Xu et al. *Advances in Difference Equations* (2019) 2019:416
https://doi.org/10.1186/s13662-019-2346-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Random attractors for the stochastic coupled suspension bridge equations of Kirchhoff type

Ling Xu[1*], Jianhua Huang[1] and Qiaozhen Ma[2]

*Correspondence: 13893414055@163.com
[1]College of Liberal Arts and Sciences, National University of Defense Technology, Changsha, P.R. China
Full list of author information is available at the end of the article

Abstract

This paper is devoted to the dynamical behavior of stochastic coupled suspension bridge equations of Kirchhoff type. For the deterministic cases, there are many classical results such as existence and uniqueness of a solution and long-term behavior of solutions. To the best of our knowledge, the existence of random attractors for the stochastic coupled suspension bridge equations of Kirchhoff type is not yet considered. We intend to investigate these problems. We first obtain the dissipativeness of a solution in higher-energy spaces $H^3(U) \times H_0^1(U) \times (H^2(U) \cap H_0^1(U)) \times H_0^1(U)$. This implies that the random dynamical system generated by the equation has a random attractor in $(H^2(U) \cap H_0^1(U)) \times L^2(U) \times H_0^1(U) \times L^2(U)$, which is a tempered random set in the space in $H^3(U) \times H_0^1(U) \times (H^2(U) \cap H_0^1(U)) \times H_0^1(U)$.

Keywords: Coupled suspension bridge equations; Random dynamical system; Random attractors; Kirchhoff-type

1 Introduction

In this paper, for simplicity, set $\Delta^2 u = u_{xxxx}$, $-\Delta u = -u_{xx}$, $\nabla u = u_x$. We consider the following stochastic coupled suspension bridge equations of Kirchhoff type:

$$\begin{cases} u_{1tt} + \alpha \Delta^2 u_1 + \delta_1 u_{1t} + k(u_1 - u_2)^+ + (p - \|\nabla u_1\|^2_{L^2(U)})\Delta u_1 + f_B(u_1) = q_B(x)\dot{W}_1, \\ \quad \text{in } U \times [\tau, +\infty), \tau \in \mathbb{R}, \\ u_{2tt} - \beta \Delta u_2 + \delta_2 u_{2t} - k(u_1 - u_2)^+ + f_S(u_2) = q_S(x)\dot{W}_2, \quad \text{in } U \times [\tau, +\infty), \tau \in \mathbb{R} \end{cases} \tag{1.1}$$

subject to boundary conditions at both ends

$$\begin{aligned} &u_1(x,t) = \Delta u_1(x,t) = 0, \quad x \in \partial U, t \geq \tau, \\ &u_2(x,t) = 0, \quad x \in \partial U, t \geq \tau, \end{aligned} \tag{1.2}$$

and the initial-value conditions

$$\begin{aligned} &u_1(x,\tau) = u_{10}(x), \qquad u_{1t}(x,\tau) = u_{11}(x), \quad x \in U, \\ &u_2(x,\tau) = u_{20}(x), \qquad u_{2t}(x,\tau) = u_{21}(x), \quad x \in U, \end{aligned} \tag{1.3}$$

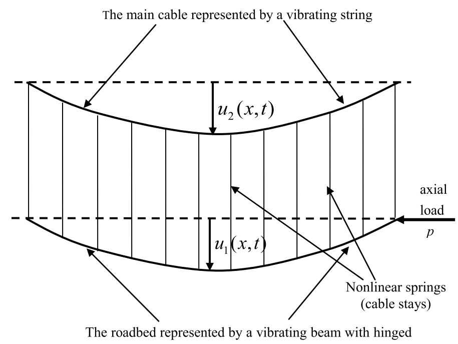

Figure 1 A model of the one-dimensional bridge represented by the coupling of the cable (a vibrating string) and the roadbed (a vibrating beam) by the stays, treated as nonlinear springs

where U is a bounded closed interval of R with a boundary $\partial U = \{0, L\}$. The first equation of (1.1) represents the vibration of the road bed in the vertical direction, and the second equation describes that of the main cable from which the road bed is suspended by the tie cables, see [1, 2]. We assume the ratios between the length of the bridge and other dimensions to be very small, which entails that the torsional motion can be ignored, so that the road bed can be simply modeled as a vibrating one-dimensional beam. In addition, by neglecting the influence of the towers and side parts of the bridge, such a beam can be assumed to have simply supported ends (see Fig. 1).

Now, we introduce the meanings of each terms in (1.1) as follows.

- $u_1 = u_1(x,t) : [0,L] \times [\tau, +\infty) \to \mathbb{R}$ represents the downward deflection of the beam mid-line (of unitary length) in the vertical plane with respect to the reference configuration.
- $u_2 = u_2(x,t) : [0,L] \times [\tau, +\infty) \to \mathbb{R}$ measures the vertical displacement of the string.
- $\alpha > 0$ and $\beta > 0$ are the flexural rigidity of the structure and coefficient of tensible strength of the cable, respectively.
- $\delta_1, \delta_2 > 0$ represent the viscous damping constants due to external resistant forces which linearly depend on the velocity.
- $k > 0$ is the spring constant, the nonlinear terms $\pm k(u_1 - u_2)^+$ model the restoring force due to the suspenders, which are assumed to behave as one-sided springs. Such a restoring force is proportional to the elongation of the suspenders if they are stretched and vanishes if they are compressed. In addition, it holds the road bed up and pulls the cable down; therefore, into the first equation, the plus sign in the front of $k(u_1 - u_2)^+$ occurs, but into the second equation, the sign in front of the same term is minus.
- $p \in \mathbb{R}$ accounts for the axial force acting at the end of the road bed of the bridge in the reference configuration: it is negative when the beam is axially stretched, positive when compressed.
- $\|\nabla u_1\|^2_{L^2(U)}$ takes into account the geometry of the beam bending due to its elongation.
- f_B and f_S are given vertical dead-load distributions acting on the deck and the main cable, respectively.

In addition to normal vehicular load, a bridge is also occasionally subject to random loads such as seismic and wind forces. Such random forces can be modeled by two noise terms $\dot{W}_1(t)$ and $\dot{W}_2(t)$ on the right-hand side of equation (1.1). $\dot{W}_i(t)$ ($i = 1, 2$) are the scalar Gaussian white noises, i.e., formally the derivative of the two-sided real-value scalar

Wiener processes $\{W_i(t)\}_{t\in\mathbb{R}}$ $(i = 1, 2)$. We assume that the functions f_B, f_S, q_B, q_S always satisfy the following assumptions.

(i) The nonlinear terms $f_B \in C^3(\mathbb{R},\mathbb{R})$ and $f_S \in C^2(\mathbb{R},\mathbb{R})$ with $f_B(0) = f_S(0) = 0$, which satisfy the following conditions.

(a) Growth conditions:

$$\left|f_B(\tau)\right|, \left|f_S(\tau)\right| \le C_0\left(1 + |\tau|^{\gamma}\right), \quad \gamma \ge 1, \forall \tau \in \mathbb{R}, \tag{1.4}$$

where C_0 is a positive constant.

(b) Dissipation conditions:

$$\begin{aligned} F_B(\tau) &:= \int_0^{\tau} f_B(r)\,dr \ge C_1\left(|\tau|^{\gamma+1} - 1\right), \\ F_S(\tau) &:= \int_0^{\tau} f_S(r)\,dr \ge C_1\left(|\tau|^{\gamma+1} - 1\right), \quad \gamma \ge 1, \forall \tau \in \mathbb{R}, \end{aligned} \tag{1.5}$$

and

$$\tau f_B(\tau) \ge C_2\left(F_B(\tau) - 1\right), \qquad \tau f_S(\tau) \ge C_2\left(F_S(\tau) - 1\right), \quad \forall \tau \in \mathbb{R}, \tag{1.6}$$

where C_1 and C_2 are positive constants.

(ii) $q_B(x) \in H^3(U) \cap H_0^1(U)$ and $q_S(x) \in H^2(U) \cap H_0^1(U)$ are not identically equal to zero.

The suspension bridge equations were presented by Lazer and McKenna as new problems in the field of nonlinear analysis [1]. Equation (1.1) models a random perturbation of the coupled suspension bridge equation of Kirchhoff type. If we ignore the effects of white noises in (1.1), that is, $q_B(x) \equiv q_S(x) \equiv 0$, then there are a lot of profound results on the dynamics of a variety of deterministic systems related to (1.1), see [3–19] and the reference therein. For example, just for a single deterministic suspension bridge equation (without white noises), in [3–9, 18], the authors investigated the existence, uniqueness, and global attractors of the solution. And for the deterministic coupled string-beam equations (without white noises) and the similar problems, Woinowsky–Krieger gave existence and uniqueness of the solution in the different spaces [17]. The authors achieved the existence of strong solutions and global attractors for both the autonomous case in [13] and the nonautonomous case in [14]. Other results have been obtained in [10–12, 15, 16, 19] and the reference therein.

From the above presentation we can see that the universal attractors for the deterministic suspension bridge equations are better investigated. On the other hand, the random dynamics of the suspension bridge equations are little considered. Ahmed [20] presented the stochastic versions of the models used in [2] and also their extensions to models that include torsional motions. This permits to study suspension bridges subject to random wind or seismic forces in addition to deterministic loads. They proved the existence, uniqueness, and regularity properties of solutions of these stochastic models on these Hilbert spaces. In case of sustained stochastic wind forces, it is conceivable that a random nonempty set as an attractor may exist and may lead to collapse of the bridge unless the attractor contains the rest state and the bridge returns to the rest state after the storm has passed. Therefore, it is necessary and meaningful to study the random attractors for the suspension bridge equation. Recently, the authors studied the existence of random attractors for both the single extensible suspension bridge equation in [21] and the coupled beam-string system in [22]. To the best of our knowledge, the existence of random attractors for the coupled suspension bridge equations of Kirchhoff type with white noises is still not considered, while it is just our concern.

The concept of random attractors was introduced in [23–25] for the infinite-dimensional random dynamical systems (RDS). A random attractor of RDS is a measurable and compact invariant random set attracting all orbits. It is the appropriate generalization of the now classical attractor from the deterministic dynamical systems to the RDS. The reason is that if such a random attractor exists, it is the smallest attracting compact set and the largest invariant set [25]. These abstract results have been successfully applied into many stochastic dissipative partial differential equations such as reaction diffusion equations, Navier–Stokes equations, and nonlinear wave equations, see [26–30] and the references therein. For instance, the existence of random attractor for a damped sine-Gordon equation was proved in [26]. Yang, Kloeden, and Duan [28, 29] studied random attractors for the stochastic semi-linear degenerate parabolic equation and the wave equation with nonlinear damping and white noise, respectively.

As we know, for a dynamical system perturbed by a white noise, there is no chance that bounded subsets of the state space remain invariant. White noise pushes the system out of every bounded set with probability one [23, 24]. It implies that the classical results of global attractors for the deterministic dynamical system are not suitable for RDS. Therefore, how to establish a compact random invariant set is the main task in studying these problems. Considering our system (1.1)–(1.3), the parameter $p \in \mathbb{R}$ and the term $-\|\nabla u\|^2_{L^2(U)}\triangle u$ also bring some difficulties, they make the calculus more complex than those in [22, 30]. In order to prove the existence of random attractors for system (1.1)–(1.3), we use the methods established by Crauel, Flandoli, Arnold, and others in [23–25], which are still vital and useful until now.

The article is organized as follows. The definition of RDS and some abstract results are stated in Sect. 2. In Sect. 3, we present the existence and uniqueness of the solution corresponding to system (1.1)–(1.3), which determines an RDS. Finally, the existence of random attractors is shown in Sect. 4.

In this paper, the letters C_i ($i \in \mathbb{N}$) below are generic positive constants which do not depend on ω, τ, and t.

2 Necessary notations and abstract results

In this section, we recall some basic concepts related to the RDS and random attractors for an RDS in [23–25], which are important for getting our main results.

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, P)$ be a probability space, where

$$\Omega = \bigl\{\omega(t) \in C\bigl(\mathbb{R}, \mathbb{R}^2\bigr) : \omega(0) = 0\bigr\}$$

is endowed with compact-open topology, P is a Wiener measure, and $\mathcal{F}$ is the P-completion of Borel σ-algebra on Ω. In addition, we write $W(t,\omega) = (W_1(t), W_2(t))^T = \omega(t)$, $t \in \mathbb{R}$, $\omega \in \Omega$ and define

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \omega \in \Omega.$$

Then $\theta_t : \Omega \to \Omega, t \in \mathbb{R}$ is a family of measure preserving and ergodic transformations such that $(t,\omega) \mapsto \theta_t\omega$ is measurable, $\theta_0 = \mathrm{id}$ and $\theta_{t+s} = \theta_t\theta_s$ for all $t, s \in \mathbb{R}$. The flow θ_t together with the probability space $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system.

Definition 2.1 Let $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. Suppose that the mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies the following properties:

(i) $\phi(0,\omega)x = x$, $x \in X$, and $\omega \in \Omega$;
(ii) $\phi(t+s,\omega) = \phi(t,\theta_s\omega) \circ \phi(s,\omega)$ for all $t, s \in \mathbb{R}^+$, $x \in X$, and $\omega \in \Omega$.

Then ϕ is called a random dynamical system (RDS). Moreover, ϕ is called a continuous RDS if ϕ is continuous with respect to x for $t \geq 0$ and $\omega \in \Omega$.

Definition 2.2 A set-valued map $D : \Omega \to 2^X$ is said to be a closed (compact) random set if $D(\omega)$ is closed (compact) for P-a.s. $\omega \in \Omega$, and $\omega \mapsto d(x, D(\omega))$ is P-a.s. measurable for all $x \in X$.

Definition 2.3 If K and B are random sets such that for P-a.s. ω there exists a time $t_B(\omega)$ such that, for all $t \geq t_B(\omega)$,

$$\phi(t,\theta_{-t}\omega)B \subset K(\omega),$$

then K is said to absorb B, and $t_B(\omega)$ is called the absorption time.

Definition 2.4 A random set $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \subset X$ is called a random attractor associated with the RDS ϕ if P-a.s.:

(i) $\mathcal{A}$ is a random compact set, i.e., $A(\omega)$ is compact for P-a.s. $\omega \in \Omega$, and the map $\omega \mapsto \mathrm{d}(x, A(\omega))$ is measurable for every $x \in X$;
(ii) $\mathcal{A}$ is ϕ-invariant, i.e., $\phi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t \geq 0$ and P-a.s. $\omega \in \Omega$;
(iii) $\mathcal{A}$ attracts every bounded (non-random) set B in X, i.e., for all bounded (and non-random) $B \subset X$,

$$\lim_{t\to\infty} \mathrm{d}\bigl(\phi(t,\theta_{-t}\omega)B, A(\omega)\bigr) = 0,$$

where $\mathrm{d}(\cdot,\cdot)$ denotes the Hausdorff semidistance:

$$\mathrm{d}(A,B) = \sup_{x\in A} \inf_{y\in B} d(x,y), \quad A, B \in X.$$

Note that $\phi(t,\theta_{-t}\omega)x$ can be interpreted as the position of the trajectory which was in x at time $-t$. Thus, the attraction property holds from $t = -\infty$.

Theorem 2.5 ([14]; Existence of a random attractor) *Let ϕ be a continuous random dynamical system on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Suppose that there exists a random compact set $K(\omega)$ absorbing every bounded non-random set $B \subset X$. Then the set*

$$\mathcal{A} = \bigl\{A(\omega)\bigr\}_{\omega\in\Omega} = \overline{\bigcup_{B\subset X} \Lambda_B(\omega)}$$

is a global random attractor for ϕ, where the union is taken over all bounded $B \subset X$, and $\Lambda_B(\omega)$ is the ω-limits set of B given by

$$\Lambda_B(\omega) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\phi(t,\theta_{-t}\omega)B}.$$

3 Existence and uniqueness of solutions

Let $(H,(\cdot,\cdot),\|\cdot\|)$ be a real Hilbert space, and let $A: D(A)\to H$ be a strictly positive self-adjoint operator. For any $r\in\mathbb{R}$, the scale of Hilbert spaces generated by the powers of A is introduced as follows:

$$H^r = D\bigl(A^{\frac{r}{4}}\bigr), \quad (u,v)_r = \bigl(A^{\frac{r}{4}}u, A^{\frac{r}{4}}v\bigr), \|u\|_r = \bigl\|A^{\frac{r}{4}}u\bigr\|.$$

When $r = 0$, the index r is omitted. In particular, we have the compact embeddings $H^{r+1} \hookrightarrow H^r$ along with the generalized Poincré inequality, there holds

$$\|u\|_{r+1}^4 \geq \lambda_1 \|u\|_r^4, \quad \forall u\in H^{r+1}, \tag{3.1}$$

where $\lambda_1 > 0$ is the first eigenvalue of A.

In order to establish more general results, we recast system (1.1)–(1.3) into an abstract setting. Without loss of generality, we define

$$\begin{aligned}
&Y_0 = L^2(U), \qquad Y_1 = H_0^1(U), \qquad Y_2 = H^2(U)\cap H_0^1(U),\\
&D(A) = \bigl\{u, \triangle u\in H^2(U) \mid u(0) = u(L) = \triangle u(0) = \triangle u(L) = 0\bigr\},
\end{aligned}$$

where $A = \Delta^2$, $A^{\frac{1}{2}} = -\Delta$. For brevity, we introduce some spaces V_1, V_2, E which are used throughout the paper, that is,

$$V_1 = Y_1\times Y_0, \qquad V_2 = Y_2\times Y_0, \qquad E = V_2\times V_1,$$

and endow spaces V_1, V_2, E with the following norms, respectively:

$$\begin{aligned}
&\|y_1\|_{V_1}^2 = \|u_1\|_2^2 + \|v_1\|^2; \qquad \|y_2\|_{V_2}^2 = \|u_2\|_1^2 + \|v_2\|^2;\\
&\|y\|_E^2 = \alpha\|u_1\|_2^2 + \|v_1\|^2 + \beta\|u_2\|_1^2 + \|v_2\|^2
\end{aligned}$$

for all $y_i = (u_i,v_i)^T\in V_i$, $y = (y_1,y_2) = ((u_1,v_1)^T,(u_2,v_2)^T)\in E$, $i = 1,2$, here T denotes the transposition.

Let $u(x,t) = (u_1(x,t),u_2(x,t))^T$, $\Phi = \begin{pmatrix}\alpha A & O\\ O & \beta A^{\frac{1}{2}}\end{pmatrix}$, $Q(x) = \begin{pmatrix} q_B(x) & 0\\ 0 & q_S(x)\end{pmatrix}$, $\Lambda = \begin{pmatrix}\delta_1 & 0\\ 0 & \delta_2\end{pmatrix}$, $f(u) = (f_B(u_1) + k(u_1-u_2)^+ - (p-\|u_1\|_1^2)A^{\frac{1}{2}}u_1, f_S(u_2) - k(u_1-u_2)^+)^T$, $W(t) = (W_1(t),W_2(t))^T$. Then system (1.1)–(1.3) can be written as

$$\begin{cases}
u_{tt} + \Phi u + \Lambda u_t + f(u) = Q(x)\dot{W}(t), \quad \text{in } U\times[\tau,+\infty), \tau\in\mathbb{R},\\
u(x,t)|_{\partial U} = (u_1(x,t)|_{\partial U}, u_2(x,t)|_{\partial U})^T = (0,0)^T,\\
\triangle u(x,t)|_{\partial U} = (\triangle u_1(x,t)|_{\partial U}, 0)^T = 0,\\
u(x,\tau) = u_0(x) = (u_{10}(x),u_{20}(x))^T,\\
u_t(x,\tau) = u_1(x) = (u_{11}(x),u_{21}(x))^T.
\end{cases} \tag{3.2}$$

Let

$$\varepsilon = \min\left\{\frac{\alpha\lambda_1\delta_1}{2\delta_1^2 + 3\alpha\lambda_1}, \frac{\beta\sqrt{\lambda_1}\delta_2}{2\delta_2^2 + 3\beta\sqrt{\lambda_1}}\right\}. \tag{3.3}$$

By the transformation, $u = u$, $z = (z_1, z_2)^T = (u_{1t} + \varepsilon u_1 - q_B(x)W_1(t), u_{2t} + \varepsilon u_2 - q_S(x) \times W_2(t))^T = u_t + \varepsilon u - Q(x)W(t)$, we obtain the following random partial differential equation (RPDE):

$$\begin{cases} \frac{du}{dt} = z - \varepsilon u + Q(x)W(t), \\ \frac{dz}{dt} = -\Lambda z + \varepsilon z - \Phi u + \varepsilon\Lambda u - \varepsilon^2 u - f(u) - \Lambda Q(x)W(t) + \varepsilon Q(x)W(t), \\ u(x,\tau) = u_0(x), \qquad z(\tau,\omega) = z(x,\tau,\omega) = u_1(x) + \varepsilon u_0(x) - Q(x)W(\tau), \quad x \in U. \end{cases} \tag{3.4}$$

In contrast to the stochastic differential equation (3.2), no stochastic differential appears here. Let $\varphi = (\varphi_1, \varphi_2) = ((u_1, z_1)^T, (u_2, z_2)^T)$. Hence equation (3.4) can be written as

$$\begin{cases} \dot{\varphi}_{1t} + L_1\varphi_1 = F_1(\varphi_1, \omega), \qquad \varphi_1(\tau,\omega) = (u_{10}, z_1(\tau,\omega))^T, \quad t \geq \tau, \\ \dot{\varphi}_{2t} + L_2\varphi_2 = F_2(\varphi_2, \omega), \qquad \varphi_2(\tau,\omega) = (u_{20}, z_2(\tau,\omega))^T, \quad t \geq \tau, \end{cases} \tag{3.5}$$

where

$$L_1 = \begin{pmatrix} \varepsilon I & -I \\ \alpha A - \varepsilon(\delta_1 - \varepsilon)I & (\delta_1 - \varepsilon)I \end{pmatrix}, \qquad L_2 = \begin{pmatrix} \varepsilon I & -I \\ \beta A^{\frac{1}{2}} - \varepsilon(\delta_2 - \varepsilon)I & (\delta_2 - \varepsilon)I \end{pmatrix},$$

$$F_1(\varphi_1, \omega) = \begin{pmatrix} q_B(x)W_1(t) \\ -k(u_1 - u_2)^+ + (p - \|u_1\|_1^2)A^{\frac{1}{2}}u_1 - f_B(u_1) - (\delta_1 - \varepsilon)q_B(x)W_1(t) \end{pmatrix},$$

and

$$F_2(\varphi_2, \omega) = \begin{pmatrix} q_S(x)W_2(t) \\ k(u_1 - u_2)^+ - f_S(u_2) - (\delta_2 - \varepsilon)q_S(x)W_2(t) \end{pmatrix}.$$

We know from [31] that L_1, L_2 are the infinitesimal generators of C_0-semigroup $e^{L_1 t}$, $e^{L_2 t}$ on V_2, V_1, respectively. It is not difficult to check that the functions $F(\cdot,\omega) = (F_1(\cdot,\omega), F_2(\cdot,\omega)) : E \mapsto E$ are locally Lipschitz continuous with respect to $\varphi = (\varphi_1, \varphi_2)$ and bounded for every $\omega \in \Omega$. Thus, by the classical semigroup theory on the local existence and uniqueness of solutions of evolution differential equations in [31], we have the following theorems.

Theorem 3.1 *Let* (1.4)–(1.6) *hold*, $q_B(x) \in H^3(U) \cap H_0^2(U)$, $q_S(x) \in H^2(U) \cap H_0^1(U)$, *then for any* $T > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *and* $\varphi(\tau,\omega) = (\varphi_1(\tau,\omega), \varphi_2(\tau,\omega)) \in E$, *RPDE* (3.5) *has a unique solution* $\varphi(\cdot,\omega) \in C([\tau, \tau + T); E)$ *in mild sense, i.e.*,

$$\begin{cases} \varphi_1(t,\omega) = e^{L_1(t-\tau)}\varphi_1(\tau,\omega) + \int_\tau^t e^{L_1(t-s)}F_1(\varphi_1(s),\omega)\,ds, \\ \varphi_2(t,\omega) = e^{L_2(t-\tau)}\varphi_2(\tau,\omega) + \int_\tau^t e^{L_2(t-s)}F_2(\varphi_2(s),\omega)\,ds. \end{cases} \tag{3.6}$$

Moreover, $\varphi(t, \varphi(\tau,\omega))$ *is continuous in* t *and* $\varphi(\tau,\omega)$.

Theorem 3.2 *Let* (1.4)–(1.6) *hold*, $q_B(x) \in H^3(U) \cap H_0^2(U)$, $q_S(x) \in H^2(U) \cap H_0^1(U)$, *then the solution* $\varphi(\cdot,\omega) \in C([\tau,\tau+T];E)$ *of system* (3.5) *generates a continuous random dynamical system* $(\theta, S_\varepsilon(t,\omega))$ *on* E *as*

$$S_\varepsilon(t,\omega): \varphi(\tau,\omega) \mapsto \varphi(t,\omega), \qquad E \to E, \tag{3.7}$$

by $S_\varepsilon(t,\omega) = \varphi(t+\tau,\tau,\varphi(\tau,\omega)) = (\varphi_1(t+\tau,\tau,\varphi(\tau,\omega)), \varphi_2(t+\tau,\tau,\varphi(\tau,\omega)))$, *where*

$$\varphi_1\big(t+\tau,\tau,\varphi(\tau,\omega)\big) = \begin{pmatrix} u_1(t+\tau,\tau,\varphi(\tau,\omega)) \\ u_{1t}(t+\tau,\tau,\varphi(\tau,\omega)) + \varepsilon u_1(t+\tau,\tau,\varphi(\tau,\omega)) - q_B(x)W_1(t) \end{pmatrix},$$

$$\varphi_2\big(t+\tau,\tau,\varphi(\tau,\omega)\big) = \begin{pmatrix} u_2(t+\tau,\tau,\varphi(\tau,\omega)) \\ u_{2t}(t+\tau,\tau,\varphi(\tau,\omega)) + \varepsilon u_2(t+\tau,\tau,\varphi(\tau,\omega)) - q_S(x)W_2(t) \end{pmatrix}.$$

To show the conjugation of the solution of the stochastic partial differential equation (3.2) and RPDE (3.5), we introduced the homeomorphism

$$R(\theta_t\omega)y = \big(y_1, y_2 - \varepsilon y_1 + Q(x)W(t)\big)^T, \quad y = (y_1,y_2)^T \in E$$

with the inverse homeomorphism being

$$R^{-1}(\theta_t\omega)y = \big(y_1, y_2 + \varepsilon y_1 - Q(x)W(t)\big)^T.$$

Then the transformation

$$S(t,\omega) = R(\theta_t\omega)S_\varepsilon(t,\omega)R^{-1}(\theta_t\omega) \tag{3.8}$$

also determines an RDS corresponding to equation (3.2).

We will also use the transformation, let

$$\eta(t) = \big(\eta_1(t),\eta_2(t)\big) = \big(\big(u_1(t), u_{1t}(t) + \varepsilon u_1(t)\big)^T, \big(u_2(t), u_{2t}(t) + \varepsilon u_2(t)\big)^T\big),$$

$$G_1(\eta_1,\omega) = \begin{pmatrix} 0 \\ -k(u_1-u_2)^+ + (p - \|u_1\|_1^2)A^{\frac{1}{2}}u_1 - f_B(u_1) + q_B(x)\dot{W}_1(t) \end{pmatrix},$$

and

$$G_2(\eta_2,\omega) = \begin{pmatrix} 0 \\ k(u_1-u_2)^+ - f_S(u_2) + q_S(x)\dot{W}_2(t) \end{pmatrix},$$

then RPDE (3.2) can be rewritten as

$$\begin{cases} \eta_{1t} + L_1\eta_1 = G_1(\eta_1,\omega), & \eta_1(\tau,\omega) = (u_{10}, u_{11} + \varepsilon u_{10})^T, \\ \eta_{2t} + L_2\eta_2 = G_2(\eta_2,\omega), & \eta_2(\tau,\omega) = (u_{20}, u_{21} + \varepsilon u_{20})^T. \end{cases} \tag{3.9}$$

We introduce the isomorphism $T_\varepsilon y = (y_1, y_2 + \varepsilon y_1)^T$, $y = (y_1,y_2)^T \in E$, which has the inverse isomorphism $T_{-\varepsilon} y = (y_1, y_2 - \varepsilon y_1)^T$. It follows that $(\theta, \bar{S}_\varepsilon(t,\omega))$ with the transformation

$$\bar{S}_\varepsilon(t,\omega) = T_\varepsilon S(t,\omega) T_{-\varepsilon} \tag{3.10}$$

is also an RDS corresponding to (3.5). Therefore, the two RDS $S(t,\omega)$ and $\bar{S}_\varepsilon(t,\omega)$ are equivalent.

4 Existence of a random attractor

In this section, we prove the existence of a random attractor for RDS (3.5) in the space E.

Lemma 4.1 *Let (1.4)–(1.6) hold, $q_B(x) \in H^3(U) \cap H_0^2(U)$, $q_S(x) \in H^2(U) \cap H_0^1(U)$. There exist a random variable $r_1(\omega) > 0$ and a bounded ball B_0 of E centered at 0 with random radius $r_0(\omega) > 0$ such that, for any bounded non-random set B of E, there exists a deterministic $T(B) \le -1$ such that the solution $\varphi(t,\omega;\varphi(\tau,\omega)) = (\varphi_1(t,\omega;\varphi_1(\tau,\omega)), \varphi_2(t,\omega;\varphi_2(\tau,\omega))) = ((u_1(t,\omega), z_1(t,\omega))^T, (u_2(t,\omega), z_2(t,\omega))^T)$ of (3.5) with the initial value $((u_{10}, u_{11} + \varepsilon u_{10})^T, (u_{20}, u_{21} + \varepsilon u_{20})^T) \in B$ satisfies, for P-a.s. $\omega \in \Omega$,*

$$\bigl\|\varphi\bigl(-1,\omega;\varphi(\tau,\omega)\bigr)\bigr\|_E \le r_0(\omega), \quad \tau \le T(B),$$

and for all $\tau \le t \le 0$,

$$\bigl\|\varphi\bigl(t,\omega;\varphi(\tau,\omega)\bigr)\bigr\|_E^2 \le R_1(t,\omega), \tag{4.1}$$

where $z_1(t,\omega) = u_{1t}(t) + \varepsilon u_1(t) - q_B(x)W_1(t)$, $z_2(t,\omega) = u_{2t}(t) + \varepsilon u_2(t) - q_S(x)W_2(t)$, and $R(t,\omega)$ is given by

$$\begin{aligned} R_1(t,\omega) = e^{-\varepsilon_1(t-\tau)}\biggl(&\|u_{10}\|_2^2 + \|u_{11} + \varepsilon u_{10}\|^2 + \|q_B\|^2\bigl|W_1(\tau)\bigr|^2 + \|u_{20}\|_1^2 + \|u_{21} + \varepsilon u_{20}\|^2 \\ &+ \|q_S\|^2\bigl|W_2(\tau)\bigr|^2 + \frac{1}{2}\bigl(\|u_{10}\|_1^2 - p\bigr)^2 + k\bigl\|(u_{10} - u_{20})^+\bigr\|^2 \\ &+ 2\int_\Omega F_B(u_{10})\,dx + 2\int_\Omega F_S(u_{20})\,dx\biggr) + Mr_1^2(\omega)). \end{aligned}$$

It is easy to deduce a similar absorption result for

$$\eta(-1) = (\eta_1,\eta_2) = \bigl(\bigl(u_1(-1), u_{1t}(-1) + \varepsilon u_1(-1)\bigr)^T, \bigl(u_2(-1), u_{2t}(-1) + \varepsilon u_2(-1)\bigr)^T\bigr)$$

instead of $\varphi(-1)$.

Proof Taking the inner product in V_2 of the first equation of (3.5) with $\varphi_1 = (u_1, z_1)^T$, and taking the inner product in V_1 of the second equation of (3.5) with $\varphi_2 = (u_2, z_2)^T$, in which $z_1 = u_{1t} + \varepsilon u_1 - q_B(x)W_1(t)$, $z_2 = u_{2t} + \varepsilon u_2 - q_S(x)W_2(t)$, then adding them, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\varphi_1\|_{V_2}^2 + (L_1\varphi_1,\varphi_1)_{V_2} + \frac{1}{2}\frac{d}{dt}\|\varphi_2\|_{V_1}^2 + (L_2\varphi_2,\varphi_2)_{V_1} \\ &\quad = \bigl(F_1(\varphi_1,\omega),\varphi_1\bigr)_{V_2} + \bigl(F_2(\varphi_2,\omega),\varphi_2\bigr)_{V_1}, \quad \forall t \ge \tau. \end{aligned} \tag{4.2}$$

Due to (3.1), using the Hölder and Young inequalities, we get

$$\begin{aligned} &(L_1\varphi_1,\varphi_1)_{V_2} + (L_2\varphi_2,\varphi_2)_{V_1} \\ &\quad = \varepsilon\|u_1\|_2^2 + (\alpha - 1)(Au_1, z_1) - \varepsilon(\delta_1 - \varepsilon)(u_1, z_1) + (\delta_1 - \varepsilon)\|z_1\|^2 \end{aligned}$$

$$\begin{aligned}
&+\varepsilon\|u_2\|_1^2+(\beta-1)\big(A^{\frac{1}{2}}u_2,z_2\big)-\varepsilon(\delta_2-\varepsilon)(u_2,z_2)+(\delta_2-\varepsilon)\|z_2\|^2\\
\geq{}&\frac{1}{2}\frac{d}{dt}\big((\alpha-1)\|u_1\|_2^2+(\beta-1)\|u_2\|_1^2\big)+\varepsilon\big(\alpha\|u_1\|_2^2+\beta\|u_2\|_1^2\big)\\
&-\frac{\varepsilon\delta_1}{\sqrt{\lambda_1}}\|u_1\|_2\|z_1\|+(\delta_1-\varepsilon)\|z_1\|^2-\frac{\varepsilon\delta_2}{\sqrt[4]{\lambda_1}}\|u_2\|_1\|z_2\|+(\delta_2-\varepsilon)\|z_2\|^2\\
&+(1-\alpha)\big(Au_1,q_B(x)W_1(t)\big)+(1-\beta)\big(A^{\frac{1}{2}}u_2,q_S(x)W_2(t)\big)\\
\geq{}&\frac{1}{2}\frac{d}{dt}\big((\alpha-1)\|u_1\|_2^2+(\beta-1)\|u_2\|_1^2\big)+\varepsilon\big(\alpha\|u_1\|_2^2+\beta\|u_2\|_1^2\big)-\frac{\varepsilon\alpha}{4}\|u_1\|_2^2\\
&-\frac{\varepsilon\delta_1^2}{\alpha\lambda_1}\|z_1\|^2+(\delta_1-\varepsilon)\|z_1\|^2-\frac{\varepsilon\beta}{4}\|u_2\|_1^2-\frac{\varepsilon\delta_2^2}{\beta\sqrt{\lambda_1}}\|z_2\|^2+(\delta_2-\varepsilon)\|z_2\|^2\\
&+(1-\alpha)\big(Au_1,q_B(x)W_1(t)\big)+(1-\beta)\big(A^{\frac{1}{2}}u_2,q_S(x)W_2(t)\big)\\
\geq{}&\frac{1}{2}\frac{d}{dt}\big((\alpha-1)\|u_1\|_2^2+(\beta-1)\|u_2\|_1^2\big)+\frac{\varepsilon}{2}\|\varphi\|_E^2+\frac{\varepsilon}{4}\big(\alpha\|u_1\|_2^2+\beta\|u_2\|_1^2\big)\\
&+\frac{\delta_1}{2}\|z_1\|^2+\frac{\delta_2}{2}\|z_2\|^2+(1-\alpha)\big(Au_1,q_B(x)W_1(t)\big)\\
&+(1-\beta)\big(A^{\frac{1}{2}}u_2,q_S(x)W_2(t)\big),
\end{aligned}\tag{4.3}$$

noting that we use (3.3) in the last inequality. In addition,

$$\begin{aligned}
&\big(F_1(\varphi_1,\omega),\varphi_1\big)_{V_2}+\big(F_2(\varphi_2,\omega),\varphi_2\big)_{V_1}\\
&\quad=\big(u_1,q_B(x)W_1(t)\big)_2+\big(u_2,q_S(x)W_2(t)\big)_1-\big(k(u_1-u_2)^+,z_1\big)+\big(k(u_1-u_2)^+,z_2\big)\\
&\qquad-\big(f_B(u_1),z_1\big)+\big((p-\|u_1\|_1^2)A^{\frac{1}{2}}u_1,z_1\big)-(\delta_1-\varepsilon)\big(q_B(x)W_1(t),z_1\big)-\big(f_S(u_2),z_2\big)\\
&\qquad-(\delta_2-\varepsilon)\big(q_S(x)W_2(t),z_2\big)\\
&\quad=-\frac{1}{2}\frac{d}{dt}k\big\|(u_1-u_2)^+\big\|^2-k\varepsilon\big\|(u_1-u_2)^+\big\|^2+\big(u_1,q_B(x)W_1(t)\big)_2+\big(u_2,q_S(x)W_2(t)\big)_1\\
&\qquad+k\big((u_1-u_2)^+,q_B(x)W_1(t)\big)-k\big((u_1-u_2)^+,q_S(x)W_2(t)\big)-(\delta_1-\varepsilon)\big(q_B(x)W_1(t),z_1\big)\\
&\qquad-(\delta_2-\varepsilon)\big(q_S(x)W_2(t),z_2\big)+\big((p-\|u_1\|_1^2)A^{\frac{1}{2}}u_1,z_1\big)\\
&\qquad-\big(f_B(u_1),z_1\big)-\big(f_S(u_2),z_2\big).
\end{aligned}\tag{4.4}$$

We deal with some terms in (4.3) and (4.4) as follows:

$$\begin{aligned}
&\alpha\big(u_1,q_B(x)W_1(t)\big)_2+\beta\big(u_2,q_S(x)W_2(t)\big)_1\\
&\quad\leq\frac{\varepsilon}{4}\big(\alpha\|u_1\|_2^2+\beta\|u_2\|_1^2\big)+\frac{\alpha}{\varepsilon}\|q_B\|_2^2\big|W_1(t)\big|^2+\frac{\beta}{\varepsilon}\|q_S\|_1^2\big|W_2(t)\big|^2;
\end{aligned}\tag{4.5}$$

$$\begin{aligned}
&\big|k\big((u_1-u_2)^+,q_B(x)W_1(t)\big)-k\big((u_1-u_2)^+,q_S(x)W_2(t)\big)\big|\\
&\quad\leq\frac{k\varepsilon}{2}\big\|(u_1-u_2)^+\big\|^2+\frac{k}{\varepsilon}\|q_B\|^2\big|W_1(t)\big|^2+\frac{k}{\varepsilon}\|q_S\|^2\big|W_2(t)\big|^2;
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
&\big|-(\delta_1-\varepsilon)\big(q_B(x)W_1(t),z_1\big)-(\delta_2-\varepsilon)\big(q_S(x)W_2(t),z_2\big)\big|\\
&\quad\leq\frac{\delta_1}{2}\|z_1\|^2+\frac{\delta_2}{2}\|z_2\|^2+\frac{\delta_1}{2}\|q_B\|^2\big|W_1(t)\big|^2+\frac{\delta_2}{2}\|q_S\|^2\big|W_2(t)\big|^2.
\end{aligned}\tag{4.7}$$

Furthermore, we get

$$\begin{aligned}
&-\big(\|u_1\|_1^2-p\big)\big(A^{\frac{1}{2}}u_1,z_1\big)\\
&\quad=-\frac{1}{4}\frac{d}{dt}\big(\|u_1\|_1^2-p\big)^2-\frac{\varepsilon}{2}\big(\|u_1\|_1^2-p\big)^2-\frac{\varepsilon}{2}\|u_1\|_1^4+\frac{\varepsilon p^2}{2}\\
&\qquad+\big(\|u_1\|_1^2-p\big)\big(A^{\frac{1}{2}}u_1,q_B(x)W_1(t)\big)\\
&\quad\le-\frac{1}{4}\frac{d}{dt}\big(\|u_1\|_1^2-p\big)^2-\frac{\varepsilon}{2}\big(\|u_1\|_1^2-p\big)^2-\frac{\varepsilon}{2}\|u_1\|_1^4+\frac{\varepsilon p^2}{2}+\frac{\varepsilon}{4}\big(\|u_1\|_1^2-p\big)^2\\
&\qquad+\frac{\varepsilon}{2}\|u_1\|_1^4+\frac{\|q_B\|_1^4}{2\varepsilon^3}\big|W_1(t)\big|^4\\
&\quad=-\frac{1}{4}\frac{d}{dt}\big(\|u_1\|_1^2-p\big)^2-\frac{\varepsilon}{4}\big(\|u_1\|_1^2-p\big)^2+\frac{\varepsilon p^2}{2}+\frac{\|q_B\|_1^4}{2\varepsilon^3}\big|W_1(t)\big|^4.
\end{aligned}\tag{4.8}$$

By means of (1.4) and (1.5), we conclude that

$$\begin{aligned}
&\big(f_B(u_1),q_B(x)W_1(t)\big)\\
&\quad\le C_0\int_U\big(1+|u_1|^\gamma\big)q_B(x)W_1(t)\,dx\\
&\quad\le C_0\|q_B\|\big|W_1(t)\big|+C_0\left(\int_U|u_1|^{\gamma+1}\,dx\right)^{\frac{\gamma}{\gamma+1}}\|q_B\|_{\gamma+1}\big|W_1(t)\big|\\
&\quad\le C_0\|q_B\|\big|W_1(t)\big|+C_0C_1^{-\frac{\gamma}{\gamma+1}}\left(\int_U\big(F_B(u_1)+C_1\big)\,dx\right)^{\frac{\gamma}{\gamma+1}}\|q_B\|_{\gamma+1}\big|W_1(t)\big|\\
&\quad\le C_0\|q_B\|\big|W_1(t)\big|+\frac{\varepsilon C_0C_1^{-1}}{2}\int_U F_B(u_1)\,dx+\frac{C_0}{2\varepsilon}\|q_B\|_{\gamma+1}^{\gamma+1}\big|W_1(t)\big|^{\gamma+1}+\frac{\varepsilon C_0|U|}{2},
\end{aligned}\tag{4.9}$$

so together (1.6) with (4.9), this yields

$$\begin{aligned}
&-\big(f_B(u_1),z_1\big)\\
&\quad=-\big(f_B(u_1),u_{1t}+\varepsilon u_1-q_BW_1(t)\big)\\
&\quad\le-\frac{d}{dt}\int_U F_B(u_1)\,dx-\varepsilon C_2\int_U F_B(u_1)\,dx+\varepsilon C_2|U|+\big(f_B(u_1),q_BW_1(t)\big)\\
&\quad\le-\frac{d}{dt}\int_U F_B(u_1)\,dx-\frac{\varepsilon(2C_2-C_0C_1^{-1})}{2}\int_U F_B(u_1)\,dx+C_0\|q_B\|\big|W_1(t)\big|\\
&\qquad+\frac{C_0}{2\varepsilon}\|q_B\|_{\gamma+1}^{\gamma+1}\big|W_1(t)\big|^{\gamma+1}+\frac{\varepsilon(C_0+2C_2)}{2}|U|.
\end{aligned}\tag{4.10}$$

Similarly, there holds w

$$\begin{aligned}
&-\big(f_S(u_2),z_2\big)\\
&\quad=-\big(f_S(u_2),u_{2t}+\varepsilon u_2-q_2W_2(t)\big)\\
&\quad\le-\frac{d}{dt}\int_U F_S(u_2)\,dx-\frac{\varepsilon(2C_2-C_0C_1^{-1})}{2}\int_U F_S(u_2)\,dx+C_0\|q_S\|\big|W_2(t)\big|\\
&\qquad+\frac{C_0}{2\varepsilon}\|q_S\|_{\gamma+1}^{\gamma+1}\big|W_2(t)\big|^{\gamma+1}+\frac{\varepsilon(C_0+2C_2)}{2}|U|.
\end{aligned}\tag{4.11}$$

Therefore, collecting with (4.2)–(4.11), we get

$$
\begin{aligned}
&\frac{d}{dt}\biggl(\|\varphi\|_E^2+\frac{1}{2}\bigl(\|u_1\|_1^2-p\bigr)^2+k\bigl\|(u_1-u_2)^+\bigr\|^2+2\int_U F_B(u_1)\,dx \\
&\quad+2\int_U F_S(u_2)\,dx+4C_1|U|\biggr)+\varepsilon\biggl(\|\varphi\|_E^2 \\
&\quad+\frac{1}{2}\bigl(\|u_1\|_1^2-p\bigr)^2+k\bigl\|(u_1-u_2)^+\bigr\|^2+\bigl(2C_2-C_0C_1^{-1}\bigr)\int_U F_B(u_1)\,dx \\
&\quad+\bigl(2C_2-C_0C_1^{-1}\bigr)\int_U F_S(u_2)\,dx+4C_1|U|\biggr) \\
&\leq M\bigl(1+\bigl|W_1(t)\bigr|+\bigl|W_1(t)\bigr|^2+\bigl|W_1(t)\bigr|^4+\bigl|W_1(t)\bigr|^{\gamma+1} \\
&\quad+\bigl|W_2(t)\bigr|+\bigl|W_2(t)\bigr|^2+\bigl|W_2(t)\bigr|^{\gamma+1}\bigr) \\
&:=M\bigl(1+E(t)\bigr),
\end{aligned}
$$

where $M=\max\{2\varepsilon(C_0+2C_2+2C_1)|U|+\varepsilon p^2, 2C_0\|q_B\|, \frac{2\alpha}{\varepsilon}\|q_B\|_2^2+(\delta_1+\frac{2k}{\varepsilon})\|q_B\|^2, \frac{\|q_B\|_1^4}{\varepsilon^3}, \frac{C_0}{\varepsilon}\|q_B\|_{\gamma+1}^{\gamma+1}, 2C_0\|q_S\|, \frac{2\beta}{\varepsilon}\|q_S\|_2^2+(\delta_2+\frac{2k}{\varepsilon})\|q_S\|^2, \frac{C_0}{\varepsilon}\|q_S\|_{\gamma+1}^{\gamma+1}\}$. Using (1.5), we have the fact $2\int_U F_1(u_1)\,dx+2\int_U F_2(u_2)\,dx+4C_1|U|\geq 0$, let

$$
\begin{aligned}
I(t)&=\|\varphi\|_E^2+\frac{1}{2}\bigl(\|u_1\|_1^2-p\bigr)^2+k\bigl\|(u_1-u_2)^+\bigr\|^2+2\int_U F_1(u_1)\,dx+2\int_U F_2(u_2)\,dx \\
&\quad+4C_1|U|\geq 0.
\end{aligned}
$$

Choosing $\varepsilon_1=\min\{\varepsilon, \frac{\varepsilon(2C_2-C_0C_1^{-1})}{2}\}$ and $C_2>\frac{C_0C_1^{-1}}{2}$, then we have that

$$
\frac{d}{dt}I(t)+\varepsilon_1 I(t)\leq M\bigl(1+E(t)\bigr). \tag{4.12}
$$

By the Gronwall lemma, we conclude that

$$
\begin{aligned}
&\bigl\|\varphi\bigl(t,\omega;\varphi(\tau,\omega)\bigr)\bigr\|_E^2 \\
&\quad\leq e^{-\varepsilon_1(t-\tau)}I(\tau)+M\int_\tau^t e^{-\varepsilon_1(t-s)}\bigl(1+E(s)\bigr)\,ds \\
&\quad=e^{-\varepsilon_1(t-\tau)}\biggl(\|u_{10}\|_2^2+\|u_{11}+\varepsilon u_{10}\|^2+\|q_B\|^2\bigl|W_1(\tau)\bigr|^2+\|u_{20}\|_1^2+\|u_{21}+\varepsilon u_{20}\|^2 \\
&\qquad+\|q_S\|^2\bigl|W_2(\tau)\bigr|^2+\frac{1}{2}\bigl(\|u_{10}\|_1^2-p\bigr)^2+k\bigl\|(u_{10}-u_{20})^+\bigr\|^2+2\int_U F_B(u_{10})\,dx \\
&\qquad+2\int_U F_S(u_{20})\,dx+4C_1|U|\biggr)+M\int_\tau^t e^{-\varepsilon_1(t-s)}\bigl(1+E(s)\bigr)\,ds.
\end{aligned} \tag{4.13}
$$

Take

$$
r_0^2(\omega)=1+\frac{M}{\varepsilon_1}+\sup_{\tau\leq -1}e^{\varepsilon_1\tau}\bigl(\|q_B\|^2\bigl|W_1(\tau)\bigr|^2+\|q_S\|^2\bigl|W_2(\tau)\bigr|^2\bigr)+M\int_{-\infty}^{-1}e^{-\varepsilon_1(-1-s)}E(s)\,ds
$$

and

$$r_1^2(\omega) = \frac{1}{\varepsilon_1} + \int_{-\infty}^{0} e^{\varepsilon_1 s} E(s)\,ds.$$

Since $\lim_{t\to\infty} \frac{W_1(t)}{t} = 0$, $\lim_{t\to\infty} \frac{W_2(t)}{t} = 0$, $r_0^2(\omega)$ and $r_1^2(\omega)$ are finite P-a.s. Given a bounded set B of E, choose $T(B) \le -1$ such that

$$\begin{aligned} &e^{-\varepsilon_1(-1-\tau)}\biggl(\|u_{10}\|_2^2 + \|u_{11}+\varepsilon u_{10}\|^2 + \|u_{20}\|_1^2 + \|u_{21}+\varepsilon u_{20}\|^2 + \frac{1}{2}\bigl(\|u_{10}\|_1^2 - p\bigr)^2 \\ &\quad + k\bigl\|(u_{10}-u_{20})^+\bigr\|^2 + 2\int_U F_B(u_{10})\,dx + 2\int_U F_S(u_{20})\,dx + 4C_1|U|\biggr) \le 1 \end{aligned} \tag{4.14}$$

for all $((u_{10}, u_{11} + \varepsilon u_{10})^T, (u_{20}, u_{21} + \varepsilon u_{20})^T) \in B$, and for all $\tau \le T(B)$.

This completes the proof of Lemma 4.1. □

In order to obtain the regularity estimates later, we decompose the solution $u(t) = (u_1(t), u_2(t))$ of system (1.1)–(1.3) with the initial value $((u_{10}, u_{11} + \varepsilon u_{10})^T, (u_{20}, u_{21} + \varepsilon u_{20})^T) \in B$ into two parts $u_1(t) = y_{11}(t) + y_{12}(t)$, $u_2(t) = y_{21}(t) + y_{22}(t)$, where $(y_{11}(t), y_{21}(t))$, $(y_{12}(t), y_{22}(t))$ satisfy

$$\begin{cases} y_{11tt} + \alpha\triangle^2 y_{11} + \delta_1 y_{11t} + (p - \|u\|_1^2)\triangle y_{11} = 0, \quad \text{in } U\times[\tau,+\infty), \tau\in\mathbb{R}, \\ y_{21tt} - \beta\triangle y_{21} + \delta_2 y_{21t} = 0, \quad \text{in } U\times[\tau,+\infty), \tau\in\mathbb{R}, \\ y_{11}(x,t) = \triangle y_{11}(x,t) = y_{21}(x,t) = 0, \quad x\in\partial U, t\ge\tau, \\ y_{11}(x,\tau) = u_{10}(x), \qquad y_{11t}(x,\tau) = u_{11}(x), \quad x\in U, \\ y_{21}(x,\tau) = u_{20}(x), \qquad y_{21t}(x,\tau) = u_{21}(x), \quad x\in U \end{cases} \tag{4.15}$$

and

$$\begin{cases} y_{12tt} + \alpha\triangle^2 y_{12} + \delta_1 y_{12t} + (p - \|u\|_1^2)\triangle y_{12} + k(u_1-u_2)^+ \\ \quad + f_B(u_1) = q_B(x)\dot{W}_1(t), \quad \text{in } U\times[\tau,+\infty), \tau\in\mathbb{R}, \\ y_{22tt} - \beta\triangle y_{22} + \delta_2 y_{22t} - k(u_1-u_2)^+ \\ \quad + f_S(u_2) = q_S(x)\dot{W}_2(t), \quad \text{in } U\times[\tau,+\infty), \tau\in\mathbb{R}, \\ y_{12}(x,t) = \Delta y_{12}(x,t) = y_{22}(x,t) = 0, \quad x\in\partial U, t\ge\tau, \\ y_{12}(x,\tau) = 0, \qquad y_{12t}(x,\tau) = 0, \quad x\in U, \\ y_{22}(x,\tau) = 0, \qquad y_{22t}(x,\tau) = 0, \quad x\in U. \end{cases} \tag{4.16}$$

Lemma 4.2 *Assume that $p \le \frac{\alpha\sqrt{\lambda_1}}{3}$, B is a bounded non-random subset of E, then for any $((u_{10}, u_{11} + \varepsilon u_{10})^T, (u_{20}, u_{21} + \varepsilon u_{20})^T) \in B$,*

$$\begin{aligned} \bigl\|Y_1(0)\bigr\|_E^2 &= \alpha\bigl\|y_{11}(0)\bigr\|_2^2 + \bigl\|y_{11t}(0) + \varepsilon y_{11}(0)\bigr\|^2 + \beta\bigl\|y_{21}(0)\bigr\|_1^2 + \bigl\|y_{21t}(0) + \varepsilon y_{21}(0)\bigr\|^2 \\ &\le \frac{e^{\frac{\varepsilon\tau}{2}}}{C(p)}\bigl(\alpha\|u_{10}\|_2^2 - p\|u_{10}\|_1^2 + \|u_{11}+\varepsilon u_{10}\|^2 + \beta\|u_{20}\|_1^2 \\ &\quad + \|u_{21}+\varepsilon u_{20}\|^2 + \|u_{10}\|_1^4\bigr), \end{aligned} \tag{4.17}$$

where $Y_1 = ((y_{11}, y_{11t} + \varepsilon y_{11})^T, (y_{21}, y_{21t} + \varepsilon y_{21})^T)$ *satisfies* (4.15), *and*

$$C(p) = \begin{cases} 1, & p \le 0, \\ 1 - \frac{p}{\alpha\sqrt{\lambda_1}}, & 0 < p \le \frac{\alpha\sqrt{\lambda_1}}{3}. \end{cases} \tag{4.18}$$

Proof Taking the scalar product in $L^2(U)$ of the first and second equation of (4.15) with $v_1 = y_{11t} + \varepsilon y_{11}$ and $v_2 = y_{21t} + \varepsilon y_{21}$, respectively, then adding them, we conclude that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\alpha\|y_{11}\|_2^2 + \|v_1\|^2 + \beta\|y_{21}\|_1^2 + \|v_2\|^2\right) + \left(p - \|u\|_1^2\right)\left(A^{\frac{1}{2}}y_{11}, v_1\right) + \varepsilon\alpha\|y_{11}\|_2^2 \\ &\quad + (\delta_1 - \varepsilon)\|v_1\|^2 - \varepsilon(\delta_1 - \varepsilon)(y_{11}, v_1) + \varepsilon\beta\|y_{21}\|_1^2 + (\delta_2 - \varepsilon)\|v_2\|^2 \\ &\quad - \varepsilon(\delta_2 - \varepsilon)(y_{21}, v_2) = 0. \end{aligned} \tag{4.19}$$

It is easy to derive that

$$\begin{aligned} &\left(\|u\|_1^2 - p\right)\left(A^{\frac{1}{2}}y_{11}, v_1\right) \\ &\quad = \frac{1}{2}\frac{d}{dt}\left(\|u\|_1^2\|y_{11}\|_1^2 - p\|u_1\|_1^2\right) + \varepsilon\left(\|u\|_1^2\|y_{11}\|_1^2 - p\|y_{11}\|_1^2\right). \end{aligned} \tag{4.20}$$

Due to (3.3), we derive that $0 < \varepsilon < \min\{\frac{\delta_1}{3}, \frac{\alpha\lambda_1}{2\delta_1}, \frac{\delta_2}{3}, \frac{\beta\sqrt{\lambda_1}}{2\delta_2}\}$, then it follows that

$$\begin{aligned} &\varepsilon\alpha\|y_{11}\|_2^2 + (\delta_1 - \varepsilon)\|v_1\|^2 - \varepsilon(\delta_1 - \varepsilon)(y_{11}, v_1) \\ &\qquad + \varepsilon\beta\|y_{21}\|_1^2 + (\delta_2 - \varepsilon)\|v_2\|^2 - \varepsilon(\delta_2 - \varepsilon)(y_{21}, v_2) \\ &\quad \ge \varepsilon\alpha\|y_{11}\|_2^2 + \frac{2\delta_1}{3}\|v_1\|^2 - \frac{\varepsilon\alpha}{2}\|y_{11}\|_2^2 - \frac{\varepsilon\delta_1^2}{2\alpha\lambda_1}\|v_1\|^2 \\ &\qquad + \varepsilon\beta\|y_{21}\|_1^2 + \frac{2\delta_2}{3}\|v_2\|^2 - \frac{\varepsilon\beta}{2}\|y_{21}\|_2^2 - \frac{\varepsilon\delta_2^2}{2\beta\sqrt{\lambda_1}}\|v_2\|^2 \\ &\quad \ge \frac{\varepsilon}{2}\left(\alpha\|y_{11}\|_2^2 + \|v_1\|^2 + \beta\|y_{21}\|_1^2 + \|v_2\|^2\right). \end{aligned} \tag{4.21}$$

Substituting (4.20) and (4.21) into (4.19) yields

$$\begin{aligned} &\frac{d}{dt}\left(\alpha\|y_{11}\|_2^2 - p\|y_{11}\|_1^2 + \|v_1\|^2 + \beta\|y_{21}\|_1^2 + \|v_2\|^2 + \|u\|_1^2\|y_{11}\|_1^2\right) \\ &\quad + \varepsilon\left(\alpha\|y_{11}\|_2^2 - 2p\|y_{11}\|_1^2 + \|v_1\|^2 + \beta\|y_{21}\|_1^2 + \|v_2\|^2 + 2\|u\|_1^2\|y_{11}\|_1^2\right) \le 0. \end{aligned} \tag{4.22}$$

Let

$$G(t) = \alpha\|y_{11}\|_2^2 - p\|y_{11}\|_1^2 + \|v_1\|^2 + \beta\|y_{21}\|_1^2 + \|v_2\|^2 + \|u\|_1^2\|y_{11}\|_1^2,$$

then (3.1) ensures that

$$G(t) \ge C(p)\|Y_1\|_E^2 > 0,$$

and

$$\varepsilon\left(\alpha\|y_{11}\|_2^2 - 2p\|y_{11}\|_1^2 + \|v_1\|^2 + \beta\|y_{21}\|_1^2 + \|v_2\|^2 + \|u\|_1^2\|y_{11}\|_1^2\right) \ge \frac{\varepsilon}{2}G(t) \quad \text{for } p \le \frac{\alpha\sqrt{\lambda_1}}{3}.$$

Thus, from (4.22) we conclude that

$$\frac{d}{dt}G(t)+\frac{\varepsilon}{2}G(t)\leq 0.$$

By the Gronwall lemma, we get

$$\begin{aligned}G(0)&\leq e^{\frac{\varepsilon\tau}{2}}G(\tau)\\&\leq e^{\frac{\varepsilon\tau}{2}}\bigl(\alpha\|u_{10}\|_2^2-p\|u_{10}\|_1^2+\|u_{11}+\varepsilon u_{10}\|^2+\beta\|u_{20}\|_1^2+\|u_{21}+\varepsilon u_{20}\|^2+\|u_{10}\|_1^4\bigr).\end{aligned}$$

Noticing that

$$\|Y_1(0)\|_E^2=\alpha\|y_{11}(0)\|_2^2+\|y_{11t}(0)+\varepsilon y_{11}(0)\|^2+\beta\|y_{21}(0)\|_1^2+\|y_{21t}(0)+\varepsilon y_{21}(0)\|^2\leq\frac{G(0)}{C(p)},$$

we complete the proof. □

Lemma 4.3 *Assume that* (1.4) *holds, there exists a random radius* $r_2(\omega)$ *such that, for P-a.s.* $\omega\in\Omega$,

$$\bigl\|A^{\frac{1}{4}}Y_2\bigl(0,\omega;Y_2(\tau,\omega)\bigr)\bigr\|_E^2\leq\frac{r_1^2(\omega)}{C(p)},\tag{4.23}$$

where $Y_2=((y_{12},y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))^T,(y_{22},y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))^T)$ *satisfies* (4.16), *provided that* $p\leq\frac{\alpha\sqrt{\lambda_1}}{3}$.

Proof Provided that

$$Y_2=(Y_{21},Y_{22})=\bigl((y_{12},y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))^T,(y_{22},y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))^T\bigr),$$

then equation (4.16) can be reduced to

$$\begin{cases}Y_{21t}+L_1Y_{21}=H_1(Y_{21},\omega), & Y_{21}(\tau)=(0,-q_B(x)W_1(\tau))^T,\quad t\geq\tau,\\ Y_{22t}+L_2Y_{22}=H_2(Y_{22},\omega), & Y_{22}(\tau)=(0,-q_S(x)W_2(\tau))^T,\quad t\geq\tau,\end{cases}\tag{4.24}$$

where

$$H_1(Y_{21},\omega)=\begin{pmatrix}q_B(x)W_1(t)\\-k(u_1-u_2)^+-(p-\|u\|_1^2)A^{\frac{1}{2}}y_{12}-f_B(u_1)-(\delta_1-\varepsilon)q_B(x)W_1(t)\end{pmatrix}$$

and

$$H_2(Y_{22},\omega)=\begin{pmatrix}q_S(x)W_2(t)\\k(u_1-u_2)^+-f_S(u_2)-(\delta_2-\varepsilon)q_S(x)W_2(t)\end{pmatrix}.$$

Taking the inner product in V_2 of the first equation of (4.24) with $A^{\frac{1}{2}}Y_{21}$, and taking the inner product in V_1 of the second equation of (4.24) with $A^{\frac{1}{2}}Y_{22}$, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}Y_{21}\|_{V_2}^2 + (L_1Y_{21}, A^{\frac{1}{2}}Y_{21})_{V_2} + \frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{4}}Y_{22}\|_{V_1}^2 + (L_2Y_{22}, A^{\frac{1}{2}}Y_{22})_{V_1} \\ &\quad = (H_1(Y_{21},\omega), A^{\frac{1}{2}}Y_{21})_{V_2} + (H_2(Y_{22},\omega), A^{\frac{1}{2}}Y_{22})_{V_1}, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} &(H_1(Y_{21},\omega), A^{\frac{1}{2}}Y_{21})_{V_2} + (H_2(Y_{22},\omega), A^{\frac{1}{2}}Y_{22})_{V_1} \\ &\quad = \bigl(-k(u_1-u_2)^+ - (p - \|u_1\|_1^2)A^{\frac{1}{2}}y_{12} - f_B(u_1) - (\delta_1-\varepsilon)q_B(x)W_1(t), \\ &\qquad A^{\frac{1}{2}}(y_{12t} + \varepsilon y_{12} - q_B(x)W_1(t))\bigr) \\ &\qquad + \bigl(k(u_1-u_2)^+ - f_S(u_2) - (\delta_2-\varepsilon)q_S(x)W_2(t), A^{\frac{1}{2}}(y_{22t} + \varepsilon y_{22} - q_S(x)W_2(t))\bigr) \\ &\qquad + \bigl(A^{\frac{1}{2}}y_{12}, q_B(x)W_1(t)\bigr)_2 + \bigl(A^{\frac{1}{2}}y_{22}, q_S(x)W_2(t)\bigr)_1. \end{aligned} \tag{4.26}$$

According to (4.3), we have

$$\begin{aligned} &(L_1Y_{21}, A^{\frac{1}{2}}Y_{21})_{V_2} + (L_2Y_{22}, A^{\frac{1}{2}}Y_{22})_{V_1} \\ &\quad \ge \frac{1}{2}\frac{d}{dt}\bigl((\alpha-1)\|A^{\frac{1}{4}}y_{12}\|_2^2 + (\beta-1)\|A^{\frac{1}{4}}y_{22}\|_1^2\bigr) + \frac{\varepsilon}{2}\|A^{\frac{1}{4}}Y_2\|_E^2 \\ &\qquad + \frac{\varepsilon}{4}\bigl(\alpha\|A^{\frac{1}{4}}y_{12}\|_2^2 + \beta\|A^{\frac{1}{4}}y_{22}\|_1^2\bigr) \\ &\qquad + \frac{\delta_1}{2}\|A^{\frac{1}{4}}(y_{12t} + \varepsilon y_{12} - q_B(x)W_1(t))\|^2 + \frac{\delta_2}{2}\|A^{\frac{1}{4}}(y_{22t} + \varepsilon y_{22} - q_S(x)W_2(t))\|^2 \\ &\qquad + (1-\alpha)\bigl(Ay_{12}, A^{\frac{1}{2}}q_B(x)W_1(t)\bigr) + (1-\beta)\bigl(A^{\frac{1}{2}}y_{22}, A^{\frac{1}{2}}q_S(x)W_2(t)\bigr). \end{aligned} \tag{4.27}$$

Thanks to the Young inequality, we obtain

$$\begin{aligned} &\alpha\bigl(Ay_{12}, A^{\frac{1}{2}}q_B(x)W_1(t)\bigr) + \beta\bigl(A^{\frac{1}{2}}y_{22}, A^{\frac{1}{2}}q_S(x)W_2(t)\bigr) \\ &\quad \le \frac{\varepsilon}{4}\bigl(\alpha\|A^{\frac{1}{4}}y_{12}\|_2^2 + \beta\|A^{\frac{1}{4}}y_{22}\|_1^2\bigr) + \frac{\alpha}{\varepsilon}\|A^{\frac{1}{4}}q_B\|_2^2|W_1(t)|^2 \\ &\qquad + \frac{\beta}{\varepsilon}\|A^{\frac{1}{4}}q_S\|_1^2|W_2(t)|^2. \end{aligned} \tag{4.28}$$

Below we estimate (4.26) one by one:

$$\begin{aligned} &-\bigl((p - \|u_1\|_1^2)A^{\frac{1}{2}}y_{12}, A^{\frac{1}{2}}(y_{12t} + \varepsilon y_{12} - q_B(x)W_1(t))\bigr) \\ &\quad \ge \frac{1}{2}\frac{d}{dt}\bigl(\|u_1\|_1^2\|y_{12}\|_2^2 - p\|y_{12}\|_2^2\bigr) + \frac{\varepsilon}{2}\bigl(\|u_1\|_1^2\|y_{12}\|_2^2 - p\|y_{12}\|_2^2\bigr) \\ &\qquad - \left(\frac{\varepsilon}{2}|p| + \frac{1}{2}\bigl(|p| + \|u_1\|_1^2\bigr)^2\right)\|y_{12}\|_2^2 - \frac{1}{2}\|A^{\frac{1}{2}}q_B\|^2|W_1(t)|^2 - \|u_1\|_2^3\|u_{1t}\|; \end{aligned} \tag{4.29}$$

$$\begin{aligned} &\bigl|-\bigl(k(u_1-u_2)^+, A^{\frac{1}{2}}(y_{12t} + \varepsilon y_{12} - q_B(x)W_1(t))\bigr) + \bigl(k(u_1-u_2)^+, \\ &\quad A^{\frac{1}{2}}(y_{22t} + \varepsilon y_{22} - q_S(x)W_2(t))\bigr)\bigr| \end{aligned}$$

$$
\begin{aligned}
&\le k\|A^{\frac{1}{4}}u_1\|\|A^{\frac{1}{4}}(y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))\| \\
&\quad + k\|A^{\frac{1}{4}}u_2\|\|A^{\frac{1}{4}}(y_{12t}+\varepsilon y_{12}-q_S(x)W_1(t))\| \\
&\quad + k\|A^{\frac{1}{4}}u_1\|\|A^{\frac{1}{4}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\| \\
&\quad + k\|A^{\frac{1}{4}}u_2\|\|A^{\frac{1}{4}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\| \\
&\le \left(\frac{4k^2}{\delta_1}+\frac{4k^2}{\delta_2}\right)\|A^{\frac{1}{4}}u_1\|^2+\frac{\delta_1}{8}\|A^{\frac{1}{4}}(y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))\|^2 \\
&\quad + \left(\frac{4k^2}{\delta_1}+\frac{4k^2}{\delta_2}\right)\|A^{\frac{1}{4}}u_2\|^2+\frac{\delta_2}{8}\|A^{\frac{1}{4}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\|^2;
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
&\big((\delta_1-\varepsilon)q_B(x)W_1(t),A^{\frac{1}{2}}(y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))\big) \\
&\qquad + \big((\delta_2-\varepsilon)q_S(x)W_2(t),A^{\frac{1}{2}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\big) \\
&\quad \le 2\delta_1\|A^{\frac{1}{4}}q_B\|^2|W_1(t)|^2+\frac{\delta_1}{8}\|A^{\frac{1}{4}}(y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))\|^2 \\
&\qquad + 2\delta_2\|A^{\frac{1}{4}}q_S\|^2|W_2(t)|^2+\frac{\delta_2}{8}\|A^{\frac{1}{4}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\|^2.
\end{aligned}
\tag{4.31}
$$

Besides, by (1.4), (4.1), and the Sobolev embedding theorem, we show that $f_B'(s), f_S'(s)$ are uniformly bounded in L^∞, that is,

$$
\left|\frac{d}{ds}f_B'(s)\right|_{L^\infty} \le C_0\big(|u|_{L^\infty}^{\gamma-1}+1\big) \le C_3\big(\|u\|_2^{\gamma-1}+1\big) \le C_4R_1^{\gamma-1}(t,\omega),
\tag{4.32}
$$

similarly,

$$
\left|\frac{d}{ds}f_S'(s)\right|_{L^\infty} \le C_4R_1^{\gamma-1}(t,\omega).
\tag{4.33}
$$

Combining with (4.32) and (4.33), it follows that

$$
\begin{aligned}
&\big|-\big(f_B(u_1),A^{\frac{1}{2}}(y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))\big)-\big(f_S(u_2),A^{\frac{1}{2}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\big)\big| \\
&\quad \le \|A^{\frac{1}{4}}f_B(u_1)\|\|A^{\frac{1}{4}}(y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))\| \\
&\qquad + \|A^{\frac{1}{4}}f_S(u_2)\|\|A^{\frac{1}{4}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\| \\
&\quad \le \frac{2C_5R_1^{2(\gamma-1)}(t,\omega)}{\sqrt{\lambda_1}\delta_1}\|u_1\|_2^2+\frac{\delta_1}{8}\|A^{\frac{1}{4}}(y_{12t}+\varepsilon y_{12}-q_B(x)W_1(t))\|^2 \\
&\qquad + \frac{C_5R_1^{2(\gamma-1)}(t,\omega)}{\delta_2}\|u_2\|_1^2+\frac{\delta_2}{4}\|A^{\frac{1}{4}}(y_{22t}+\varepsilon y_{22}-q_S(x)W_2(t))\|^2.
\end{aligned}
\tag{4.34}
$$

Thus, collecting all (4.26)–(4.34) from (4.25) yields, for $\tau \le T(B)$,

$$
\begin{aligned}
&\frac{d}{dt}\big(\|A^{\frac{1}{4}}Y_2\|_E^2-p\|y_{12}\|_2^2+\|u_1\|_1^2\|y_{12}\|_2^2\big)+\varepsilon\big(\|A^{\frac{1}{4}}Y_2\|_E^2-p\|y_{12}\|_2^2+\|u_1\|_1^2\|y_{12}\|_2^2\big) \\
&\quad \le \left(\frac{8k^2+4C_5R_1^{2(\gamma-1)}(t,\omega)}{\sqrt{\lambda_1}\delta_1}+\frac{8k^2}{\sqrt{\lambda_1}\delta_2}+\varepsilon|p|+\big(|p|+\|u_1\|_1^2\big)^2\right)\|u_1\|_2^2 \\
&\qquad + \left(\frac{8k^2+2C_5R_1^{2(\gamma-1)}(t,\omega)}{\delta_2}+\frac{8k^2}{\delta_1}\right)\|u_2\|_1^2+2\|u_1\|_2^3\|u_{1t}\|
\end{aligned}
$$

$$+\left(\frac{2\alpha}{\varepsilon}\|A^{\frac{1}{4}}q_B\|_2^2+4\delta_1\|A^{\frac{1}{4}}q_B\|^2+\|A^{\frac{1}{2}}q_B\|^2\right)|W_1(t)|^2$$
$$+\left(\frac{2\beta}{\varepsilon}\|A^{\frac{1}{4}}q_S\|_1^2+4\delta_2\|A^{\frac{1}{4}}q_S\|^2\right)|W_2(t)|^2.$$

We have

$$\begin{aligned}&\frac{d}{dt}H(t)+\varepsilon H(t)\\&\quad\le C_6R_1^{2\gamma}(t,\omega)+C_7R_1^6(t,\omega)+\left(\frac{2\alpha}{\varepsilon}\|A^{\frac{1}{4}}q_B\|_2^2+4\delta_1\|A^{\frac{1}{4}}q_B\|^2+\|A^{\frac{1}{2}}q_B\|^2\right)|W_1(t)|^2\\&\qquad+\left(\frac{2\beta}{\varepsilon}\|A^{\frac{1}{4}}q_S\|_1^2+4\delta_2\|A^{\frac{1}{4}}q_S\|^2\right)|W_2(t)|^2,\quad\tau\le t\le 0,\end{aligned}$$

where

$$H(t)=\|A^{\frac{1}{4}}Y_2\|_E^2-p\|y_{12}\|_2^2+\|u_1\|_1^2\|y_{12}\|_2^2.$$

Noting (3.1) and $C(p)$ defined as in Lemma 4.2, it follows that $H(t)\ge C(p)\|A^{\frac{1}{4}}Y_2\|_E^2>0$. Applying the Gronwall lemma, we arrive at

$$\begin{aligned}H(0)&\le e^{\varepsilon\tau}\left(\|A^{\frac{1}{4}}q_B\|^2|W_1(\tau)|^2+\|A^{\frac{1}{4}}q_S\|^2|W_2(\tau)|^2\right)\\&\quad+\int_\tau^0 e^{\varepsilon s}\left(C_6R_1^{2\gamma}(t,\omega)+C_7R_1^6(t,\omega)\right)ds\\&\quad+\left(\frac{2\alpha}{\varepsilon}\|A^{\frac{1}{4}}q_B\|_2^2+4\delta_1\|A^{\frac{1}{4}}q_B\|^2+\|A^{\frac{1}{2}}q_B\|^2\right)\int_\tau^0 e^{\varepsilon s}|W_1(s)|^2\,ds\\&\quad+\left(\frac{2\beta}{\varepsilon}\|A^{\frac{1}{4}}q_S\|_1^2+4\delta_2\|A^{\frac{1}{4}}q_S\|^2\right)\int_\tau^0 e^{\varepsilon s}|W_2(s)|^2\,ds.\end{aligned}\tag{4.35}$$

It is easy to see that

$$A^{\frac{1}{4}}Y_2\big(0,\omega;Y_2(\tau,\omega)\big)\big\|_E^2\le\frac{H(0)}{C(p)}.\tag{4.36}$$

Set

$$\begin{aligned}r_1^2(\omega)&=\|A^{\frac{1}{4}}q_B\|^2\sup_{\tau\le 0}e^{\varepsilon\tau}|W_1(\tau)|^2+\|A^{\frac{1}{4}}q_S\|^2\sup_{\tau\le 0}e^{\varepsilon\tau}|W_2(\tau)|^2\\&\quad+\int_{-\infty}^0 e^{\varepsilon s}\left(C_6R_1^{2\gamma}(t,\omega)+C_7R_1^6(t,\omega)\right)ds\\&\quad+\left(\frac{2\alpha}{\varepsilon}\|A^{\frac{1}{4}}q_B\|_2^2+4\delta_1\|A^{\frac{1}{4}}q_B\|^2+\|A^{\frac{1}{2}}q_B\|^2\right)\int_{-\infty}^0 e^{\varepsilon s}|W_1(s)|^2\,ds\\&\quad+\left(\frac{2\beta}{\varepsilon}\|A^{\frac{1}{4}}q_S\|_1^2+4\delta_2\|A^{\frac{1}{4}}q_S\|^2\right)\int_{-\infty}^0 e^{\varepsilon s}|W_2(s)|^2\,ds.\end{aligned}\tag{4.37}$$

Since $\lim_{t\to\infty}\frac{W_1(t)}{t}=0$, $\lim_{t\to\infty}\frac{W_2(t)}{t}=0$, $r_1^2(\omega)$ is finite P-a.s. By (4.35)–(4.37), we have

$$\|A^{\frac{1}{4}}Y_2\big(0,\omega;Y_2(\tau,\omega)\big)\big\|_E^2\le\frac{r_1^2(\omega)}{C(p)}$$

for all $((u_{10},u_{11}+\varepsilon u_{10})^T,(u_{20},u_{21}+\varepsilon u_{20})^T)\in B$, and all $\tau\le T(B)$. □

Theorem 4.4 *Let (1.4)–(1.6) hold, $p \le \frac{\alpha\sqrt{\lambda_1}}{3}$, $q_B(x) \in H^3(U)\cap H_0^2(U)$, and $q_S(x) \in H^2(U)\cap H_0^1(U)$. Then the random dynamical system $S(t,\omega)$ possesses a nonempty compact random attractor $\mathcal{A}$.*

Proof Let $B_1(\omega)$ be the ball of $H^3(U)\times H_0^1(U)\times (H^2(U)\cap H_0^1(U))\times H_0^1(U)$ of radius $\frac{r_1(\omega)}{\sqrt{C(p)}}$. From the compact embedding $H^3(U)\times H_0^1(U)\times (H^2(U)\cap H_0^1(U))\times H_0^1(U)$ into E, it follows that $B_1(\omega)$ is compact in E for every bounded non-random set B of E and any $\varphi(0)\in \bar{S}_\varepsilon(t,\theta_{-t}\omega)B$. From Lemma 4.2, we know that $Y_2(0)=\varphi(0)-Y_1(0)\in B_1(\omega)$, where $Y_2(t,\omega)$ is given by (4.24). Therefore, by means of Lemma 4.3, for $\tau\le 0$,

$$\begin{aligned}
&\inf_{l(0)\in B_1(\omega)}\|\varphi(0)-l(0)\|_E^2\\
&\quad\le \|Y_1(0)\|_E^2\\
&\quad\le \frac{e^{\frac{\varepsilon\tau}{2}}}{C(p)}\bigl(\alpha\|u_{10}\|_2^2-p\|u_{10}\|_1^2+\|u_{11}+\varepsilon u_{10}\|^2+\beta\|u_{20}\|_1^2+\|u_{21}+\varepsilon u_{20}\|^2\bigr).
\end{aligned}$$

Thus, for all $t\ge 0$,

$$\begin{aligned}
&\mathrm{d}\bigl(\bar{S}_\varepsilon(t,\theta_{-t}\omega)B,B_1(\omega)\bigr)\\
&\quad\le \frac{e^{-\frac{\varepsilon t}{2}}}{C(p)}\bigl(\alpha\|u_{10}\|_2^2-p\|u_{10}\|_1^2+\|u_{11}+\varepsilon u_{10}\|^2+\beta\|u_{20}\|_1^2+\|u_{21}+\varepsilon u_{20}\|^2\bigr).
\end{aligned}$$

Finally, from relation (3.9) between $S(t,\omega)$ and $\bar{S}_\varepsilon(t,\omega)$, one can easily obtain that, for any non-random bounded $B\subset E$ P-a.s.,

$$\mathrm{d}\bigl(S(t,\theta_{-t}\omega)B,B_1(\omega)\bigr)\to 0 \quad \text{as } t\to+\infty.$$

Hence, the RDS $S(t,\omega)$ associated with (3.7) possesses a uniformly attracting compact set $B_1(\omega)\subset E$. Then, applying Theorem 2.5, we complete the proof. □

Acknowledgements
The authors would like to thank the reviewers and the editors for their valuable suggestions and comments.

Funding
This work was supported by the NSF of China (11771449, 11561064), the NSF of Gansu Province (17JR5RA069), the University Project of Gansu Province (2017B-90), the Project of Northwest Normal University (NWNU-LKQN-16-16, NWNU-LKQN-18-14), and China Postdoctoral Science Foundation (2017M623380).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]College of Liberal Arts and Sciences, National University of Defense Technology, Changsha, P.R. China. [2]College of Mathematics and Statistics, Northwest Normal University, Lanzhou, P.R. China.

Publisher's Note

Received: 24 June 2019 Accepted: 21 September 2019 Published online: 01 October 2019

References

1. Lazer, A.C., McKenna, P.J.: Large-amplitude periodic oscillations in suspension bridges: some new connection with nonlinear analysis. SIAM Rev. **32**(4), 537–578 (1990)
2. Ahmed, N.U., Harbi, H.: Mathematical analysis of dynamical models of suspension bridges. SIAM J. Appl. Math. **58**(3), 853–874 (1998)
3. Humphreys, L.D.: Numerical mountain pass solutions of a suspension bridge equation. Nonlinear Anal. **28**(11), 1811–1826 (1997)
4. McKenna, P.J., Walter, W.: Nonlinear oscillation in a suspension bridges. Arch. Ration. Mech. Anal. **98**, 167–177 (1987) Results: Nonlinear Anal. **39**, 731–743 (2000)
5. Zhong, C.K., Ma, Q.Z., Sun, C.Y.: Existence of strong solutions and global attractors for the suspension bridge equations. Nonlinear Anal. **67**, 442–454 (2007)
6. Park, J.Y., Kang, J.R.: Global attractors for the suspension bridge equations with nonlinear damping. Q. Appl. Math. **69**, 465–475 (2011)
7. Park, J.Y., Kang, J.R.: Pullback $\mathcal{D}$-attractors for non-autonomous suspension bridge equations. Nonlinear Anal. **71**, 4618–4623 (2009)
8. Park, J.Y., Kang, J.R.: Uniform attractor for non-autonomous suspension bridge equations with localized damping. Math. Methods Appl. Sci. **34**, 487–496 (2011)
9. Litcanu, G.: A mathematical model of suspension bridges. Appl. Math. **49**(1), 39–55 (2004)
10. Malik, J.: Mathematical modelling of cable-stayed bridge: existence, uniqueness, continuous dependence on data, homogenization of cable systems. Appl. Math. **49**(1), 1–38 (2004)
11. Holubová, G., Matas, A.: Initial-boundary value problem for the nonlinear string-beam system. J. Math. Anal. Appl. **288**, 784–802 (2003)
12. Ma, Q.Z., Zhong, C.K.: Existence of global attractors for the coupled suspension bridge equations. J. Math. Anal. Appl. **308**, 365–379 (2005)
13. Ma, Q.Z., Zhong, C.K.: Existence of strong solutions and global attractors for the coupled suspension bridge equations. J. Differ. Equ. **246**, 3755–3775 (2009)
14. Ma, Q.Z., Wang, S.P., Chen, X.B.: Uniform attractors for the coupled suspension bridge equations. Appl. Math. Comput. **217**, 6604–6615 (2011)
15. Ma, Q.Z., Wang, B.L.: Existence of pullback attractors for the coupled suspension bridge equations. Electron. J. Differ. Equ. **2011**, 16 (2011)
16. Kang, J.R.: Pullback attractors for the non-autonomous coupled suspension bridge equations. Appl. Math. Comput. **219**, 8747–8758 (2013)
17. Woinowsky-Krieger, S.: The effect of an axial force on the vibration of hinged bars. J. Appl. Mech. **17**, 35–36 (1950)
18. Bochicchio, I., Giorgi, C., Vuk, E.: Long-term damped dynamics of the extensible suspension bridge. Int. J. Differ. Equ. **2010**, 383420 (2010)
19. Bochicchio, I., Giorgi, C., Vuk, E.: Asymptotic dynamical of nonlinear coupled suspension bridge equations. J. Math. Anal. Appl. **402**, 319–333 (2013)
20. Ahmed, N.U.: A general mathematical framework for stochastic analysis of suspension bridges. Nonlinear Anal., Real World Appl. **1**, 457–483 (2000)
21. Ma, Q.Z., Xu, L.: Random attractors for the extensible suspension bridge equation with white noise. Comput. Math. Appl. **70**, 2895–2903 (2015)
22. Ma, Q.Z., Xu, L.: Random attractors for the coupled suspension bridge equations with white noises. Appl. Math. Comput. **306**, 38–48 (2017)
23. Crauel, H. Flandoli, F.: Attractors for random dynamical systems. Probab. Theory Relat. Fields **100**, 365–393 (1994)
24. Crauel, H., Debussche, A., Flandoli, F.: Random attractors. J. Dyn. Differ. Equ. **9**, 307–314 (1997)
25. Arnold, L.: Random Dynamical Systems. Springer, New York (1998)
26. Fan, X.M.: Random attractor for a damped sine-Gordon equation with white noise. Pac. J. Math. **216**, 63–76 (2004)
27. Fan, X.M., Wang, Y.G.: Fractal dimensional of attractors for a stochastic wave equation with nonlinear damping and white noise. Stoch. Anal. Appl. **25**, 381–396 (2007)
28. Yang, M.H., Kloeden, P.E.: Random attractors for stochastic semi-linear degenerate parabolic equation. Nonlinear Anal., Real World Appl. **12**, 2811–2821 (2011)
29. Yang, M.H., Duan, J.Q., Kloeden, P.E.: Asymptotic behavior of solutions for random wave equation with nonlinear damping and white noise. Nonlinear Anal., Real World Appl. **12**, 464–478 (2011)
30. Ma, W.J., Ma, Q.Z.: Attractors for stochastic strongly damped plate equations with additive noise. Electron. J. Differ. Equ. **2013**, 111 (2013)
31. Pazy, A.: Semigroup of Linear Operators and Applications to Partial Differential Equations. Appl. Math. Sci. Berlin. Springer, New York (1983)