frac{2|u(t)|}{1+2|u(t)|}\\ \qquad+\frac{e^{-\pi t}\sin\pi t}{1+t^3}(1+\frac{2|(\varphi_1u)(t)+{}^cD_q^{\frac{1}{3}}u(t)|}{1+2|(\varphi_1u)(t)+{}^cD_q^{\frac{1}{3}}u(t)|})\\ \qquad+\frac{1+\cos^2\pi t}{3(t^{\frac{4}{3}}+6)}(3(\varphi_2u)(t)+\frac{2|{}^cD_q^{\frac{2}{5}}u(t)|}{1+2|{}^cD_q^{\frac{2}{5}}u(t)|})], \end{cases} \tag{9}$$

under boundary conditions $u(0)+u(1)=0$ and $u'(0)+u'(1)=0$, where $(\varphi_1u)(t)$ and $(\varphi_2u)(t)$ are defined by $\frac{1}{10}\int_0^t e^{-2(s-t)}u(s)\,d_qs$ and $\frac{1}{10}\int_0^t e^{-(s-t)/4}u(s)\,d_qs$, respectively, with

$$\begin{aligned} {}_0\lambda_1&=\sup_{t\in J}\left|\int_0^t\frac{e^{-2(s-t)}}{10}\,d_qs\right|=\sup_{t\in J}\left|t(1-q)\sum_{k=0}^{\infty}q^k\frac{e^{2t(1-q^k)}}{10}\right|\\ &=|1-q|\sum_{k=0}^{\infty}\left|q^k\frac{e^{2(1-q^k)}}{10}\right| \end{aligned}$$

and

$$\begin{aligned} {}_0\lambda_2&=\sup_{t\in I}\left|\int_0^t\frac{e^{-(s-t)/4}}{10}\,d_qs\right|=\sup_{t\in I}\left|t(1-q)\sum_{k=0}^{\infty}q^k\frac{e^{t(1-q^k)/4}}{10}\right|\\ &=|1-q|\sum_{k=0}^{\infty}\left|q^k\frac{e^{(1-q^k)/4}}{10}\right|. \end{aligned}$$

Then we have

$$\begin{aligned} |F_{u(t)}-F_{v(t)}|\leq{}&\frac{1}{30\sqrt{\pi}}\bigl(\bigl|u(t)-v(t)\bigr|\\ &+\bigl|(\varphi_1u)(t)-(\varphi_1v)(t)\bigr|+\bigl|(\varphi_2u)(t)-(\varphi_2v)(t)\bigr|\\ &+\bigl|{}^cD_q^{\frac{1}{3}}u(t)-{}^cD_q^{\frac{1}{3}}v(t)\bigr|+\bigl|{}^cD_q^{\frac{2}{5}}u(t)-{}^cD_q^{\frac{2}{5}}v(t)\bigr|\bigr), \end{aligned}$$

where

$$F_{u(t)} = w\big(t, u(t), (\varphi_1 u)(t), (\varphi_2 u)(t), {}^cD_q^{\frac{1}{3}} u(t), {}^cD_q^{\frac{2}{5}} u(t)\big),$$
$$F_{v(t)} = w\big(t, v(t), (\varphi_1 v)(t), (\varphi_2 v)(t), {}^cD_q^{\frac{1}{3}} v(t), {}^cD_q^{\frac{2}{5}} v(t)\big).$$

Take $\ell(t) = \frac{1}{30\sqrt{\pi}}$ belongs to $L^{\frac{1}{5}}(J, \mathbb{R}^+)$, $\kappa = \frac{1}{5}$ and

$$\ell^* = \left(\int_0^1 \left(\frac{1}{30\sqrt{\pi}}\right)^5 d_q s\right)^{\frac{1}{5}} = \left((1-q)\sum_{k=0}^{\infty} \frac{q^k}{(30\sqrt{\pi})^5}\right)^{\frac{1}{5}}.$$

For different values of q, which are shown in Tables 1, 2 and 3, by using Algorithm 6, we obtain

$$\begin{aligned}
\Delta &= (1 + {}_0\lambda_1 + {}_0\lambda_2)\bigg[\frac{3\ell^*}{2\Gamma_q(\alpha)}k_1 + \frac{\ell^*}{4\Gamma_q(\alpha-1)}k_2 \\
&\quad + \frac{\Gamma_q(\alpha-\kappa)\ell^*}{\Gamma_q(\alpha-1)}k_2\left(\frac{1}{\Gamma_q(\alpha-\beta_1-\kappa+1)} + \frac{1}{\Gamma_q(\alpha-\beta_2-\kappa+1)}\right) \\
&\quad + \frac{\ell^*}{2\Gamma_q(\alpha-1)}k_2\left(\frac{1}{\Gamma_q(2-\beta_1)} + \frac{1}{\Gamma_q(2-\beta_2)}\right)\bigg] \\
&= (1 + {}_0\lambda_1 + {}_0\lambda_2)\bigg[\frac{3\ell^*}{2\Gamma_q(\frac{8}{5})}\left(\frac{4}{7}\right)^{\frac{4}{5}} + \frac{\ell^*}{4\Gamma_q(\frac{3}{5})}(2)^{\frac{4}{5}} \\
&\quad + \frac{\Gamma_q(\frac{7}{5})\ell^*}{\Gamma_q(\frac{3}{5})}(2)^{\frac{4}{5}}\left(\frac{1}{\Gamma_q(\frac{31}{15})} + \frac{1}{\Gamma_q(2)}\right) \\
&\quad + \frac{\ell^*}{2\Gamma_q(\frac{3}{5})}(2)^{\frac{4}{5}}\left(\frac{1}{\Gamma_q(\frac{5}{3})} + \frac{1}{\Gamma_q(\frac{8}{5})}\right)\bigg] \\
&< 1,
\end{aligned}$$

where $k_1 = (\frac{1-\kappa}{\alpha-\kappa})^{1-\kappa}$ and $k_2 = (\frac{1-\kappa}{\alpha-\kappa-1})^{1-\kappa}$. Now, by using Algorithms 1 and 2, we calculated ${}_0\lambda_1$, ${}_0\lambda_2$, ℓ^*, $\Gamma_q(\frac{8}{5})$, $\Gamma_q(\frac{3}{5})$, $\Gamma_q(\frac{7}{5})$, $\Gamma_q(\frac{31}{15})$ and $\Gamma_q(2)$ for some values $n \in \mathbb{N}$ and $q \in (0,1)$.

Table 1 Some numerical results for calculation of Δ with $q = \frac{1}{3}$ and $n = 15$ of Algorithm 6

n	${}_0\lambda_1$	${}_0\lambda_2$	ℓ^*	Ω	Δ
1	0.084304	0.026252	0.013921	0.073438	0.081557
2	0.128131	0.035503	0.014745	0.077555	0.090246
3	0.145073	0.038644	0.014983	0.078740	0.093205
4	0.151006	0.039698	0.015059	0.079118	0.094206
5	0.153017	0.040050	0.015084	0.079242	0.094541
6	0.153691	0.040167	0.015092	0.079282	0.094651
7	0.153916	0.040206	0.015095	0.079297	0.094690
8	0.153991	0.040219	0.015096	0.079302	0.094703
9	0.154016	0.040224	0.015096	0.079302	0.094705
10	0.154024	0.040225	0.015097	0.079307	0.094712
11	0.154027	0.040226	0.015097	0.079307	0.094712
12	0.154028	0.040226	0.015097	0.079307	0.094713
13	0.154028	0.040226	0.015097	0.079307	0.094713
14	0.154028	0.040226	0.015097	0.079307	0.094713
15	0.154028	0.040226	0.015097	0.079307	0.094713

Table 2 Some numerical results for calculation of Δ with $q = \frac{1}{2}$ and $n = 19$ of Algorithm 6

n	${}_0\lambda_1$	${}_0\lambda_2$	ℓ^*	Ω	Δ
1	0.067957	0.028329	0.014253	0.071709	0.078613
2	0.123978	0.043407	0.015456	0.077908	0.090949
3	0.159944	0.051185	0.015940	0.080482	0.097475
4	0.180322	0.055135	0.016162	0.081684	0.100917
5	0.191168	0.057126	0.016268	0.082263	0.102688
6	0.196763	0.058125	0.016320	0.082548	0.103589
7	0.199605	0.058626	0.016346	0.082691	0.104045
8	0.201037	0.058876	0.016359	0.082763	0.104274
9	0.201755	0.059002	0.016365	0.082796	0.104386
10	0.202115	0.059064	0.016369	0.082818	0.104448
11	0.202296	0.059096	0.016370	0.082823	0.104473
12	0.202386	0.059111	0.016371	0.082829	0.104488
13	0.202431	0.059119	0.016371	0.082829	0.104493
14	0.202453	0.059123	0.016372	0.082834	0.104502
15	0.202465	0.059125	0.016372	0.082834	0.104503
16	0.202470	0.059126	0.016372	0.082834	0.104503
17	0.202473	0.059126	0.016372	0.082834	0.104504
18	0.202475	0.059127	0.016372	0.082834	0.104504
19	0.202475	0.059127	0.016372	0.082834	0.104504

Table 1 shows these calculated values. So, from Theorem 4, the problem (9) has a unique solution. In Tables 1, 2 and 3, we put

$$\begin{aligned}\Omega &= \frac{3\ell^*}{2\Gamma_q(\alpha)}k_1 + \frac{\ell^*}{4\Gamma_q(\alpha-1)}k_2 \\ &\quad + \frac{\Gamma_q(\alpha-\kappa)\ell^*}{\Gamma_q(\alpha-1)}k_2\left(\frac{1}{\Gamma_q(\alpha-\beta_1-\kappa+1)} + \frac{1}{\Gamma_q(\alpha-\beta_2-\kappa+1)}\right) \\ &\quad + \frac{\ell^*}{2\Gamma_q(\alpha-1)}k_2\left(\frac{1}{\Gamma_q(2-\beta_1)} + \frac{1}{\Gamma_q(2-\beta_2)}\right).\end{aligned}$$

Algorithm 6 shows the technique of calculation Δ which was introduced in Eq. (6). Tables 1, 2 and 3 show variables of Δ when $q = \frac{1}{3}$, $q = \frac{1}{2}$ and $q = \frac{4}{5}$, respectively. As it is seen, always $\Delta < 1$ for all n and $q \in (0,1)$. In addition, when values q are close to one, Δ is obtained with more values of n in comparison with other rows. It is shown by underlined rows. They have been underlined in line 10 of Table 1, line 14 of Table 2 and line 31 of Table 3.

Example 2 Consider the multi-term nonlinear fractional q-integro-differential equation

$$\begin{cases} {}^cD_q^{\frac{7}{4}}u(t) = \frac{\lambda e^{-2\pi t}}{\sqrt{1+t^3}} + \frac{\sin\pi t}{\sqrt{2\pi+|u(t)|+|{}^cD_q^{\frac{1}{2}}u(t)|}}(u(t))^{\sigma_1} \\ \qquad + \frac{e^{-2\pi t}(1+\cos^2 u(t))}{(t+6)^2}((\varphi_1 u)(t))^{\sigma_2} \\ \qquad + \frac{tu(t)}{(5+t^2)(1+|u(t)|)}((\varphi_2 u)(t))^{\sigma_3} \\ \qquad + \frac{(1+\alpha)(t-\frac{1}{2})^2}{\Gamma_q(\alpha)(1+|u(t)+{}^cD_q^{\frac{3}{2}}u(t)|)}\sum_{k=1}^{4}(\frac{\sin k\pi t}{2^k})({}^cD_q^{\beta_k}u(t))^{\delta_k}, \end{cases} \tag{10}$$

Table 3 Some numerical results for calculation of Δ with $q = \frac{4}{5}$ and $n = 35$ of Algorithm 6

n	${}_0\lambda_1$	${}_0\lambda_2$	ℓ^*	Ω	Δ
1	0.023869	0.016820	0.013036	0.056019	0.058298
2	0.050166	0.030826	0.014662	0.064329	0.069539
3	0.077341	0.042394	0.015581	0.069665	0.078006
4	0.104022	0.051889	0.016186	0.073504	0.084964
5	0.129167	0.059642	0.016613	0.076388	0.090810
6	0.152100	0.065947	0.016925	0.078595	0.095732
7	0.172475	0.071058	0.017158	0.080303	0.099860
8	0.190201	0.075189	0.017337	0.081647	0.103315
9	0.205366	0.078522	0.017474	0.082698	0.106175
10	0.218168	0.081207	0.017581	0.083531	0.108538
11	0.228858	0.083366	0.017665	0.084192	0.110479
12	0.237710	0.085101	0.017731	0.084716	0.112064
13	0.244988	0.086493	0.017783	0.085133	0.113353
14	0.250940	0.087610	0.017824	0.085464	0.114397
15	0.255786	0.088506	0.017857	0.085730	0.115246
16	0.259718	0.089224	0.017883	0.085941	0.115929
17	0.262900	0.089799	0.017904	0.086111	0.116482
18	0.265468	0.090259	0.017920	0.086243	0.116922
19	0.267537	0.090628	0.017933	0.086350	0.117277
20	0.269202	0.090923	0.017944	0.086438	0.117566
21	0.270540	0.091160	0.017952	0.086505	0.117794
22	0.271614	0.091349	0.017959	0.086561	0.117980
23	0.272476	0.091500	0.017964	0.086603	0.118125
24	0.273168	0.091621	0.017968	0.086637	0.118241
25	0.273722	0.091718	0.017972	0.086668	0.118340
26	0.274166	0.091796	0.017975	0.086692	0.118418
27	0.274521	0.091858	0.017977	0.086709	0.118477
28	0.274806	0.091907	0.017979	0.086724	0.118527
29	0.275034	0.091947	0.017980	0.086734	0.118564
30	0.275217	0.091979	0.017981	0.086743	0.118594
31	0.275363	0.092004	0.017982	0.086751	0.118620
32	0.275480	0.092025	0.017983	0.086758	0.118642
33	0.275573	0.092041	0.017983	0.086760	0.118654
34	0.275648	0.092054	0.017984	0.086766	0.118670
35	0.275708	0.092064	0.017984	0.086767	0.118678

under boundary conditions $u(0) + \frac{1}{4}u(1) = 0$ and $u'(0) + \frac{3}{4}u'(1) = 0$, here $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{3}{5}$, $\beta_3 = \frac{1}{2}$, $\beta_4 = \frac{1}{6}$, $\lambda \in [0, \infty)$,

$$(\varphi_1 u)(t) = \int_0^t \frac{se^{-(s-t)}u(s)}{s^2 + 4}\, d_q s, \qquad (\varphi_2 u)(t) = \int_0^t \frac{16(t-s)^4 u(s)}{\sqrt{1+s^2}}\, d_q s.$$

Hence, we obtain

$$\begin{aligned} |F_{u(t)}| \le{}& \ell(t) + \frac{1}{\sqrt{2\pi}}\bigl|u(t)\bigr|^{\sigma_1} + \frac{1}{18}\bigl|(\varphi_1 u)(t)\bigr|^{\sigma_2} + \frac{1}{5}\bigl|(\varphi_2 u)(t)\bigr|^{\sigma_3} \\ &+ \sum_{k=1}^{4} \frac{1+\alpha}{\Gamma_q(\alpha)2^{k+2}}\bigl|{}^cD_q^{\beta_k}u(t)\bigr|^{\delta_k}, \end{aligned}$$

where

$$F_{u(t)} = w\bigl(t, u(t), (\varphi_1 u)(t), (\varphi_2 u)(t), {}^cD_q^{\beta_1}u(t), {}^cD_q^{\beta_2}u(t), {}^cD_q^{\beta_3}u(t), {}^cD_q^{\beta_4}u(t)\bigr),$$

Algorithm 6 The proposed method for calculation of Δ

Input: $n, m, q \in (0,1), \gamma_1(t,s), \gamma_2(t,s), \ell(t), \alpha, \kappa, \beta_1, \beta_2, a, b$

$t \leftarrow 1$
for $j = 1$ to m **do**
  $w \leftarrow 0$
  for $k = 1$ to j **do**
    $w \leftarrow w + |q^k * \gamma_1(t, tq^k)|$
  end for
  ${}_0\lambda_1 \leftarrow |1 - q| * w$
end for
for $j = 1$ to m **do**
  $w \leftarrow 0$
  for $k = 1$ to j **do**
    $w \leftarrow w + |q^k * \gamma_2(t, tq^k)|$
  end for
  ${}_0\lambda_2 \leftarrow |1 - q| * w$
end for
for $j = 1$ to m **do**
  $w \leftarrow 0$
  for $k = 1$ to j **do**
    $w \leftarrow w + q^k * (\ell(tq^k))^\kappa$
  end for
  $\mu^* \leftarrow ((1-q) * w)^{1/\kappa}$
end for
$A_1 \leftarrow ((1-\kappa)/(\alpha-\kappa))^{1-\kappa}$
$A_2 \leftarrow ((1-\kappa)/(\alpha-\kappa-1))^{1-\kappa}$
$W \leftarrow 0$
for $i = 1$ to n **do**
  $W \leftarrow W + b_2\mu^* \Gamma_q(\alpha-\kappa)/\Gamma_q(\alpha-\beta_i-\kappa+1) + a_3 b_2 \ell^*/\Gamma_q(2-\beta_i)$
end for
$W \leftarrow A_2 W$
$\Delta_q \leftarrow A_1(b_1(1+2|a|)\ell^*)/|1+a|) + a_2 b_2 \ell^* A_2 + W$

Output: Δ

and $m(t) = \frac{\lambda e^{-\pi t}}{\sqrt{1+t^2}}$ for t belongs to J. Also, if $l = \frac{1}{2}$ and $\lambda = 1$, then we have

$$\ell^* = \left(\int_0^1 \left(\ell(t)\right)^{\frac{1}{\kappa}} d_q s\right)^\kappa = \left((1-q)\sum_{k=0}^{\infty}\left(\frac{\lambda q^k e^{-\pi q^k}}{\sqrt{1+q^{2k}}}\right)^2\right)^{\frac{1}{2}}.$$

Table 4 shows the variables of $\Gamma_q(\alpha)$, $\Gamma_q(\alpha-1)$, A_0, ℓ^* and K_0 when $q = \frac{1}{3}$ and $m = 1, \dots, 40$. Since $0 < \sigma_j$, for $j = 1,2,3$, and $\delta_i < 1$, for $i = 1,2,3,4$, the assumption (7) holds. At present, if $\lambda = 0$, $\delta_i > 1$ and $\sigma_j > 1$ for $i = 1,2,3,4$ and $j = 1,2,3$, respectively, the second condition, (8) of Theorem 5 holds. Thus, problem (10) has at least one solution. Note the features of the q-Gamma function, for values of q close to one, the results are obtained at a greater rate of m.

Table 4 Some numerical results for calculation of $\Gamma_q(\alpha)$, $\Gamma_q(\alpha-1)$, A, M and K in Theorem 5 with $q=\frac{1}{3}$ and $m=40$

n	$\Gamma_q(\alpha)$	$\Gamma_q(\alpha-1)$	A_0	ℓ^*	K_0
1	1.054210	1.106942	6.752182	0.065737	0.174695
2	0.996499	1.137694	6.991433	0.102154	0.271472
3	0.970276	1.152228	7.125250	0.118079	0.313793
4	0.957751	1.159303	7.195379	0.123517	0.328243
5	0.951628	1.162796	7.231213	0.125129	0.332527
6	0.948600	1.164531	7.249318	0.125570	0.33370
7	0.947094	1.165395	7.258417	0.125686	0.334007
8	0.946343	1.165827	7.262978	0.125715	0.334086
9	0.945968	1.166043	7.265261	0.125723	0.334105
10	0.945781	1.166150	7.266403	<u>0.125725</u>	0.334111
11	0.945687	1.166204	7.266975	0.125725	<u>0.334112</u>
12	0.945641	1.166231	7.267261	0.125725	0.334112
13	0.945617	1.166245	7.267403	0.125725	0.334112
14	0.945606	1.166251	7.267475	0.125725	0.334112
15	0.945600	1.166255	7.267511	0.125725	0.334112
16	0.945597	1.166256	7.267528	0.125725	0.334112
17	0.945595	1.166257	7.267537	0.125725	0.334112
18	0.945595	1.166258	7.267542	0.125725	0.334112
19	<u>0.945594</u>	<u>1.166258</u>	7.267544	0.125725	0.334112
20	0.945594	1.166258	7.267545	0.125725	0.334112
21	0.945594	1.166258	<u>7.267546</u>	0.125725	0.334112
22	0.945594	1.166258	7.267546	0.125725	0.334112
23	0.945594	1.166258	7.267546	0.125725	0.334112
24	0.945594	1.166258	7.267546	0.125725	0.334112

Funding
Not applicable.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
All authors contributed equally and significantly in this manuscript and they read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of Ioannina, Ioannina, Greece. [2]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia. [3]Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan, Iran.

Publisher's Note

Received: 6 October 2019 Accepted: 8 November 2019 Published online: 15 November 2019

References

1. Jackson, F.H.: *q*-difference equations. Am. J. Math. **32**, 305–314 (1910). https://doi.org/10.2307/2370183
2. Carmichael, R.D.: The general theory of linear *q*-difference equations. Am. J. Math. **34**, 147–168 (1912)
3. Mason, T.E.: On properties of the solution of linear *q*-difference equations with entire function coefficients. Am. J. Math. **37**, 439–444 (1915)
4. Adams, C.R.: The general theory of a class of linear partial *q*-difference equations. Trans. Am. Math. Soc. **26**, 283–312 (1924)
5. Adams, C.R.: Note on the integro-*q*-difference equations. Trans. Am. Math. Soc. **31**(4), 861–867 (1929)
6. Trjitzinsky, W.J.: Analytic theory of linear *q*-difference equations. Acta Math. **61**, 1–38 (1933). https://doi.org/10.1007/BF02547785

7. Al-Salam, W.A.: Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. **15**, 135–140 (1966–1967). https://doi.org/10.1017/S0013091500011469
8. Agarwal, R.P.: Certain fractional q-integrals and q-derivatives. Proc. Camb. Philos. Soc. **66**, 365–370 (1969). https://doi.org/10.1017/S0305004100045060
9. Annaby, M.H., Mansour, Z.S.: q-Fractional Calculus and Equations. Springer, Cambridge (2012). https://doi.org/10.1007/978-3-642-30898-7
10. Rajković, P.M., Marinković, S.D., Stanković, M.S.: Fractional integrals and derivatives in q-calculus. Appl. Anal. Discrete Math. **1**, 311–323 (2007)
11. Stanković, M.S., Rajković, P.M., Marinković, S.D.: On q-fractional derivatives of Riemann–Liouville and Caputo type (2009). e-prints arXiv:0909.0387
12. Ahmad, B., Ntouyas, S.K., Tariboon, J.: Quantum Calculus. New Concepts, Impulsive IVPs and BVPs, Inequalities. Trends in Abstract and Applied Analysis, vol. 4. World Scientific, Hackensack (2016). https://doi.org/10.1142/10075
13. Goodrich, C., Peterson, A.: Discrete Fractional Calculus. Springer, Berlin (2015). https://doi.org/10.1007/978-3-319-25562-0
14. Ahmad, B., Ntouyas, S.K.: Existence of solutions for nonlinear fractional q-difference inclusions with nonlocal robin (separated) conditions. Mediterr. J. Math. **10**, 133–1351 (2013). https://doi.org/10.1007/s00009-013-0258-0
15. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
16. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000). https://doi.org/10.1142/3779
17. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
18. Ahmad, B., Sivasundaram, S.: On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order. Appl. Math. Comput. **217**(2), 480–487 (2010). https://doi.org/10.1016/j.amc.2010.05.080
19. Ahmad, B., Nieto, J.J.: Riemann–Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. **2011**, 36 (2011). https://doi.org/10.1186/1687-2770-2011-36
20. Anguraj, A., Karthikeyan, P., Trujillo, J.J.: Existence of solutions to fractional mixed integrodifferential equations with nonlocal initial condition. Adv. Differ. Equ. **2011**(1), 690653 (2011). https://doi.org/10.1155/2011/690653
21. Rezapour, S., Shabibi, M.: A singular fractional differential equation with Riemann–Liouville integral boundary condition. J. Adv. Math. Stud. **8**(1), 80–88 (2015)
22. Shabibi, M., Rezapour, S., Vaezpour, S.M.: A singular fractional integro-differential equation. UPB Sci. Bull., Ser. A **79**(1), 109–118 (2017)
23. Ntouyas, S.K., Etemad, S.: On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions. Appl. Math. Comput. **266**, 235–243 (2017). https://doi.org/10.1016/j.amc.2015.05.036
24. Samei, M.E., Khalilzadeh Ranjbar, G.: Some theorems of existence of solutions for fractional hybrid q-difference inclusio. J. Adv. Math. Stud. **12**(1), 63–76 (2019)
25. Samei, M.E., Ranjbar, G.K., Hedayati, V.: Existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions. J. Inequal. Appl. **2019**, 273 (2019). https://doi.org/10.1186/s13660-019-2224-2
26. Ahmad, B., Ntouyas, S.K., Purnaras, I.K.: Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. Adv. Differ. Equ. **2012**, 140 (2012). https://doi.org/10.1186/1687-1847-2012-140
27. Zhao, Y., Chen, H., Zhang, Q.: Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions. Adv. Differ. Equ. **2013**, 48 (2013). https://doi.org/10.1186/1687-1847-2013-48
28. Ahmad, B., Nieto, J.J., Alsaedi, A., Al-Hutami, H.: Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. J. Franklin Inst. **351**, 2890–2909 (2014). https://doi.org/10.1016/j.jfranklin.2014.01.020
29. Ahmad, B., Alsaedi, A., Al-Hutami, H.: A study of sequential fractional q-integro-difference equations with perturbed anti-periodic boundary conditions. In: Fractional Dynamics, pp. 110–128. De Gruyter, Berlin (2015). https://doi.org/10.1515/9783110472097-007
30. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: Existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions. Adv. Differ. Equ. **2014**, 257 (2014). https://doi.org/10.1186/1687-1847-2014-257
31. Ahmad, B., Etemad, S., Ettefagh, M., Rezapour, S.: On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions. Bull. Math. Soc. Sci. Math. Roum. **59(107)**(2), 119–134 (2016)
32. Abdeljawad, T., Alzabut, J., Baleanu, D.: A generalized q-fractional Gronwall inequality and its applications to non-linear delay q-fractional difference systems. J. Inequal. Appl. **2016**, 240 (2016). https://doi.org/10.1186/s13660-016-1181-2
33. Kac, V., Cheung, P.: Quantum Calculus. Universitext. Springer, New York (2002). https://doi.org/10.1007/978-1-4613-0071-7-1
34. Ferreira, R.A.C.: Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2010**, 70 (2010)
35. Agarwal, R.P., O'Regan, D., Sahu, D.R.: Fixed Point Theory for Lipschitzian-Type Mappings with Applications. Springer, Dordrecht (2009). https://doi.org/10.1007/978-0-387-75818-3